1. Determinacy

We introduce the axiom of determinacy, which provides a powerful strengthening of ZF. Although the full axiom is inconsistent with AC, various restictions of it are, and provide a way to extexd the ZF theory of the reals presented earlier. We begin with the basic terminology. We emphasize that out background theory throughout this discussion is ZF set theory.

Let X be a set. By a *game* on X we mean a setting where two players, called I and II alternate playing elements from the set X as follows:

$$\begin{array}{lcccccccc} \text{I} & x_0 & & x_2 & & x_4 & & \dots & \\ \text{II} & & x_1 & & x_3 & & x_5 & & \dots \end{array}$$

The sequence $\vec{x} = (x_0, x_1, \dots)$ that they jointly produce is called a *run* of the game. The winning condition for the game is given by a *payoff set* $A \subseteq X^\omega$. We say I wins the run of the game if $\vec{x} \in A$, and otherwise II wins the run. We frequently denote the game with payoff set A by G_A.

A *strategy* for player I (respectively player II) is a function σ from the sequences in $X^{<\omega}$ of even length (respectively odd length) into X. If σ is a strategy for I, we say a run $\vec{x}$ is according to σ (or I has followed σ) if for all even $n \in \omega$, $x_n = \sigma(\vec{x} \restriction n)$. The definition for a strategy for II is similar, using "odd" instead of "even." If σ is a strategy for I, and $\vec{z} \in X^\omega$, we let $\sigma(\vec{z}) \in X^\omega$ be the run according to σ where II has made moves $\vec{z} = (z_0, z_1, \dots)$. Thus,

$$\sigma(\vec{z}) = (\sigma(\emptyset), z_0, \sigma(\sigma(\emptyset), z_0), z_1, \dots).$$

We likewise define $\tau(\vec{z})$ when τ is a strategy for II. We let $\sigma_0(\vec{z})$ denote the sequence of moves I makes against II play of $\vec{z}$, and likewise $\tau_1(\vec{z})$ denotes II's moves against I's play of $\vec{z}$.

Exercise 1. Let d be the standard metric on ω^ω, that is, $d(x, y) = \frac{1}{2^n}$ where $n \in \omega$ is least such that $x(n) \neq y(n)$ (and of course $d(x, y) = 0$ if $x = y$). Let σ be a strategy for I or II in an integer game. Show that the map $x \mapsto \sigma(x)$ from ω^ω to ω^ω is one-to-one and Lipschitz continuous, that is, $d(\sigma(x), \sigma(y)) \leq d(x, y)$.

Exercise 2. Show that for any strategy σ for I or II in an intger game that $\sigma[\omega^\omega] \doteq \{\sigma(x) \colon x \in \omega^\omega\}$ perfect. Show that $\sigma_0[\omega^\omega]$ is $\boldsymbol{\Sigma}^1_1$. Give an example to show $\sigma_0[\omega^\omega]$ is not necessarily Borel [hint: use the fact that every $\boldsymbol{\Sigma}^1_1$ set A in ω^ω is the projection of a closed set in $\omega^\omega \times \omega^\omega$. It may help to assume that A is dense in ω^ω.]

We say a strategy for I (respectively II) is a *winning strategy* if for all runs $\vec{x}$ according to σ, $\vec{x} \in A$ (respectively, $\vec{x} \notin A$). We say the game G_A is *determined* if one of the players has a winning strategy.

Exercise 3. Show that for any game G_A on any set X, that it cannot be the case that both players have a winning strategy.

If X cannot be wellordered in ZF (e.g., $X = \mathbb{R}$), then the notion of a winning strategy is too strong. One uses instead the notion of a *quasi-strategy*. A quasi-strategy σ for I is a function with domain the set of $s \in X^{<\omega}$ of even length and such that $\sigma(s)$ is a non-empty subset of X. The definition for II is similar. Thus, a quasi-strategy is like a strategy except that instead of giving a single move for the player, it produces a non-empty set of possible moves. We say $\vec{x} = (x(0), x(1), \dots)$ is a run according to σ if for all even n, $x(n) \in \sigma(x \restriction n)$. Likewise for II. The notion of a winning quasi-strategy is then defined in the obvious manner. We say $A \subseteq X^\omega$ is *quasi-determined* if one of the players has a winning quasi-strategy. We sometimes still use the term "determined" in this case. To illustrate the difference, suppose there is a relation $R \subseteq \omega^\omega \times \omega^\omega$ which has no uniformization (this will be the case, for example, assuming $\mathsf{AD} + V = L(\mathbb{R})$). Consider the game on $X = \mathbb{R}$ where I plays x in the first move, and II plays y in the next move. All moves after this are irrelavant. II wins the run iff $R(x, y)$. Then II has a winning quasi-strategy, namely R itself, but has no winning strategy as a winning strategy for II produces a uniformization for R.

Exercise 4. We say a game on a set X is *finite* if there is an $n \in \omega$ such that the payoff only depends on $\vec{x} \restriction n$. Show that any finite game on any set X is determined. [hint: Say the game is of length n which is even. Then I has a winning (quasi) strategy iff $\exists x_0\, \forall x_1\, \exists x_2\, \cdots \forall x_{n-1}\, (x_0, \dots, x_{n-1}) \in A$. If this fails, consider the negation and pass the negation through the quantifiers.]

A case of particular interest in when $X = \omega$, that is, both players play integers, in which case the run $\vec{x}$ they produce is an elemnt of the Baire space ω^ω. In this case, the game can be identified with a subset $A \subseteq \omega^\omega$ of the Baire space.

Definition 1.1. The axiom of determinacy, AD, is the assertion that for every $A \subseteq \omega^\omega$ the game G_A is determined.

So, AD asserts that every two-player integer game is determined. We will see below that AD contradicts AC. However, restricted form are consistent with AC.

Definition 1.2. Let $\boldsymbol{\Gamma} \subseteq \mathcal{P}(\omega^\omega)$ be any collection of subsetes of ω^ω. By $\det(\boldsymbol{\Gamma})$ we mean the assertion that G_A is determined for all $A \in \boldsymbol{\Gamma}$.

Usually $\boldsymbol{\Gamma}$ will be a poinclass, in which case we interpret $\det(\boldsymbol{\Gamma})$ as meaning that G_A is determined for all $A \in \boldsymbol{\Gamma}$ which are subsets of ω^ω.

The fact that AD contradicts AC does not mean that the full axiom AD loses interest. It simply means we must restrict out attention to an inner model of V in which choice fails. A natural such model, which we discuss later, is the inner model $L(\mathbb{R})$, the smallest inner-model of set theory containg the reals. On the one hand, AD suffices to give a reasonably complete theory for this model, while on the other hand this model contains all the sets of real which are reasonably definable. In particular $L(\mathbb{R})$ contains all the projective sets, and much more.

We next discuss some of the basic facts about games and determinacy. First we show that AD is inconsistent with ZFC. Note that a strategy for an integer game is a subset of $\omega^{<\omega} \times \omega$, and can thus be identified with a real. We frequently implicitly make this identification.

Lemma 1.3. AD *is inconsistent with* ZFC.

Proof. By AC, let $\{x_\alpha\}_{\alpha < 2^\omega}$ enumerate the reals ω^ω (and hence all the strategies for either I or II in integer games). We define inductively sets $A_\alpha, B_\alpha \subseteq \omega^\omega$ which are monotonically increasing (i.e., $\alpha < \beta \to A_\alpha \subseteq A_\beta$) and disjoint at each step (i.e., $A_\alpha \cap B_\alpha = \varnothing$). We also assume inductively that $|A_\alpha|, |B_\alpha| < 2^\omega$. We think of the reals in A_α as being reals we wish to add to the set we are building, and the reals in B_α as those we wish to forbid from being in our set. At stage α, first let $A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta$ and $B_{<\alpha} = \bigcup_{\beta<\alpha} B_\beta$. Note that $A_{<\alpha} \cap B_{<\alpha} = \varnothing$, assuming the above inductive hypotheses. Consider then x_α. If x_α is not a strategy for I or II, let $A_\alpha = A_{<\alpha}$, $B_\alpha = B_{<\alpha}$. Since $x \mapsto \sigma(x)$ is one-to-one, $C \doteq \{z \colon \sigma(z) \in A_{<\alpha} \cup B_{<\alpha}\}$ has size $< 2^\omega$. Let $z \in \omega^\omega - C$. If x_α is a strategy for I, let $A_\alpha = A_{<\alpha}$, $B_\alpha = B_{<\alpha} \cup \{x_\alpha(z)\}$. If x_α is a strategy for II, let $A_\alpha = A_{<\alpha} \cup \{x_\alpha(z)\}$, $B_\alpha = B_{<\alpha}$. Since $z \notin C$, this maintains our disjointness hypothesis. Clearly x_α cannot be a winning strategy for I for any set disjoint from B_α, and cannot be a winning strategy for II for any set containing A_α. Let $A = \bigcup_{\alpha<2^\omega} A_\alpha$. So, $A \cap B = \varnothing$, where $B = \bigcup_{\alpha<2^\omega} B_\alpha$. So, no $x \in \omega^\omega$ can be a winning strategy for either I or II in G_A. □

In view of lemma 1.3, in working with the full axiom of determinacy AD we work in the background theory $\mathsf{ZF}+\mathsf{AD}$. In fact, a weak form of choice, DC, is frequently also added to the backgroung theory; we discuss this in more detail below. Actually, AD implies a weak form of choice, nemrly countable choice for reals. We recall the definitions.

Definition 1.4. For any cardinal κ, AC_κ is the statement that for any κ-sequence $\{A_\alpha\}_{\alpha<\kappa}$ of non-empty sets, there is a function f with domain κ such that $\forall \alpha < \kappa\ (f(\alpha) \in A_\alpha)$. *Countable choice* is the statement AC_ω. Countable choice for reals is the statement AC_ω restricted to sequences $\{A_n\}_{n\in\omega}$ for which $A_n \subseteq \omega^\omega$ for all n.

Lemma 1.5. AD *implies countable choice for reals.*

Proof. Let $\{A_n\}_{n\in\omega}$ be given where each $A_n \subseteq \omega^\omega$ is non-empty. consider the game

I	n				
II		$x(0)$	$x(1)$	$x(2)$	$\dots$

where I plays an integer n and II plays out a real x. II wins the run if $x \in A_n$. Clearly I cannot have a winning strategy σ since as soon as I plays n, can play any $x \in A_n$ to defeat σ. A winning strategy τ for II gives a choice function. □

A small variation of this argument shows the following.

Lemma 1.6. *Assume* AD. *Then countable choice holds in* $L(\mathbb{R})$.

Proof. There is a definable map π in $L(\mathbb{R})$ from $\omega^\omega \times \mathrm{On}$ onto $L(\mathbb{R})$. Given the sequence $\{A_n\}_{n\in\omega}$ of non-empty sets, let $A'_n = \{(x,\alpha) \in \omega^\omega \times \mathrm{On} \colon \pi(x,\alpha) \in A_n\}$. It clearly suffices to get a choice function for the A'_n. For each n, let $\alpha_n \in \mathrm{On}$ be least such that $\exists x\ (x, \alpha_n) \in A'_n$. Let $A''_n = \{x \in \omega^\omega \colon (x,\alpha_n) \in A'_n\}$. From lemma‘1.5, let $\langle x_n\rangle_{n\in\omega}$ be a choice function for the A''_n. Then $\langle \pi(x_n,\alpha_n)\rangle_{n\in\omega}$ is a choice function for the A_n. □

The axiom of dependent choice is a strengthening of countable choice.

Definition 1.7. DC is the following statement.Let X be a set and $R \subseteq X^{<\omega}$ such that for all $s \in R$ there is an $x \in X$ with $s^\frown x \in R$. Then there is a $\vec{x} \in X^\omega$ such that $\forall n\ \vec{x} \restriction n \in R$. We let DC_X be the statement of DC for sets $R \subseteq X^{<\omega}$.

Exercise 5. Show that DC implies countable choice.

In fact, AD implies that DC holds in $L(\mathbb{R})$ (Kechris). This shows that if AD is consistent, then so is $\mathsf{AD}+\mathsf{DC}$. The proof of this theorem requires more techniques.

DC is equivalent to the assertion that every ill founded tree on a set X has a branch. In fact the following exercise shows a bit more.

Exercise 6. Show that DC is equivalent to the assertion that every relation R on a set X which is illfounded has an infinite decreasing sequence. Recall a relation is wellfounded if every non-empty subset of X has an R-minimal element. [hint: If R is illfounded, let $S \subseteq X^{<\omega}$ be the set of sequences $(x_0, x_1, \dots, x_n)$ such that $x_n R x_{n-1} R \dots R x_1 R x_0$ (i.e., the chain is R-decreasing) and $(\{x \colon x R x_n\}, R)$ is illfounded. Apply DC to S.]

If X cannot be wellordered in ZF, then we need DC to even produce a run according to a given quasi-strategy. More precisely, we need DC_X to produce such a run.

We consider next the situation for other X. We let AD_X denote the assertion that every game on X (i.e., $A \subseteq X^\omega$) is determined. So, $\mathsf{AD} = \mathsf{AD}_\omega$. First note the following simple fact.

Exercise 7. Show that if $X \subseteq X'$ then $\mathsf{AD}_{X'} \Rightarrow \mathsf{AD}_X$.

In particular, $\mathsf{AD}_\omega \Rightarrow \mathsf{AD}_2$. The converse is also true by the following lemma.

Lemma 1.8. $\mathsf{AD}_2 \Rightarrow \mathsf{AD}_\omega$.

Proof. Assume AD_2, and let $A \subseteq \omega^\omega$. We show that G_A is determined. We define an $A' \subseteq 2^\omega$ which simulates the game G_A, and such that whichever player has a winning strategy in $G_{A'}$ has one in G_A. Call a finite sequence $s \in 2^{<\omega}$ good if it is an initial segment of a sequence of the form

$$0^{2a_0} {}^\frown (1,0) {}^\frown 0^{2a_1} {}^\frown (0,1) {}^\frown 0^{2a_2} {}^\frown (1,0) {}^\frown 0^{2a_3} {}^\frown (0,1) {}^\frown \cdots .$$

In other words, s is good if it is a partial play of a game on $\{0,1\}$ of the form:

$$\begin{array}{lcccccccc}
\mathrm{I} & \overbrace{0,0,\dots,0}^{a_0} & 1 & 0,0,\dots,0 & 0 & \overbrace{0,0,\dots,0}^{a_2} & 1 & & \dots \\
\mathrm{II} & 0,0,\dots,0 & 0 & \underbrace{0,0,\dots,0}_{a_1} & 1 & 0,0,\dots,0 & 0 & & \dots
\end{array}$$

We now define $G_{A'}$. The first player to play so that the sequence is not good loses. Suppose they produce $x \in 2^\omega$, and all initial segments of x are good. If x is eventually equal to 0, then the last player to play a 1 wins (if they both play all 0's then II wins). Otherwise, consider the sequence $y = (a_0, a_1, \dots) \in \omega^\omega$. Then I wins the run of $G_{A'}$ iff $y \in A$. Suppose σ' is a winning strategy for I in $G_{A'}$ (the case for II is similar). We define a strategy σ for I in G_A as follows. To define $\sigma(\varnothing)$, have I follow σ' in $G_{A'}$ where II plays 0's. I must eventually play a 1 as otherwise I loses by definition. If there have been a_0 rounds of 0's played before the round where I plays a 1, then we let $\sigma(\varnothing) = a_0$.

To define $\sigma(a_0, a_1)$, have II play a_1 more 0's followed by a 1. Following σ', I must must respond with a 0 to all of these moves, as otherwise I loses, contradicting σ' being a winning strategy. Have II then continue to play 0's until I plays a 1, which I must do as otherwise I loses by definition. This defines $a_2 = \sigma(a_0, a_1)$ as the above diagram illustrates. Continuing in this manner defines the strategy σ.

For any run $x = (a_0, a_1, \dots)$ of σ, there is a corresponding run x' of σ' where x' is good and both players have played infinitely many 1's. Furthermore, x' and x correspond as in the diagram above. So, by definition, $x' \in a'$ iff $x \in A$. Since σ' is a winning strategy for I in $G_{A'}$, $x' \in A'$. Thus, $x \in A$. $\square$

Corollary 1.9. AD *is equivalent to* AD_X *for any countable set* X *with at least* 2 *elements.*

Determinacy is not symmetrical between the two players in the sense that I having a winning strategy for a game G_A is not the same as II having a winning strategy for the complement G_{A^c}. However, the asymmetry is minor according to the following lemma.

Lemma 1.10. *If* $\mathbf{\Gamma}$ *is any pointclass then* $\det(\mathbf{\Gamma}) \Leftrightarrow \det(\check{\mathbf{\Gamma}})$. *In fact for any* $A \subseteq \omega^\omega$ *there is a* B *recursively reducible to* $\omega^\omega - A$ *such that I (resp. II) has a winning strategy in* G_B *iff II (resp. I) has a winning strategy in* G_A.

Proof. Given $A \subseteq \omega^\omega$, let $B = \{x \colon x' \in \omega^\omega - A\}$, where $x'(n) = x(n+1)$. If σ is a winning strategy for I in G_A, then σ' is winning for II in G_B, where $\sigma'(a_0, a_1, a_1, \dots, a_{2n}) = \sigma(a_1, \dots, a_{2n})$, that is, II ignores I's first move and then follows σ. Also, if σ' is winning for II in $G_{A'}$, then σ is winning for I in G_A.

Similarly, if τ is winning for II in G_A iff τ' is winning for I in $G_{A'}$ where τ' makes an arbitrary first move, and then follows τ.

Thus, G_A is determined iff B_B is determined. The first statement of the lemma follows. $\square$

Exercise 8. Show assuming ZFC that there are two determined games A and B such that $A \cap B$ is not determined, and likewise for unions. [hint: Let $C \subseteq \omega^\omega$ be a non-determined game. Let $A = \{x \colon x(0) \text{ is even} \vee x' \in C\}$. Let $B = \{x \colon x(0) \text{ is odd} \vee x' \in C\}$. Then I wins both G_A and G_B, but $G_{A \cap B}$ is not determined.

Two natural generalizations of $\mathsf{AD} = \mathsf{AD}_\omega$ suggest themselves. One is allowing X to be a larger ordinal (we must go to at least ω_1 to have something potentially different). Another is to allow the players to play reals (since integers can be viewed as simple reals).

Regarding the ordinal case, we have the following lemma. In the proof, we will borrow from an upcoming fact, namelt that AD implies the perfect set property for all sets of reals.

Lemma 1.11 (ZF)**.** *For any uncountable well-ordered set* X*, the axiom* AD_X *is inconsistent.*

Proof. Assuming ZF, it suffices to show that AD_{ω_1} is inconsistent (from exercise 7). Suppose AD_{ω_1}. From exercise 7 we also have AD. Consider the following game on ω_1:

$$\begin{array}{llllll} \mathrm{I} & \alpha & & & \\ \mathrm{II} & x(0) & x(1) & x(2) & \cdots \end{array}$$

where I plays an ordinal $\alpha < \omega_1$, and plays integers $x(0, x(1), \dots,$ therby playing out a real $x \in \omega^\omega$. II wins iff $x \in \mathrm{WO}$ and $|x| = \alpha$. First note that I cannot have a winning strategy σ, since if σ calls for I to play α, II can defeat σ by playing any $x \in \mathrm{WO}$ with $|x| = \alpha$. So, since we are assuming this game is determined, II must have a winning strategy τ. Let $A = \{\tau(\alpha) \colon \alpha < \omega_1\}$. Then A is a wellordered subsete of WO os size ω_1 (for each $\alpha < \omega_1$ there is a unique $x \in A$ with $|x| = \alpha$). We will show in theorem 2.2 below that under AD, every uncountable subset of ω^ω contains a perfect set. So, let $P \subseteq A$ be perfect. In particular A is closed in ω^ω. But every $\boldsymbol{\Sigma}^1_1$ subset of WO codes only boundedly many ordinals since $\phi(x) = |x|$ is a $\boldsymbol{\Pi}^1_1$-norm on WO. $\square$

Although the full axiom AD_κ for uncountable κ is inconsistent, nevertheless ordinal games are very useful in determinacy theory. We will see later that important classes of ordinal games are determined.

When $X = \mathbb{R}$, the axiom AD_X becomes a powerful strengthening of AD. $\mathsf{AD}_\mathbb{R}$, as we will see later, is strictly stronger than AD. Its extra strength, however, does not generally become apparent until one goes beyound the model $L(\mathbb{R})$, and so will not directly concern us for a while.

The next result is one of the most basic results in determinacy theory and is used in many arguments. For X a set, we topologixe X^ω by giving X the discrete topology and X^ω the corresponding product topology. So, a basic open set is of the form $N_s = \{\vec{x} \in X^\omega \colon (\vec{x} \restriction \mathrm{lh}(s)) = s\}$ where $s \in X^{<\omega}$.

Theorem 1.12 (Gale-Stewart)**.** *Assume* ZF. *Then for any set X, and any open (or closed) $A \subseteq X^\omega$, the game G_A is (quasi) determined.*

Proof. Let E be the set of all $s = (s(0), \dots, s(2n-1))$ of even length. We define a subset $W \subseteq E$ of *winning posirions* for I. Let $W_0 = \{s \in e \colon N_S \subseteq A\}$. In general, for $\alpha \in \mathrm{On}$, first let $W_{<\alpha} = \bigcup_{\beta<\alpha} W_\beta$. Then set

$$W_\alpha = W_{<\alpha} \cup \{s \in E \colon \exists x \in X\ \forall y \in X\ (s^\frown x^\frown y \in W_{<\alpha})\}.$$

Let θ be the least ordinal such that $W_\theta = W_{<\theta}$. So, $W_\alpha = W_\theta$ for all $\alpha \geq \theta$. Let $W = W_\theta$. For $s \in W$, let $|s|$ be the least ordinal $\alpha < \theta$ such that $s \in W_\alpha$. Note that if $s \in W$ and $|s| > 0$, then $\exists x\ \forall y\ (|s^\frown x^\frown y| < |s|)$.

Suppose first that $\varnothing \in W$. Then I has a winning strategy σ in G_A. Namely, let $(s(0), s(1), \dots, s(2n-1)) \in \Sigma$ iff for all $i < n$ either $s \restriction (2i) \in W_0$ or $|s \restriction 2i| > |s \restriction (2i+2)|$. From the fact mentioned at the end of the previous paragraph, σ is a quasi-strategy for I. From the wellfoundedness of On it follows that I is a winning quasi-strategy. that is, for any $\vec{s}$ a run according to Σ, we have the ranks $|s \restriction 2i|$ decrease until we reach an i such that $s \restriction 2i \in W_0$, and thus $\vec{s} \in A$.

Suppose next that $\varnothing \notin W$. We claim that II has a winning quasi-strategy Σ in G_A. Namely, let $\Sigma = E - W$. Note that $\varnothing \in \Sigma$. Also, if $s \in \Sigma$ then $\forall x\ \exists y\ (s^\frown x^\frown y \in \sigma)$. For if $s \in \sigma$ and $\exists x\ \forall y\ (s^\frown x^\frown y \notin \sigma)$, then $\exists x\ \forall y\ (s^\frown x^\frown y \in W)$. So, $s \in W_{\theta+1} = W_\theta = W$, a contradiction. So, Σ is a quasi-strategy for II. If $\vec{s} \in X^\omega$ is a run according to Σ, then $\vec{\notin} A$. For if $\vec{s} \in A$, then for some i we would have $N_{\vec{s}\restriction 2i} \subseteq A$ as A is open. But then $\vec{s} \restriction 2i \in W_0 \subseteq W$, a contradiction (as $\vec{s} \restriction 2i \in \Sigma = E - W$). $\square$

As an immediate corollary we have that any open or closed game on an ordinal is determined.

The proof of theorem 1.12 produces for any open or closed game G_A a *canonical winning quasi-strategy* for G_A. This does not use any form of choice.

An important and deep theorem of Martin says that every Borel game on any set is determined. We will give this proof later, but for now we just extend theorem 1.12 one step further.

Theorem 1.13 (Wolff). *Assume* ZF. *Every* $\boldsymbol{\Sigma}^0_2$ *game on a set* X *is (quasi) determined.*

Proof. Let $A \subseteq X^\omega$ be $\boldsymbol{\Sigma}^0_2$, say $A = \bigcup_n F_n$ where each F_n is closed in X^ω. Let T_n be the canonical tree on X such that $F_n = [T_n]$. We again define a set W of winning positions for I. Again let E be the set of all $s \in X^{<\omega}$ of even length. Let W_0 be the set of all $s \in E$ such that there is an $n \in \omega$ such that I has a winning quasi-strategy starting at s for the game F_n. For general α, let $W_{<\alpha} = \bigcup_{\beta<\alpha} W_\beta$ and then define W_α to be $W_{<\alpha}$ together with the set of $s \in E$ such that for some i, I has a winning quasi-strategy starting at s for $A^i_{<\alpha}$, where $A^i_{<\alpha}$ is the set of $x \in X^\omega$ such that either $\forall j\ x \restriction 2j \in T_i$ or for the least j such that $x \restriction 2j \notin T_i$ we have $x \restriction 2j \in W_{<\alpha}$. Note that $A^i_{<\alpha}$ is a closed set, so is quasi-determined.

Define θ and $W = W_\theta = W_{<\theta}$ as in theorem 1.12. Also as before define, for $s \in W$, $|s|$ to be the least $\alpha < \theta$ such that $s \in W_\alpha$.

First suppose $\varnothing \in W$, and we define a winning quasi-strategy Σ for I in G_A. We give an informal description of Σ, the underlying formal definition will be apparent. $s = (s(0), s(1), \dots, s(2n-1))$ is according to Σ provided the following holds. Let n_0 be least such that I has a winning quasi-strategy starting at $\varnothing$ in $A^{n_0}_{<\theta}$. I follows the canonical winning quasi-strategy for this closed game until an i_0 is reached (if it is) such that $s \restriction 2i_0 \notin T_{n_0}$. If this happens, then $s \restriction 2i_0 \in W_{<\theta}$. Let $\theta_0 < \theta$ be least such that $s \restriction 2i_0 \in W_{\theta_0}$. Let n_1 be least such that I has a winning quasi-strategy in $A^{n_1}_{<\theta_0}$ starting from $s \restriction 2i_0$. I then follows the canonical winning quasi-strategy for this game until an $i_1 > i_0$ is reached (if it is) such that $s \restriction 2i_1 \notin T_{n_1}$. Thus, $s \restriction 2i_1 \in W_{<\theta_0}$. Let $\theta_1 < \theta_0$ be least such that $s \restriction 2i_1 \in W_{\theta_1}$. Σ continues in this manner. If $\vec{s}$ is a run according to Σ, then for some k we have that $\vec{s}$ is a run starting from $\vec{s} \restriction 2i_k$ which stays in the tree T_{n_k} (as otherwisw we get an infinite decreasing sequence of θ_i). Thus, $\vec{s} \in F_{n_k}$, so $\vec{s}$ is a win for I.

Suppose next that $\varnothing \notin W$. We describe a winning quasi-strategy for II in G_A. Start with $n = 0$. Since $\varnothing \notin W_\theta$, II has a canonical winning strategy for the game $A^0_{<\theta}$ starting from $\varnothing$. II follows this strategy until a least i_0 is reached such that $s \restriction 2i_0 \notin T_0$ and $s \restriction 2i_0 \notin W_{<\theta} = W_\theta$. This must happen as II is winning for $A^0_{<\theta}$. Consider then $n = 1$. Since $s \restriction 2i_0 \notin W_\theta$, II has a winning quasi-strategy for the game $A^1_{<\theta}$ starting from $s \restriction 2i_0$. II follows this canonical strategy until an i_1 is reached such that $s \restriction 2i_1 \notin T_1$ and $s \restriction 2i_1 \notin W_{<\theta} = W_\theta$. This describes Σ. If $\vec{s}$ is a run according to Σ, then clearly $\vec{s} \notin [T_n] = F_n$ for all n, and thus II wins the run. $\square$

2. Regularity Results

In this section we show that AD implies regularity properties for sets of reals, that is, AD eliminates the pathological sets produced from AC.

First we consider the perfect set property. The basic game argument is best illustrated on the Cantor space.

Lemma 2.1 (ZF+AD)**.** *Every $A \subseteq 2^\omega$ is either countable or else contains a perfect subset.*

Proof. Given $A \subseteq 2^\omega$ we consider the following game G^*_A:

I	s_0		s_1		s_2		$\dots$	
II		i_0		i_1		i_2		$\dots$

where I plays sequences $s_k \in 2^{<\omega}$ (possibly empty) and II plays $i_k \in \{0,1\}$. I wins the run iff $x = s_0{}^\frown i_0{}^\frown s_1{}^\frown i_1{}^\frown \cdots$ is in A. If I has a winning strategy σ, then clearly $\sigma[2^\omega] = \{x = s_0{}^\frown i_0{}^\frown s_1{}^\frown i_1{}^\frown \dots : (s_0, i_0, \dots) \text{ is according to } \sigma\}$ is a perfect set contained in A. Suppose then that II has a winning strategy τ. Consider $x \in A$. We say a partial run $(s_0, i_0, \dots, s_n, i_n)$ is *good* for x if it is according to τ and $(s_0{}^\frown i_0{}^\frown \cdots {}^\frown s_n{}^\frown i_n)$ is an intitial segment of x. There must be a maximal good sequence, since if every good sequence had a good extension then there would be a run $(s_0, i_0, \dots)$ according to τ with $x = (s_0{}^\frown i_0{}^\frown \cdots)$. This contradicts τ being winning for II. Let $s = (s_0, i_0, \cdots, s_n, i_n)$ be maximal good for x. Say $s_0{}^\frown i_0{}^\frown \cdots {}^\frown s_n{}^\frown i_n = x \restriction j$. So, for every $t \in 2^{<\omega}$, $(x \restriction j){}^\frown t{}^\frown \tau(s{}^\frown t)$ is incompatible with x. This, however, allows us to compute x from s and τ. Namely, $x(j) = 1 - \tau(s{}^\frown \varnothing)$ (by $s{}^\frown \varnothing$ we mean the partial run consisting of s and the one extra move by I consisting of $s_{n+1} = \varnothing$). Then $x(j+1) = 1 - \tau(s{}^\frown(x(j))$, etc. Since the set of possible s is countable (and τ is a single fixed real), this shows that A is countable. $\square$

As a consequence we have the following.

Theorem 2.2 (ZF + AD)**.** *Let X be a Polish soace. Then every $A \subseteq X$ is either countable or else contains a perfect subset.*

Proof. We may clearly assume X is uncountable, and thus is Borel isomorphic to 2^ω. Let $\pi \colon 2^\omega \to X$ be a Borel isomorphism. If $A \subseteq X$ is uncountable, then so is $B = \pi^{-1}(A)$. Let $P \subseteq B$ be perfect. Then $\pi(P) \subseteq A$ is uncountable and Borel, and so contains a perfect subset. $\square$

We can also prove theorem 2.2 directly on an arbitrary Polish space X by a game argument. We sketch this alternate proof. Let $\mathcal{U}$ be a countable base for X. Consider the game G played as follows.

I	U^0_0, U^0_1		U^1_0, U^1_1		U^2_0, U^2_1		$\dots$	
II		i_0		i_1		i_2		$\dots$

For the first move, I plays a pair $U^0_0, U^0_1 \in \mathcal{U}$ where $U^0_0 \cap U^0_1 = \varnothing$, and $\operatorname{diam}(U^0_0)$, $\operatorname{diam}(U^0_1) < \frac{1}{2^0}$. II plays an $i_0 \in \{0,1\}$ signifying a choice of one of these sets. At the next round, I again plays a pair U^1_0, U^1_1 with $U^1_0 \cap U^1_1 = \varnothing$, $\operatorname{diam}(U^1_0), \operatorname{diam}(U^1_1) < \frac{1}{2^1}$, and $\overline{U^1_0} \subseteq U^0_{i_0}$, $\overline{U^1_1} \subseteq U^0_{i_0}$. II plays $i_1 \in \{0,1\}$ which picks one of these sets. The play continues in this manner, and I wins the run iff $x \in A$ where $x = \bigcap_n U^n_{i_n}$. If I has a winning strategy σ, it is clear that following σ against all possible plays of II produces a perfect subset of A. Suppose II has a winning strategy τ. Say a partial run of the game

$$(U_0^0, U_1^0), i_0, \dots, (U_0^n, U_1^n), i_n$$

is x-good if it is a run according to τ where I has followed the rules above and $x \in U_{i_n}^n$. Since τ is winning for II, there is a maximal x-good seuqence, say $s = (U_0^0, U_1^0), i_0, \dots, (U_0^n, U_1^n), i_n$. So, for any pair (U_0^{n+1}, U_1^{n+1}) satisfying the requirements for I, if $i_{n+1} = \tau(s^\frown(U_0^{n+1}, U_1^{n+1}))$, then $x \notin U_{i_{n+1}}^{n+1}$. Let W be the union of all open sets $U_{i_{n+1}}^{n+1}$ which are attained in this manner. So, $W \subseteq U_{i_n}^n$. Also, $x \notin W$. We claim that $U_{i_n} - W$ is a singleton. To see this, suppose $x \neq y$ are both in $U_{i_n}^n - W$. Let U_0^{n+1}, U_1^{n+1} be disjoint basic open sets of diameters $< \frac{1}{2^{n+1}}$ whose closures are contained in $U_{i_n}^n$, and with $x \in U_0^{n+1}$, $y \in U_1^{n+1}$. Let i_{n+1} be II's response. If $i_{n+1} = 0$, then $x \in W$, and if $i_{n+1} = 1$, then $y \in W$, a contradiction.

The determinacy used in theorem 2.2 was local, that is, to get the perfect set property for a pointclass $\boldsymbol{\Gamma}$ requires only $\det(\boldsymbol{\Gamma})$. In fact, we can improve this a little using an "unfolding argument" for an existential quantifier.

Theorem 2.3. *Let $\boldsymbol{\Gamma}$ be a pointclass and assume $\det(\boldsymbol{\Gamma})$. Then for any Polish space X and every $A \subseteq X$ in $\exists^{\omega^\omega}\boldsymbol{\Gamma}$, A is either countable or else contains a perfect set.*

Proof. We give the proof in the case $X = 2^\omega$, leaving the general case to an exercise. Let $A \subseteq 2^\omega$ be in $\exists^{\omega^\omega}\boldsymbol{\Gamma}$. Let $B \subseteq 2^\omega \times \omega^\omega$ be such that $A(x) \leftrightarrow \exists y\, B(x, y)$. Consider the following "unfolded perfect set game":

$$\begin{array}{lcccccccc}
\text{I} & \overset{y(0)}{s_0} & & \overset{y(1)}{s_1} & & \overset{y(2)}{s_2} & & \dots & \\
\text{II} & & i_0 & & i_1 & & i_2 & & \dots
\end{array}$$

where I plays sequence $s_n \in 2^{<\omega}$ and integers $y(n) \in \omega$, and II plays $i_n \in \{0, 1\}$. Let $x = s_0{}^\frown i_0{}^\frown s_1{}^\frown i_1{}^\frown \cdots$, and $y = (y(0), y(1), \dots)$. Then I wins the run iff $(x, y) \in B$. The payoff set is in $\boldsymbol{\Gamma}$, and so the game is determined. If I has a winning strategy, then we clearly get a perfect set contained in A ($\{(\sigma(z))_0 \colon z \in 2^\omega\}$ is a perfect set conained in A). Suppose that II has a winning strategy τ. Fix for the moment $x \in A$, and fix $y \in \omega^\omega$ such that $(x, y) \in B$. We say a partial run $\vec{s}$ according to τ is x-good if it is of the form $\vec{s} = ((y(0), s_0), i_0, \dots, (y(n), s_n), i_n)$ where $s_0{}^\frown i_0{}^\frown \dots {}^\frown s_n{}^\frown i_n = x \restriction j$ is an initial segment of x. Since τ is winning for II, there is a maximal x-good sequence. Say $\vec{s} = ((y(0), s_0), i_0, \dots, (y(n), s_n), i_n)$ is maximal good. Thus, if I plays $(y(n+1), s_{n+1})$ as the next move, where $s_{n+1} = (x(j), \dots, x(\ell - 1))$, and τ responds with i_{n+1}, then $x(\ell) = 1 - i_{n+1}$. This again gives an algorithm for computing x from $\vec{s}$ and the fixed strategy τ. Since there are only countable many possible $\vec{s}$, this shows A is countable. $\square$

Exercise 9. Give a direct game proof of theorem 2.3 on a general Polish space. [hint: modify the previously given perfect set game on a general X to include the witness y.]

If A is $\boldsymbol{\Sigma}_1^1$, then then there is a closed $B \subseteq X \times \omega^\omega$ such that $A(x) \leftrightarrow \exists y\, B(x, y)$. In this case, the unfolded perfect set game is closed, and so it is determined in ZF.

This gives another proof of the perfect set property for $\mathbf{\Sigma}^1_1$. On the other hand, $\det(\mathbf{\Pi}^1_1)$ suffices to show the perfect set property for $\mathbf{\Sigma}^1_2$. We will investigate the strength of $\det(\mathbf{\Pi}^1_1)$ later. Note that projective determinacy give the perfect set property for all projective sets.

We next consider the Baire property, which is the topological notion of regularity. Recall from definition **??** that a set A in a topological space is meager if it a countable union of nowhere dense sets (equivalently, contained in a countable union of closed nowhere dense sets). Also, A has the Baire property if there is an open set U such that $A \triangle U$ is meager. An easy argument (c.f. lemma **??**) shows every Borel set has the Baire property, and in theorem **??** we showed that every $\mathbf{\Sigma}^1_1$ (or $\mathbf{\Pi}^1_1$) set has the Baire property. This last fact will also follow by a game argument below.

We again first consider the case $X = \omega^\omega$ where the basic ideas are more clear. Let $A \subseteq \omega^\omega$. We consider the game G^{**}_A played as follows:

$$\begin{array}{lcccccccc} \text{I} & s_0 & & s_2 & & s_4 & & \dots & \\ \text{II} & & s_1 & & s_3 & & s_5 & & \dots \end{array}$$

where each player playes $s_i \in \omega^{<\omega}$. II wins the run iff $x \in A$ where $x = s_0{}^\frown s_1{}^\frown \cdots$.

Lemma 2.4. *II has a winning strategy in G^{**}_A iff A is comeager in ω^ω. I has a winning strategy in G^{**}_A iff there is a neighborhood on which $\omega^\omega - A$ is comeager.*

Proof. First suppose A is comeager in ω^ω. Say $A \supseteq \bigcap_n D_n$ where D_n is dense open. Define the strategy τ for II as follows. Let $\tau(s_0, s_1, \dots, s_{2n})$ be the least sequence s_{2n+1} such that $N_s \subseteq D_n$, where $s = s_0{}^\frown s_1{}^\frown \dots {}^\frown s_{2n}$. This exists since D_n is dense open. Clearly τ is a winning strategy for II in G^{**}_A.

Suppose next that II has a winning strategy τ in G^{**}_A. We define a sequence D_n of dense open sets. Say a sequence $s \in \omega^{<\omega}$ is 1-good if it is of the form $s = s_0{}^\frown \tau(s_0)$ for some s_0. In general, say s is n-good if it is of the form $s = s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{2n-1}$ where $(s_0, s_1, \dots, s_{2n-1})$ is a partial run according to τ.

Let M_1 be a maximal set of pairwise incompatible 1-good elements. Let $D_1 = \cup\{N_s \colon s \in M_1\}$. To see that D_1 is dense, consider a basic open set N_t. Then $u = t^\frown \tau(t)$ is 1-good, so there is an $s \in M_1$ compatible with u. So, $N_s \cap N_u \neq \varnothing$, and so $N_s \cap N_t \neq \varnothing$. For each 1-good sequence s, pick a canonical partial run $(s_0, \tau(s_0))$ with $s_0{}^\frown \tau(s_0) = s$.

In general, suppose an antichain M_n of n-good sequences has been defined and $D_n = \cup\{N_s \colon s \in M_n\}$ is dense. Assume also that for each $s \in M_n$ we have defined a canonical partial run $(s_0, s_1, \dots, s_{2n-1})$ according to τ with $s = s_0{}^\frown \cdots {}^\frown s_{2n-1}$, and that for all $m < n$, $(s_0, s_1, \dots, s_{2m-1})$ is the canonical run associated to $s_0{}^\frown \cdots {}^\frown s_{2m-1}$. Let M_{n+1} be maximal subject to being an antichain, every $s \in M_{n+1}$ is $n+1$-good, and every $s \in M_{n+1}$ is of the form $s'^\frown t^\frown \tau(s_0, s_1, \dots, s_{2n-1}, t)$ for some $s' \in M_n$ with associated partial run $(s_0, \dots, s_{2n-1})$. Let $D_{n+1} = \cup\{N_s \colon s \in M_{n+1}\}$. To see that D_{n+1} is dense, consider a basic open set N_t. Let $s' \in M_n$ be compatible with t and let $(s_0, s_1, \dots, s_{2n-1})$ be the partial run associated to s' (so $s' = s_0{}^\frown \cdots {}^\frown s_{2n-1}$). Note that $u = s'^\frown (t - s')^\frown \tau(s_0, s_1, \dots, s_{2n-1}, t - s')$ is $n+1$-good and extends t (where $t - s'$ is the sequence such that $s'^\frown (t - s') = t$)

if $\mathrm{lh}(s') \leq \mathrm{lh}(t)$, and otherwise $t - s' = \varnothing$). By maximality, there is an $s \in M_{n+1}$ such that $s \parallel u$. Then $N_s \cap N_u \neq \varnothing$, and so $N_s \cap N_t \neq \varnothing$.

If $x \in \bigcap_n D_n$, then for each n there is an initial segment $x \restriction i_n$ of x in M_n. $x \restriction i_n$ is of the form $s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{2n-1}$ where $(s_0, s_1, \ldots, s_{2n-1})$ is the canonical partial run assosiated to $x \restriction i_n$. Note that $(s_0, \ldots, s_{2n-3})$ is the canonical partial run associated to $x \restriction i_{n-1}$ since M_{n-1} is an atichain. Thus, the s_n give a run according to τ which produces the real x. Hence, $x \in A$.

If there is a neighborhood on which $\omega^\omega - A$ is comeager, let s_0 be such that $\omega^\omega - A$ is comeager on N_{s_0}. Have I play s_0 for the first move, and then follow a strategy for II to get into $\omega^\omega - A$ as in the first part of the proof. Conversely, if I has a winning strategy σ, let $s_0 = \sigma(\varnothing)$. σ then gives a strategy for II in the $**$ game starting from s_0 to get into $\omega^\omega - A$. The first part of the proof shows that $\omega^\omega - A$ is comeager on N_{s_0}. □

Lemma 2.4 says that every A is either comeager or else there is a basic open set U on which it is meager. Applying this to $\omega^\omega - A$ gives that every A is either meager or else there is a basic open set U on which it is comeager (i.e., $U - A$ is meager).

As a corollary we have the following.

Theorem 2.5 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Every $A \subseteq \omega^\omega$ has the Baire property.*

Proof. Let $A \subseteq \omega^\omega$. Let $\{U_i\}_{i \in \omega}$ be a maximal, pairwise disjoint collection of basic open sets such that A is comeager of U_i (i.e., $U_i - A$ is meager). By countable additivity, A is comeager on $U = \bigcup_i U_i$. It suffices to show that $A - U$ is meager. If $A - U$ is not meager, then by lemma 2.4, there is a basic open set V on which $A - U$ is comeager. By maximality, $V \cap U_i \neq \varnothing$ for some i. Let $W \subseteq V \cap U_i$ be basic open. Then $\omega^\omega - U$is comeager on W, a contradiction since $U_i \subseteq U$ is nonmeager. □

We can also define an analog of the $**$ game on a general Polish space X. In fact, it makes sense to define this game on a general topological space. We assume for this discussion that X has a wellorderable base $\mathcal{U}$ (which, of course, holds if X is second countable). Given $A \subseteq X$, X a topological space, the general game G^{**}_A is played as follows.

I	U_0		U_2		U_4		$\ldots$	
II		U_1		U_3		U_5		$\ldots$

where the U_i are basic open sets and $U_i \subseteq U_{i-1}$ (the first player to violate this rule loses). II wins the run iff $\bigcap_n U_n \subseteq A$. If A is comeager, then easily II has a winning strategy as before. Assume now II has a winning strategy τ in G^{**}_A. Let M_1 be a maximal pairwise disjoint collection of basic open sets U_1 such that for some U_0, (U_0, U_1) is a partial run of τ. As before, $\cup M_1$ is dense in X. For each $U_1 \in M_1$, let $s_1(U_1) = (U_0, U_1)$ be a partial run of τ ending with U_1. We can do this since $\mathcal{U}$ is wellordered. In general, assume M_{2n-1} has been defined and is a pairwise disjoint collection of open sets U_{2n-1}, and each U_{2n-1} is the last set played in a partial run following τ. In fact, assume for each $U_{2n-1} \in M_{2n-1}$, a canonical run $s_{2n-1}(U_{2n-1}) = (U_0, U_1, \ldots, U_{2n-3}, U_{2n-2}, U_{2n-1})$ according to

τ is given, where each $U_{2j-1} \in M_{2j-1}$ and $s_{2n-1}(U_{2n-1})$ extends $s_{2n-3}(U_{2n-3})$. Assume also that $\cup M_{2n-1}$ is dense in X. Let then $M_{2n+1} \subseteq \mathcal{U}$ be maximal subject to being an antichain and for every $U_{2n+1} \in M_{2n+1}$, there is a partial run $(U_0, U_1, \dots, U_{2n-1}, U_{2n}, U_{2n+1})$ according to τ such that $U_{2n-1} \in M_{2n-1}$ and $s_{2n-1}(U_{2n-1}) = (U_0, \dots, U_{2n-1})$. As before, $\cup M_{2n+1}$ is dense. Using the wellordering of $\mathcal{U}$, we can easily define s_{2n+1} (the sequence of functions $\langle s_n \rangle_{n\in\omega}$ is actually being constructed from the wellordering of $\mathcal{U}$).

Let $D_{2n+1} = \cup M_{2n+1}$. Suppose $x \in \bigcap_n D_{2n+1}$. For each n, let $U_{2n+1} \in M_{2n+1}$ be such that $x \in U_{2n+1}$. Let $s_{2n+1} = s_{2n+1}(U_{2n+1})$. The antichain property of the M_{2i+1} gives that $s_1 \subseteq s_3 \subseteq \cdots \subseteq s_{2n+1}$ (that is, these sequences extend each other). Thus, there is a run of τ where the odd moves are the U_{2n+1}. Since τ is winning for II, $\bigcap_n U_{2n+1} \subseteq A$, and so $x \in A$. Thus, A is comeager.

The same argument essentially works when I has a winning strategy, provided X satisfies a technical assumption $(*)$: there are $\mathcal{U}_n \subseteq \mathcal{U}$, each $\mathcal{U}_n$ a base, such that if $U_n \in \mathcal{U}_n$ for all n, then $\bigcap_n U_n$ contains at nost one point. For example, any metric space staisfies $(*)$ since we can take $\mathcal{U}_n = \{u \in \mathcal{U} \colon \mathrm{diam}(U) < \frac{1}{n}\}$. To see this, suppose I has a winning strategy σ. Let $U_0 = \sigma(\varnothing)$. I's strategy σ gives a strategy τ for II in the game stating from U_0 to produce a sequence $U_0, U_1, \dots$ such that $\bigcap_n U_n \nsubseteq A$. From τ we can easily get a strategy τ' for II which is also winning for II in this game starting from U_0, and with the additional property that for every U_{2n} which τ plays at stage n, there is a $U \in \mathcal{U}_n$ such that $U_n \subseteq U$ [after I plays U_{2n-1}, two picks the least basic open set $V \subseteq U_{2n-1}$ with $V \in \mathcal{U}_n$ and follows τ against I's last move of V.] Define the sets D_{2n} as before using τ', so the D_{2n} are open and dense in U_0. If $x \in \bigcap_n D_{2n}$, then as before there is a run $(U_0, U_1, \dots)$ following τ' such that $\forall n\ (x \in U_{2n})$ and $\bigcap_n U_{2n} \nsubseteq A$. From $(*)$ we have $\bigcap_n U_{2n} = \{x\}$. Thus, $x \notin A$.

So, for X satisfying $(*)$, and $A \subseteq X$, II has a winning strategy iff A is comeager, and I has a winning strategy iff A is meager on some non-empty open U_0. If X is second countable, then the game G_A^{**} is essentially an integer game, and thus determined. Thus, for every $A \subseteq X$, either A is comeager or else meager in some open set U. Equivalently, every $A \subseteq X$ is either meager or else comeager in some open U. This gives the following.

Theorem 2.6 ($\mathsf{ZF}+\mathsf{AD}$)**.** *Every subset A of a second countable space X satisfying $(*)$ has the Baire property.*

The proof is exactly as before (theorem 2.5), using the following exercise.

Exercise 10. Assume $\mathsf{ZF}+\mathsf{AD}$, and let X be second countable. Show that a countable union of meager sets is meager. [hint: Let A_n be meager. A sequence of closed, nowhere sets whose union is A_n can be coded by a real. This reduces the choice needed to countable choice for reals, which follows from AD.]

Exercise 11. Assume $\mathsf{ZF}+\mathsf{AC}_\kappa$. Let $\{U_\alpha\}_{\alpha<\kappa}$ be a pairwise disjoint collection of non-empty open sets in a topological space X. Suppose $\{A_\alpha\}_{\alpha<\kappa}$ are such that each $A_\alpha \subseteq U_\alpha$ is meager. Show that $\bigcup_{\alpha<\kappa} A_\alpha$ is meager. [hint: use AC_κ to write each A_α as a union $A_\alpha = \bigcup_n F_n^\alpha$, where F_n^α is closed nowhere dense in X. Show that $\bigcup_\alpha F_n^\alpha$ is nowhere dense for each n.]

Recall a function $f \colon X \to Y$ is said to be Baire measurable if for every open $U \subseteq Y$, $f^{-1}(U)$ has the Baire property in X.

Lemma 2.7. *Let X, Y be Polish and $F\colon X \to Y$ have the Baire property. Then there is a comeager $A \subseteq X$ such that $f \restriction A$ is continuous.*

Proof. Let $\{V_i\}_{i\in\omega}$ be a base for Y. For each i, let $U_i \subseteq X$ be open such that $M_i \doteq U_i \triangle f^{-1}(V_i)$ is meager. Let $M = \bigcup_i M_i$, so M is meager in X, and let $A = X - M$. Then $(f \restriction A)^{-1}(V_i) = U_i \cap A$ and so $f \restriction A$ is continuous. $\square$

As a corollary we have the following theorem.

Theorem 2.8 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let $f\colon X \to Y$, where X, Y are Polish. Then there is a comeager $A \subseteq X$ such that $f \restriction A$ is continuous.*

We next prove a general unfolding theorem for the $**$ game. We do this for the spaces $X = Y = \omega^\omega$, though we can generalize to arbitrary Polish spaces. Let $A \subseteq \omega^\omega$, $B \subseteq \omega^\omega \times \omega^\omega$, and suppose $\forall x \in A\ \exists y \in \omega^\omega\ B(x, y)$ (i.e., $A \subseteq p[B]$). We have the G^{**}_A game defined above in which the player s play $s_0, s_1, \dots$ and II wins iff $x = s_0{}^\frown s_1{}^\frown \cdots \in A$. The *unfolded* version of this game $\bar{G}^{**}_A$ is the following game:

$$\begin{array}{lccccccc}
\mathrm{I} & s_0 & & s_2 & & s_4 & \dots & \\
\mathrm{II} & & \begin{matrix} s_1 \\ y(0) \end{matrix} & & \begin{matrix} s_3 \\ y(1) \end{matrix} & & \begin{matrix} s_5 \\ y(2) \end{matrix} & \dots
\end{array}$$

where $s_i \in \omega^{<\omega}$, $y(i) \in \omega$. II wins the run of $\bar{G}^{**}_A$ iff $(x, y) \in B$.

Theorem 2.9 ($**$-unfolding)**.** *Assume the determinacy of $\overline{G}^{**}_A$. Then Player I wins $\overline{G}^{**}_A$ iff I wins G^{**}_A iff and likewise II wins $\overline{G}^{**}_A$ iff II wins G^{**}_A.*

Proof. To simplify the following argument a bit notationally, we assume $Y = 2^\omega$, that is, $y(i) \in \{0, 1\}$ (the case $Y = \omega^\omega$ is similar).

Since we are assuming the determinacy of $\overline{G}^{**}_A$, it suffices to show that whoever wins this unfolded game also wins the game G^{**}_A. If II wins $\overline{G}^{**}_A$ it is clear that II wins G^{**}_A (just by using the same strategy and ignoring the $y(i)$ moves). So, suppose I has a winning strategy $\overline{\sigma}$ in the unfolded game. We define a strategy σ for I in G^{**}_A. The strategy σ will have the property that for any $x \in \omega^\omega$ which is the result of a play according to σ, for all $y \in 2^\omega$ there is a play according to $\overline{\sigma}$ for which (x, y) is the pair of real produced. This will show that σ is winning for I in G^{**}_A [If x the the result of a run by σ, we must have $x \notin A$ since otherwise we can fix a y with $B(x, y)$ and (x, y) is the result of a run by $\overline{\sigma}$. This will contradict $\overline{\sigma}$ being a winning strategy for I in the unfolded game.].

We define σ as follows. Let $s_0 = \overline{\sigma}(\varnothing)$ be I's first move in G^{**}_A. That is, $\sigma(\varnothing) = \overline{\sigma}(\varnothing)$. Let s_1 be II's first move in G^{**}_A. We define I's response s_2 in two steps by defining $u_0 \subseteq u_1 = s_2$. Let $u_0 = \overline{\sigma}(s_0{}^\frown(s_1, 0))$. Let $u_1 = u_0{}^\frown\overline{\sigma}(s_0, (s_1{}^\frown u_0, 1))$. Note that s_2 has the property that for either value of $y(0) \in \{0, 1\}$, there is an initial segment s of $s_0{}^\frown s_1{}^\frown s_2$ and a partial play according to $\overline{\sigma}$ which produces $(s, y(0))$.

Assume inductively that we have defined I's responses in G^{**}_A up through the moves $s_0, s_1, \dots, s_{2n}$, and II now moves s_{2n+1} in G^{**}_A. Assume inductively that $s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{2n}$ has the property that for every $t = (y(0), \dots, y(n-1)) \in 2^n$, there is an initial segment s_t of $s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{2n}$ and a partial play $p = p(t)$ of the game $\overline{G}^{**}_A$ in which II has played t for the $y(i)$ moves and this play has resulted in the sequences (s,t). We define s_{2n+2} in 2^{n+1} steps, one for each $t \in 2^{n+1}$. We will define $u_0 \subseteq u_1 \subseteq \cdots \subseteq u_{2^{n+1}-1}$ and then let $s_{2n+2} = u_{2^{n+1}-1}$. Let $t_0, t_1, \dots, t_{2^{n+1}-1}$ enumerate 2^{n+1}. Given u_i, let s be the initial segment of $s_0{}^\frown s_1{}^\frown \cdots {}^\frown s_{2n}$ such that there is a partial play p according to $\overline{\sigma}$ which produces $(s, t_{i+1} \restriction n)$. Let $s_0{}^\frown \cdots {}^\frown s_{2n} = s^\frown v$. Then let $u_{i+1} = u_i{}^\frown \overline{\sigma}(p, (v^\frown s_{2n+1}{}^\frown u_i, t_{i+1}(n)))$. Clearly $u_i \subseteq u_{i+1}$ and there is a partial play p' of $\overline{\sigma}$ which extends the partial play p and which produces $(s_0{}^\frown \cdots {}^\frown s_{2n+1}{}^\frown u_{i+1}, t_{i+1})$.

This defines the strategy σ for I in G^{**}_A. Suppose x is a run according to σ. Say $x = s_0{}^\frown s_1{}^\frown \cdots$ where $(s_0, s_1, s_2, \dots)$ is a run of σ. Let $y \in \omega^\omega$ be arbitrary. We define a run $\bar{s}_0, (\bar{s}_1, y(0)), \bar{s}_2, \dots$ according to $\overline{\sigma}$ in which II has played y and also $\bar{s}_0{}^\frown \bar{s}_1{}^\frown \cdots = x$. Suppose $(\bar{s}_0, (\bar{s}_1, y(0)), \dots, \bar{s}_{2n})$ has been defined, which is a partial play according to $\overline{\sigma}$ and $\bar{s}_0{}^\frown \cdots {}^\frown \bar{s}_{2n}$ is an initial segment of $s_0{}^\frown \cdots {}^\frown s_{2n}$. Let $s_0{}^\frown \cdots s_{2n} = \bar{s}_0{}^\frown \cdots {}^\frown \bar{s}_{2n}{}^\frown v$. Let $t_{i+1} = (y(0), \dots, y(n))$. By definition of s_{2n+2} we have that $s_0{}^\frown \cdots {}^\frown s_{2n+2}$ extends

$$\bar{s}_0{}^\frown \cdots {}^\frown \bar{s}_{2n}{}^\frown v^\frown s_{2n+1}{}^\frown u_i{}^\frown \overline{\sigma}(\bar{s}_0, (\bar{s}_1, y(0)), \dots, \bar{s}_{2n}, (v^\frown s_{2n+1}{}^\frown u_i, y(n)))$$

which show we may extend the play by $\overline{\sigma}$ and continue to build up (x, y). $\square$

Exercise 12. Show the version of the unfolding Theorem 2.9 for any Polish space X. In the unfolded game, the players play basic open sets which are decreasing and II also plays the integers $y(i)$ as before.

As a corollary of the unfolding we get the Baire property for $\mathbf{\Sigma}^1_1$ (and $\mathbf{\Sigma}^1_1$).

Theorem 2.10 (ZF). *Every $\mathbf{\Sigma}^1_1$ (or $\mathbf{\Pi}^1_1$) set in a Polish space X has the Baire property.*

Proof. Let $A \subseteq X$ be $\mathbf{\Pi}^1_1$. Let $U = \cup\{N_i \colon N_i - A \text{ is meager }\}$ where $\{N_i\}$ is a base for X. Clearly $U - A$ is meager. We show $A - U$ is also meager. Suppose $A - U$ is non-meager. Note that $A - U$ is still $\mathbf{\Pi}^1_1$. Write $x \in (A-U)^c \leftrightarrow \exists y \in \omega^\omega \ (x,y) \in B$ where $B \subseteq X \times \omega^\omega$ is closed. Using this B we define the unfolded game $G^{**}_B(X)$, which is a closed game for II, hence determined. If II wins this game, then II also wins $G^{**}_{(A-U)^c}$ which gives that $(A-U)^c$ is comeager, a contradiction since we are assuming $A - U$ is non-meager. If I wins the unfolded game, them I also wins $G^{**}_{(A-U)^c}$, and so there is a basic neighborhood V such that $(A-U)$ is comeager on V. By definition of U we have $V \subseteq U$, and so $(A-U) \cap V = \varnothing$, a contradiction. $\square$

Another important consequence of AD is comeager uniformization.

Theorem 2.11 (ZF + AD). *Let X, Y be Polish and $R \subseteq X \times Y$ be such that $\forall x \ \exists y \ R(x,y)$. Then there is a comeager $A \subseteq X$ and a function $f \colon A \to Y$ such that $\forall x \in A \ R(x, f(x))$.*

Proof. We give the proof for the case $X = \omega^\omega$, $Y = 2^\omega$ leaving the general case to an exercise below. So, suppose $R \subseteq \omega^\omega \times 2^\omega$ and $\mathrm{dom}(R) = \omega^\omega$. Consider the "unfolded" variation of the $**$ game.

$$\begin{array}{lccccccc}
\mathrm{I} & s_0 & & s_2 & & s_4 & \dots & \\
\mathrm{II} & & \underset{y(0)}{s_1} & & \underset{y(1)}{s_3} & & \underset{y(2)}{s_5} & \dots
\end{array}$$

where $s_i \in \omega^{<\omega}$ and $y(i) \in \{0,1\}$. Let $x = s_0 {}^\frown s_1 {}^\frown \cdots$, and let $y = (y(0), y(1), \dots)$. Then II wins the run iff $R(x,y)$. Applying Theorem 2.9 to the set ω^ω we see that II has a winning strategy τ in the unfolded game G^{**}_R. We define a sequence of dense open sets D_i as in the proof of lemma 2.4. Let $M_1 \subseteq \omega^{<\omega}$ be a maximal antichain of 1-good sequences, where s is 1-good if it is of the form $s = t {}^\frown \tau_0(t)$ for some t (here τ_0 denotes the sequence part of τ's response). Let $D_1 = \cup\{N_s \colon s \in M_1\}$, so D_1 is dense open as in lemma 2.4. Suppose the maximal antichain M_i has been defined and consists of i-good sequences, where s is i-good if there is a partial run $(s_0, (s_1, y(0)), \dots, s_{2i-1}, (s_{2i-1}, y(i-1)))$ according to τ with $s = s_0 {}^\frown s_1 {}^\frown \dots {}^\frown s_{2i-1}$. Assume we have associated to each $s \in M_i$ a canonical such partial run of τ.

Let M_{i+1} be maximal subject to being an antichain and every for $s \in M_{i+1}$ there is a (unique) $t \in M_i$ and associated partial run $(s_0, (s_1, y(0)), \dots, s_{2i-1}, (s_{2i-1}, y(i-1)))$ (so $t = s_0 {}^\frown \dots {}^\frown s_{2i-1}$) such that $s = t {}^\frown s_{2i} {}^\frown s_{2i+1}$ where

$$(s_0, (s_1, y(0)), \dots, s_{2i-1}, (s_{2i-1}, y(i-1)), s_{2i}, (s_{2i+1}, y(i)))$$

is according to τ (for some $y(0), \dots, y(i)$). As in lemma 2.4, each M_i is a maximal antichain, and so $D_i = \cup\{N_s \colon s \in M_i\}$ is dense. If $x \in \bigcap_i D_i$, then there is a unique run according to τ in which x is the concatenation of the sequences played. Let y be the corresponding real resulting from the integer moves that τ makes in this run. Since τ is winning for II, $R(x,y)$. Note, in fact, that if $A = \bigcap_i D_i$, then the map $x \in A \mapsto y$ is continuous.

$\square$

Corollary 2.12 (ZF+AD)**.** *Let $R \subseteq X \times Y$ with X, Y Polish. Suppose $\forall x\, \exists y\, R(x,y)$. Then there is a comeager $A \subseteq X$ and a continuous function $f \colon A \to Y$ such that $\forall x \in A\ R(x, f(x))$.*

Exercise 13. Prove theorem 2.11 for arbitrary Polish spaces. [hint: Play the game where I and II plau basic open sets U_i with $U_0 \supseteq U_1 \supseteq \cdots$, $\operatorname{diam}(U_i) < \frac{1}{2^i}$, and $\overline{U}_i \subseteq U_{i-1}$. II also plays basic open sets V_i satisfying these same requirements. Let $\{x\} = \bigcap_i U_i$, and $\{y\} = \bigcap_i V_i$. II wins the run iff $R(x,y)$. If II has a winning strategy, get the dense open sets D_i as in the discussion after theorem 2.5. Argue that I cannot have a winning strategy as in the proof of theorem 2.11.

Let us recall the Kuratowski-Ulam theorem, which is the "Fubibni" theorem for category. If X is a topological space we use the abbreviation $\forall^* x \in X\ A(x)$ to abbreviate "for comeager many x in X we have $x \in A$".

Theorem 2.13 (Kuratowski-Ulam)**.** *Let X, Y be topologocal spaces with Y second countable. If $R \subseteq X \times Y$ is comeager (in the product topology on $X \times Y$) then $\forall^* x \in X\ \forall^* y \in Y\ (x,y) \in R$. Furthermore, if A has the Baire property and X, Y are Baire spaces, then the reverse implication holds.*

Proof. Suppose $R \supseteq \bigcap_n D_n$ where $D_n \subseteq X \times Y$ is dense open. We claim that for each dense open $D_n \subseteq X \times Y$, there is a comeager set $A_n \subseteq X$ such that for all $x \in A_n$, $(D_n)_x = \{y \colon (x,y) \in D_n\}$ is dense open in Y. To see this, let $V_i \subseteq Y$ be basic open. Given an open set $U \subseteq X$, the set $U \times V_i$ is open in $X \times Y$, and since D_n is dense open there is an open set $U' \subseteq U$ and an open set $V' \subseteq V_i$ such that $U' \times V' \subseteq D_n$. This shows that A^i_n is dense open in X, where A^i_n is the set of x such that there is a neighborhood U of x and a neighborhood $V \subseteq V_i$ such that $U \times V \subseteq D_n$. So, $A_n = \bigcap_i A^i_n$ is comeager in X, and for all $x \in A_n$ we have that

$(D_n)_x$ is dense (and, of course, open). Let $A = \bigcap_n A_n$, so $A \subseteq X$ is comeager. If $x \in A$ then all sections $(D_n)_x$ are dense open so R_x is comeager.

Assume now R has the Baire poperty and $\forall^* x\, \forall^* y\, R(x, y)$. If R is not comeager, then by the Baire poperty there is a basic open set $U \times V$ on which R^c is comeager. By the first part we have that $\forall^* x \in U\, \forall^* y \in V\, R^c(x, y)$. We can then choose $x \in U$ such that $\forall^* y \in V\, R^c(x, y)$ and also $\forall y\, R(x, y)$ by intersecting two comeager sets. We then get $y \in V$ such that $R^c(x, y)$ and $R(x, y)$, a contradiction. $\square$

Note that from AD and the Kuratowski-Ulam theorem we may switch comeager quantifiers, that is, $\forall^* x\, \forall^* y\, A(x, y)$ holds iff $\forall^* y\, \forall^* x\, A(x, y)$ holds.

From these results on category we now have the following basic fact.

Theorem 2.14 ($\mathsf{ZF} + \mathsf{AD}$)**.** *There does not exist an uncountable wellordered set of reals.*

Proof. Suppose $\{x_\alpha\}_{\alpha<\omega_1} \subseteq \omega^\omega$ was a sequence of distinct reals. Let $P \subseteq \{x_\alpha\}_{\alpha<\omega_1}$ be perfect. Since P is in bijection (in fact, homeomorphic to) to 2^ω, we may assume that we have a wellordering $\prec$ of 2^ω of length ω_1. Consider $\prec$ as a subset of $2^\omega \times 2^\omega$. For every y, $\{x \colon x \prec y\}$ is countable, and therefore is meager. So, $\forall^* y\, \forall^* x\, \neg(x \prec y)$. Since $\prec$ has the Baire property, by Kuratowski-Ulam we have $(2^\omega \times 2^\omega) - \prec$ is comeager, and thus $\forall^* x\, \forall^* y\, \neg(x \prec y)$. This is a contradiction since for any x, $\{y \colon x \prec y\}$ is co-countable and thus comeager. $\square$

Remark 2.15. The proof of theorem 2.14 shows in ZF that if there is a wellordering of the reals, then there is a set without the Baire property. See also remark 2.22.

Exercise 14. Show from AD that there does not exists an ω_1 sequence of distinct closed or open sets in a Polish space. [remark: the same is actually true for any level of the Borel hierarchy; we'll show this later.]

Another application of this argument gives the following important fact, the full aditivity of the category ideal.

Theorem 2.16 ($\mathsf{ZF} + \mathsf{AD}$)**.** *A wellordered union of meager sets is meager.*

Proof. Suppose $\{A_\alpha\}_{\alpha<\theta}$ is given with each A_α meager, but $A = \bigcup_{\alpha<\theta} A_\alpha$ non-meager. Let $\rho \leq \theta$ be the least ordinal such that $\bigcup_{\alpha<\rho} A_\alpha$ is non-meager. Since A has the Baire property, there is a basic open set U in X such that A is comeager on U. Let

$$\prec = \{(x, y) \in U \times U \colon \exists \alpha < \beta\ (x \in A_\alpha - \bigcup_{\alpha'<\alpha} A_{\alpha'} \wedge y \in A_\beta - \bigcup_{\beta'<\beta} A_{\beta'})\}.$$

For all x in U, $\{y \in U \colon x \prec y\}$ is comeager in U. So, for comeager many $y \in U$, $\{x \in U \colon x \prec y\}$ is comeager in U. However, the last set is meager by the minimality of ρ. $\square$

We now discuss the situation concerning measure.

Exercise 15. Let μ be a Borel probability measure on 2^ω. Let A be a μ measurable set. Show that for any $\epsilon > 0$, there are basic open sets N_j (in the usual base for 2^ω) such that $A \subseteq \bigcup_j N_j$ and $|\mu(A) - \sum_j \mu(N_j)| < \epsilon$. [hint: Since μ is regular, it is enough to show this for open A. Write A as a countable disjont union of basic open sets.]

Theorem 2.17. *Let μ be a Borel probability measure on a Polish space X. Then every $A \subseteq X$ is μ-measurable.*

Proof. We first argue that it is enough to prove the theorem in the case $X = 2^\omega$. Let μ be a Borel measure on the Polish space X, and let $\pi \colon X \to 2^\omega$ be a Borel bijection (we assume X is uncountable, or the result is trivial). Let $\nu = \pi(\mu)$. That is, ν is the measure on 2^ω given by $\nu(B) = \mu(\pi^{-1}(B))$. Suppose $A \subseteq X$ is given. Let $B = \pi[A]$. By the 2^ω case of the theorem, there is a Borel set $C \subseteq 2^\omega$ such that $Z \doteq B \triangle C$ has ν meqasure 0. So, there is a Borel set D with $Z \subseteq D$ and $\nu(D) = 0$. Then, $\pi^{-1}(C)$ and $\pi^{-1}(D)$ are Borel sets in X and $\mu(\pi^{-1}(D)) = 0$. Since π is a bijection, $A \triangle \pi^{-1}(C) = \pi^{-1}(D)$. Thus, A is μ-measurable.

Suppose now $A \subseteq 2^\omega$, and μ is a Borel probability measure on 2^ω. Let $\mathcal{M}$ be a maximal collection of Borel sets of positive measure which are pairwise almost disjoint, and almost contained in A ("almost" refers to the measure μ). $\mathcal{M}$ must be countable, say $\mathcal{M} = \{B_i\}_{i<\omega}$. Let $B = \bigcup_i B_i$. So, $\mu(B - A) = 0$ and every Borel subset of $A - B$ has μ measure 0. It suffices to show that $A - B$ has measure 0, as then $A = (A - B) \cup (B - (B - A))$, which is a union of a measure 0 set and a Borel set minus a measure 0 set.

Changing notation, let us assume that $A \subseteq 2^\omega$ and every Borel subset of A has μ measure 0. We must show that A has μ measure 0. Note that in fact every μ-measurable subset of A has μ measure 0.

For every fixed $\epsilon > 0$ we consider the following "covering" game $G_\epsilon(A)$ (due to Harrington).

$$\begin{array}{lcccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots & \\ \mathrm{II} & & a_0 & & a_1 & & a_2 & & \dots \end{array}$$

where I plays $x(i) \in \{0,1\}$ building a real $x \in 2^\omega$, and II plays integers $a_i \in \omega$ which we think of as coding a finite sequence $N^i_0, \dots, N^i_{k_i}$ of basic open sets in 2^ω. II must must play so that $\sum_{j \le k_i} \mu(N^i_j) \le \frac{\epsilon}{4^n}$ (otherwise II loses). If II follows the rules, then II wins the run iff $(x \in A \to x \in \bigcup_i \bigcup_{j \le k_i} N^i_j)$.

Suppose I had a winning strategy σ. Note that $B = \sigma[\omega^\omega] \subseteq A$, and $B \in \mathbf{\Sigma}^1_1$. In particular, B is μ measurable, and so $\mu(B) = 0$. Let $N_1, N_2, \dots$ be a sequence of basic open sets in 2^ω with $A \subseteq \bigcup_j N_j$ and such that $\sum_j \mu(N_j) < \epsilon$. Let a_0 code the first k_0 of the Nj, where k_0 is large enough so that $\sum_{j > k_0} \mu(N_j) < \frac{\epsilon}{4}$. Let a_1 code the sets $N_{k_0+1}, \dots N - k_1$ where k_1 is large enogh so that $\sum_{j > k_1} \mu(N_j) < \frac{\epsilon}{4^2}$. Continue in this manner to define a_i, so that the sum of the $\mu(N_j)$ for j coded by a_i is less than $\frac{\epsilon}{4^i}$. If II plays the a_i, then II wins the run, a contradicion.

Suppose now that τ is a winning strategy for II. For each i, let U_i be the union of all the $N^i_j = N^i_j(s)$ coded by some a_i of the form $\tau(s)$ for $s \in 2^i$. The μ measure of U_i is at most $2^i \cdot \frac{\epsilon}{4^i} = \frac{\epsilon}{2^i}$. Then $U = \bigcup_i U_i$ contains A and $\mu(U) \le \sum_i \frac{\epsilon}{2^i} = 2\epsilon$.

So, for every $\epsilon > 0$, A can be covered by an open set of μ measure less than ϵ. This shows that A has μ measure 0. □

Many of the sets constructed with AC give rise to sets which are non-measurable and without the Baire property. For example, consider an ultrafilter $\mathcal{U}$ on ω. We view $\mathcal{U}$ as a subset of 2^ω by identifying an $A \subseteq \omega$ with its characteristic function $\chi_A \in 2^\omega$. We then have the following.

Theorem 2.18 (ZF)**.** *Suppose $\mathcal{U}$ is a non-principal ultrafilter on ω. Then $\mathcal{U}$ is non-measurable and does not have the Baire property (measurable here refers to the standard Bernoulli measure on 2^ω).*

Proof. Suppose first that $\mathcal{U}$ has the Baire property. Note that for any $x, y \in 2^\omega$, if $\exists n\ \forall m \geq n\ (x(m) = y(m))$, then $x \in \mathcal{U}$ iff $y \in \mathcal{U}$ (by $x \in \mathcal{U}$ we mean $\{n : x(n) = 1\} \in \mathcal{U}$). This is because every co-finite set is in the ultrafilter. If $s \in 2^{<\omega}$ is such that $\mathcal{U}$ is comeager on N_s, then it follows that $\mathcal{U}$ is comeager on every N_t with $\mathrm{lh}(t) = \mathrm{lh}(s)$. This is because the natural bijection between N_s and N_t is a homeomorphism, and so preserves category. Thus, either $\mathcal{U}$ is meager or else comeager. Consider now the map $\pi : 2^\omega \to 2^\omega$ defined by $\pi(x)(n) = 1 - x(n)$. Clearly π is a homeomorphism, and so $\mathcal{U}$ is meager (or comeager) iff $\pi(A)$ is meager (or comeager). However, $x \in \mathcal{U}$ iff $\pi(x) \notin \mathcal{U}$, and so $\mathcal{U}$ is meager (or comeager) iff $2^\omega - \mathcal{U}$ is meager (or comeager), a contradiction.

Suppose next that $\mathcal{U}$ is measurable. If $\mathcal{U}$ has positive measure, then for any $\epsilon > 0$ there is a basic open set N_s such that $\mu(\mathcal{U} \cap N_s) > (1-\epsilon)\mu(N_s)$ (Lebesgue density theorem). Again, the natural bijection between N_s and N_t, where $\mathrm{lh}(s) = \mathrm{lh}(t)$ is measure preserving, and so $\mu(\mathcal{U} \cap N_t) > (1-\epsilon)\mu(N_t)$. It follows that $\mu(\mathcal{U}) \geq 1 - \epsilon$. Since this holds for all $\epsilon > 0$, we have $\mu(\mathcal{U}) = 1$. So, either $\mu(\mathcal{U}) = 0$ or $\mu(\mathcal{U}) = 1$. Consider again the map π, and note that π is also measure preserving. So, $\mu(\mathcal{U}) = 0$ (or $= 1$) iff $\mu(\pi(\mathcal{U})) = 0$ (or $= 1$). Since π flips menbership in $\mathcal{U}$, we have that $\mu(\mathcal{U}) = 0$ (or $= 1$) iff $\mu(2^\omega - \mathcal{U})) = 0$ (or $= 1$), a contradiction. □

As a consequence of this, we have the following.

Theorem 2.19 (AD)**.** *Every ultrafilter on a set X is countably additive.*

Proof. Let $\mathcal{U}$ be an ultrafilter on X. We may assume $\mathcal{U}$ is non-principal since principal untrafilters are arbitrarily additive. Suppose $\mathcal{U}$ is not countable additive, say $A_n \subseteq X$ are such that $A_n \notin \mathcal{U}$ but $A \doteq \bigcup_n A_n \in \mathcal{U}$. We may assume the A_n are increasing by finite additivity of $\mathcal{U}$. Define $f : x \to \omega$ by $f(x) =$ the least n such that $x \in A_n$. Let $\mu = f(\mathcal{U})$, so μ is an ultrafilter on ω. μ is non-principal since each $A_n \notin \mathcal{U}$. From theorem 2.18, μ does not have the Baire property as a subset of 2^ω. This contradicts theorem 2.5. □

Definition 2.20. A *measure* on a set X is a countably aditive ultrafilter on X.

So, assuming AD, every ultrafilter on a set X is a measure. As another excample we have the following.

Lemma 2.21. *If there is a wellordering of the reals, then there is a set without the Baire property, and there is a set which is not measurable.*

Proof. Suppose $\{x_\alpha\}_{\alpha<c}$ is a wellordering of 2^ω. suppose that every subset of 2^ω has the Baire property. Let $\rho \leq c$ be least such that $A_\rho \doteq \{x_\alpha\}_{\alpha<\rho}$ is non-meager. Let $\prec = \{(x_\alpha, x_\beta) : \alpha < \beta < \rho\}$. We have that $\forall y \in 2^\omega\ \forall^* x \in 2^\omega\ ((x, y) \notin \prec)$. Since $\prec$ has the Baire property, by Kuratowski-Ulam we have $\forall^* x\ \forall^* y\ ((x, y) \notin \prec)$. Since A_ρ is non-meager, there is an $x \in A_\rho$ such that $\forall^* y\ ((x, y) \notin \prec)$. Say $x = x_\alpha$, where $\alpha < \rho$. Since $\{x_\beta : \alpha < \beta < \rho\}$ must be non-meager, there is an x_β in this set such that $(x_\alpha, x_\beta) \notin \prec$, contradicting the definition of $\prec$.

The argument for measure is similar, using Fubibi's theorem instead of Kuratowski-Ulam. □

Remark 2.22. Shelah has shown in $\mathsf{ZF} + \mathsf{DC}$ that if there is an uncountable wellordered set of reals, then there is a non-measurable set. The proof of this also shows that if every $\mathbf{\Sigma}^1_3$ is measurable, then $\forall x\ (\omega_1^{L[x]} < \omega_1)$. Thus, the statement that every $\mathbf{\Sigma}^1_3$ set is measurable has the consistency strength of an inaccessible cardinal. On the other hand, Shelah has shown that if ZF is consistent then so is ZF+"every set of reals has the Baire property". So, even the Baire property for all sets of reals doesn't have consistency strength beyond ZF. This is an interesting asymmetry between measure and category.

3. Turing Degrees

If x, y are in ω^ω or 2^ω, then we say x is *Turing* reducible to y, $x \leq_T y$ iff x can be effectively computed from y. This can be made precise in several different ways. For example, one use the notion of computation from an oracle. In this approach, the ordinary notion of a Turing machine is modified to include an auxiliary tape (the "oracle") which is read-only, and which the machine is allowed to scan at any time (in the current position). The action the machine takes at any step is allowed to depend on this scanned value as well as the value on the read-write tape (and the state of the machine as usual). Thus, when we say "x is computed from y," we mean that there is a Turing machine which when started with input n on the read-write tape and y on the read-only tape will terminate with the correct value of $x(n)$.

We identify subsets of ω with elements of 2^ω via their characteristic functions, and we frequently pass between the two points of view. We say an x in ω^ω or 2^ω (ar a subset of ω) is recursive if x is computable from the constant 0 sequence (i.e., x is just outright computable). We say $x \equiv_T y$ if $x \leq_T y$ and $y \leq_T x$. It is clear that $\equiv_T$ is an equivalence relation on the reals and $\leq_T$ is a partial order on the equivalence classes. By a *Turing degree* we mean an equivalence class $[x]_T = \{y \colon y \equiv_T x\}$. Clearly each Turing degree is a countable set of reals. We sometimes write 0 for the Turing equivalence class of the contsnt 0 real, that is, the class of all recursive reals. We write $\mathcal{D}$ for the set of Turing degrees. We frequently use d to denote a Turing degree (i.e., $d = [x]_T$ for some $x \in \omega^\omega$).

By the *cone above* x, we mean the set $\{y \colon x \leq_T y\}$. This is clearly a set of Turing degrees. We have the following fundamental theorem of Martin.

Theorem 3.1 (Martin)**.** *Every set of Turing degrees either contains or omits a cone of degrees.*

Proof. Let $A \subseteq \mathcal{D}$ be a set of degrees. Consider the usual game G_A:

$$\begin{array}{lccccccc} \mathrm{I} & x_0 & & x_2 & & x_4 & \dots & \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & \dots \end{array}$$

So, I wins the run iff $x \in A$. Suppose that I has a winning strategy σ. Let $d \in \mathcal{D}$ and $\sigma \leq_T d$ (i.e., d is in the cone above σ). Consider the run of the game where I follows σ and II plays any x with $d = [x]_T$. The resulting run $\sigma(x)$ is computable from σ and x, and thus computable from x (since $\sigma \leq_T x$). Clearly x is also computable from $\sigma(x)$ ($x = (\sigma(x))_1$). So, $d = [x]_T = [\sigma(x)]_T$. Since σ is winning for I, $\sigma(x) \in A$, and since A is a set of degrees, $d \in A$. So, A contains the cone above σ. A similar argument shows that if II has a winning strategy τ, then $\mathcal{D} - A$ contains the cone above τ. □

In view of theorem‘3.1 we make the following definition.

Definition 3.2. The Martin measure on the set of turing degrees $\mathcal{D}$ is the measure defined by cones, that is, $A \subseteq \mathcal{D}$ has measure one iff A contains a cone of degrees.

Theorem 3.1 says that the Martin measure is in fact a measure (i.e., a countably additive ultrafilter) on $\mathcal{D}$ (recall that from AD every ultrafilter is countably additive, although here it is obvious directly that the Martin measure is countably additive).

We can give an improvement to theorem 3.1. To state this, we make the following definition.

Definition 3.3. A tree T on $\{0, 1\}$ (or on ω) is said to be *pointed* if for any $x \in [T]$, $T \leq_T x$.

Thus, a tree T is pointed if T is computable from any of its branches. We now state our improvement to theorem 3.1.

Theorem 3.4 ($\mathsf{ZF} + \mathsf{AD}$). *Let $A \subseteq 2^\omega$ (or ω^ω). Then there is a perfect pointed tree T on $\{0, 1\}$ (or on ω) such that $[T] \subseteq A$ or $[T] \subseteq 2^\omega - A$.*

Proof. Suppose $A \subseteq 2^\omega$, and consider again the basic game G_A. Suppose I has a winning strategy σ (the case for II is similar). For any $x = (x(0), x(1), \dots) \in 2^\omega$, let $\sigma(x) = (y(0), x(0), y(1), x(1), \dots)$ be the real produced when II plays x and I follows σ. So, for any $x \in 2^\omega$, $\sigma(x) \in A$. consider the set $C \subseteq 2^\omega$ of those x such that for all n, $x(2n) = 1$ iff $\sigma(n) = 1$ (viewing σ as an element of 2^ω). Clearly $\sigma[C] = [T]$ for some perfect tree T. So, $[T] \subseteq A$. If $z \in [T]$, then $((z)_1)_0 = \sigma$, and so $\sigma \leq_T z$. However, T is computable from σ, and so $T \leq_T z$. Thus, T is pointed. □

We note that theorem 3.4 is indeed a strengthening of theorem 3.1. For suppose $A \subseteq \mathcal{D}$, and T is perfect pointed with $[T] \subseteq A$. Suppose $x \geq_T T$. Define a branch y of T as follows. Let $y = s_0{}^\frown x(0){}^\frown s_1{}^\frown x(1)\dots$, where s_0 is the least splitting node of T (i.e., such that $s_0{}^\frown 0$, $s_0{}^\frown 1$ are both in T), s_1 is the least splitting node extending $s_0{}^\frown x(0)$, etc. Clearly y is computable from T an x. Since $x \geq_T T$, we have $y \leq_T x$. On the other hand, x is computable from y and T. Since T is pointed, $T \leq_T y$, and so $x \leq_T y$. Thus, $x \equiv_T y$. Since $y \in [T]$, $y \in A$ and thus $x \in A$ as $A \subseteq \mathcal{D}$. So, A containsa the cone above T.

To give an application of the Martin measure we use the following fact, whose proof uses the coding lemma, which we give later (in fact, the coding lemma provides a much stronger version of the fact).

Fact 3.5 ($\mathsf{ZF} + \mathsf{AD}$). *If there is a map from ω^ω onto an ordinal λ, then there is a map from ω^ω onto $\mathcal{P}(\omega^\omega)$.*

We make the following important definition.

Definition 3.6. Θ is the supremum of lengths of the prewellorderings of ω^ω.

Of course, it makes no difference in the definition of Θ whether we use ω^ω, 2^ω, or any other uncountable polish space (as they are all isomorphic). Easily Θ is a limit ordinal. Equivalently, we can say that Θ is the supremum of the ordinals λ such that there is a surjection from ω^ω onto λ. Assuming AC, where ω^ω has a welldefined cardinality, $\Theta = (2^\omega)^+$ (so $\Theta = \omega_2$ assuming CH). However, the definition is meant primarily in the determinacy context, and provides an AD version of the cardinality of the reals.

Assuming countable choice gives that $\text{cof}(\Theta) > \omega$ according to the following exercise.

Exercise 16. Assume $\mathsf{ZF}+$ countable choice. Show that $\text{cof}(\Theta) > \omega$. [hint: suppose $\{\alpha_n\}_{n\in\omega}$ were a cofinal ω sequence in Θ. By countable choice, let $\preceq_n$ by a prewellordering of ω^ω of length α_n. Glue the $\preceq_n$ together to get a prewellordering of length Θ, a contradiction.]

Theorem 3.7 ($\mathsf{ZF}+\mathsf{AD}$)**.** *Every countably additive filter $\mathcal{F}$ on an ordinal $\lambda < \Theta$ can be extended to a measure on λ.*

Proof. Since $\lambda < \Theta$, from fact 3.5 there is a map $\pi\colon \omega^\omega \xrightarrow{onto} \mathcal{P}(\lambda)$. For every degree d, define

$$A_d = \bigcap\{\pi(x)\colon x \in d \wedge \pi(x) \in \mathcal{F}\}.$$

Let $f(d)$ be the least element of A_d, which makes sense since $\mathcal{F}$ is countably additive. So, $f\colon \mathcal{D} \to \lambda$. Let $\mu = f(\nu)$, where ν is the Martin measure on $\mathcal{D}$. So, μ is a measure on λ. If $F \in \mathcal{F}$, then consider x such that $\pi(x) = F$. If $d \in \mathcal{D}$ with $x \leq_T d$, then $A_d \subseteq F$, and so $f(d) \in F$. So, $\forall^*_\nu d\ (f(d) \in F)$. Thus, $\mu(F) = 1$. so, μ extends the filter $\mathcal{F}$. $\square$

Corollary 3.8. *If $\lambda < \Theta$ and cof$(\lambda) > \omega$, then there is a measure μ on λ such that $\mu([0,\alpha]) = 0$ for all $\alpha < \lambda$.*

Proof. Let $\mathcal{F}$ be the co-bounded filter on λ. Since $\text{cof}(\lambda) > \omega$, $\mathcal{F}$ is countably additive. From theorem 3.7, there is a measure μ on λ extending $\mathcal{F}$. Since μ extends $\mathcal{F}$, every co-bounded set has μ measure 1, and so every bounded in λ set has μ measure 0. $\square$

As another application of the Martin measure, we give the following theorem of Kunen.

Theorem 3.9 ($\mathsf{ZF}+\mathsf{AD}$)**.** *Let $\lambda < \Theta$. Then the set of measures on λ is wellorderable.*

Proof. Again let $\pi\colon \xrightarrow{onto} \mathcal{P}(\lambda)$. Suppose μ_1, μ_2 are measures on λ. For μ a measure on λ and $d \in \mathcal{D}$, again let

$$A^\mu_d = \bigcap\{\pi(x)\colon x \in d \wedge \mu(\pi(x)) = 1\}.$$

Let also $f^\mu(d)$ be the least element of $A^\mu(d)$. Define $\mu_1 \prec \mu_2$ iff $\forall^*_\nu d\ (f^{\mu_1}(d) < f^{\mu_2}(d))$. For any two measure μ_1, μ_2 we have either $\mu_1 \prec \mu_2$ or $\mu_2 \prec \mu_1$. To see this, let $A \subseteq \lambda$ such that (without loss of generality) $\mu_1(A) = 0$, $\mu_2(A) = 1$. Let $\pi(x_0) = A$, $\pi(x_1) = \lambda - A$. If $x_0, x_1 \leq_T d \in \mathcal{D}$, then $f^{\mu_1}(d) \in \lambda - A$ and $f^{\mu_2}(d) \in A$, and thus $f^{\mu_1}(d) \neq f^{\mu_2}(d)$. So, $\forall^*_\nu d\ (f^{\mu_1}(d) \neq f^{\mu_2}(d))$, and it follows that that either $\mu_1 \prec \mu_2$ or $\mu_2 \prec \mu_1$. Easily $\prec$ is transitive and irreflexive. So, $\prec$ is a linearordering of the measures on λ. To see it is wellfounded, suppose $\mu_{n+1} \prec \mu_n$ for all n. Let

$A_n \subseteq \mathcal{D}$ be such that for all $d \in A_n$, $f^{\mu_{n+1}}(d) < f^{\mu_n}(d)$. By countable additivity of ν, $\bigcap_n A_n \neq \varnothing$. Let $d \in \bigcap_n A_n$. Then $f^{\mu_1}(d) > f^{\mu_2}(d) > \cdots$, a contradiction. □

In view of theorem 3.9, we make the following definition.

Definition 3.10 ($\mathsf{ZF}+\mathsf{AD}$)**.** For $\lambda < \Theta$, let $\beta(\lambda)$ denote the cardinality of the set of measures on λ.

We will give estimates for $\beta(\lambda)$ later.

4. Wadge Degrees

We now turn to a discussion of the basic facts concerning Wadge degrees. The objects of study now are not reals but sets of reals. It is most convenient for this discussion to work in the space ω^ω, which we do for the rest of this section. We will frequently write A^c for the complement $\omega^\omega - A$.

Recall that a function f from a metric space (X, d) to a metric space (Y, ρ) is said to be Lipschitz continuous, with Lipschitz constant C, if $\rho(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. In the case $X = Y = \omega^\omega$, and $d = \rho =$ the usual metric: $d(x, y) = \frac{1}{2^n}$, where n is least such that $x(n) \neq y(n)$ (and $d(x, y) = 0$ if $x = y$), to say that f is Lipschitz continuous (with constant 1) simply means that if $x \restriction n = y \restriction n$, then $f(x) \restriction n = f(y) \restriction n$. We simply call such a function Lipschitz continuous. Note that Lipschitz continuous functions are essentially strategies for II in integer games (a strategy for I is also Lipschitz continuous, and in fact slightly better).

Definition 4.1. Lat A, $B \subseteq \omega^\omega$. We say A is Wadge reducible to B, $A \leq_w B$, if there is a continuous function $f \colon \omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$. We say A is Lipschitz reducible to B, $A \leq_\ell B$, if there is a Lipschitz continuous function $f \colon \omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$.

Note that to say $A = f^{-1}(B)$ means that for all x we have $x \in A$ iff $f(x) \in B$. Thus f reduces the question of membership in A to that of membership in B. Trivially, if $A \leq_\ell B$ then $A \leq_w B$. Both $\leq_\ell$ and $\leq_w$ are clearly reflexive and transitive (the composition of two Lipschitz functions is Lipschitz), that is, both are partial orders. Note also that $\mathbf{\Gamma} \subseteq \mathcal{P}(\omega^\omega)$ is a pointclass iff $\mathbf{\Gamma}$ is closed under Wadge reduction.

Note that $A \leq_\ell B$ iff $A^c \leq_\ell B^c$, and likewise for $\leq_w$.

AD gives us direct information about the partial order $\leq_\ell$. The following fundamental lemma, due to Wadge, is known as Wadge's lemma.

Lemma 4.2 ($\mathsf{ZF}+\mathsf{AD}$)**.** *Let $A, B \subseteq \omega^\omega$. Then either $A \leq_\ell B$ or $B \leq_\ell A^c$.*

Proof. Consider the game $G_{A,B}$ where I plays out $x \in \omega^\omega$ and II plays out $y \in \omega^\omega$, and where II wins the run iff ($x \in A \leftrightarrow y \in B$). If II has a winning strategy for $G_{A,B}$, then a winning strategy τ gives a Lipschitz continuous function, which we also call τ, from ω^ω to ω^ω such that $x \in A$ iff $\tau(x) \in B$. Thus, $A \leq_\ell B$. A winning strategy σ for I also gives a (slightly better than) Lipschitz continuous function from ω^ω to ω^ω such that for all x, $x \in B$ iff $\sigma(x) \notin A$. Thus, $B \leq_\ell A^c$. □

We can also define a game $G^w_{A,B}$ corresponding to Wadge reduction. In this game, the players make integer moves but II is also allowed to pass. If II makes a terminal sequence of passes, II loses. Otherwise, the payoff is exactly as in the Lipschitz game $G_{A,B}$. It is easy to see that $A \leq_w B$ iff II has a winning strategy in the game $G_{A,B}$.

Of course, it follows immediately that for any A, B that either $A \leq_w B$ or $B \leq_w A$.

So, $\leq_\ell$ and $\leq_w$ are not linear orders on $\mathcal{P}(\omega^\omega)$, but if we amalgamate sets with their complements, then they are. We make this precise in the next definition.

Definition 4.3. By a *Lipschitz degree* we mean the equivalence class $\{A, A^c\}$ of a set $A \subseteq \omega^\omega$ together with its complement A^c under the relation $\{A, A^c\} \equiv_\ell \{B, B^c\}$ iff A is Lipschitz reducible to either B or B^c, and B is Lipschitz reducible to either A or A^c. Likewise, we define a *Wadge degree* to be the equivalence class of a pair $\{A, A^c\}$ using $\leq_w$ instead of $\leq_\ell$.

Note that $\leq_\ell$ is welldefined on the Lipschitz degrees, and likewise $\leq_w$ is welldefined on the Wadge degrees. That is, $\{A, A^c\} \leq_\ell \{B, B^c\}$ iff $A \leq_\ell B$ or $A \leq_\ell B^c$. We write $[A]_\ell$ and $[A]_w$ for the Lipschitz and Wadge degrees of a set A (that is, $[A]_\ell$ is the equivalence clas of $\{A, A^c\}$). From Wadge's lemma it follows that $\leq_\ell$, and thus also $\leq_w$, is a linear ordering of the Lipschitz (resp. Wadge) degrees. As usual, we write $[A]_\ell < [B]_\ell$ to mean $[A]_\ell \leq [B]_\ell$ and $[B]_\ell \not\leq [A]_\ell$.

We next present the following important result of Martin and Monk which states that the Lipschitz and Wadge degrees are actually wellordered by these orders.

Theorem 4.4 ($\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$)**.** *The Lipschitz degrees are wellordered under $\leq_\ell$. Likewise, the Wadge degrees are wellordered by $\leq_w$.*

Proof. Suppose $[A_0]_\ell > [A_1]_\ell > [A_2]_\ell > \cdots$. For each n, II does not have a strategy in the Lipschitz game $G_{A_n, A_{n+1}}$, and so I has a winning strategy σ_n for this game. So, $\sigma_n(x) \in A_n$ iff $x \notin A_{n+1}$. Likewise, II does not win $G_{A_n, A^c_{n+1}}$, so let τ_n be a winning strategy for I in this game. Thus, $\tau_n(x) \in A_n$ iff $x \in A_{n+1}$.

So, σ_n "flips" membership between A_{n+1} and A_n, and τ_n preserves menbership. It is important for the following argument that all of the σ_n, τ_n are strategies for I. For $z \in 2^\omega$ we fill in the following diagram in such a way that for all n, $x_n = \sigma_n(x_{n+1})$ if $z(n) = 1$ and $x_n = \tau_n(x_{n+1})$ if $z(n) = 0$.

$$\begin{array}{|c|c|c|c|c|c}
A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\
\hline
x_0(0) & x_1(0) & x_2(0) & x_3(0) & x_4(0) & \cdots \\
x_0(1) & x_1(1) & x_2(1) & x_3(1) & x_4(1) & \cdots \\
x_0(2) & x_1(2) & x_2(2) & x_3(2) & x_4(2) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
x_0 & x_1 & x_2 & x_3 & x_4 & \cdots
\end{array}$$

For n such that $z(n) = 1$, let $x_n(0) = \sigma_n(\varnothing)$, and for n such that $z(n) = 0$ let $x_n(0) = \tau_n(\varnothing)$. In a similar manner we fill in all of the $x_n(i)$. Let $x_0(z)$ be the value of x_0 produced as described above using $z \in 2^\omega$. Let $B_0 = \{z \colon x_0(z) \in A_0\}$, and $B_1 = \{z \colon x_0(z) \notin A_0\}$. At least one of B_0, B_1 must be non-meager. Say N_s be a basic open set in 2^ω on which one of these sets is comeager. Say, B_0 is comeager on N_s (the other case is similar). So, B_0 is comeager on $N_{s^\frown 0}$ and on $N_{s^\frown 1}$. Consider the map π from $N_{s^\frown 0}$ to $N_{s^\frown 1}$ obtained by flipping the $lh(s)$ digit.

So, π is a homeomorphism between $N_{s^\frown 0}$ and $N_{s^\frown 1}$. Since B_0 is comeager in $N_{s^\frown 0}$, $\pi(B_0)$ is comeager in $N_{s^\frown 1}$. However, for any z, $x_0(z) \in A_0$ iff $x_0(\pi(z)) \notin A_0$ (as τ_n does not flip membership but σ_n does). Thus, $\pi(B_0 \cap N_{s^\frown 0}) = B_1 \cap N_{s^\frown 1})$. Thus, B_1 is also comeager in $N_{s^\frown 1}$, a contradiction (as B_0, B_1 are disjoint).

The proof also works for Wadge degrees, since if $[A_0]_w >_w [A_1]_w >_w \cdots$, then it is also the case that for each n that $[A_n]_\ell >_\ell [A_{n+1}]_\ell$. □

Variations of the Martin-Monk method are frequently used in the abstract theory of pointclasses.

As an immediate consequence of Wadge's lemma we have the following.

Theorem 4.5 ($\mathsf{ZF} + \mathsf{AD}$). *If $\mathbf{\Gamma}$ is a non-selfdual pointclass, then $\mathbf{\Gamma} \restriction \omega^\omega$ has a universal set.*

Proof. Let $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$. If $B \in \mathbf{\Gamma}$, then we must have $B \leq_\ell A$, as otherwise we would have $A \leq_\ell B^c$ which would give $A \in \check{\mathbf{\Gamma}}$. Define then $U(x,y) \leftrightarrow x(y) \in A$, where we view every real x as coding a Lipschitz continuous map from ω^ω to ω^ω (for example $x(s) = m$, for $s \in \omega^{<\omega}$, iff $x(\langle s \rangle) = m$). The map $(x,y) \mapsto x(y)$ is continuous, and so $U \in \mathbf{\Gamma}$. Since every $B \in \mathbf{\Gamma}$ is Lipschitz reducible to A, it follows that U is universal for $\mathbf{\Gamma} \restriction \omega^\omega$. □

Definition 4.6. If $A \subseteq \omega^\omega$, we define the Wadge rank of A, $o(A)$ to be the rank of $\{A, A^c\}$ in $\leq_w$. Likewise we define the Lipschitz rank $o_\ell(A)$. If $\mathbf{\Gamma}$ is a pointclass, we define $o(\mathbf{\Gamma}) = \sup\{o(A) \colon A \in \mathbf{\Gamma}\}$.

So, if $\mathbf{\Gamma}$ is non-selfdual, then $o(\mathbf{\Gamma}) = o(A)$ for any $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$.

We develop some of the facts concerning the Lipschitz and Wadge hierarchies. Note that every Wadge degree is a union of Lipschitz degrees. The following lemma is another connection.

Definition 4.7. We say $\{A, A^c\}$ is a selfdual Lipschitz (or Wadge) degree if $A \equiv_\ell A^c$, otherwise we say the degree is non-selfdual.

Note that if $A \leq_\ell A^c$ then $A^c \leq_\ell A$, and so $A \equiv_\ell A^c$. The following important lemma is due to Steel.

Lemma 4.8 ($\mathsf{ZF} + \mathsf{AD}$). *$A \equiv_w A^c$ iff $A \equiv_\ell A^c$.*

Proof. Suppose $A \leq_w A^c$. Let $f \colon \omega^\omega \to \omega^\omega$ be continuous such that $x \in A$ iff $f(x) \in A^c$. Suppose $A \not\leq_\ell A^c$. Then I wins the game G_{A,A^c}, say by σ. For each $z \in 2^\omega$, we consider the diagram as in the proof of theorem 4.4, where for n such that $z(n) = 1$ we use σ (that is $x_n = \sigma(x_{n+1})$, and if $z(n) = 0$ we use f (i.e., $x_n = f(x_{n+1})$. Since f is just assumed to be continuous, it is no longer the case that the diagram can always be filled in. However, we claim that for comeager many $z \in 2^\omega$, the diagram can be filled in. This is because the the set of z such that $x_0 \restriction n, \dots, x_n \restriction n$ are defined is dense open in 2^ω. To see this, note that for any $s \in 2^{<\omega}$, there is a t extending s of the form $t = s^\frown 1^m$ such that a sufficiently large initial segment of x_n is defined so that $x_0 \restriction n, \dots, x_n \restriction n$ are all defined. Let $C \subseteq 2^\omega$ be the comeager set of z such that the diagram can be completely filled in. Let $x_i(z)$ denote the corresponding reals produced. Note that σ preserves membership in A and f flips membership.

The argument is now similar to that of theorem 4.4. Let $C_0 = \{z \in C \colon x_0(z) \in A\}$, and $C_1 = \{z \in C \colon x_0(z) \notin A\}$. At least one of the C_i is nonmeager, and so comeager on some neighborhood N_s of 2^ω. So, C_i is comeager in $N_{s^\frown 0}$ and $N_{s^\frown 1}$. Let π be the natural homeomorphism between $N_{s^\frown 0}$ and $N_{s^\frown 1}$. Let $D = N_{s^\frown 0} \cap C_i \cap \pi^{-1}(C)$, so D is comeager in $N_{s^\frown 0}$. Thus, $\pi(D)$ is comeager in $N_{s^\frown 1}$, and $\pi(D) \subseteq C_{1-i}$, a contradiction since C_i is also comeager in $N_{s^\frown 1}$ and C_i, C_{1-i} are disjoint. □

In view of lemma 4.8, we may speak unambiguously of a degree being selfdual or non-selfdual.

Definition 4.9. If $A, B \subseteq \omega^\omega$, their *join* is defined by

$$A \oplus B = \{x \colon (x(0) \text{ is even and } x' \in A) \vee (x(0) \text{ is odd and } x' \in B)\},$$

where $x'(n) = x(n+1)$. Similarly, if $A_n \subseteq \omega^\omega$ are given for each n, their join is

$$\oplus_n A_n = \{x \colon x' \in A_{x(0)}\}.$$

Lemma 4.10 ($\mathsf{ZF} + \mathsf{AD}$)**.** *For any $A, B \subseteq \omega^\omega$, $A, B \leq_\ell A \oplus B$. Furthermore, if $\{C, C^c\} <_\ell A \oplus B$ then $\{C, C^c\} \leq_\ell A$ or $\{C, C^c\} \leq_\ell B$.*

Proof. Clearly $A, B \leq_\ell A \oplus B$. For example, to see that $A \leq_\ell A \oplus B$, have II play first play a 0 in the game $G_{A, A\oplus B}$, and then copy I's moves. Suppose now that $C \leq_\ell A \oplus B$, but $A \oplus B \not\leq_\ell C$. Thus, I wins the game $G_{A\oplus B, C}$ by σ. suppose σ's first move is even (the other case is similar). Then σ gives a winning strategy for II in the game G_{C, A^c}. Thus, $C \leq_\ell A^c$. So, $\{C, C^c\} \leq_\ell \{A, A^c\}$. So, $o(C) \leq \max\{o(A), o(B)\}$. □

Note that lemma 4.10 does not say that $o_\ell(A \oplus B) = \max\{o_\ell(A), o_\ell(B)\}$ (it does say that $o_\ell(A \oplus B) \leq \max\{o_\ell(A), o_\ell(B)\} + 1$). We do get the following.

Lemma 4.11. *If A is non-selfdual, then $A \oplus A^c$ is the least ℓ-degree strictly above the degree of A. Also, $A \oplus A^c \not\leq_w A$, so $A \oplus A^c$ is also the least Wadge degree strictly above A. Furthermore, $A \oplus A^c$ is selfdual.*

Proof. First note that $A \oplus A^c \not\leq_w A$. For if $A \oplus A^c \leq_w A$, then $A^c \leq_w A$. This contradicts A being non-selfdual. To see that II wins $G_{A\oplus A^c, (A\oplus A^c)^c}$, if I plays i for the first move, have II play $i+1$ (to change the parity), and then copy I's moves. Thus, $A \oplus A^c$ is selfdual. So, $A \oplus A^c$ is selfdual and $o(A \oplus A^c) > o(A)$. Finally, suppose $\{B, B^c\} <_\ell A \oplus A^c$. So, I has a winning strategy σ in $G_{A\oplus A^c, B}$. If I's first move is even (the other case is similar), then σ gives a strategy for II in the game G_{B, A^c}, and so $o(B) \leq o(A)$. □

Corollary 4.11 identifies the next Lipschitz (in fact Wadge) degree after $\{A, A^c\}$ when A is non-selfdual. We now identify the next Lipschitz degree when A is selfdual.

Lemma 4.12 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Suppose A is selfdual. Then $A' \doteq \{0^\frown x \colon x \in A\}$ is the next Lipschitz degree above A. Also, A' is selfdual.*

Proof. Clearly $A \leq_\ell A'$. Suppose $A' \leq_\ell \{A, A^c\}$. Since A is selfdual, we have $A' \leq_\ell A^c$. So, II has a winning strategy τ in the game G_{A', A^c}. From the definition of A', it follows that τ give a strategy for I in the game $G_{A,A}$ (make a 0 for the first move, and then follow τ as a strategy for I). II can defeat this, however, by copying. To see that II wins $G_{A'(A')^c}$, if I plays 0, then have II play 0 and then follow a strategy for II in G_{A,A^c}. If I make a first move other than 0, have II play a 0, and then any sequence in A. Thus, A' is selfdual. Finally, supose $\{B, B^c\} <_\ell A'$.

So, I has a winning strategy σ in $G_{A',B}$. σ must make a first move of 0 as otherwise II could defeat σ by playing a real not in B. Ignoring I's first move, σ then gives a winning strategy for II in the game G_{B,A^c}, and so $o(B) \leq o(A)$. $\square$

Exercise 17. Show that if A is selfdual, then $A' \equiv_w A$ (where A' is as in lemma 4.12).

Exercise 18. Show that if A is non-selfdual, then $A' \equiv_\ell A$. [hint: Since A is non-selfdual, I wins the game G_{A,A^c}. This gives a winning strategy for II in the game $G_{A',A}$.]

Lemmas 4.11, 4.12 identify the Lipschitz degrees at successor stages in the Lipschitz hierarchy. We next consider limit stages. First we consider limit stages of cofinality ω.

Lemma 4.13. *Suppose $\{A_n\}_{n\in\omega}$ is given with $o_\ell(A_n) < o_\ell(A_{n+1})$ for all n. Then $o_\ell(\oplus_n A_n) = \sup_n o_\ell(A_n)$. Also, $\oplus_n A_n$ is selfdual. Furthermore, if $A_n \equiv_w A_0$ for all n, then $\oplus_n A_n \equiv_w A_0$.*

Proof. Let $A = \oplus_n A_n$. Clearly $A_n \leq_\ell A$ for all n. To see that A is selfdual, we show that II wins G_{A,A^c}. If I makes first move n, II plays any m such that $o_\ell(A_n) < o_\ell(A_m)$. II then follows any winning strategy for II in G_{A_n,A^c_m}. Suppose $\{B, B^c\} <_\ell A$. Since $A \not\leq_\ell B$, I has a winning strategy σ in $G_{A,B}$. From the definition of join, this gives a winning strategy for II in the game G_{B,A^c_n}, where $n = \sigma(\varnothing)$. Thus, $\{B, B^c\} \leq_\ell \{A_n, A^c_n\}$. Thus, $o_\ell(A) = \sup_n o_\ell(A_n)$.

Finally, suppose $A_n \equiv_w A_0$ for all n. II has a winning strategy in G^w_{A,A_0} defined as follows. If I makes first move n, II passes, and then follows a strategy for II in the game $G^w_{A_n,A_0}$. $\square$

Thus, at limit stages of cofinality ω in the Lipschitz hierarchy, there is a selfdual degree. From lemma 4.13 it follows that after a non-selfdual degree $\{A, A^c\}$, the next ω_1 Lipschitz degrees are all selfdual and all of the same Wadge degree as $\{A, A^c\}$.

Now we consider stages in the Lipschitz hierarchy of uncountable cofinality.

Lemma 4.14. *Suppose cof(α) $> \omega$. Then the A such that $o_\ell(A) = \alpha$ is non-selfdual.*

Proof. Suppose $o(A) = \alpha$, and A is selfdual. For each $n \in \omega$, let $A_n = \{x \colon n^\frown x \in A\}$. Thus, $A = \oplus_n A_n$. Let $\alpha_n = o(A_n)$. If $\alpha_n < \alpha$ for all α, then $o_\ell(A) = \sup_n \alpha_n < \alpha$, a contradiction. So, fix n such that $o_\ell(A_n) = o_\ell(A)$, in particular, A_n is selfdual. Thus, II has a winning strategy in G_{A,A^c_n}. If we fix I's first move as n, and then copy II's moves, we defeat II's winning strategy. $\square$

From the previous lemmas, we now have a comnplete picture of the Lipschitz and Wadge hierarchies.

Theorem 4.15 ($\mathsf{ZF} + \mathsf{AD}$)**.** *The non-selfdual and the selfdual Wadge degrees alternate. At limit stages of countable cofinality there is a selfdual Wadge degree which is the degree of the join of Waadge degrees from a cofinal ω sequence. At limit stages of uncountable cofinality there is a non-selfdual Wadge degree. Every non-selfdual Lipschitz degree is non-selfdual as a Wadge degree. Every selfdual Wadge degree consists of an ω_1 block of selfdual Lipschitz degrees.*

Along the way, we have also obtained some additional information about the degrees. For $A \subseteq \omega^\omega$ and $n \in \omega$, recall $A_n = \{x \colon n^\frown x \in A\}$. Also, for $s \in \omega^{<\omega}$, let $A_s = \{x \colon s^\frown x \in A\}$.

Lemma 4.16. *If A is a selfdual degree, then for every $n \in \omega$, $A_m <_\ell A$. Also, for every $x \in \omega^\omega$ there is an n such that $A_{x \restriction n} <_w A$. If A is non-selfdual, then there is an $x \in \omega^\omega$ such that for all n, $A_{x \restriction n} \equiv_w A$ (thus, this property characterizes the non-selfdual degrees).*

Proof. Clearly $A_s \leq_\ell A$ for any s and any A. If A is selfdual, then we showed in the proof of lemma 4.14 that for every n that $A_n <_\ell A$. So, for any $x \in \omega^\omega$, the sequence $A_{x\restriction 0}$, $A_{x \restriction 1}, \dots$ is strictly decreasing as Lipschitz degrees until we hit a non-selfdual degree $A_{x\restriction n}$ (which we must eventually do by wellfoundedness). We then have that $A_{x \restriction n} <_w A$ as otherwise A is non-selfdual.

Suppose now that A is non-selfdual. If for each n we had $A_n <_w A$, then $A = \oplus_n A_n$ would be selfdual. Namely, II could win $G^w(A, A^c)$ as follows: I plays n, II passes, and then II follows a strategy Wadge reducing A_n to A^c. So, for some n we have $A_n \equiv_w A$, and in particular A_n is also non-selfdual. Repeating the argument gives an x such that for all n, $A_{x\restriction n} \equiv_w A$. □

Exercise 19. Show the property of lemma 4.16 for Wadge degrees directly, that is without mentioning Lipschitz degrees. [hint: suppose A is selfdual, and suppose $x \in \omega^\omega$ were such that for all n, $A_{x\restriction n} \equiv_w A$. For each n, fix winning strategies σ_n, τ_n for II in the Wadge games $G^w(A, A_{x\restriction n})$ and $G^w(A, A^c_{x\restriction n})$. Define a sequence of integers k_n inductively so that for all n, if we set $s_n = x \restriction k_n$ and for $m < n$ let $s_m = (x \restriction k_m)^\frown \rho_{k_m}(s_{m+1})$, then $\mathrm{lh}(s_0) \geq n$ (for all choices of $\rho_{k_m} \in \{\sigma_{k_m}, \tau_{k_m}\}$). For every $z \in 2^\omega$, this defines a filling-in of a diagram producing $x_0, x_1, \dots$ with $x_n \restriction k_n = x \restriction k_n$, and where $x_n = \sigma_{k_n}(x_{n+1})$ if $x(n) = 0$ and $x_n = \tau_{k_n}(x_{n+1})$ if $x(n) = 1$. This gives a contradiction as in theorem 4.4.]

The analysis of Wadge degrees (Theorem 4.15) gives immediately an analysis of the pointclasses.

Theorem 4.17 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let $\mathbf{\Gamma}$ be a pointclass. then one of the following cases holds.*

1. *There is a non-selfdual Wadge degree $[(A, A^c)]_w$ with $\mathbf{\Gamma} = \{B \colon B \leq_w A\}$ (note: this case is symmetrical between A and A^c).*
2. *There is a selfdual Wadge degree $[(A)]_w$ such that $\mathbf{\Gamma} = \{B \colon B \leq_w A\}$.*
3. *There is a non-selfdual Wadge degree $[(A, A^c)]_w$ with $\mathbf{\Gamma} = \{B \colon B \leq_w A\} \cup \{B \colon B \leq_w A^c\}$.*
4. *There is a limit ordinal α such that $\mathbf{\Gamma} = \{B \colon o_w(B) < \alpha\}$.*

In the first case, $\mathbf{\Gamma}$ is non-selfdual, and in the last three cases $\mathbf{\Gamma}$ is selfdual.

Proof. Consider $I = \{\alpha \in \mathrm{On} \colon \exists A \in \mathbf{\Gamma}\ o_w(A) = \alpha\}$. Clearly I is an ordinal (if $A \in \mathbf{\Gamma}$ and $(B, B^c) <_w (A, A^c)$, then $B, B^c \in \mathbf{\Gamma}$ by Wadge). If I is a limit ordinal (that is, there is no largest α in I), then we have that $\mathbf{\Gamma}$ is selfdual (if $A \in \mathbf{\Gamma}$, let $B \in \mathbf{\Gamma}$ with $(A, A^c) <_w (B, B^c)$, then $A^c \leq B$ and so $A^c \in \mathbf{\Gamma}$). Suppose then that there is a largest ordinal α in I. Let $o_w(A) = \alpha$. If A is selfdual, then for every $B \in \mathbf{\Gamma}$ we have $B \leq_w A$ and so $\mathbf{\Gamma} = \{B \colon B \leq_w A\}$. As A is selfdual, so is $\mathbf{\Gamma}$ in this case. If (A, A^c) is a non-selfdual Wadge degree, at least one of A, A^c must be in $\mathbf{\Gamma}$ (as α is in I). If exactly one of these sets, say A, is in $\mathbf{\Gamma}$, then for every

$B \in \mathbf{\Gamma}$ we must have $B \leq_w A$ as otherwise, by Wadge, $A^c \leq_w B$ would be in $\mathbf{\Gamma}$. So, $\mathbf{\Gamma} = \{B \colon B \leq_w A\}$. $\mathbf{\Gamma}$ is clearly non-selfdual in this case. If $\mathbf{\Gamma}$ contains both A and A^c, then $\mathbf{\Gamma} \supseteq \{B \colon B \leq_w A\} \cup \{B \colon B \leq_w A^c\}$. We must have equality here as otherwise α is not the largest element of I. Clearly $\mathbf{\Gamma}$ is selfdual then. □

5. Theory Of Pointclasses

We continue with the AD theory of Wadge degrees, developing an abstract theory of pointclasses. Recall the definitions of the separation and reduction properties.

Definition 5.1. Let $\mathbf{\Gamma}$ be a pointclass. $\mathbf{\Gamma}$ has the *separation* property if for all $A, B \in \mathbf{\Gamma}$ with $A \cap B = \varnothing$, there is a $C \in \mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ with $A \subseteq C$ and $C \cap B = \varnothing$

Note that this definition is symmetrical between A and B. We say the set C (as in the definition) *separates* A from B. We let $\mathrm{sep}(\mathbf{\Gamma})$ denote the statement that $\mathbf{\Gamma}$ has the separation property.

Definition 5.2. Let $\mathbf{\Gamma}$ be a pointclass. $\mathbf{\Gamma}$ has the *reduction* property if for all $A, B \in \mathbf{\Gamma}$ there are $A', B' \in \mathbf{\Gamma}$ with $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \varnothing$, and $A' \cup B' = A \cup B$.

Note that (in the notation of the definition), $A - B \subseteq A'$ and $B - A \subseteq B'$. the points in $A \cap B$ can go in either A' or B' (but only one of them). We let $\mathrm{red}(\mathbf{\Gamma})$ denote the reduction property for $\mathbf{\Gamma}$.

The reduction property is a stronger property in the sense of the following.

Fact 5.3 (ZF)**.** $\mathrm{red}(\mathbf{\Gamma}) \to \mathrm{sep}(\check{\mathbf{\Gamma}})$.

Proof. Asssume $\mathrm{red}(\mathbf{\Gamma})$, and let $A, B \in \check{\mathbf{\Gamma}}$ with $A \cap B = \varnothing$. Let $A_1 = B^c$, $B_1 = A^c$, so $A_1, B_1 \in \mathbf{\Gamma}$ and $A_1 \cup B_1 = \omega^\omega$. □

First we consider the separation property. We have the following general result of Steel and Van-Wesep.

Theorem 5.4 (ZF+AD)**.** *For every non-selfdual pointclass* $\mathbf{\Gamma}$*, exactly one of* $\mathrm{sep}(\mathbf{\Gamma})$ *or* $\mathrm{sep}(\check{\mathbf{\Gamma}})$ *holds.*

Steel showed [**?**] that for every non-selfdual $\mathbf{\Gamma}$ that either $\mathrm{sep}(\mathbf{\Gamma})$ or $\mathrm{sep}(\check{\mathbf{\Gamma}})$ holds. Van-wesep [**?**] showed that both sides of a non-selfdual class cannot have the separation property.

We first prove Steel's theorem, which we state separately in the following.

Theorem 5.5 (Steel)**.** *Assume* ZF + AD*. For every non-selfdual pointclass* $\mathbf{\Gamma}$*, either* $\mathrm{sep}(\mathbf{\Gamma})$ *or* $\mathrm{sep}(\check{\mathbf{\Gamma}})$*.*

Proof. Let $\mathbf{\Gamma}$ be non-selfdual, and let $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$. We say a pair of sets $A, B \subseteq \omega^\omega$ is $\mathbf{\Delta}$-inseparable if there is no $\mathbf{\Delta}$ set C separating them, that is, $A \subseteq C$, $C \cap B = \varnothing$ (this is symmetric in A and B).

Lemma 5.6. *Let* A_0, A_1 *be sets which are* $\mathbf{\Delta}$ *inseparable. Then for any pair* B_0, B_1 *of disjoint sets both of which in* $\mathbf{\Gamma}$ *or both of which are in* $\check{\mathbf{\Gamma}}$*, there is a strategy* σ *for I (i.e., a Lipschitz* $\frac{1}{2}$ *function) such that for all* $x \in \omega^\omega$*,* $(x \in B_0 \to \sigma(x) \in A_0)$ *and* $(x \in B_1 \to \sigma(x) \in A_1)$*.*

Proof. Play the game where I plays out $z \in \omega^\omega$ and II plays out $x \in \omega^\omega$ and I wins the run iff

$$(x \in B_0 \to z \in A_0) \wedge (x \in B_1 \to z \in A_1).$$

If I has a winning strategy σ, then we are done. If II has a winning strategy τ, then note that $\forall z\ (\tau(z) \in B_0 \cup B_1)$. Also, if $z \in A_0$, then $\tau(z) \in B_1$, and if $z \in A_1$ then $\tau(z) \in A_1$. So, $\{z \colon \tau(z) \in B_1\} = \omega^\omega - \{z \colon \tau(z) \in B_0\}$, and so these are both $\boldsymbol{\Delta}$ sets. This gives a separation of A_0 and A_1, a contradiction. □

To prove theorem 5.5, suppose toward a contradiction that $\neg\,\mathrm{sep}(\boldsymbol{\Gamma})$ and $\neg\,\mathrm{sep}(\check{\boldsymbol{\Gamma}})$. Let A_0, A_1 be $\boldsymbol{\Gamma}$ sets which are $\boldsymbol{\Delta}$ inseparable, and let B_0, B_1 be $\check{\boldsymbol{\Gamma}}$ sets which are $\boldsymbol{\Delta}$-inseparable. From the lemma (applied to the $\boldsymbol{\Delta}$-inseparable pair B_0, B_1) there is a continuous function f such that $(x \in A_0 \to f(x) \in B_0)$ and $(x \in A_1 \to f(x) \in B_1)$. Let $C_0 = f^{-1}(B_0)$, $C_1 = f^{-1}(B_1)$. Thus, $C_0, C_1 \in \check{\boldsymbol{\Gamma}}$, $A_0 \subseteq C_0$, $A_1 \subseteq C_1$, and $C_0 \cap C_1 = \varnothing$.

From the lemma (appplied to the $\boldsymbol{\Delta}$-inseparable pair A_0, A_1) there is a strategy σ_0 for I such that $(x \in C_0 \to \sigma_0(x) \in A_1)$, and $(x \in C_1 \to \sigma_0(x) \in A_0)$. There is also a strategy σ_1 for I such that $(x \in A_0 \to \sigma_1(x) \in A_0)$ and $(x \in C_0^c \to \sigma_1(x) \in A_1)$. Finally, there is a strategy σ_2 for I such that $(x \in A_1 \to \sigma_2(x) \in A_1)$, and $(x \in C_1^c \to \sigma_2(x) \in A_0)$. For any $z \in 3^\omega$ there is a filling-in of the diagram to produce reals $x_0, x_1, \dots$ such that $x_n = \rho(x_{n+1})$ where $\rho = \sigma_0$ or σ_1 or σ_2 if $z(n) = 0$ or 1 or 2 respectively. Let $x_0(z)$ be the value of x_0 produced for this particular z. Note that all of the σ_i have the property that if $x \in A_0 \cup A_1$ then $\sigma_i(x) \in A_0 \cup A_1$.

We now get the usual contradiction an in theorem 4.4. Namely, let $E_0 = \{z \colon x_0(z) \in C_0\}$, $E_1 = \{z \colon x_0(z) \in C_1\}$, and $E_2 = \{z \colon x_0(z) \notin C_0 \cup C_1\}$. Let $s \in \omega^{<\omega}$ be such that one of the E_i is comeager on N_s. Suppose E_0 is comeager on N_s. So, E_0 is comeager on the neighborhood determined by $t = s^\frown 0^{k\frown} s$ for any k. By choosing k of the appropriate parity, we have that for almost all z in N_t that $x_0(z) \in A_1 \subseteq C_1$, a contradiction. The argument in the case E_1 is comeager on N_s is identical. Finally, suppose E_2 is comeager on N_s. Then E_2 is also comeager on N_t, for $t = s^\frown 1^\frown s$. However, for almost all z in N_t we have $x_0(z) \in A_0 \cup A_1$ from the definition of σ_1 (we use here the fact that if $x \notin C_0$, then $\sigma_1(x) \in A_1$, and so all further σ_i applied to this point stay in $A_0 \cup A_1$). This is a contradiction and completes the proof of theorem 5.5. □

We now give the other half of theorem 5.4.

Theorem 5.7 (Van Wesep)**.** *Assume* $\mathsf{ZF} + \mathsf{AD}$. *If* $\boldsymbol{\Gamma}$ *is non-selfdual, then* $\mathrm{sep}(\boldsymbol{\Gamma})$, $\mathrm{sep}(\check{\boldsymbol{\Gamma}})$ *cannot both hold.*

Proof. Suppose toward a contradiction that $\mathrm{sep}(\boldsymbol{\Gamma})$, $\mathrm{sep}(\check{\boldsymbol{\Gamma}})$ both hold. Let $A \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$. We again regard every real $x \in \omega^\omega$ as coding a Lipschitz continuous function from ω^ω to ω^ω. Define $A_0(x, y) \leftrightarrow ((x)_0(y) \in A)$ and $A_1(x, y) \leftrightarrow ((x)_1(y) \in A)$. Then $A_0, A_1 \in \boldsymbol{\Gamma}$ and form a $\boldsymbol{\Gamma}$-universal pair. That is, for every pair B_0, B_1 of $\boldsymbol{\Gamma}$ sets, there is an x such that $B_0 = (A_0)_x$, $B_1 = (A_1)_x$ (recall C_x denotes the section $\{y \colon C(x, y)\}$ of the set C). We cannot have that $A_0 - A_1$ and $A_1 - A_0$ can be separated by a $\boldsymbol{\Delta}$ set. This is the same argument that $\mathrm{red}(\boldsymbol{\Gamma}) \to \neg\,\mathrm{sep}(\boldsymbol{\Gamma})$. Here briefly is the argument again. Suppose $C \in \boldsymbol{\Delta}$ and $A_0 - A_1 \subseteq C$, $C \cap (A_1 - A_0) = \varnothing$. Then $C \in \boldsymbol{\Delta}$ and is universal for $\boldsymbol{\Delta}$ sets, a contradiction (since we then define $D(x) \leftrightarrow \neg C(x, x)$, so $D \in \boldsymbol{\Delta}$, and for all x, $D \neq C_x$). Since we are assuming sep($\check{\mathbf{\Gamma}}$), it follows that $A_0 - A_1$ and $A_1 - A_0$ cannot be separated by disjoint $\check{\mathbf{\Gamma}}$ sets (that is, there does not exist disjoint $\check{\mathbf{\Gamma}}$ sets D_1, D_2 with $A_0 - A_1 \subseteq D_0$ and $A_1 - A_0 \subseteq D_1$).

Consider the game G_0 where I plays out x, II plays out y, and II wins the run iff

$$\begin{aligned}&(y \in A_0 \cup A_1)\\ &\wedge ((x \in A_0 - A_1) \to (y \in A_0 - A_1))\\ &\wedge ((x \in A_1 - A_0) \to (y \in A_1 - A_0)).\end{aligned}$$

If II had a winning strategy τ, then $D_0 = \tau^{-1}(A_1^c)$ and $D_1 = \tau^{-1}(A_0^c)$ would be disjoint $\check{\mathbf{\Gamma}}$ sets with $A_0 - A_1 \subseteq D_0$, $A_1 - A_0 \subseteq D_1$, a contradiction. Let σ_0 be a winning strategy for I in G_0. Then $y \in (A_0 - A_1) \to (\sigma_0(y) \in A_1 - A_0)$, $y \in (A_1 - A_0) \to (\sigma_0(y) \in A_0 - A_1)$, and $y \in (A_0 \cup A_1) \to \sigma_0(y) \in ((A_0 - A_1) \cup (A_1 - A_0))$.

Now apply the same argument to to the pair A_0^c, A_1^c, which is a universal pair for $\check{\mathbf{\Gamma}}$. As before, we cannot have that $A_0^c - A_1^c = A_1 - A_0$ and $A_1^c - A_0^c = A_0 - A_1$ can be separated by disjoint $\mathbf{\Gamma}$ sets. We consider the game G_1 where I plays out x, II plays out y, and II wins the run iff

$$\begin{aligned}&(y \in A_0^c \cup A_1^c)\\ &\wedge ((x \in A_0 - A_1) \to (y \in A_1 - A_0))\\ &\wedge ((x \in A_1 - A_0) \to (y \in A_0 - A_1)).\end{aligned}$$

If II had a winning strategy τ, then $D_0 = \tau^{-1}(A_1)$, $D_0 = \tau^{-1}(A_0)$ would be disjoint $\mathbf{\Gamma}$ sets separating $A_0 - A_1$ and $A_1 - A_0$. Let σ_1 be a winning strategy for I in G_1. Then $y \in (A_0 - A_1) \to (\sigma_0(y) \in A_0 - A_1)$, $y \in (A_1 - A_0) \to (\sigma_0(y) \in A_1 - A_0)$, and $y \in (A_0^c \cup A_1^c) \to \sigma_0(y) \in ((A_0 - A_1) \cup (A_1 - A_0))$.

Notice that σ_0 flips menbership between $A_0 - A_1$ and $A_1 - A_0$, while σ_1 preserves menbership. Also, σ_0 maps $A_0 \cup A_1$ into the union of these two sets, while σ_1 maps $(A_0 \cap A_1)^c$ into these two sets.

We get the usual Martin-Monk contradiction. For every $z \in 2^\omega$, consider the filling-in to produce $x_0, x_1, \dots$ where $x_n = \sigma_0(x_{n+1})$ if $z(n) = 0$, and $x_n = \sigma_1(x_{n+1})$ if $z(n) = 1$. Suppose that for nonmeager many z that $x_0(z) \notin (A_0 - A_1) \cup (A_1 - A_0)$. Say for nonmeager many z that $x_0(z) \in (A_0 \cup A_1)^c$ (the case where for nonmeager many z we have $x_0(z) \in A_0 \cap A_1$ is similar). Say $E = \{z \colon x_0(z) \notin A_0 \cup A_1\}$ is comeager on N_s. So, E is also comeager on N_t, where $t = s^\frown 1^\frown s$. But $y \notin (A_0 \cup A_1)$ implies $\sigma_1(y) \in (A_0 - A_1) \cup (A_1 - A_0)$. So, for comeager in N_t many z we have that $x_0(z) \in (A_0 - A_1) \cup (A_1 - A_0)$, a contradiction. So, for comeager many z we have $x_0(z) \in (A_0 - A_1) \cup (A_1 - A_0)$. Suppose without loss of generality that $F = \{z \colon x_0(z) \in A_0 - A_1\}$ is comeager on N_s. Then F is comeager on N_t, where $t = s^\frown 0^{k\frown} s$, and this is a contradiction as σ_0 flips membership between $A_0 - A_1$ and $A_1 - A_0$, provided we take k of the appropriate parity. $\square$

This completes the proof of theorem 5.4.

One use of the separation property is to transfer closure properties from $\mathbf{\Delta}$ to $\mathbf{\Gamma}$.

Theorem 5.8 (Steel). *Assume* $\mathsf{ZF}+\mathsf{AD}$. *Let* $\mathbf{\Gamma}$ *be non-selfdual and assume* sep($\check{\mathbf{\Gamma}}$). *Then:*

(1) *If* $\mathbf{\Delta}$ *is closed under finite (countable) unions, then* $\check{\mathbf{\Gamma}}$ *is closed under finite (resp. countable) unions.*
(2) *If* $\mathbf{\Delta}$ *is closed under* $\exists^{\omega^\omega}$, *then* $\check{\mathbf{\Gamma}}$ *is closed under* $\exists^{\omega^\omega}$.

Proof. To prove (1), let A, $B \in \check{\mathbf{\Gamma}}$ and assume $A \cup B \notin \check{\mathbf{\Gamma}}$. By Wadge, every $\mathbf{\Gamma}$ set is Wadge reducible to $A \cup B$, and thus every $\mathbf{\Gamma}$ set can be written as the union of two $\check{\mathbf{\Gamma}}$ sets. Say A', B' are $\check{\mathbf{\Gamma}}$ sets with $A' \cup B' \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$. By $\mathrm{sep}(\check{\mathbf{\Gamma}})$, let $C \in \mathbf{\Delta}$ and separate A' from $(A' \cup B')^c$, and let $D \in \mathbf{\Delta}$ separate B' from $(A' \cup B')^c$. Then $A' \cup B' = C \cup D \in \mathbf{\Delta}$, a contradiction.

To prove (2), suppose $\exists^{\omega^\omega} \check{\mathbf{\Gamma}} \nsubseteq \check{\mathbf{\Gamma}}$. By Wadge, $\mathbf{\Gamma} \subseteq \exists^{\omega^\omega} \check{\mathbf{\Gamma}}$. We will show $\mathrm{sep}(\mathbf{\Gamma})$, a contradiction. Let A, B be disjoint $\mathbf{\Gamma}$ sets. Let $A(x) \leftrightarrow \exists y\ A'(x,y)$, and $B(x) \leftrightarrow \exists y\ B'(x,y)$, where A', B' are $\check{\mathbf{\Gamma}}$ subseteq of $\omega^\omega \times \omega^\omega$. Define $A''(x,y,z) \leftrightarrow A'(x,y)$ and $B''(x,y,z) \leftrightarrow B'(x,z)$. Clearly A'', B'' are disjoint $\check{\mathbf{\Gamma}}$ sets. Let $A'' \subseteq D \subseteq (B'')^c$, wirh $D \in \mathbf{\Delta}$, from $\mathrm{sep}(\check{\mathbf{\Gamma}})$. Let $E(x) \leftrightarrow \exists y\ \forall z\ D(x,y,z)$. Then $E \in \mathbf{\Delta}$ and $A \subseteq E \subseteq B^c$. [Note that if $x \in B \cap E$ then $\exists z\ \forall y\ B''(x,y,z)$ and $\exists y\ \forall z\ D(x,y,z)$.Fix z, y witnessing these two existential statements. Then $B''(x,y,z)$ and $D(x,y,z)$, a contradiction as B'' and D are disjoint.] $\square$

Remark 5.9. Using the coding lemma one has that if $\exists^{\omega^\omega} \mathbf{\Delta} \subseteq \mathbf{\Delta}$ and $\alpha < \mathrm{cof}(o(\mathbf{\Delta}))$, then $\mathbf{\Delta}$ is closed under α-length unions. The proof of (1) then generalizes (using the coding lemma) to show that $\check{\mathbf{\Gamma}}$ is closed under α-length unions.

An example due to Van Wesep shows that there is a non-selfdual class with neither sider having the reduction property. However, we have the following.

Theorem 5.10 (Steel, Van Wesep)**.** *Assume* $\mathsf{ZF} + \mathsf{AD}$. *Suppose* $\mathbf{\Gamma}$ *is non-selfdual,* $\mathrm{sep}(\check{\mathbf{\Gamma}})$*, and the intersection of two* $\mathbf{\Delta}$ *sets is in* $\mathbf{\Gamma}$*. Then* $\mathrm{red}(\mathbf{\Gamma})$*.*

Proof. The proof of (1) of theorem 5.8 shows that $\check{\mathbf{\Gamma}}$ is closed under finite unions (so $\mathbf{\Gamma}$ is closed under finite intersections). Since we are assuming $\mathrm{sep}(\check{\mathbf{\Gamma}})$, we have $\neg\,\mathrm{sep}(\mathbf{\Gamma})$. Let C, D be a disjoint pair of $\mathbf{\Gamma}$ sets which are not separable by a $\mathbf{\Delta}$ set. Let A, B be a pair of $\mathbf{\Gamma}$ sets. Consider the game where I plays out x, II plays out y, and II wins iff $y \in A \cup B$, $(x \in C \to y \in (A - B))$, and $(x \in D \to y \in (B - A))$. If II had a winning strategy τ, then $\tau^{-1}(B^c)$, $\tau^{-1}(A^c)$ would be disjoint $\check{\mathbf{\Gamma}}$ sets with $C \subseteq \tau^{-1}(B^c)$, $D \subseteq \tau^{-1}(A^c)$. From $\mathrm{sep}(\check{\mathbf{\Gamma}})$ we would then have a $\mathbf{\Delta}$ set separating C from D, a contradiction. Let σ be a winning strategy for I. So, $y \in (A \cup B) \to \sigma(y) \in (C \cup D)$, $(y \in (A - B) \to \sigma(y) \in D)$, and $(y \in (B - A) \to \sigma(y) \in C)$. Then $A' = \{y \in A \colon \sigma(y) \in D\}$ and $B' = \{y \in B \colon \sigma(y) \in C\}$ are in $\mathbf{\Gamma}$ (since $\mathbf{\Gamma}$ is closed under intersections) and reduce A, B. $\square$

Corollary 5.11. *If* $\mathbf{\Gamma}$ *is non-selfdual and* $\mathbf{\Delta}$ *is closed under finite intersections (equivalently, finite unions), then red*$(\mathbf{\Gamma})$ *or* $\mathrm{red}(\check{\mathbf{\Gamma}})$*.*

Corollary 5.12. *If* $\mathbf{\Gamma}$ *is non-selfdual,* $\mathrm{sep}(\check{\mathbf{\Gamma}})$*, and* $\mathbf{\Gamma}$ *is closed under finite intersections, then* $\mathrm{red}(\mathbf{\Gamma})$*.*

In particular, if $\mathbf{\Gamma}$ is non-selfdual and $\mathbf{\Gamma}$ is closed under finite unions and intersections, then $\mathrm{red}(\mathbf{\Gamma})$ or $\mathrm{red}(\check{\mathbf{\Gamma}})$ holds.

6. The Coding Lemma

The coding lemma of Moschovakis is a basic tool in determinacy theory. It provides a choice-like principle which holds assuming AD. We first need to some abstract recursion theoretic or "lightface" notions.

Let $\mathbf{\Gamma}$ denote a non-selfdual pointclass. Recall that we may define a univer set $U \subseteq \omega^\omega \times \omega^\omega$ for the $\mathbf{\Gamma}$ subsets of ω^ω by fixing a set $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$ and then leting

$$U(x,y) \leftrightarrow x(y) \in A,$$

where every $x \in \omega^\omega$ is viewed as giving a Lipschitz continuous function (strategy for II) from ω^ω to ω^ω.

Recall we also have recursive coding and decoding functions on ω^ω which we collectively denote by $(x, y) \mapsto \langle x, y\rangle$, $(x_0, x_1, \dots) \mapsto \langle x_0, x_1, \dots\rangle$, $z \mapsto ((z)_0, (z)_1)$, $z \mapsto ((z)_0, (z)_1, \dots)$, etc. Exactly which of these coding/decoding functions the notation refers to will be clear from the context.

In effective ("lightface") descriptive set theory, the existence of continuous (even recursive) so-called *s-m-n* functions is a basic starting point for much of the theory. We first show that this set-up can be developed for an arbitrary non-selfdual $\boldsymbol{\Gamma}$. This is the content of the next lemma. For the lemma, we only require that $\boldsymbol{\Gamma}$ have a universal set (which under AD holds for all non-selfdual $\boldsymbol{\Gamma}$). By a "product space" we mean the smallest collection of spaces containing ω^ω and closed under finite products, e.g., $X = (\omega^\omega \times \omega^\omega \times \omega^\omega) \times (\omega^\omega \times \omega^\omega)$. The coding and decoding functions extend naturally to these product spaces.

We assume $\boldsymbol{\Gamma}$ is a pointclass for products spaces as defined above.

Lemma 6.1 (*s-m-n* theorem)**.** *Let $\boldsymbol{\Gamma}$ be a pointclass and assume $\boldsymbol{\Gamma}$ has a universal set. Then for every product space X there is a universal set $U_X \subseteq \omega^\omega \times X$ such the following holds:*

(s-m-n property) for every pair of product spaces of the form $X = X_1 \times \cdots \times X_n$, $Y = X_1, \times \cdots, X_n \times \cdots \times X_m$ $(m > n)$ there is a continuous function $s_{X,Y} : \omega^\omega \times X \to \omega^\omega$ such that

$$U_Y(y, x_1, \dots, x_n, \dots, x_m) \leftrightarrow U_{X'}(s_{X,Y}(y, x_1, \dots, x_n), x_{n+1}, \dots, x_m),$$

where $X' = X_{n+1} \times \cdots \times X_m$.

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ be the universal $\boldsymbol{\Gamma}$ set for subseteq of ω^ω as mentioned above. The idea is to define all the U_X by referring to this single set U. For $X = X_1 \times X_2 \times \cdots \times X_n$ we define

$$U_X(y, x_1, \dots, x_n) \leftrightarrow U((y)_0, \langle (s)_1, x_1, \dots, x_n\rangle).$$

Let $Y = X_1 \times \cdots \times X_n \times X_{n+1} \times \cdots \times X_m$. So we also have

$$U_Y(y, x_1, \dots, x_n, \dots, x_m) \leftrightarrow U((y)_0, \langle (y)_1, x_1, \dots, x_n, \dots, x_m\rangle)$$

and

$$U_{X'}(s, x_{n+1}, \dots, x_m) \leftrightarrow U((s)_0, \langle (s)_1, x_{n+1}, \dots, x_m\rangle).$$

So, we take

$$s_{X,Y}(y, x_1, \dots, x_n) = \langle \epsilon, \langle y, x_1, \dots, x_n\rangle\rangle$$

where $\epsilon \in \omega^\omega$ is a fixed real such that

$$U(\epsilon, \langle\langle y, x_1, \dots, x_n\rangle, x_{n+1}, \dots, x_m\rangle) \leftrightarrow U((y)_0, \langle (y)_1, x_1, \dots, x_m\rangle)$$

holds for all $y, x_1, \dots, x_m$. This can be done by choosing ϵ such that

$$U(\epsilon, z) \leftrightarrow U(z_{0,0,0}, \langle z_{0,0,1}, z_{0,1}, \dots, z_{0,n}, z_1, \dots, z_{m-n}\rangle),$$

where in this last equation we have simplified the notation, so that $z_{0,0,0}$ means $(((z)_0)_0)_0$, etc. $\square$

Exercise 20. Show that the *s-m-n* functions of Lemma 6.1 may be taken to be recursive. [hint: we need to show that there is a recursive function $f : z = \langle y, x_1, \dots, x_n\rangle \mapsto \epsilon$ such that for all $z, x_{n+1}, \dots, x_m \in \omega^\omega$ we have $(f(z))(\langle z, x_{n+1}, \dots, x_m\rangle) = z_{0,0}(w)$, where $w = \langle z_{0,1}, z_1, \dots, z_n, x_{n+1}, \dots, x_m\rangle$. It suffices to get a recursive f such that for all $z, x_{n+1}, \dots, x_m, u \in \omega^\omega$ we have $(f(z))(\langle u, x_{n+1}, \dots, x_m\rangle) = z_{0,0}(w)$, where $w = \langle u_{0,1}, u_1, \dots, u_n, x_{n+1}, \dots, x_m\rangle$. Argue this directly, assuming our coding and decoding functions are reasonable, in particular, the first k digits of $\langle a_0, \dots, a_\ell\rangle$ determine the first k digits of $a_0, \dots, a_\ell$ for all of our coding functions.]

An important consequence of the *s-m-n* theorem is the recursion theorem. Again, this is a classical result in recursion theory (due to Kleene) which also generalizes to arbitrary non-selfdual pointclasses. The recursion theorem says the remarkable fact that in defining a $\mathbf{\Gamma}$ set, we may actually use the code for the set we are defining in its definition. This can also be viewed as a fixed-point property on the set of codes for $\mathbf{\Gamma}$ sets.

Theorem 6.2 (recursion theorem)**.** *Let $\mathbf{\Gamma}$ have a universal set. Then for every product space $X = X_1 \times \cdots \times X_n$ and every $\mathbf{\Gamma}$ set $A \subseteq \omega^\omega \times X$, there is a $y^* \in \omega^\omega$ such that $U_X(y^*, x_1, \dots, x_n)$ iff $A(y^*, x_1, \dots, x_n)$ for all $x_1, \dots, x_n$.*

Proof. Let $X = X_1 \times \cdots \times X_n$ and fix $A \subseteq \omega^\omega \times X$ in $\mathbf{\Gamma}$. Let $s\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ be the *s-m-n* function corresponding to the product spaces ω^ω and $\omega^\omega \times X$. Fix $\epsilon \in \omega^\omega$ such that $U_{\omega^\omega \times X}(\epsilon, y, x) \leftrightarrow A(s(y, y), x)$.

We thus have for all x, y:

$$U_X(s(\epsilon, y), x) \leftrightarrow U_{\omega^\omega \times X}(\epsilon, y, x) \leftrightarrow A(s(y, y), x).$$

We then set $y = \epsilon$, that is we take $y^* = s(\epsilon, \epsilon)$. □

We first prove a version of the coding lemma which applies to prewellorderings, and then prove a more general version which applies to wellfounded relations.

Theorem 6.3 (AD)**.** *Let $\mathbf{\Gamma}$ be a nonselfdual pointclass closed under $\exists^{\omega^\omega}$ and $\wedge$ (and assume $\mathbf{\Gamma} \supseteq \mathbf{\Pi}^0_1$). Let $\preceq$ be a prewellordering with both $\preceq$ and the strict part $\prec$ in $\mathbf{\Gamma}$. Let $R \subseteq \mathrm{dom}(\preceq) \times \omega^\omega$ be a relation with $\mathrm{dom}(R) = \mathrm{dom}(\preceq)$. Then there is a $\mathbf{\Gamma}$ relation $R' \subseteq R$ such that $\mathrm{dom}(R')$ meets every $[x]$ for $x \in \mathrm{dom}(\preceq)$ (here $[x] = \{y \colon y \preceq x \wedge x \preceq y\}$).*

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be universal for $\mathbf{\Gamma} \restriction \omega^\omega \times \omega^\omega$, and we assume all universal sets are good, that is, we have continuous *s-m-n* functions. We prove the coding lemma by induction on the length $|\preceq|$ of the prewellordering $\preceq$. If $|\preceq| = \alpha + 1$ is a successor ordinal, the result follows easily by induction. Namely, given $R \subseteq \mathrm{dom}(\preceq) \times \omega^\omega$, apply induction to $\preceq'$ where $x \preceq' y \leftrightarrow (x \preceq y \wedge y \prec z)$, where $[z]$ is maximal in $\preceq$. Note that $\preceq'$ and the strict part $\prec'$ are in $\mathbf{\Gamma}$. Applying induction to $\preceq'$ and $S = R \cap (\mathrm{dom}(\preceq') \times \omega^\omega)$ given an $S' \subseteq S \in \mathbf{\Gamma}$. Let $R' = S' \cup \{\langle z, y\rangle\}$ where $R(z, y)$. Since by induction $\mathrm{dom}(S')$ meets every class below $[z]$, $\mathrm{dom}(R')$ meets every class of $\preceq$. Also, $R' \in \mathbf{\Gamma}$ since $S' \in \mathbf{\Gamma}$, $\mathbf{\Gamma}$ is closed under $\vee$ (since it is closed under $\exists^{\omega^\omega}$), and $\mathbf{\Gamma}$ contains all closed sets.

So, assume that $|\preceq| = \lambda$ is a limit. If $\alpha \leq \lambda$ we say $\epsilon \in \omega^\omega$ is α-good if $U_\epsilon \subseteq R$ and for all for all $x \in \mathrm{dom}(\preceq)$ with $|x| < \alpha$ we have $\mathrm{dom}(U_\epsilon) \cap [x] \neq \emptyset$ (i.e., U_ϵ is a choice relation below α). Note that if ϵ is α-good for some α, then there is a maximal $\alpha \leq \lambda$ such that ϵ is α-good wheich we denote by $\alpha(\epsilon)$. So, $\alpha(\epsilon)$ is defined iff $U_\epsilon \subseteq R$. By induction, for any $\alpha < \lambda$ there is an ϵ which is α-good. To see this, pick $z \in \mathrm{dom}(\preceq)$ with $|z| = \alpha$. Let $\preceq' = \preceq \cap \{(x, y) \colon y \prec z\}$, so $\preceq'$ and the strict part $\prec'$ are in $\mathbf{\Gamma}$. Apply induction to $\preceq'$ and $S = R \cap \{(x, w) \colon x \prec z\}$ to produce S'. If $U_\epsilon = S'$, then ϵ is α-good.

Consider now the following game G: I and II play out reals ϵ, δ respectively in ω^ω. II wins the run provided: $(U_\epsilon \subseteq R) \to (U_\delta \subseteq R \wedge \alpha(\delta) > \alpha(\epsilon))$.

Suppose first that I has a winning strategy σ for G. Let $A = \sigma[\omega^\omega]$. For every $\epsilon \in A$, $U_\epsilon \subseteq R$. Furthermore $\{\alpha(\epsilon) : \epsilon \in A\}$ is unbounded in λ since if δ is such that $\alpha(\delta) = \beta$, then $\alpha(\sigma(\delta)) \geq \beta$, and there are δ which are β-good for arbitrarily large $\beta < \lambda$. Define then $R'(x, y) \leftrightarrow \exists \epsilon \in A \ (U_\epsilon(x, y))$. $R' \in \mathbf{\Gamma}$ from our assumed closure properties, and clearly R' is λ-good so we are done.

Assume next that II has a winning strategy τ for G. The relation

$$S(\epsilon, z, x, w,) \leftrightarrow U_\epsilon(x, w) \wedge (x \prec z)$$

is in $\mathbf{\Gamma}$, and so $S(\epsilon, z, x, w) \leftrightarrow U(a, \epsilon, z, x, w) \leftrightarrow U(s(a, \epsilon, z), x, w)$ for some $a \in \omega^\omega$, and where s is our continuous s-m-n function. Let $f(\epsilon, z) = s(a, \epsilon, z)$, so f is continuous. Note that if z is in the field of $\preceq$, then $U_{f(\epsilon,z)} = U_\epsilon \cap \{(x, w) : x \prec z\}$. If z is not in the field of $\preceq$, then $U_{f(\epsilon,z)} = \varnothing$. In particular, for any z and ϵ, if $U_\epsilon \subseteq R$, then $U_{f(\epsilon,z)} \subseteq R$.

From the recursion theorem, let ϵ be such that

$$U_\epsilon(x, w) \leftrightarrow \exists z \ (z \preceq x \wedge x \preceq z \wedge U(\tau(f(\epsilon, z)), x, w).$$

We claim that $R' = U_\epsilon$ works. Note that $U_\epsilon(x, w) \to x \in \mathrm{dom}(\preceq)$. We show by induction on $|x|$ that $U_\epsilon(x, w) \to R(x, w)$. For any $z \in [x]$, by induction we have $U_{f(\epsilon,z)} = U_\epsilon \cap \{(x, w) : x \prec z\} \subseteq R$. It follows that $U_{\tau(f(\epsilon,z))} \subseteq R$ as well, and so $R(x, w)$.

Finally, we prove by induction on $|x|$ that for all $x \in \mathrm{dom}(\preceq)$ that $\mathrm{dom}(R') \cap [x] \neq \varnothing$. Let $x \in \mathrm{dom}(\preceq)$, and by induction we may assume that $\mathrm{dom}(R') \cap [x'] \neq \varnothing$ for all $x' \prec x$. It follows that for any $z \in [x]$ that $\mathrm{dom}(U_{f(\epsilon,z)}) \cap [x'] \neq \varnothing$ as well. That is, $U_{f(\epsilon,z)}$ is $|x|$-good. It follows that $\tau(f(\epsilon, x))$ is $> |x|$ good, that is, $U_{\tau(f(\epsilon,z))} \cap [x] \neq \varnothing$. Since the definition of U_ϵ involves unioning of all possible $z \in [x]$, it follows that $U_\epsilon \cap [x] \neq \varnothing$. □

Remark 6.4. The hypothesis that both $\preceq$ and $\prec$ are in $\mathbf{\Gamma}$ from theorem 6.3 is more than necessary. The result holds just assuming the strict part $\prec$ is in $\mathbf{\Gamma}$, although a different proof must be given. Theorem 6.5 below contains this stronger result, and also requires $\prec$ to only be a wellfounded relation.

Theorem 6.5 (AD)**.** *Let $\mathbf{\Gamma}$ be a nonselfdual pointclass closed under $\exists^{\omega^\omega}$ and $\wedge$ (and assume $\mathbf{\Gamma} \supseteq \mathbf{\Pi}^0_1$). Let $\prec$ be a wellfounded relation in $\mathbf{\Gamma}$. Let $R \subseteq \mathrm{dom}(\preceq) \times \omega^\omega$ be a relation with $\mathrm{dom}(R) = \mathrm{dom}(\prec)$. Then there is a $\mathbf{\Gamma}$ relation $R' \subseteq R$ such that for every $\beta < |\prec|$, $\mathrm{dom}(R')$ meets $\{x \in \mathrm{dom}(\prec) : |x|_\prec = \beta\}$.*

Proof. Let $\prec_\beta$ denote $\{x \in \mathrm{dom}(\prec) : |x|_\prec = \beta\}$. We again proceed by induction on $|\prec|$, and we may assume $|\prec|$ is a limit. Similar to before, we say ϵ is α-good, for $\alpha \leq \lambda = |\prec|$, if $U_\epsilon \subseteq R$ and for all $\beta < \alpha$ with have $\mathrm{dom}(U_\epsilon) \cap \prec_\beta \neq \varnothing$. Also as before, if $U_\epsilon \subseteq R$, then we let $\alpha(\epsilon)$ be the largest $\alpha \leq \lambda$ such that ϵ is α-good. As before we consider the game G where I and II play out reals ϵ, δ respectively in ω^ω and II wins the run provided: $(U_\epsilon \subseteq R) \to (U_\delta \subseteq R \wedge \alpha(\delta) > \alpha(\epsilon))$. Note that by induction, considering $\prec \restriction z = \{(x, y) : x \prec y \wedge y \prec z\}$ for $z \in \mathrm{dom}(\prec)$, we have that there are ϵ which are α-good for arbitrarily large α below λ.

Suppose first that I has a winning strategy σ for G. Let $A = \sigma[\omega^\omega]$. So, for $\epsilon \in A$, $U_\epsilon \subseteq R$. Also, $\sup\{\alpha(\epsilon) : \epsilon \in A\} = \lambda$. Define R' by $R'(x, w) \leftrightarrow \exists \epsilon \in A \ (U_\epsilon(x, w))$. Then $R' \subseteq R$ and $\mathrm{dom}(R') \cap \prec_\beta \neq \varnothing$ for all $\beta < \lambda$.

Suppose next that II has a winning strategy τ for G. Here we must argue a little differently from theorem 6.3. We attempt to define using the recursion theorem a relation U_ϵ with $\mathrm{dom}(U_\epsilon) = \mathrm{dom}(\prec)$ and such that $U_\epsilon(x, u)$ implies $U_u \subseteq R$ and $\alpha(u) > |x|_\prec$. Consider the relation

$$S(\epsilon, x, y, w) \leftrightarrow \exists z\ \exists u\ (z \prec x \wedge U_\epsilon(z, u) \wedge U_u(y, w)).$$

From the s-m-n theorem, let $s\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ be continuous such that $S(\epsilon, x, y, w) \leftrightarrow U(s(\epsilon, x), y, w)$. Define then, using the recursion theorem,

$$U_\epsilon(x, u) \leftrightarrow (x \in \mathrm{dom}(\prec)) \wedge u = \tau(s(\epsilon, x)).$$

Clearly $\mathrm{dom}(U_\epsilon) = \mathrm{dom}(\prec)$, and for every $x \in \mathrm{dom}(\prec)$ there is exactly one u such that $U_\epsilon(x, u)$. We prove by induction on $|x|_\prec$ that $U_\epsilon(x, u)$ implies that $U_u \subseteq R$ and $\alpha(u) > |x|_\prec$. By induction, it follows that $U_{s(\epsilon,x)}$ is a union of relations which are β-good for various β whose supremum is at least $|x|_\prec$. Thus, $s(\epsilon, x)$ is $|x|_\prec$-good. Since τ is winning for II, $u = \tau(s(\epsilon, x))$ is $|x|_\prec + 1$-good.

Finally, define R' by:

$$R'(x, w) \leftrightarrow \exists y\ \exists u\ (y \in \mathrm{dom}(\prec) \wedge U_\epsilon(y, u) \wedge U_u(x, w)).$$

Clearly $R' \subseteq R$ since $U_\epsilon(y, u)$ implies $U_u \subseteq R$. Also, since $\alpha(u) > |y|_\prec$, it follows that $\mathrm{dom}(R') \cap \prec_\beta \neq \varnothing$ for all $\beta < \lambda$. □

As a corollary it follows that if we can map the reals onto an ordinal λ, then we can map the reals onto $\mathcal{P}(\lambda)$. First we note a general fact, Suppose $A \subseteq \omega^\omega$. Define the pointclass $\mathbf{\Sigma}^1_1(A)$ to be collection of B which can be written in the form

$$B(x) \leftrightarrow C(x) \vee \exists y\ (\forall n\ (y)_n \in A \wedge D(\langle x, y\rangle)),$$

where $C, D \subseteq \omega^\omega$ are $\mathbf{\Sigma}^1_1$. We likewise define $\mathbf{\Sigma}^1_1(A) \restriction (\omega^\omega)^n$ for any n, using the same formula and our recursive bijection between $(\omega^\omega)^n$ and ω^ω.

Exercise 21. Show that $\mathbf{\Sigma}^1_1(A)$ is a pointclass which contains A and is closed under $\exists^{\omega^\omega}$, $\wedge$ and $\vee$. Also, $\mathbf{\Sigma}^1_1(A)$ has a universal set. [hint: closure under continuous preimages is immediate. Note that $A(x) \leftrightarrow \exists y\ (\forall n\ (y)_n \in A \wedge D(\langle x, y\rangle))$ where $D(z) \leftrightarrow \forall i, j\ (((z)_1)_i = ((z)_1)_j \wedge x = ((z)_1)_0)$. Thus, $A \in \mathbf{\Sigma}^1_1(A)$. To see closure under $\vee$ notice that $\exists y\ (\forall n\ (y)_n \in A \wedge D(\langle x, y\rangle)) \vee \exists z\ (\forall n\ (z)_n \in A \wedge D'(\langle x, y\rangle)) \leftrightarrow \exists w\ (\forall n\ (w)_n \in A \wedge (D(\langle x, w\rangle) \vee D'(\langle x, w\rangle)))$. To see closure under $\wedge$, note that $C(x) \wedge \exists y\ (\forall n\ (y)_n \in A \wedge D(\langle x, y\rangle)) \leftrightarrow \exists y\ (\forall n\ (y)_n \in A \wedge D(\langle x, y\rangle) \wedge C(x))$. Also, $\exists y\ (\forall n\ (y)_n \in A \wedge D(\langle x, y\rangle)) \wedge \exists z\ (\forall n\ (z)_n \in A \wedge D'(\langle x, z\rangle)) \leftrightarrow \exists w\ (\forall n\ (w)_n \in A \wedge \exists u\ \exists v\ (D(\langle x, u\rangle) \wedge D'(\langle x, v\rangle) \wedge \forall j\ ((u)_j = (w)_{2j} \wedge (v)_j = (w)_{2j+1}))$. To see closure under $\exists^{\omega^\omega}$, suppose that $B'(x) \leftrightarrow \exists z\ B(\langle x, z\rangle) \leftrightarrow \exists z\ [C(\langle x, z\rangle) \vee \exists y\ (\forall n\ (y)_n \in A \wedge D(\langle\langle x, z\rangle, y\rangle)]$. Then $B'(x) \leftrightarrow \exists z\ C(\langle x, z\rangle) \vee \exists y\ (\forall n\ (y)_n \in A \wedge \exists z\ D(\langle\langle x, z\rangle, y\rangle))$. Finally, if $U \subseteq \omega^\omega \times \omega^\omega$ is universal for $\mathbf{\Sigma}^1_1 \restriction \omega^\omega$, then define $V(\epsilon, x) \leftrightarrow U((\epsilon)_0, x) \vee \exists y\ (\forall n\ (y)_n \in A \wedge U((\epsilon)_1, \langle x, y\rangle))$. Then V is universal for $\mathbf{\Sigma}^1_1(A) \restriction \omega^\omega$.]

Definition 6.6. The ordinal Θ is the supremum of the lengths of the prewellorderings of ω^ω.

The ordinal Θ plays a role in AD models similar to that of the continuum in ZFC models. Note that for any $\alpha < \Theta$ thete is a map from ω^ω onto α (since there is a prewellordering of length α). With ZFC we clearly have $\Theta = (2^\omega)^+$. Note that Θ is a limit ordinal.

Corollary 6.7 (AD)**.** *If $\lambda < \Theta$, then there is a map f from ω^ω onto $\mathcal{P}(\lambda)$.*

Proof. Suppose $\lambda < \Theta$, and let $\preceq$ be a prewelordring of ω^ω of length λ. Let $\mathbf{\Gamma}$ be a nonselfdual pointclass closed under $\exists^{\omega^\omega}$, $\wedge$ and with $\preceq, \prec \in \mathbf{\Gamma}$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for $\mathbf{\Gamma} \restriction \omega^\omega$. If $A \subseteq \lambda$, then the coding lemma applied to the characteristic function of A show that the relation $R(x, y) \leftrightarrow (x \in \mathrm{dom}(\preceq) \wedge ((|x| \in A \wedge y = \bar{1}) \vee (|x| \notin A \wedge y = \bar{0})))$ is in $\mathbf{\Gamma}$. In particular, the code set of A, $C_A = \{x \in \mathrm{dom}(\preceq) : |x| \in A\}$ is in $\mathbf{\Gamma}$. Define $f(\epsilon) = A$ if U_ϵ is the code set for A, and $f(\epsilon) = \varnothing$ otherwise. Then, f is onto $\mathcal{P}(\lambda)$. □

Corollary 6.8 (AD)**.** *Θ is a limit cardinal.*

Proof. Suppose $\alpha < \Theta$ but $\Theta \leq \alpha^+$. Since $\alpha < \theta$, there is a prewellordering of length α, and hence a map from ω^ω onto α. From corollary 6.7 there is a map π from ω^ω onto $\mathcal{P}(\alpha)$. Since there is map from $\mathcal{P}(\alpha)$ onto α^+ (send $A \subseteq \alpha$ to $|A|$ if A is a wellordering, and to 0 otherwise), we have that there is a map from ω^ω onto α^+. Since the supremum defining Θ is not atained, we have $\alpha^+ < \Theta$. □

Exercise 22. Show that countable choice implies that $\mathrm{cof}(\Theta) > \omega$.

Theorem 6.9 (AD)**.** $\Theta = \sup\{|A|_w : A \subseteq \omega^\omega\}$ *is the supremum of the wadge ranks of sets of reals.*

Proof. If $A \subseteq \omega^\omega$, then there is a map π from ω^ω onto $|A|_w$, namely, $\pi(x) = |\tau^{-1}(A)|_w$, where $x \in \omega^\omega$ codes the strategy τ_x as before. Thus, $\Theta \geq \sup\{|A|_w : A \subseteq \omega^\omega\}$.

For the other direction, fix a prewellordering $\preceq$ of ω^ω. There is a function $F : \mathcal{P}(\omega^\omega) \to \mathcal{P}(\omega^\omega)$ such that for all $A \subseteq \omega^\omega$, $F(A) >_w A$. Namely, let $F(A) = A'$ be defined by $A'(x) \leftrightarrow (x(0) = 0 \wedge \tau_{x'}(x) \notin A) \vee (x(0) = 1 \wedge \tau_{x'}(x) \in A)$, where as before $x'(n) = x(n+1)$ and τ_y is the continuous function from ω^ω to ω^ω coded by y (in some fixed coding scheme of the continuous funcions). By construction, A' is not Wadge reducible to either A or A^c, so has strictly higher Wadge degree [if say τ_y reduced A' to A then $0^\frown y \in A'$ iff $\tau_y(0^\frown y) \in A$, however by definition $0^\frown y \in A'$ iff $\tau_y(0^\frown y) \notin A$].

Now define by induction on $\alpha < |\preceq|$ a set A_α by $A_0 = \varnothing$, and for $\alpha > 0$:

$$A_\alpha = F(\{\langle x, y\rangle : (x \in \mathrm{dom}(\preceq)) \wedge (|x|_\preceq = \alpha) \wedge (y \in A_{|x|_\preceq})\}).$$

A straightforward induction on α shows that $|A_\alpha|_w \geq \alpha$. □

Using the recursion theorem we can prove an effective strengthening of Theorem 6.9.

Theorem 6.10 (AD)**.** *Let $\preceq$ be a prewellordering of ω^ω, and let $\mathbf{\Gamma}$ be the pointclass (see exercise 21) $\mathbf{\Gamma} = \mathbf{\Sigma}^1_1(\preceq \oplus \preceq^c \oplus (\mathrm{dom}(\preceq))^c)$. Then there is a sequence $\{A_\alpha : \alpha < |\preceq|\}$ of sets in $\mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ which is strictly increasing in Wadge degree.*

Proof. Let $\mathbf{\Gamma} = \mathbf{\Sigma}^1_1(\preceq \oplus \preceq^c \oplus (\mathrm{dom}(\preceq))^c)$, so $\mathbf{\Gamma}$ is non-selfdual and closed under $\exists^{\omega^\omega}, \wedge$ (and so also $\vee$). Also, $\preceq \in \mathbf{\Delta}$. Let $\prec$ denote the strict part of $\preceq$, that is, $x \prec y$ iff $x \preceq y \wedge \neg(y \preceq x)$. Note that $\preceq$, $\prec$, $\mathrm{dom}(\preceq)$ are also in $\mathbf{\Delta}$. Note that $\mathbf{\Delta}$ is closed under $\wedge$, $\vee$, $\neg$. Let U denote the universal $\mathbf{\Gamma}$ sets with continuous s-m-n functions. The idea is to define a $\mathbf{\Gamma}$ relation U_ϵ with domain $\mathrm{dom}(\preceq)$ (which will actually be a function) such that $U_\epsilon(x, w)$ implies that w_0, w_1 code $\mathbf{\Gamma}$ sets which are complements (so they jointly code a $\mathbf{\Delta}$ set) which is the "jump" (as in the proof of Theorem 6.9) of the $\mathbf{\Delta}$ set $B(y, z)$ of those (y, z) such that $y \prec x$ and z in the $\mathbf{\Delta}$ set coded by $U_\epsilon(y)$.

Consider the relations S_1, S_2 defined by:

$$S_1(\epsilon, x, y, z) \leftrightarrow (x \in \mathrm{dom}(\preceq) \wedge (y \prec x) \wedge \exists w (U(\epsilon, y, w) \wedge U(w_0, z)))$$
$$S_2(\epsilon, x, y, z) \leftrightarrow (x \notin \mathrm{dom}(\preceq) \vee [\neg(y \prec x) \vee \exists w (U(\epsilon, y, w) \wedge U(w_1, z))])$$

From the closure properties of $\mathbf{\Gamma}$ we see that $S_1, S_2 \in \mathbf{\Gamma}$. Let $S_1(\epsilon, x, y, z) \leftrightarrow U(a_1, \epsilon, x, y, z) \leftrightarrow U(s(a_1, \epsilon, x), y, z)$, and let $s_1(\epsilon, x) = s(a_1, \epsilon, x)$. Similarly define s_2 using S_2 (and some real a_2). Note that if $x \in \mathrm{dom}(\preceq)$, and we assume ϵ is such that for all $y \prec x$ there is a unique w such that $U(\epsilon, y, w)$, and this w codes a $\mathbf{\Delta}$ set (i.e., $U_{w_0} = (U_{w_1})^c$), then for this fixed ϵ and x, $S_2(\epsilon, x, y, z)$ iff $\neg S_1(\epsilon, x, y, z)$ for all y, z. Roughly speaking, $(S_1)_{\epsilon, x}$ is the "join" of the $\mathbf{\Delta}$ sets corresponding to $y \prec x$.

Define

$$T_1(w, u) \leftrightarrow [(u(0) = 0 \wedge U(w_1, \tau_u(u'))) \vee (u(0) = 1 \wedge U(w_0, \tau_u(u')))]$$
$$T_2(w, u) \leftrightarrow [(u(0) \notin \{0, 1\}) \vee (u(0) = 0 \wedge U(w_0, \tau_u(u'))) \vee (u(0) = 1 \wedge U(w_1, \tau_u(u')))]$$

Clearly $T_1, T_2 \in \mathbf{\Gamma}$. Let $T_1(w, u) \leftrightarrow U(b_1, w, u) \leftrightarrow U(s(b_1, w), u)$. Let $t_1(w) = s(b_1, w)$. Similarly define $t_2(w)$ using T_2 (and some real b_2). Note that if w codes a $\mathbf{\Delta}$ set, then $t_1(w)$ codes the jump of this $\mathbf{\Delta}$ set and $t_2(w)$ codes the complement of the jump. So, $\langle t_1(w), t_2(w) \rangle$ is a $\mathbf{\Delta}$ code for the jump.

Finally, using the recursion theorem let ϵ be such that

$$U(\epsilon, x, w) \leftrightarrow [(x \in \mathrm{dom}(\preceq)) \wedge w = \langle t_1(\langle s_1(\epsilon, x), s_2(\epsilon, x) \rangle), t_2(\langle s_1(\epsilon, x), s_2(\epsilon, x) \rangle) \rangle].$$

Clearly $U \in \mathbf{\Gamma}$, $\mathrm{dom}(U) = \mathrm{dom}(\preceq)$, and U_ϵ is a function. By induction on $|x|_\preceq$ we have that if $U(\epsilon, x, w)$ then w codes a $\mathbf{\Delta}$ set A_x (that is $U_{w_0} = (U_{w_1})^c$, which we take to be A_x), and $|A_x|_W \geq |x|_\preceq$. For $\alpha < |\preceq|$ we can then let $A_\alpha = \{z \colon (|z_0|_\preceq = \alpha) \wedge z_1 \in A_{z_0}\}$.

$\square$

As another application of the coding lemma we prove the following.

Theorem 6.11. *Suppose $\mathbf{\Gamma}$ is a nonselfdual pointclass closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$. Suppose also $\mathrm{pwo}(\mathbf{\Gamma})$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal set for $\mathbf{\Gamma} \restriction \omega^\omega$. Let ϕ be a $\mathbf{\Gamma}$-norm on U. Then $|\phi| = \delta(\mathbf{\Gamma}) \doteq$ the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings = the supremum of the lengths of the $\check{\mathbf{\Gamma}}$ welfounded relations.*

Proof. Let ϕ be a $\mathbf{\Gamma}$-norm on U. All the initial segments of the prewellordering are in $\mathbf{\Delta}$ since π is a $\mathbf{\Gamma}$-norm. Thus, $|\phi| \leq \delta(\mathbf{\Gamma})$. It suffices therefore to show that any $\check{\mathbf{\Gamma}}$ wellfounded relation $\prec$ has length less than $|\phi|$. From the recursion theorem let ϵ be such that

$$U_\epsilon(x) \leftrightarrow \forall y\ (y \prec x \rightarrow (\epsilon, y) <^*_\phi (\epsilon, x)),$$

where $<^*$ is the norm relation corresponding to ϕ (recall for a norm ψ on a set A that $x <^*_\psi y$ iff $(x \in A \wedge (y \notin A \vee \psi(x) < \psi(y))))$. U_ϵ is attempting to define an embedding from $\prec$ into $|\phi|$ by $x \mapsto |(\epsilon, x)|$.

We show by induction on $|x|_\prec$ that $U_\epsilon(x)$ and that for all $y \prec x$ that $\phi(\epsilon, y) < \phi(\epsilon, x)$. Let x be least such that the stated property fails. So, for all $y \prec x$ we have $U_\epsilon(y)$. If $\neg U_\epsilon(x)$, then by definition of $<^*_\phi$ we have that for all $y \prec x$ that $y <^*_\phi x$. From the definition of U_ϵ it now follows that $U_\epsilon(x)$, a contradiction. So, $U_\epsilon(x)$. From the definition of U_ϵ it now follows that for all $y \prec x$ that $\phi(\epsilon, y) < \phi(\epsilon, x)$. Thus, $x \mapsto \phi(\epsilon, x)$ is orderpreserving from $\prec$ to $|\phi|$, so $|\prec| \leq |\phi|$. We must in fact have $|\prec| < |\phi|$ as given any $\check{\mathbf{\Gamma}}$ wellfounded relation $\prec$ we can easily get another $\check{\mathbf{\Gamma}}$ welfounded relation $\prec'$ with $|\prec'| > |\prec|$. $\square$

As another application we have the following.

Theorem 6.12. *Let $\mathbf{\Gamma}$ be a nonselfdual pointclas closed under $\exists^{\omega^\omega}$, $\wedge$. Then $\delta \doteq$ the supremum of the lengths of the $\mathbf{\Gamma}$ wellfounded relations is a regular cardinal.*

Proof. Suppose $\rho = \mathrm{cof}(\delta) < \delta$. Let $f\colon \rho \to \delta$ be cofinal. Let $\prec$ be a $\mathbf{\Gamma}$ wellfounded relation of length ρ. Let $U \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be a universal $\mathbf{\Gamma}$ set. Apply the coding lemma to the relation $R(x,y) \leftrightarrow (x \in \mathrm{dom}(\prec)) \wedge (U_y \text{ is wellfounded } \wedge |U_y| \geq f(|x|_\prec))$. Let $R' \subseteq R$ with $R' \in \mathbf{\Gamma}$ be from the coding lemma, so $\mathrm{dom}(R') \cap |\prec|_\beta \neq \varnothing$ for all $\beta < \rho$ (recall $|\prec|_\beta = \{z\colon |z|_\prec = \beta\}$). Let $A(y) \leftrightarrow \exists x\ (x \in \mathrm{dom}(\prec) \wedge R'(x,y))$, so $A \in \mathbf{\Gamma}$ and consists of codes of $\mathbf{\Gamma}$ wellfounded relations whose lengths are unbounded in δ. Define then $(y,z) \ll (y',z') \leftrightarrow (y = y' \wedge y \in A) \wedge (U_y(z,z'))$. Then $\ll$ is a wellfounded relation in $\mathbf{\Gamma}$, and $|\ll| = \delta$. $\square$

7. The Prewellordering Property

The prewellordering property is one the important structural properties of poinclasses. It is also an important ingrediant in the scale property. The prewellordering property for $\mathbf{\Gamma}$ says that every set in $\mathbf{\Gamma}$ can be written as an increasing union of sets in $\mathbf{\Delta}$, in an effective manner. We recall the precise definition. There are two almost equivalent definitions of the notion of a $\mathbf{\Gamma}$-norm. We take as our official definition the slightly stronger version.

Definition 7.1 ($\mathbf{\Gamma}$-norm)**.** Let $\mathbf{\Gamma}$ be a pointclass, and $A \subseteq X$. We say a norm ϕ on A is a $\mathbf{\Gamma}$-norm if the relations $<^*$, $\leq^*$ are in $\mathbf{\Gamma}$, where

$$x <^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) < \phi(y)))$$
$$x \leq^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) \leq \phi(y)))$$

If $A \in \mathbf{\Gamma}$, and ϕ is a $\mathbf{\Gamma}$-norm on A, then we can consider the *componenets* of A with respect to the norm. Say (without loss of generality) that ϕ is regular, so $\phi\colon A \to \lambda$ is onto. For $\alpha < \lambda$, let $A_\alpha = A^\phi_\alpha = \{x \in A\colon \phi(x) = \alpha\}$, and let $A_{\leq\alpha} = \{x \in A\colon \phi(x) \leq \alpha\}$. If $x \in A$ is such that $\phi(x) = \alpha$, then we have

$$A_{\leq\alpha} = \{y\colon y \leq^* x\} = \{y\colon \neg(x <^* y)\}$$

which shows that $A_{\leq\alpha} \in \mathbf{\Delta}$. Thus $A = \bigcup_{\alpha<\lambda} A_{\leq\alpha}$ writes A as an increasing union of $\mathbf{\Delta}$ sets, the length of the union being the length of the norm ϕ.

A similar argument shows that each set $A_{<\alpha} = \{x \in a\colon \phi(x) < \alpha\}$ is in $\mathbf{\Delta}$. So we can also write $A = \bigcup_{\alpha\leq\lambda} A_{<\alpha}$. In practice, λ will always be a limit ordinal, so we can write $A = \bigcup_{\alpha<\lambda} A_{<\alpha}$. Written this way, the sets are continuous with respect to union at limit ordinals.

Definition 7.2. A pointclass $\mathbf{\Gamma}$ has the *prewellordering property* (written $\mathrm{pwo}(\mathbf{\Gamma})$) if every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-norm.

so, if $\mathrm{pwo}(\mathbf{\Gamma})$ holds, then every $\mathbf{\Gamma}$ (or $\check{\mathbf{\Gamma}}$) set admits a representation as an increasing union (or intersection) of simpler, anmely $\mathbf{\Delta}$, sets.

We will restrict our attention now to *Levy* pointclasses.

Definition 7.3. $\mathbf{\Gamma}$ is a Levy pointclass if it is non-selfdual and either $\exists^{\omega^\omega}\mathbf{\Gamma} \subseteq \mathbf{\Gamma}$ or $\forall^{\omega^\omega}\mathbf{\Gamma} \subseteq \mathbf{\Gamma}$.

So, a Levy pointclass is non-selfdual and is closed under either existential or universal quantification over the reals (possibly closed under both). The first few Levy pointclasses are the levels of the projective hierarchy, the $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ sets. The Levy classes are the important pointclasses in descriptive set theory. Although there are certainly poinclasses between the Levy classes, they are of less significance.

Our goal in this section completely classify the prewellordering property among the Levy pointclasses. We will see that for every Levy pointclass $\mathbf{\Gamma}$, either $\mathrm{pwo}(\mathbf{\Gamma})$ or $\mathrm{pwo}(\check{\mathbf{\Gamma}})$.

We first prove two transfer results which allow us to propagate the prewellordering property up a projective-like hierarchy, that is, by applications of real quantifiers. This is the simplest case of the so-called *periodicity theorems* of Moschovakis and Martin.

First we do the easy case of transfer by $\exists^{\omega^\omega}$, and then we give the first periodicity theorem, which gives the transfer by $\forall^{\omega^\omega}$. The transfer under $\exists^{\omega^\omega}$ just uses the inf.

Theorem 7.4 (ZF)**.** *Let $\mathbf{\Gamma}$ be a pointclass and assume* $\mathrm{pwo}(\mathbf{\Gamma})$*. Then every set in $\exists^{\omega^\omega}\mathbf{\Gamma}$ admits a $\exists^{\omega^\omega}\forall^{\omega^\omega}\mathbf{\Gamma}$-norm.*

Proof. Let $A(x) \leftrightarrow \exists y \in \omega^\omega\ B(x,y)$, where $B \in \mathbf{\Gamma}$. Let ϕ br a $\mathbf{\Gamma}$-norm on B, with norm relations $<^*_\phi$, $\leq^*_\phi$. Define the norm ψ on A by:

$$\psi(x) = \min\{\phi(x,y) \colon (x,y) \in B\}.$$

Let $<^*_\psi$, $\leq^*_\psi$ be the corresponding norm relations on A. We have $x_1 <^*_\psi x_2$ if $\exists y_1\ \forall y_2\ (x_1,y_1) <^*_\phi (x_2,y_2)$. Likewise, $x_1 \leq^*_\psi x_2$ iff $\exists y_1\ \forall y_2\ (x_1,y_1) \leq^*_\phi (x_2,y_2)$. □

Exercise 23. Show that for any non-selfdual pointclass $\mathbf{\Gamma}$ that $\exists^\omega\mathbf{\Gamma}$ admits a $\exists^\omega\forall^\omega\mathbf{\Gamma}$-norm. Deduce that $\mathrm{pwo}(\mathbf{\Sigma}^0_\alpha)$ holds for all $\alpha \geq 2$ for general Polish X, and for all $\alpha \geq 1$ if $X = \omega^\omega$.

We next do the transfer under $\forall^{\omega^\omega}$. Here we use the "fake sup" which is defined using a game.

Theorem 7.5 ($\mathbf{\Delta}$-determinacy)**.** *Let $\mathbf{\Gamma}$ be a non-selfdual pointclass, and assume* $\mathrm{pwo}(\mathbf{\Gamma})$*. Then every set in $\forall^{\omega^\omega}\mathbf{\Gamma}$ admits a $\forall^{\omega^\omega}\exists^{\omega^\omega}\mathbf{\Gamma}$-norm.*

Proof. Let $A(x) \leftrightarrow \forall y \in \omega^\omega\ B(x,y)$, and let ϕ be a $\mathbf{\Gamma}$-norm on B with corresponding norm relations $<^*_\phi$, $\leq^*_\phi$. We define a relation $\leq_A$ on A as follows. Let $x_1, x_2 \in A$. Consider the following game $G_A(x_1,x_2)$:

$$\begin{array}{lllllllll}
x_1 & \mathrm{I} & y_1(0) & & y_1(1) & & y_1(2) & & \dots \\
x_2 & \mathrm{II} & & y_2(0) & & y_2(1) & & y_2(3) & \dots
\end{array}$$

II wins the run iff $\phi(x_1,y_1) \leq \phi(x_2,y_2)$. Note that this makes sense as $B(x_1,y_1)$, $B(x_2,y_2)$ since $x_1, x_2 \in A$. Note also the game G_A is determined ae payoss set is $\{(y_0,y_1) \colon (x_0,y_0) \leq^*_\phi (x_1,y_1)\} = \{(y_0,y_1) \colon \neg(x_1,y_1) <^*_\phi (x_1,y_1)\}$ (using here that $x_0, x_1 \in A$).

Claim. $\leq_A$ is reflexive.

Proof. Have II copy I's moves. □

$$\begin{tikzcd}[row sep=small]
 & x_0 & y_0(0) \arrow[dd, bend left, "\tau_0"] & y_0(1) \arrow[dd, bend left] & y_0(2) \arrow[dd, bend left] & \cdots \\
G_A(x_0, x_1) & & & & & \\
 & x_1 & y_1(0) \arrow[dd, bend left, "\tau_1"] & y_1(1) \arrow[dd, bend left] & y_1(2) \arrow[dd, bend left] & \cdots \\
G_A(x_1, x_2) & & & & & \\
 & x_2 & y_2(0) & y_2(1) & y_2(2) & \cdots
\end{tikzcd}$$

Figure 1. Showing transitivity

$$\begin{tikzcd}[row sep=small]
 & x_0 & y_0(0) & y_0(1) & y_0(2) & \cdots \\
G_A(x_0, x_1) & & & & & \\
 & x_1 & y_1(0) \arrow[uur, "\sigma_0"] & y_1(1) \arrow[uur] & y_1(2) & \cdots \\
G_A(x_1, x_2) & & & & & \\
 & x_2 & y_2(0) \arrow[uur, "\sigma_1"] & y_2(1) \arrow[uur] & y_2(2) & \cdots \\
G_A(x_2, x_3) & & & & & \\
 & x_3 & y_3(0) \arrow[uur, "\sigma_2"] & y_3(1) \arrow[uur] & y_3(2) & \cdots
\end{tikzcd}$$

Figure 2. Showing wellfoundedness

Claim. $\leq_A$ is transitive.

Proof. Suppose $x_0, x_1, x_2 \in A$ and $x_0 \leq_A x_1$, $x_1 \leq_A x_2$. We must show that $x_0 \leq_A x_2$, that is, II wins the game $G_A(x_0, x_2)$. To do that, we fix strategies τ_0, τ_1 for II in $G_A(x_0, x_1)$ and $G_A(x_1, x_2)$, and then move as shown in Figure 7.

Claim. $\leq_A$ is connected, that is, $\forall x_0, x_1 \in A\ (x_0 \leq_A x_1 \vee x_1 \leq_A x_0)$.

Proof. Let $x_0, x_1 \in A$. If II wins $G_A(x_0, x_1)$ then $x_0 \leq_A x_1$ by definition. So, suppose I wins $G_A(x_0, x_1)$. Note that since x_0, x_1 are in A, the payoff condition for I, namely $\neg((x_0, y_0) \leq^*_\phi (x_1, y_1))$, is equivalent to saying $(x_1, y_1) <^*_\phi (x_0, y_0)$. However, if I wins this game, then II can clearly win the same game (use the same strategy but just ignore tha last move of the opponent). So, $x_1 \leq_A x_0$. $\square$

At this point we have shown that $\leq_A$ is a prelinearorder on the set A. We next show that it is wellfounded, so that it is actually a prewellorder (and thus gives a norm on A).

Claim. $\leq_A$ is wellfounded, that is, there does not exist a sequence $x_0 >_A x_1 >_A x_2 \cdots$ with all $x_i \in A$ (recall $x <_A y$ means $x \leq_A y$ and $\neg(y \leq_A x)$).

Proof. Suppose $x_i \in A$, and $x_0 >_A x_1 >_A x_2 \cdots$. For each i we have that $\neg(x_i \leq_A x_{i+1})$, so I wins the game $G_A(x_i, x_{i+1})$ for each i. Fix (countable choice for reals) winning strategies σ_i for I in $G_A(x_i, x_{i+1})$. We play these strategies against each other as shown in Figure 7.
Since $x_i \in A$, all of the (x_i, y_i) are in B. So, since $\neg((x_i, y_i) \leq^*_\phi (x_{i+1}, y_{i+1}))$ we have that $(x_i, y_i) >^*_\phi (x_{i+1}, y_{i+1})$ for all i, a contradiction. $\square$

At this point we know that $\leq_A$ is a prewellordering on the set A. Let $\psi \colon A \to \mathrm{On}$ be the corresponding norm on A. We must show that the corresponding norm relations $\leq^*_\psi$, $<^*_\psi$ are in $\forall^{\omega^\omega} \exists^{\omega^\omega} \boldsymbol{\Gamma}$.

Consider first $\leq_\psi^*$. We claim that $x_0 \leq_\psi^* x_1$ iff II wins the game $G_A(x_0, x_1)$ (recall that the payoff condition for this game is $(x_0, y_0) \leq_\phi^* (x_1, y_1)$). Suppose first that $x_0 \leq_\psi^* x_1$. In particular, $x_0 \in A$. If $x_1 \in A$ as well, then $x_0 \leq_A x_1$, and so by definition we have that II wins $G_A(x_0, x_1)$. If $x_1 \notin A$, Then II can win $G_A(x_0, x_1)$ by playing any real y_1 such that $(x_1, y_1) \notin B$. Suppose next that II wins the game $G_A(x_0, x_1)$. We must have that $x_0 \in A$ as otherwise I could win this game by playing a real y_0 such that $(x_0, y_0) \notin B$. If $x_1 \notin A$ then by definition we have $x_0 \leq_\psi^* x_1$. If $x_1 \in A$, then since II wins the game we have by definition that $x_0 \leq_A x_1$, that is $\psi(x_0) \leq \psi(x_1)$. so, $x_0 \leq_\psi^* x_1$ holds in this case as well.

So, $x_0 \leq_\psi^* x_1$ iff II has a winning strategy in the game $G_A(x_0, x_1)$. This last condition is easily a $\forall^{\omega^\omega} \exists^{\omega^\omega} \boldsymbol{\Gamma}$ condition.

Consider next the relation $x_0 <_\psi^* x_1$. We claim that $x_0 <_\psi^* x_1$ iff I wins the game $G'_A(x_0, x_1)$, which is the game where I plays the $y_1(i)$, II plays the $y_0(i)$, and the payoff condition for I to win is $(x_0, y_0) <_\phi^* (x_1, y_1)$. This will again show that $<_\psi^*$ is a $\forall^{\omega^\omega} \exists^{\omega^\omega} \boldsymbol{\Gamma}$ relation.

Suppose first that $x_0 <_\psi^* x_1$. In particular, $x_0 \in A$. If $x_1 \notin A$, I wins G'_A by playing any y_1 such that $(x_1, y_1) \notin B$. If $x_1 \in A$ as well, then $\neg(\psi(x_1) \leq \psi(x_0))$, so I wins $G_A(x_1, x_0)$. Since both x_0, x_1 are in A, this gives a strategy for I in $G'_A(x_0, x_1)$.

Suppose next that I wins $G'_A(x_0, x_1)$, say by σ. We must have $x_0 \in A$ as otherwise II could win this game by playing any y_0 such that $(x_0, y_0) \notin B$. If $x_1 \notin A$ then we are done, so assume $x_1 \in A$ also. II can clearly win the game $G_A(x_0, x_1)$ (by ignoring the last move of I), so $\psi(x_0) \leq \psi(x_1)$. We must show strict inequality. Suppose $\psi(x_1) \leq \psi(x_0)$. This says that II can win $G_A(x_1, x_0)$, say by τ. Playing σ and τ against each other gives a contradiction. □

This completes the proof of Theorem 7.5, the first periodicity theorem.

□

Corollary 7.6 (PD). *The pointclasses $\boldsymbol{\Pi}^1_{2n+1}$, $\boldsymbol{\Sigma}^1_{2n+2}$ have the prewellordering property for all $n \geq 0$.*

Our next goal is to classify the pointclasses having the prewellordering property among the Levy pointclasses. We will see that for Levy classes $\boldsymbol{\Gamma}$, either pwo($\boldsymbol{\Gamma}$) or pwo($\check{\boldsymbol{\Gamma}}$).

First we state a useful general fact.

Definition 7.7. A *projective algebra* is a pointclass Λ closed under $\exists^{\omega^\omega}$, $\neg$, $\wedge$, $\vee$ (and so also $\forall^{\omega^\omega}$).

Exercise 24. Show that a projective algebra Λ does not have a largest Wadge degree in it. [hint: if it did, it would either have successor Wadge rank and so be a join of a nonselfdual degree immediately below it, or else have limit rank of cofinality ω. In the first case Λ would not be closed under finite unions, and in the second case would not be closed under $\exists^\omega$.]

Lemma 7.8. *Let Λ be a projective algebra. Then $o(\Lambda) = \sup\{|A|_w \colon A \in \Lambda\}$ is equal to $\delta(\Lambda) = \sup\{| \preceq | \colon \preceq \in \Lambda$ is a prewellordering $\}$.*

Proof. If $\alpha < o(\Lambda)$, then let $A \in \Lambda$ with $|A|_w = \alpha$. The initial segment of A in the Wadge hierarchy then determines a prewellordering of length α which is in Λ from the closure properties of Λ. Namely, let $x \preceq y$ iff $x^{-1}(A) \leq_w y^{-1}(A)$ where x, y are viewed as coding continuous functions from ω^ω to ω^ω. This is easily projective over A, and so in Λ. So, $\alpha < \delta(\Lambda)$, and so $o(\Lambda) \leq \delta(\Lambda)$.

For the other direction, suppose $\alpha < \delta(\Lambda)$, and fix a prewellordering $\preceq$ in Λ of length α. It suffices to show that there is an α increasing sequence of Wadge degrees. Given a set $A \subseteq \omega^\omega$, let A' be the set defined by:

$$A'(x) \leftrightarrow (x(0) = 0 \wedge f_{x'}(x) \in A) \vee (x(0) > 0 \wedge f_{x'}(x) \notin A)$$

where f_x is the continuous function from ω^ω to ω^ω coded by x. Easily $A' >_w A$. For if $f_{x'}$ were a reduction of A' to A, then considering $x = 1 ^\frown x'$ gives a contraction, and if $f_{x'}$ were a reduction of A' to A^c, the considering $x = 0 ^\frown x'$ gives a contradiction.

Now we define by induction on $\alpha < |\preceq|$ a set A_α. Let $A_0 = \emptyset$. Let $A_{\alpha+1} = (A_\alpha)'$. For α limit let $A_\alpha(x) \leftrightarrow (|(x)_0|_\preceq < \alpha \wedge (x)_1 \in A_{|(x)_0|_\preceq})$. A straightforward induction shows that the A_α are strictly increasing in Wadge degree.

Finally, we show that each $A_\alpha \in \Lambda$. Let $R(x, y) \leftrightarrow x \in \mathrm{dom}(\preceq) \wedge y \in A_{|x|_\preceq}$. It suffices to show that $R \in \Lambda$. Consider the relation $W(x, y, i, z, w, j)$ where $i, j \in \{0, 1\}$, and $x, y, z, w \in \omega^\omega$. The intention is that $i = 1$ and (z, w, j) witnesses that $R(x, y)$, or $i = 0$ and (z, w, j) witnesses that $\neg R(x, y)$. We define $W(x, y, i, z, w, j)$ to hold if one of the following cases holds:

(1) $i = 1$, x is an immediate successor of z in $\preceq$, and either $y(0) > 0$, $w = f_{y'}(y)$, and $j = 0$, or $y(0) = 0$, $w = f_{y'}(y)$, and $j = 1$.
(2) $i = 0$, and either $x \notin \mathrm{dom}(\preceq)$ or x is an immediate successor of z in $\preceq$, and either $y(0) > 0$, $w = f_{y'}(y)$, and $j = 1$, or $y(0) = 0$, $w = f_{y'}(y)$, and $j = 0$.
(3) $i = 1$, x has limit rank in $\preceq$, $(y)_0 \prec x$, $(y)_0 = z$, $w = (y)_1$, and $j = 1$.
(4) $i = 0$, and either $x \notin \mathrm{dom}(\preceq)$ or x has limit rank in $\preceq$ and the following: $\neg(y)_0 \prec x$, or $(z = (y)_0 \wedge w = (y)_1 \wedge j = 0)$.
(5) $i = 0$, $x \notin \mathrm{dom}(\preceq)$, or $|x|_\preceq = 0$.

W is easily in Λ. We then have that

$$\begin{aligned} R(x, y) \leftrightarrow \exists z, w, r \in \omega^\omega\ [&(z)_0 = x \wedge (w)_0 = y \wedge r(0) = 1 \\ &\wedge \forall i\ W((z)_i, (w)_i, r(i), (z)_{i+1}, (w)_{i+1}, r(i+1))]. \end{aligned}$$

$\square$

Lemma 7.9 (Martin)**.** *Let* $\mathbf{\Gamma}$ *be nonselfdual, closed under* $\forall^{\omega^\omega}$, $\wedge$, $\vee$, *and assume* $\mathrm{pwo}(\mathbf{\Gamma})$. *Let* $\delta = \delta(\mathbf{\Gamma}) = \sup\{|\preceq| \colon \preceq \in \mathbf{\Delta}$ *is a prewellordering* $\}$. *Then* $\mathbf{\Delta}$ *is closed under* $< \delta$ *unions and intersections.*

Proof. Assume not, and let $\rho < \delta$ be the least ordinal such that there is a ρ increasing sequence $\{A_\alpha\}_{\alpha<\rho}$ of sets in $\mathbf{\Delta}$ with $A = \bigcup_{\alpha<\delta} \notin \mathbf{\Delta}$. As $\rho < \delta$, the coding lemma implies that $A \in \check{\mathbf{\Gamma}}$ (which is closed under $\exists^{\omega^\omega}, \wedge$). By Wadge, every set if $\check{\mathbf{\Gamma}}$ can be written as a ρ union of $\mathbf{\Delta}$ sets. Given a set $B \in \check{\mathbf{\Gamma}} - \mathbf{\Gamma}$, let $B = \bigcup_{\alpha<\delta} B_\alpha$ where $B_\alpha \in \mathbf{\Delta}$, and we may assume the sequence in increasing by minimality of ρ. Let φ be the norm on B given by $\varphi(x) = \mu\alpha\ (x \in B_\alpha)$. The norm relations $\leq^*_\varphi$, $<^*_\varphi$ are both in $\check{\mathbf{\Gamma}}$ as they can be written as ρ unions of $\mathbf{\Delta}$ sets. This shows $\mathrm{pwo}(\check{\mathbf{\Gamma}})$, a contradiction to $\mathrm{pwo}(\mathbf{\Gamma})$ (as both $\mathbf{\Gamma}$, $\check{\mathbf{\Gamma}}$ now have the reduction property, and so both sides have the separation property). $\square$

We first show that every Levy pointclass falls into a *projective-like hierarchy*, which can be one of four types. We then analyze the prewellordering property within each of these four possible types.

Fix a Levy pointclass $\mathbf{\Gamma}$.

Definition 7.10. Let

$$\Lambda = \Lambda(\mathbf{\Gamma}) = \bigcup\{\mathbf{\Delta}\colon (\mathbf{\Delta} \subseteq \mathbf{\Gamma}) \wedge \exists^{\omega^\omega}\mathbf{\Delta} \subseteq \mathbf{\Delta} \wedge (\neg\Delta \subseteq \Delta) \wedge (\cap\Delta \subseteq \Delta)\}$$

So, $\Lambda \subseteq \mathbf{\Gamma}$ and Λ is a *projective algebra*, that is, Λ is closed under $\exists^{\omega^\omega}$, $\neg$, and finite unions and intersections (and so also $\forall^{\omega^\omega}$). $\Lambda(\mathbf{\Gamma})$ is the largest projective algebra contained in $\mathbf{\Gamma}$.

We show that $\Lambda(\mathbf{\Gamma})$ is the base of a projective-like hierarchy containing $\mathbf{\Gamma}$. Let $\lambda = o(\Lambda) = \sup\{|A|_w\colon A \in \Lambda\}$ be the supremum of the Wadge ranks of the sets in Λ (we also call λ the Wadge ordinal of Λ). We consider the following cases.

Case 1: $\mathrm{cof}(\lambda) = \omega$.

Let A_n be such that $|A_n|_w = \alpha_n < \lambda$ and $\sup_n \alpha_n = \lambda$. We also assume that $|A_n|_w < |A_{n+1}|_w$. Let $A = \oplus_n A_n$ be the countable join of the A_n. Recall that $|A|_w = \sup_n |A_n|_w = \lambda$. Also, A is selfdual. Let $\mathbf{\Sigma}_0 = \bigcup_\omega \Lambda$ be the pointclass of sets which are countable unions of sets in Λ. Note that $A \in \mathbf{\Sigma}_0$ from the definition of the join.

$\mathbf{\Sigma}_0$ is closed under countable unions by definition. Also $\mathbf{\Sigma}_0$ is closed under $\exists^{\omega^\omega}$. For suppose $B(x) \leftrightarrow \exists y\ C(x,y)$ with $C \in \mathbf{\Sigma}_0$. Say $C = \bigcup_n C_n$ with $C_n \in \Lambda$. Then

$$\begin{aligned} B(x) &\leftrightarrow \exists y\ C(x,y) \\ &\leftrightarrow \exists y\ \exists n\ C_n(x,y) \\ &\leftrightarrow \exists n\ \exists y\ C_n(x,y) \end{aligned}$$

so $B = \bigcup_n B_n$ where $B_n(x) \leftrightarrow \exists y\ C_n(x,y)$ is in Λ. note also that $\mathbf{\Sigma}_0$ is closed under finite intersections (since Λ is).

Finally, $\mathbf{\Sigma}_0$ is nonselfdual. To see this, note that we may assume that each of the A_n is nonselfdual (recall the selfdual and nonselfdual Wadge degrees alternate). Let U_n be universal for $\mathbf{\Gamma}'_n = \{B\colon B \le_w A_n\}$. Let $U(x,y) \leftrightarrow \exists n\ U_n((x)_n, y)$. Then easily U is in $\mathbf{\Sigma}_0$ and is universal for $\mathbf{\Sigma}_0$.

So, $\mathbf{\Sigma}_0$ is a Levy class containing Λ, which is closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$. Let $\mathbf{\Pi}_0 = \check{\mathbf{\Sigma}}_0$ be the dual class, so $\mathbf{\Pi}_0$ is nonseldual and closed under $\forall^{\omega^\omega}$ (and countable intersections), $\wedge$, $\vee$. Note that any Levy class which contains Λ must contain either $\mathbf{\Sigma}_0$ or $\mathbf{\Pi}_0$, since any nonselfdual class which is closed under $\exists^{\omega^\omega}$ is closed under countable unions.

Clearly $\mathbf{\Sigma}_0$ is not closed under countable intersections (as it is nonselfdual, so doesn't contain $\mathbf{\Pi}_0$) so not closed under $\forall^\omega$, so certainly not closed under $\forall^{\omega^\omega}$. We thus the projective-like hierarchy starting from $\mathbf{\Sigma}_0$, $\mathbf{\Pi}_0$ in the usual way: $\mathbf{\Sigma}_{n+1} = \exists^{\omega^\omega}\mathbf{\Pi}_n$, $\mathbf{\Pi}_{n+1} = \check{\mathbf{\Sigma}}_{n+1} = \forall^{\omega^\omega}\mathbf{\Sigma}_n$. for $n \ge 1$, the $\mathbf{\Sigma}_n$, $\mathbf{\Pi}_n$ are closed under countable unions and intersections, just as for the projective hierarchy. We must have $\mathbf{\Gamma} \subseteq \bigcup_n \mathbf{\Sigma}_n$, as otherwise $\bigcup_n \mathbf{\Sigma}_n$ would be a projective algebra contained in $\mathbf{\Gamma}$ which is larger that Λ. An easy argument using that $\mathbf{\Gamma}$ is a Levy class now shows that $\mathbf{\Gamma} = \mathbf{\Sigma}_n$ or $\mathbf{\Gamma} = \mathbf{\Pi}_n$ for some n [Let $n-1$ be largest so that $\mathbf{\Gamma}$ contain both $\mathbf{\Sigma}_{n-1}$ and $\mathbf{\Pi}_{n-1}$. As $\mathbf{\Gamma}$ is a Levy class, it must contain either $\mathbf{\Sigma}_n$ or $\mathbf{\Pi}_n$. If, say, $\mathbf{\Sigma}_n \subseteq \mathbf{\Gamma}$ but $\mathbf{\Pi}_n \not\subseteq \mathbf{\Gamma}$, then by Wadge $\mathbf{\Gamma} = \mathbf{\Sigma}_n$.].

We now place the prewellordering property within the $\mathbf{\Sigma}_n$, $\mathbf{\Pi}_n$ classes.

Claim. pwo($\mathbf{\Sigma}_0$).

Proof. Let $A \in \mathbf{\Sigma}_0$, say $A = \bigcup_n A_n$, an increasing union with $A_n \in \Lambda$. For $x \in A$ let $\varphi(x) = \mu n\ [x \in A_n]$. The relations $<^*_\varphi$, $\leq^*_\varphi$ are easily both countable unions of sets in Λ, hence in $\mathbf{\Sigma}_0$. $\square$

From first periodicity it now follows that pwo($\mathbf{\Sigma}_{2n}$, pwo($\mathbf{\Pi}_{2n+1}$) for all $n \geq 0$. So, we either have pwo($\mathbf{\Gamma}$) or pwo($\check{\mathbf{\Gamma}}$).

This ends Case 1.

In the remaining cases we have $\text{cof}(\lambda) > \omega$. Recall in this case that there is a nonselfdual pointclass $\mathbf{\Gamma}'$ of Wadge rank λ. So, $\mathbf{\Delta}' = \mathbf{\Gamma}' \cap \check{\mathbf{\Gamma}}' = \Lambda$. Also, exactly one of $\mathbf{\Gamma}'$, $\check{\mathbf{\Gamma}}'$ has the separation property, so choose $\mathbf{\Gamma}'$ so that sep($\check{\mathbf{\Gamma}}'$). From Theorem 5.8 it follows that $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$.

We refer to $\mathbf{\Gamma}'$ as the *Steel* pointlcass associated to Λ. We emphasize that $\mathbf{\Gamma}'$, in general, is not closed under both $\wedge$ and $\vee$. Of course, $\mathbf{\Gamma}'$, being nonseldual and closed under $\forall^{\omega^\omega}$ is closed under countable intersections, and likewise $\check{\mathbf{\Gamma}}'$ is closed under $\exists^{\omega^\omega}$ and so countable unions.

Before considering further cases, we prove some results about the Steel pointclass $\mathbf{\Gamma}'$ in general. We will assume that Λ is not closed under arbitrary length wellordered unions (equivalently intersections). This follows immediately from the hypothesis that there is no proper (i.e., $\neq \mathcal{P}(\omega^\omega)$) pointclass closed under complements and arbitrary length wellorderded unions and intersection. This, in turn, follows from AD^+, as one of the axioms of this theory asserts that every set is ∞-Borel.

Lemma 7.11. *$\rho = cof(\lambda)$ is the least ordinal such that Λ is not closed under wellorderded unions of length ρ.*

Proof. Since Λ is a proper pointclass, fix a Λ prewellordering $\preceq \in \Lambda$ with $|\preceq| = \rho$.

Let δ be the least ordinal such that there is a δ increasing seqquence of sets in Λ, say $\{A_\alpha\}_{\alpha<\delta}$ with $A = \bigcup_{\alpha<\delta} A_\alpha$ not in Λ. Easily δ is a regular cardinal. Suppose first that $\rho < \delta$. Since $\rho = \text{cof}(\lambda)$ and δ is regular, we may thin the sequence A_α to a δ length subsequnce where all the A_α now have Wadge degree $\leq \lambda' < \lambda$. Assume for the moment that $\delta < \lambda$. We may then fix a nonselfdual pointclass $\mathbf{\Gamma}' \subseteq \Lambda$ closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$, with all the $A_\alpha \in \mathbf{\Gamma}'$ and such that there is a δ length prewellordering in $\mathbf{\Gamma}'$. The coding lemma then gives that $A = \bigcup_{\alpha<\delta} A_\alpha \in \mathbf{\Gamma}'$, a contradiction. If $\delta > \lambda$, then from the minimality of δ we have that there is a λ prewellordering in Λ, a contradiction. If $\delta = \lambda$, then λ is regular, so $\rho = \delta = \lambda$, and we are done.

Thus, $\delta \leq \rho$. Suppose that $\delta < \rho \leq \lambda$. As $\rho = \text{cof}(\lambda)$, there is a $\lambda' < \lambda$ such that all the A_α have Wadge rank $< \lambda'$. As $\delta < \lambda$, the coding lemma again shows that $A \in \Lambda$, a contradiction. $\square$

We next see how to generate the Steel pointclass from Λ. Let Λ be a projective algebra with $\rho = \text{cof}(\lambda) > \omega$ (where $\lambda = o(\Lambda)$), and let $\mathbf{\Gamma}$ be the Steel pointlcass (so $|\mathbf{\Gamma}|_w = \lambda$, sep($\check{\mathbf{\Gamma}}$)). Recall $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$. From Lemma 7.11 we have that $\bigcup_\rho \Lambda \supsetneq \Lambda$. We cannot have $\bigcup_\rho \Lambda = \check{\mathbf{\Gamma}}$ as then Martin's argument would show pwo($\check{\mathbf{\Gamma}}$) (c.f. Lemma 7.9), and since $\check{\mathbf{\Gamma}}$ would also be closed under $\vee$, $\wedge$ (as $\bigcup_\rho \Lambda$ is) we would then have red($\check{\mathbf{\Gamma}}$) which contradicts sep($\check{\mathbf{\Gamma}}$). So, by Wadge we have $\bigcup_\rho \Lambda \supseteq \mathbf{\Gamma}$. But since Λ is closed under $\exists^{\omega^\omega}$, we also have that $\bigcup_\rho \Lambda$ is closed under $\exists^{\omega^\omega}$, and so we have $\bigcup_\rho \Lambda \supseteq \exists^{\omega^\omega} \mathbf{\Gamma}$.

Let $\mathbf{\Gamma}'$ be the pointclass of $\mathbf{\Sigma}^1_1$-bounded ρ length unions of Λ sets. By this we mean A can be written as $A = \bigcup_{\alpha<\rho} A_\alpha$, where $A_\alpha \in \Lambda$, is an increasing union, and if $S \subseteq A$ is $\mathbf{\Sigma}^1_1$ then $\exists \alpha < \rho \; (A \subseteq A_\alpha)$.

Claim. $\mathbf{\Gamma}' = \mathbf{\Gamma}$.

Proof. First note that $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$. To see this, let $A(x) \leftrightarrow \forall y \; B(x,y)$ where $B \in \mathbf{\Gamma}'$. Let $B = \bigcup_{\alpha<\rho} B_\alpha$ be an increasing $\mathbf{\Sigma}^1_1$ union of Λ sets. If $A(x)$, then there is an $\alpha < \rho$ such that $\forall y \; B_\alpha(x,y)$ as $\{(x,y)\colon y \in \omega^\omega\}$ is a $\mathbf{\Sigma}^1_1$ set. So, $A = \bigcup_{\alpha<\delta} A_\alpha$ where $A_\alpha(x) \leftrightarrow \forall y \; B_\alpha(x,y)$. Each $A_\alpha \in \Lambda$ by the closure of Λ. Also, $\{A_\alpha\}$ forms a $\mathbf{\Sigma}^1_1$-bounded union, since if $S \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_1$, then so is $S' = \{(x,y)\colon x \in S\}$.

Let $A \in \mathbf{\Gamma}$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\mathbf{\Sigma}^1_1$ set. Let $B = \{x \colon U_x \subseteq A\}$. So, $B(x) \leftrightarrow \forall y \; (U(x,y) \to A(y))$, which shows $B \in \mathbf{\Gamma}$ provided we know that $\mathbf{\Gamma}$ is closed under disjunctions with $\mathbf{\Pi}^1_1$ sets. If $P \in \mathbf{\Pi}^1_1$, say $P(x) \leftrightarrow \forall y \; U(x,y)$ where U is open. If $C \in \mathbf{\Gamma}$ then then $x \in P \cup C$ iff $\forall y \; (U(x,y) \vee C(x))$ and since $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$, it suffices to know that $\mathbf{\Gamma}$ is closed under disjunction with open sets. To see this, let $C \in \mathbf{\Gamma}$ and U be open. We define a strategy τ Wadge reducing $C \cup U$ to C. τ will copy I's moves as long as I has played so far an $s \in \omega^{<\omega}$ such that $N_s \cap (C \cup U)$ is not Borel (if $C \cup U$ is Borel, the result is immediate). Suppose I plays $s^\frown a_n$ which is the first position to violate this condition. Since $N_s \cap (C \cup U)$ is not Borel, it follows that $N_s \cap C$ is not Borel. Thus $N_s \cap C$ is Wadge above $N_{s^\frown a_n} \cap (C \cup U)$. In this case, τ then switches to a strategy which reduces $N_{s^\frown a_n} \cap (C \cup U)$ to $N_s \cap C$. τ is easily a continuous reduction of $C \cup U$ to C.

So, $B \in \mathbf{\Gamma} \subseteq \bigcup_\rho \Lambda$. Write $B = \bigcup_{\alpha<\rho} B_\alpha$ with each $B_\alpha \in \Lambda$. Define $A_\alpha(y) \leftrightarrow \exists x \in B_\alpha(U(x,y))$. Each A_α is in Λ, and clearly $A = \bigcup_{\alpha<\rho} A_\alpha$ (each singleton is a $\mathbf{\Sigma}^1_1$ set). Also, $\{A_\alpha\}_{\alpha<\rho}$ is $\mathbf{\Sigma}^1_1$-bounded [If $S \subseteq A$ is $\mathbf{\Sigma}^1_1$, then $S = U_x$ for some x, and $x \in B$. So, $x \in B_\alpha$ for some α, and thus $S \subseteq A_\alpha$.]. This shows $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}'$.

We claim that $\bigcup_\rho \Lambda \subseteq \exists^{\omega^\omega} \mathbf{\Gamma}$. If $\rho < \lambda$, this follows immediately from the coding lemma (as there is a $\Lambda \subseteq \exists^{\omega^\omega} \mathbf{\Gamma}$ prewellordering of length ρ in this case; note here that $\exists^{\omega^\omega} \mathbf{\Gamma}$ is closed under $\wedge$, $\exists^{\omega^\omega}$). So assume $\rho = \lambda$, so λ is regular. It suffices by the coding lemma to show that there is a λ length wellfounded relation in $\exists^{\omega^\omega} \mathbf{\Gamma}$. We show, in fact, there is a $\mathbf{\Gamma}$ wellfounded relation of length λ. Fix $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$ and write $A = \bigcup_{\alpha<\lambda} A_\alpha$, an increasing $\mathbf{\Sigma}^1_1$ union of sets in Λ. Play the game where I plays out x, II plays out y and II wins iff

$$(x \in A \to (\sigma_{(y)_0}(A) = (\sigma_{(y)_1}(A))^c \text{ is a prewellordering of length } \geq |x|),$$

where $|x| = \mu\alpha \; (x \in A_\alpha)$. In this game I is playing out an element of A and II is playing a code of a Λ prewellordering of length at least $|x|$.

II wins this game by $\mathbf{\Sigma}^1_1$ boundedness of the A_α. Let τ be a winning strategy for II. Define then

$$(x_0, y_0) \prec (x_1, y_1) \leftrightarrow (x_0 = x_1 \in A) \wedge (y_0 \prec_{\tau(x)} y_1).$$

Here $\prec_{\tau(x)}$ is the Λ wellfounded relation coded by $\tau(x)$, that is, $\prec_{\tau(x)} = (\sigma_{(\tau(x))_0})^{-1}(A)$. Thus, $\prec$ is the "join" of the wellfounded relation $\prec_{\tau(x)}$ for $x \in A$. Since each $\prec_{\tau(x)}$ is wellfounded, so is $\prec$. Clearly $|{\prec}| \geq \lambda$ (in fact $= \lambda$). Finally, as $\mathbf{\Gamma}$ is closed under $\wedge$ we see that $\prec \in \mathbf{\Gamma}$. This completes the proof that $\bigcup_\rho \Lambda \subseteq \exists^{\omega^\omega} \mathbf{\Gamma}$.

To summarize, we know that $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}' \subseteq \bigcup_\rho \Lambda \subseteq \exists^{\omega^\omega}\mathbf{\Gamma}$. If $\mathbf{\Gamma}$ is closed under $\exists^{\omega^\omega}$ (and so closed under both quantifiers, then clearly $\mathbf{\Gamma} = \mathbf{\Gamma}'$. If $\mathbf{\Gamma} \subsetneq \exists^{\omega^\omega}\mathbf{\Gamma}$, Then as $\mathbf{\Gamma}'$ is closed under $\forall^{\omega^\omega}$ we must have $\mathbf{\Gamma}' = \mathbf{\Gamma}$. For if $\mathbf{\Gamma}'$ is properly larger than $\mathbf{\Gamma}$, then by Wadge it must contain $\check{\mathbf{\Gamma}}$, and so must contain $\forall^{\omega^\omega}\check{\mathbf{\Gamma}}$, which is the dual class of $\exists^{\omega^\omega}\mathbf{\Gamma}$. This contradicts $\mathbf{\Gamma}' \subseteq \exists^{\omega^\omega}$ and the fact that $\exists^{\omega^\omega}\mathbf{\Gamma}$ is nonselfdual (since $\mathbf{\Gamma}$ has a universal set, so does $\exists^{\omega^\omega}\mathbf{\Gamma}$). This completes the proof that $\mathbf{\Gamma} = \mathbf{\Gamma}'$. □

To summarize, we have shown that the Steel pointclass corresponding to a projective algebra Λ (defined when $\mathrm{cof}(\lambda) > \omega$, where $\lambda = o(\Lambda)$) can be described as the collection of $\mathbf{\Sigma}^1_1$-bounded, ρ-length increasing unions of sets in Λ, where $\rho = \mathrm{cof}(\lambda)$ is the least ordinal such that Λ is not closed under ρ-length unions.

Theorem 7.12. *Let Λ be a projective algebra with $\lambda = \mathrm{cof}(o(\Lambda))$, and assume $\mathrm{cof}(\lambda) > \omega$. Let $\mathbf{\Gamma}$ be the Steel pointclass corresponding to Λ (so $|\mathbf{\Gamma}|_w = \lambda$ and* $\mathrm{sep}(\check{\mathbf{\Gamma}})$*). Then* $\mathrm{pwo}(\mathbf{\Gamma})$*.*

Proof. Let $\rho = \mathrm{cof}(\lambda)$. Let $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$, and write $A = \bigcup_{\alpha<\rho} A_\alpha$, an increasing, $\mathbf{\Sigma}^1_1$-bounded union with each $A_\alpha \in \Lambda$. Let φ be the norm on A given by $\varphi(x) = \mu\alpha\ (x \in A_\alpha)$. The norm relation $<^*_\varphi$ can be written as $<^*_\varphi = \bigcup_{\alpha<\rho} B_\alpha$ where $B_\alpha(x,y) \leftrightarrow x \in A_\alpha \wedge y \notin A_\alpha$. Clearly $B_\alpha \in \Lambda$, and we see that the sequence $\{B_\alpha\}_{\alpha<\rho}$ is $\mathbf{\Sigma}^1_1$-bounded. To see this, suppose $S \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Sigma}^1_1$ and $S \subseteq <^*_\varphi$. In particular, if $S(x,y)$ then $x \in A$. As the $\{A_\alpha\}_{\alpha<\rho}$ is $\mathbf{\Sigma}^1_1$-bounded there is an $\alpha_0 < \rho$ such that $S(x,y)$ implies $x \in A_{\alpha_0}$. If $x <^*_\varphi y$ we then have that there is an $\alpha \leq \alpha_0$ such that $x \in A_\alpha$ and $y \notin A_\alpha$, and so $(x,y) \in B_\alpha$ for some $\alpha < \alpha_0$. Thus, $<^*_\varphi$ is a $\mathbf{\Gamma}$ relation. A similar computation show that $\leq^*_\varphi$ is a $\mathbf{\Gamma}$ relation, so φ is a $\mathbf{\Gamma}$-norm. □

It is worth recording another fact about $\mathbf{\Gamma}$-prewellorderings.

Lemma 7.13. *Let Λ, $\mathbf{\Gamma}$ be as in Theorem 7.12. Then there is a $\mathbf{\Gamma}$ prewellordering of length $\lambda = o(\Lambda)$. Furthermore, all strict initial segments of this prewellordering are in Λ (in particular, there is λ increasing sequence of Λ sets).*

Proof. Let $\rho = \mathrm{cof}(\lambda)$. Assume first that $\rho < \lambda$. Let $\{A_\alpha\}_{\alpha<\rho}$ be a $\mathbf{\Sigma}^1_1$-bounded increasing sequence of sets in Λ with $A = \bigcup_{\alpha<\rho} A_\alpha \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$. We must have that $\{|A_\alpha|_w \colon \alpha < \rho\}$ is cofinal in λ (as otherwise the coding lemma shows that $A \in \Lambda$). Let $f(\alpha) = |A_\alpha|_w$. By thinning the sequence, we may assume that f is strictly increasing and discontinuous. Also, we may assume that for each $\alpha < \rho$ there is a prewellordering of length $f(\alpha)$ which is Wadge reducible to any set of Wadge degree $f(\alpha+1)$. We may also assume that there is a prewellordeing of length ρ in $\mathbf{\Gamma}_0$. Let $\mathbf{\Gamma}_\alpha = \mathbf{\Sigma}^1_1(A_{\alpha+1})$, and let U_α be a universal set for $\mathbf{\Gamma}_{\alpha+1}$ (this is defined uniformly from $A_{\alpha+1}$, see exercise 21). So, $\mathbf{\Gamma}_\alpha$ is nonselfdual, closed under $\exists^{\omega^\omega}$, $\wedge$, and has a prewellordering of length $f(\alpha)$.

Fix for the moment an $\alpha < \rho$. By the coding lemma applied to $\mathbf{\Gamma}_\alpha$, there is a $\mathbf{\Gamma}_\alpha$ set E such that

(1) Every $z \in E$ codes, relative to U_α a set B_z of Wadge degree $g(z) \in [\sup_{\beta<\alpha} f(\beta), f(\alpha))$. Here $B_z = \sigma^{-1}_{(z)_0}(U_\alpha) = (\sigma^{-1}_{(z)_1}(U_\alpha))^c$.
(2) For every $\gamma \in [\sup_{\beta<\alpha} f(\beta), f(\alpha))$ there is an $z \in E$ with $g(z) = \gamma$.

By the coding lemma again, there is a $D \subseteq \omega^\omega \times \omega^\omega \in \mathbf{\Gamma}_0 \subseteq \Lambda$ (using now $\rho < \lambda$) such that

(1) $\mathrm{dom}(D) \subseteq A \times \omega^\omega$.
(2) For all $\alpha < \rho$ there is an $(x, y) \in D$ with $x \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta$.
(3) If $D(x, y)$ and $x \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta$, then y codes, relative to U_α, a $\mathbf{\Gamma}_\alpha$ set E_y satisfying (1) and (2) above (for the set E) and this α.

For $\alpha < \rho$ define E_α by:

$$z \in E_\alpha \leftrightarrow \exists x\ \exists y\ (D(x, y) \wedge x \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta \wedge z \in E_y)$$

where "$z \in E_y$" means $\sigma_y(z) \in U_\alpha$. So, each E_α is a $\mathbf{\Gamma}_\alpha$ set of codes (relative to U_α) for sets of Wadge degrees in $[\sup_{\beta<\alpha} f(\beta), f(\alpha))$ (and every ordinal in this interval is coded by a real in E_α).

We now define an increasing sequence $\{C_\gamma\}_{\gamma<\lambda}$ of sets in Λ as follows. Fix $\gamma < \lambda$, and let α be such that $\gamma \in [\sup_{\beta<\alpha} f(\beta), f(\alpha))$. Let

$$\begin{aligned}(x, z) \in C_\gamma \leftrightarrow \exists \alpha_0 \leq \alpha[(x \in A_{\alpha_0} - \bigcup_{\beta<\alpha_0} A_\beta) \wedge z \in E_{\alpha_0} \wedge ((\alpha_0 < \alpha) \\ \vee (\sigma^{-1}_{(z)_0}(U_\alpha) = (\sigma^{-1}_{(z)_1}(U_\alpha))^c \text{ has Wadge degree } \leq \gamma))]\end{aligned}$$

As $\mathbf{\Gamma}_{\alpha+1}$ is closed under $f(\alpha)$ unions (by the coding lemma), we easily have that each C_γ is in $\mathbf{\Gamma}_{\alpha+1}$. The C_γ clearly form a λ-length increasing sequence. Let ψ be the norm corresponding the sequence $\{C_\gamma\}_{\gamma<\lambda}$, so ψ has length λ. We may write

$$<^*_\psi = \bigcup_{\alpha<\rho} \bigcup_{\gamma<f(\alpha)} (C_\gamma \times (\omega^\omega - C_\gamma)))$$

This is a ρ union of sets, say F_α, each of which is easily in Λ (since for $\gamma < f(\alpha)$, C_γ has Wadge degree $< f(\alpha + 1)$). Also, the sequence $\{F_\alpha\}_{\alpha<\rho}$ is a $\mathbf{\Sigma}^1_1$-bounded union, since if $S \subseteq \bigcup_{\alpha<\rho} F_\alpha = <^*_\psi$ is $\mathbf{\Sigma}^1_1$, then $\{x \colon \exists y\ (x, y) \in S\}$ is a $\mathbf{\Sigma}^1_1$ subset of A, and hence is contained in some fixed A_α. This completes the proof in the case $\rho < \lambda$.

The case $\rho = \lambda$ is immediate, since in this case we may write a set $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$ as as λ increasing, $\mathbf{\Sigma}^1_1$-bounded, union of Λ sets, and this gives a $\mathbf{\Gamma}$ prewellordering of length λ. $\square$

Case 2: $\mathrm{cof}(\lambda) > \omega$ and $\mathbf{\Gamma}$ is not closed under $\vee$.

First we note that this case includes the case where λ is singular (and $\mathrm{cof}(\lambda) > \omega$). To see this, let $\rho = \mathrm{cof}(\lambda)$, so $\omega < \rho < \lambda$. Let $A = \bigcup_{\alpha<\rho} A_\alpha$ where $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$ and each $A_\alpha \in \Lambda$. Let $\mathbf{\Gamma}_0$ be a pointclass closed under $\exists^{\omega^\omega}$, $\wedge$ such that there is a $\mathbf{\Gamma}_0$ prewellordering of length ρ. By the coding lemma there is a $\mathbf{\Gamma}_0$ set B such that each $x \in B$ is a $\check{\mathbf{\Gamma}}$ code for one of the A_α and each A_α is coded by some $x \in B$. We then have

$$y \in A \leftrightarrow \exists x\ [(x \in B) \wedge \sigma_x(y) \in A^c]$$

and since $\check{\mathbf{\Gamma}}$ is closed under $\exists^{\omega^\omega}$, this shows that $\check{\mathbf{\Gamma}}$ is not closed under intersection with $\mathbf{\Gamma}_0$ sets.

Since $\mathbf{\Gamma}$ is not closed under $\vee$, it is not closed under $\exists^{\omega^\omega}$. Let $\mathbf{\Pi}_1 = \mathbf{\Gamma}$ be the Steel pointclass, and define the $\mathbf{\Sigma}_n$, $\mathbf{\Pi}_n$ classes for $n \geq 1$ as usual. Note that these classes are all distinct, for example $\mathbf{\Sigma}_n$ is not closed under $\forall^{\omega^\omega}$ as otherwise, since $\mathbf{\Pi}_n = \forall^{\omega^\omega} \mathbf{\Sigma}_{n-1}$, we would have that $\mathbf{\Sigma}_n \supseteq \mathbf{\Pi}_n$ and so $\mathbf{\Sigma}_n = \mathbf{\Pi}_n$, a contradiction as all the $\mathbf{\Sigma}_n$ are nonselfdual.

We have pwo($\boldsymbol{\Pi}_1$), and by periodicity we have pwo($\boldsymbol{\Pi}_{2n+1}$), pwo($\boldsymbol{\Sigma}_{2n+2}$) for all $n \geq 0$. Note that for $n \geq 2$, $\boldsymbol{\Sigma}_n$, $\boldsymbol{\Pi}_n$ are closed under countable unions and intersections.

Case 3: $\mathrm{cof}(\lambda) > \omega$, and $\boldsymbol{\Gamma}$ is closed under $\vee$ but not $\exists^{\omega^\omega}$.

As we remarked above, in this case λ is regular. As in Case 2, let $\boldsymbol{\Pi}_1 = \boldsymbol{\Gamma}$ and define the $\boldsymbol{\Sigma}_n$, $\boldsymbol{\Pi}_n$ as usual. Since by assumption $\boldsymbol{\Pi}_1$ is not closed under $\exists^{\omega^\omega}$, we again have that all the $\boldsymbol{\Sigma}_n$, $\boldsymbol{\Pi}_n$ are distinct. As in Case 2, by periodicity we again have pwo($\boldsymbol{\Pi}_{2n+1}$), pwo($\boldsymbol{\Sigma}_{2n+2}$) for all $n \geq 0$.

Case 4: $\mathrm{cof}(\lambda) > \omega$ and $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$.

In this case $\boldsymbol{\Gamma}$ is closed under $\forall^{\omega^\omega}$, $\exists^{\omega^\omega}$, and hence also countable unions and intersections. As we noted in Case 2, we must have λ is regular in this case. We have pwo($\boldsymbol{\Gamma}$), but in this case we cannot propagate from $\boldsymbol{\Gamma}$ by periodicity as $\boldsymbol{\Gamma}$ is closed under quantifiers.

Let $\boldsymbol{\Pi}_1 = \boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}}$, $\boldsymbol{\Sigma}_1 = \check{\boldsymbol{\Pi}}_1$ and define the $\boldsymbol{\Sigma}_n$, $\boldsymbol{\Pi}_n$ as usual. Note that $\boldsymbol{\Pi}_1$ is closed under $\forall^{\omega^\omega}$ and countable intersections, but not $\vee$ (as otherwise it would contain $\boldsymbol{\Sigma}_1$, which it does not as it is nonselfdual, as it easily has a universal set). In particular $\boldsymbol{\Pi}_1$ is not closed under $\exists^{\omega^\omega}$, and as before we have that all the $\boldsymbol{\Sigma}_n$, $\boldsymbol{\Pi}_n$ are distinct. Thus, we have a projective-like hierarchy defined over $\boldsymbol{\Pi}_1$.

We show pwo($\boldsymbol{\Pi}_1$). For this, we use the following representation of $\boldsymbol{\Pi}_1$ sets.

Claim. $\boldsymbol{\Pi}_1$ is the collection of $\boldsymbol{\Sigma}^1_1$-bounded increasing unions of $\check{\boldsymbol{\Gamma}}$ sets of length λ.

Proof. Let $\boldsymbol{\Pi}'_1$ be the collection of $\boldsymbol{\Sigma}^1_1$-bounded increasing unions of $\check{\boldsymbol{\Gamma}}$ sets of length λ. Suppose $A \in \boldsymbol{\Pi}_1$, say $A = B \cap C$ where $B \in \boldsymbol{\Gamma}$, $C \in \check{\boldsymbol{\Gamma}}$. Write $B = \bigcup_{\alpha<\lambda} B_\alpha$, an increasing, $\boldsymbol{\Sigma}^1_1$-bounded union of Λ sets. Then $A = \bigcup_{\alpha<\lambda}(B_\alpha \cap C)$. This is easily a $\boldsymbol{\Sigma}^1_1$-bounded union of $\check{\boldsymbol{\Gamma}}$ sets (note here $\check{\boldsymbol{\Gamma}}$ is closed under intersections with Λ sets, in fact, $\boldsymbol{\Gamma}$ and $\check{\boldsymbol{\Gamma}}$ are closed under unions and intersections). So, $\boldsymbol{\Pi}_1 \subseteq \boldsymbol{\Pi}'_1$.

Since $\check{\boldsymbol{\Gamma}}$ is closed under $\forall^{\omega^\omega}$, the argument of Claim 7 shows that $\boldsymbol{\Pi}'_1$ is closed under $\forall^{\omega^\omega}$. On the other hand, since there is a $\boldsymbol{\Gamma}$ prewellordering of length λ, the coding lemma shows that $\bigcup_\lambda \check{\boldsymbol{\Gamma}} \subseteq \exists^{\omega^\omega}(\boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}}) = \boldsymbol{\Sigma}_2$. Since $\boldsymbol{\Pi}_1 \subseteq \boldsymbol{\Pi}'_1 \subseteq \boldsymbol{\Sigma}_2$ and $\boldsymbol{\Pi}'_1$ is closed under $\forall^{\omega^\omega}$, by Wadge it follows that $\boldsymbol{\Pi}'_1 = \boldsymbol{\Pi}_1$.

Let $A \in \boldsymbol{\Pi}_1$, and as above write $A = B \cap C = \bigcup_{\alpha<\lambda}(B_\alpha \cap C)$, where $C \in \check{\boldsymbol{\Gamma}}$ and the B_α form a $\boldsymbol{\Sigma}^1_1$-bounded union of $\boldsymbol{\Delta}$ sets. Let $A_\alpha = B_\alpha \cap C$, so $A = \bigcup_{\alpha<\lambda} A_\alpha$ is $\boldsymbol{\Sigma}^1_1$-bounded union of $\check{\boldsymbol{\Gamma}}$ sets. Let φ be the norm on A associated to this union, that is, $\varphi(x) = \mu x\ (x \in A_\alpha)$. We show that φ is a $\boldsymbol{\Pi}_1$-norm. Write $\omega^\omega - C = \bigcup_{\beta<\lambda} C_\beta$, a $\boldsymbol{\Sigma}^1_1$-bounded union of $\boldsymbol{\Delta}$ sets. Note that $\omega^\omega - A_\alpha = \bigcup_{\beta<\lambda}(C_\beta \cup B^c_\alpha)$. Note also that the C_β form a $\check{\boldsymbol{\Gamma}}$-bounded union. This is because the norm ψ associated to the C_β is a $\boldsymbol{\Gamma}$-norm (as shown above) and since $\boldsymbol{\Gamma}$ is closed under $\wedge$, $\vee$ the usual boundedness argument shows every $\check{\boldsymbol{\Gamma}}$ subset of $\omega^\omega - C$ is bounded in the ψ norm. From this it follows that each sequence $\{C_\beta \cup B^c_\alpha\}_{\beta<\lambda}$ is actually a $\check{\boldsymbol{\Gamma}}$-bounded union (since if $S \subset \bigcup_\beta (C_\beta \cup B^c_\alpha) = C^c \cup B^c_\alpha$ is $\check{\boldsymbol{\Gamma}}$, then $S - B^c_\alpha \subseteq C^c$ is in $\check{\boldsymbol{\Gamma}}$).

We then have

$$x <^*_\varphi y \leftrightarrow \exists \beta < \lambda\ \exists \alpha \leq \beta\ [(x \in A_\alpha) \wedge (y \in C_\beta \cup B^c_\alpha)].$$

For a fixed $\beta < \lambda$, the rest of the above equation after the $\exists\beta$ quantifier defines a $\check{\boldsymbol{\Gamma}}$ set (it is a β-union of $\check{\boldsymbol{\Gamma}}$ sets, and $\check{\boldsymbol{\Gamma}}$ is closed under $< \lambda$ unions by the coding lemma). So, $<^* = \bigcup_\beta E_\beta$, where $E_\beta \in \check{\boldsymbol{\Gamma}}$. Moreover, this is a $\boldsymbol{\Sigma}^1_1$-bounded union. To see this, let $S \subseteq <^*$ be $\boldsymbol{\Sigma}^1_1$. Then $S_1 = \{x \colon \exists y\ (x, y) \in S\} \in \boldsymbol{\Sigma}^1_1$ and $S_1 \subseteq A$, and so there is an $\alpha_0 < \lambda$ such that $S_1 \subseteq A_{\alpha_0}$. For $\alpha \leq \alpha_0$, let $S_\alpha = \{(x, y) \in S \colon x \in A_\alpha - \bigcup_{\gamma<\alpha} A_\gamma\} = \{(x, y) \in S \colon x \in C \cap (B_\alpha - \bigcup_{\gamma<\alpha} B_\gamma)\}$ which shows that $S_\alpha \in \check{\mathbf{\Gamma}}$. By $\check{\mathbf{\Gamma}}$-boundedness of the $\{B^c_\alpha \cup C_\beta\}_{\beta<\lambda}$ we have that there is a $\beta(\alpha) < \lambda$ such that for all $(x, y) \in S_\alpha$, $y \in (B^c_\alpha \cup \bigcup_{\beta<\beta(\alpha)} C_\beta)$. Since λ is regular, $\beta_0 = \sup_{\alpha\leq\alpha_0} \beta(\alpha) < \lambda$. We then have that $S \subseteq E_{\beta_0}$. This shows that $<^*_\varphi$ is in $\mathbf{\Pi}_1$. A similar computation shows that $\leq^*_\varphi$ is in $\mathbf{\Pi}_1$.

This completes the proof of pwo($\mathbf{\Pi}_1$). By periodicity we then have pwo($\mathbf{\Pi}_{2n+1}$), pwo($\mathbf{\Sigma}_{2n+2}$) for all $n \geq 0$. □

8. Wellordered Unions

In this section we investigate wellordered unions from a pointclass $\mathbf{\Gamma}$. the main closure theorem is the following.

Theorem 8.1. *Let $\mathbf{\Gamma}$ be nonselfdual, closed under $\exists^{\omega^\omega}$, and assume* pwo($\mathbf{\Gamma}$)*. Then $\mathbf{\Gamma}$ is closed under wellordered unions.*

Before starting the proof, we note that there is no harm in adding the assumption that $\mathbf{\Gamma}$ is also closed under $\wedge$. For inspecting the hierarchy analysis for the Levy poinclasses of §7 we see that in all cases if $\mathbf{\Gamma}$ is nonseldudal and closed under $\exists^{\omega^\omega}$ and pwo($\mathbf{\Gamma}$), then $\mathbf{\Gamma}$ is also closed under $\wedge$. For if $\Lambda = \Lambda(\mathbf{\Gamma})$ is of type I, then $\mathbf{\Gamma}$ must be of the form $\mathbf{\Sigma}_{2n}$ for some $n \geq 0$, and all these classes are closed under $\wedge$. If Λ is of types II or III, then $\mathbf{\Gamma}$ must be of the form $\mathbf{\Sigma}_{2n}$ for $n \geq 1$), since in these cases the Steel class $\mathbf{\Gamma} = \mathbf{\Pi}_1$ is not closed under $\exists^{\omega^\omega}$. All of these classes are closed under $\wedge$, in fact, closed under countable unions and intersections (if $\mathbf{\Gamma}' = \exists^{\omega^\omega}\mathbf{\Gamma}$ where $\mathbf{\Gamma}$ is nonselfdual and closed under $\forall^\omega$, then $\mathbf{\Gamma}'$ is closed under countable unions and intersections by the same argument for $\mathbf{\Sigma}^1_1$). If Λ is of type IV (that is, the Steel class is closed under both quantifiers), then either $\mathbf{\Gamma}$ is the Steel class (since we are assuming pwo($\mathbf{\Gamma}$), $\mathbf{\Gamma}$ must be the Steel class, not the dual class), or $\mathbf{\Gamma}$ is of the form $\mathbf{\Sigma}_{2n}$ for $n \geq 1$, where recall $\mathbf{\Sigma}_2 = \exists^{\omega^\omega}(\mathbf{\Gamma} \wedge \check{\mathbf{\Gamma}})$ (since pwo($\mathbf{\Pi}_1$) where $\mathbf{\Pi}_1 = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$). All of these classes are closed countable unions and intersections by the above comment.

So, in all cases, a nonselfdual $\mathbf{\Gamma}$ closed under $\exists^{\omega^\omega}$ with pwo($\mathbf{\Gamma}$) is also closed under $\wedge$ (of course, $\mathbf{\Gamma}$ is also closed under $\vee$ as it is nonselfdual and closed under $\exists^\omega$). Note also we showed above that $\mathbf{\Gamma}$ is actually closed under countable unions and intersections in all cases except one: Λ is of type I and $\mathbf{\Gamma} = \mathbf{\Sigma}_0 = \bigcup_\omega \Lambda$.

The proof of Theorem 8.1 breaks into two cases depending on whether $\check{\mathbf{\Gamma}}$ is closed under $\forall^{\omega^\omega}$ or not. In both cases we are also assuming $\mathbf{\Gamma}$ is nonselfdual, $\mathbf{\Gamma}$ is closed under $\exists^{\omega^\omega}$, and pwo($\mathbf{\Gamma}$), and by the above comments also closed under $\wedge$, $\vee$.

<u>Case 1:</u> $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$.

In this case (the type IV case), $\mathbf{\Gamma}$ is closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$, and so also countable unions and intersections. So, $\check{\mathbf{\Gamma}}$ is also closed under these operations. Going by contradiction, let ρ be the least ordinal such that there is a ρ-union of $\mathbf{\Gamma}$ sets which is not in $\mathbf{\Gamma}$. Easily, ρ is a regular cardinal. Let $\delta = \sup\{|\prec| \colon \prec \in \check{\mathbf{\Gamma}}$ is a wellfounded relation$\}$. $\check{\mathbf{\Gamma}}$ is closed under $< \delta$ length unions by the coding lemma, so $\delta \leq \rho$. Since we are assuming $\bigcup_\rho \mathbf{\Gamma} \not\subseteq \mathbf{\Gamma}$ we have $\check{\mathbf{\Gamma}} \subseteq \bigcup_\rho \mathbf{\Gamma}$ by Wadge. We have, in fact, that every $\check{\mathbf{\Gamma}}$ set can be written as a $\mathbf{\Sigma}^1_1$-bounded ρ-union of sets in $\mathbf{\Gamma}$. To see this, let $B \in \check{\mathbf{\Gamma}}$, and let U be a universal $\mathbf{\Sigma}^1_1$ set. Let $C(x) \leftrightarrow \forall y\ (U_x(y) \to B(y))$, so $C \in \check{\mathbf{\Gamma}}$ by the closure properites of $\check{\mathbf{\Gamma}}$. Write $C = \bigcup_{\alpha<\rho} C_\alpha$, an increasing union with $C_\alpha \in \mathbf{\Gamma}$. Then let $B_\alpha(y) \leftrightarrow \exists x[(x \in C_\alpha) \wedge (y \in U_x)]$. Then $B = \bigcup_{\alpha<\rho} B_\alpha$, each $B_\alpha \in \mathbf{\Gamma}$, and the B_α form a $\mathbf{\Sigma}^1_1$-bounded union.

Fix now an increasing $\mathbf{\Sigma}^1_1$-bounded union $A = \bigcup_{\alpha<\rho} A_\alpha$ with each $A_\alpha \in \mathbf{\Gamma}$ and $A \in \check{\mathbf{\Gamma}} - \mathbf{\Gamma}$. For $x \in A$ let $|x| = \mu\alpha\ (x \in A_\alpha)$. Consider the game where I plays out x, II plays out y, z, and II wins the run iff

$$(x \in A \to ((y \in A) \wedge (|x| < |y|) \wedge A_{|y|} = V_z)),$$

where V denotes a universal $\mathbf{\Gamma}$ set. By $\mathbf{\Sigma}^1_1$-boundedness, II has a winning strategy τ. Consider the relation $\prec$ defined by

$$x_0 \prec x_1 \Leftrightarrow (x_0, x_1 \in A) \wedge (\tau(x_1)_0 \notin V_{(\tau(x_0))_1})$$

which is clearly a $\check{\mathbf{\Gamma}}$ relation. Note that for $x_0, x_1 \in A$ we have $x_0 \prec x_1$ iff $\varphi(x_0) < \varphi(x_1)$ where $\varphi(x) = |(\tau(x))_0|$. Thus, $\prec$ is a prewellordering. Since ρ is regular, it follows that $\prec$ has length at least ρ [By induction on $\alpha < \rho$ there is a $\beta(\alpha) < \rho$ such that every $x \in A$ with $|x| \geq \beta(\alpha)$ has $\prec$-rank $\geq \alpha$. At limit stages we use the regularity of ρ, and at successor stages the result easily follows.] So, $\prec$ is a $\check{\mathbf{\Gamma}}$ prewellordering on length ρ. This shows that $\rho < \delta$, a contradiction.

Case 2: $\mathbf{\Gamma}$ is not closed under $\forall^{\omega^\omega}$, but closed under $\forall^\omega$.

Going by contradiction again, let ρ be the least ordinal such that there is a ρ-union of $\mathbf{\Gamma}$ sets which is not in $\mathbf{\Gamma}$. Again, ρ is a regular cardinal. Let $\mathbf{\Gamma}' = \bigcup_{\alpha<\rho} \mathbf{\Gamma}$. By minimality of ρ, any ρ-length union of $\mathbf{\Gamma}$ sets has the same union as a ρ-length increasing union. We are assuming $\mathbf{\Gamma}' \nsubseteq \mathbf{\Gamma}$, and so by Wadge, $\check{\mathbf{\Gamma}} \subseteq \mathbf{\Gamma}'$. Since $\mathbf{\Gamma}'$ is clearly closed under $\exists^{\omega^\omega}$ we have $\exists^{\omega^\omega} \check{\mathbf{\Gamma}} \subseteq \mathbf{\Gamma}'$. Let

$$\delta_1 = \sup\{|\prec| \colon \prec \in \mathbf{\Gamma} \text{ is a wellfounded relation}\},$$

and let $\delta_2 = \sup\{|\prec| \colon \prec \in \exists^{\omega^\omega} \check{\mathbf{\Gamma}} \text{ is a wellfounded relation}\}$. From the coding lemma, both δ_1 and δ_2 are regular cardinals. Also, $\delta_1 < \delta_2$ since we may put all the $\mathbf{\Gamma}$ wellfounded relations together into a single $\mathbf{\Gamma} \wedge \check{\mathbf{\Gamma}} \subseteq \exists^{\omega^\omega} \check{\mathbf{\Gamma}}$ relation (this uses the closure of $\mathbf{\Gamma}$ under $\forall^\omega$).

Suppose first that $\rho < \delta_2$. By periodicity we have $\mathrm{pwo}(\forall^{\omega^\omega} \mathbf{\Gamma})$, and it follows by Theorem 7.9 that $\mathbf{\Delta}_1 = \exists^{\omega^\omega} \check{\mathbf{\Gamma}} \cap \forall^{\omega^\omega} \mathbf{\Gamma}$ is closed under $< \delta_2$ unions and intersections. This would give that $\bigcup_\rho \mathbf{\Gamma} \subseteq \mathbf{\Delta}_1$, contradicting $\exists^{\omega^\omega} \mathbf{\Gamma} \subseteq \bigcup_\rho \mathbf{\Gamma}$.

Suppose next that $\rho \geq \delta_2$. Let $\prec$ be a $\exists^{\omega^\omega} \check{\mathbf{\Gamma}}$ wellfounded relation of rank $> \delta_1$. Write $\prec = \bigcup_{\alpha<\rho} A_\alpha$ an increasing unions with $A_\alpha \in \mathbf{\Gamma}$. For $x \in \mathrm{dom}(\prec)$, let $|x|_\alpha = |x|_{A_\alpha}$ be the rank of x in the wellfounded relation A_α. So, $|x|_\alpha < \delta_1$. Since ρ is regular, there is an $\alpha(x) < \delta_1$ such that for all $\alpha \geq \alpha(x)$, $|x|_\alpha = |x|_{\alpha(x)}$. The map $x \mapsto |x|_{\alpha(x)}$ is an order-preserving map from $\prec$ to δ_1, a contradiction.

This completes the proof of Theorem 8.1. $\square$

Case 3: $\mathbf{\Gamma}$ is not closed under $\forall^\omega$.

Inspecting the hierarchy analysis we see that there is a type I projective algebra Λ, that is $\mathrm{cof}(\lambda) = \omega$ such that $\mathbf{\Gamma} = \mathbf{\Sigma}_0 = \bigcup_\omega \Lambda$. Recall that in this case $\mathbf{\Sigma}_0$ is closed under $\wedge$, $\vee$, $\exists^{\omega^\omega}$, and we have $\mathrm{pwo}(\mathbf{\Sigma}_0)$ and also $\mathbf{\Sigma}_0$ is not closed under $\forall^\omega$, so this case does occur.

9. Scales

For the basic definition and facts about semi-scales, Suslin representations, and scales, see the notes on Polish spaces. We cary on from that discussion.

For T a tree on $\omega \times \kappa$ ($\kappa \in \mathrm{On}$) with $A = p[T]$, let $\vec{\varphi}^T$ be the coresponding semi-scale on A, so each norm φ_n^T maps into α. Similarly, given a semi-scale $\vec{\varphi}$ on the set A, let T^φ denote the tree of $\vec{\varphi}$ (we drop the vector notation when it causes no confusion). So, T^φ is a tree with $A = p[T]$.

In general the maps $\varphi \mapsto T(\varphi) = T^\varphi$ and $T \mapsto \varphi(T) = \varphi^T$ are not inverses of each other.

Fact 9.1. *If $\vec{\varphi}$ is a scale on A, then $\varphi = \varphi(T(\varphi))$.*

Proof. Let $X \in A$, and let $\vec{\psi}$ denote the semi-scale $\varphi(T(\varphi))$. Then $\psi_n(x)$ is the nth digit of the leftmost branch of $T(\varphi)_x$. Clearly $\varphi_0(x), \varphi_1(x), \dots)$ is a branch through $T(\varphi)_x$, and since $\vec{\psi}$ is a scale, it is the (true) leftmost brach of $T(\varphi)_x$. So, $\psi_n(x) = \varphi_n(x)$. $\square$

Fact 9.2. *If T is a tree, then $T(\varphi(T)) \subseteq T$.*

Proof. If $(s, \vec{\alpha}) \in T(\varphi(T))$, then there is an x extending s in $A = p[T]$ such that $\varphi_n(T)(x) = \alpha(n)$ for all $n < \mathrm{ln}(s)$. So, for all such n we have that the leftmost branch of T_x begins with $\vec{\alpha} \restriction n$. In particular, $(s, \vec{\alpha}) \in T$. $\square$

From the last fact we have the following.

Fact 9.3. *For any tree T we have $\varphi(T(\varphi(T))) = \varphi(T)$.*

Proof. We have that $T(\varphi(T)) \subseteq T$, and both of these trees project to the same set $A = p[T]$. To see that $\varphi(T(\varphi(T))) = \varphi(T)$ it is enough to observe that for any $x \in A$ that the leftmost branch of $T(\varphi(T)))_x$ is the leftmost branch of T_x. For this it is enough to see that the leftmost branch of T_x is a branch through $T(\varphi(T)))_x$. If f is the leftmost branch of T_x, then for all n, $(x \restriction n, f \restriction n)$ is in $T(\varphi(T)))$ since we can use x as the witness (as x extends s and its leftmost branch f starts out with $f \restriction n$). This shows $\varphi(T(\varphi(T))) = \varphi(T)$. $\square$

Corollary 9.4. *The maps $\Sigma\colon \varphi \mapsto \varphi(T(\varphi))$ and $\Pi\colon T \mapsto T(\varphi(T))$ are idempotent (that is, $\Sigma^2 = \Sigma$, $\Pi^2 = \Pi$).*

Proof. Since $\varphi(T(\varphi(T))) = \varphi(T)$ holds for any tree T, we have in particular that $\varphi(T(\varphi(T(\varphi)))) = \varphi(T(\varphi))$, which says that $\Sigma^2 = \Sigma$. Similarly, we can "apply T" to both sides of the equation to get $T(\varphi(T(\varphi(T)))) = T(\varphi(T))$, which says that $\Pi^2 = \Pi$. $\square$

The scale property and basic facts about it were discussed in the other set of notes, we just recall here without proof some of these facts.

Definition 9.5. A $\mathbf{\Gamma}$-scale on a set $A \subseteq \omega^\omega$ is a scale $\{\varphi_n\}_{n\in\omega}$ on A such that each norm φ_n is a $\mathbf{\Gamma}$-norm. Likewise we define $\mathbf{\Gamma}$-semi-scale, $\mathbf{\Gamma}$-good scale, etc. We say $\mathbf{\Gamma}$ has the *scale property* if every $\mathbf{\Gamma}$ set $A \subseteq \omega^\omega$ has a $\mathbf{\Gamma}$-scale.

We write scale($\mathbf{\Gamma}$) to say that $\mathbf{\Gamma}$ has the scale property. Clearly scale($\mathbf{\Gamma}$) implies pwo($\mathbf{\Gamma}$).

Fact 9.6. *If $\mathbf{\Gamma}$ is closed under $\wedge$, $\vee$, and* scale($\mathbf{\Gamma}$)*, then every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-excellent scale.*

Recall that if pwo($\mathbf{\Gamma}$) and $\mathbf{\Gamma}$ is closed under $\forall^\omega$, then $\mathbf{\Gamma}$ has the number uniformization property. We state next the corresponding result for the full uniformization property.

Fact 9.7. *If* scale($\mathbf{\Gamma}$) *and* $\mathbf{\Gamma}$ *is closed under* $\forall^{\omega^\omega}$, $\wedge$, $\vee$, *then every* $R \subseteq \omega^\omega \times \omega^\omega$ *in* $\mathbf{\Gamma}$ *has a* $\mathbf{\Gamma}$ *uniformization.*

Proof. From Fact 9.6 we may assume $\{\varphi_n\}$ is an excellent scale on R. As in Lemma 2.28 of the other notes, this gives a uniformization of R which is also in $\mathbf{\Gamma}$ using the closure of $\mathbf{\Gamma}$ under $\forall^{\omega^\omega}$. $\square$

Fact 9.8. *If* $\mathbf{\Gamma}$ *is a pointclass and* unif($\mathbf{\Gamma}$), *then* unif($\exists^{\omega^\omega}\mathbf{\Gamma}$).

Also (from the other notes) we have the following fundamental fact.

Fact 9.9 (ZF)**.** $\mathbf{\Pi}^1_1$ *has the scale property.*

Corollary 9.10 (ZF)**.** $\mathbf{\Pi}^1_1$ *and* $\mathbf{\Sigma}^1_2$ *have the uniformization property.*

We now begin the analysis of the scale property assuming AD. The arguments will be similar in some respects to the analysis of the prewellordering property given before, but new ingredients arise.

First we give the *periodicity theorems* which propagate the scale property under quantifiers, similar to what the first periodicity theorem did for norms. As with the prewellordering property, the propagation by the $\exists^{\omega^\omega}$ quantifier is easy and done in ZF (see Lemma 2.48 in the other notes).

Fact 9.11. *Let* $\mathbf{\Gamma}$ *be a pointclass closed under* $\forall^{\omega^\omega}$, $\wedge$, $\vee$ *and assume* scale($\mathbf{\Gamma}$). *Then* scale($\exists^{\omega^\omega}\mathbf{\Gamma}$).

Next we have the second periodicity theorem which transfers the scale property by the $\forall^{\omega^\omega}$ quantifier.

Theorem 9.12 (Second Periodicity)**.** *Let* $\mathbf{\Gamma}$ *be closed under* $\exists^{\omega^\omega}$, $\wedge$, $\vee$ *and assume* $\mathbf{\Delta}$ *determinacy. If* scale($\mathbf{\Gamma}$) *then* scale($\forall^{\omega^\omega}\mathbf{\Gamma}$). *More generally (without assuming the closure properties of* $\mathbf{\Gamma}$*), if* $B \subseteq \omega^\omega \times \omega^\omega$ *admits a very good scale in* $\mathbf{\Gamma}$, *then* $A = \forall^{\omega^\omega} B$ *admits a scale in* $\forall^{\omega^\omega}\exists^{\omega^\omega}\mathbf{\Gamma}$.

Let $A \in \forall^{\omega^\omega}\mathbf{\Gamma}$, say $A(x) \leftrightarrow \forall y\; B(x,y)$ where $B \in \mathbf{\Gamma}$. Let $\{\varphi_n\}$ be a scale on B. We define norms $\{\psi_n\}$ on A as follows. Let $\{s_n\}$ enumerate $\omega^{<\omega}$ with all s_n preceding any of its proper extensions (so $s_0 = \varnothing$). To define the norm ψ_n we consider, for $x_0, x_1 \in A$, the following game:

$$
G_n(x_0,x_1) \qquad
\begin{array}{cc|cccc}
s_n & \mathrm{I} & y_0(0) & y_0(1) & y_0(2) & \cdots \\
 & & & & & \\
s_n & \mathrm{II} & y_1(0) & y_1(1) & y_1(2) & \cdots
\end{array}
$$

where I and II play out y_0 and y_1 respectively and II wins the run iff $(x_0, s_n{}^\frown y_0) \leq^*_{\varphi_n} (x_1, s_n{}^\frown y_1)$. Since x_0, x_1 are in A, this game is in $\mathbf{\Delta}$ and hence determined. We define $x_0 \leq_n x_1$ iff II has a winning strategy in $G_n(x_0,x_1)$. Exactly as in the first periodicity theorem, $\leq_n$ is a reflexive, transitive, connected, wellfounded relation on A, that is, it defines a prewellordering on A. Let ψ_n be the corresponding norm on A.

We next show that $\{\psi_n\}$ is a scale on A. Suppose $\{x_m\} \subseteq A$, $X_m \to x$, and for each n the norms $\psi_n(x_m)$ are eventually equal to λ_n. First we show that $x \in A$. Fix $y \in \omega^\omega$, and we show $B(x,y)$. Let $i_0 < i_1 < \cdots$ be a sequence such that

$\forall m \geq i_k(\psi_l(x_m) = \lambda_l$, where $s_l = (y(0), \ldots, y(k-1))$ (for $k = 0$ we stabilize ψ_0, that is, $l = 0$). For each k, let τ_k be a winning strategy for II in the game $G_l(x_{i_{k+1}}, x_{i_k})$, with l as above. Using the τ_k we fill in the following diagram.

$$
\begin{array}{cc|cccccc}
 & x_{i_0} & y_{i_0}(0) & y_{i_0}(1) & y_{i_0}(2) & y_{i_0}(3) & \cdots \\
\tau_0 & & & & & & \\
 & x_{i_1} & y(0) & y_{i_1}(1) & y_{i_1}(2) & y_{i_1}(3) & \cdots \\
\tau_1 & & & & & & \\
 & x_{i_2} & y(0) & y(1) & y_{i_2}(2) & y_{i_2}(3) & \cdots \\
\tau_2 & & & & & & \\
 & x_{i_3} & y(0) & y(1) & y(2) & y_{i_3}(3) & \cdots \\
\tau_3 & & & & & & \\
 & x_{i_4} & y(0) & y(1) & y(2) & y(3) & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\hline
 & x & y(0) & y(1) & y(2) & y(3) & \cdots
\end{array}
$$

This produces reals y_{i_k} which converge to the real y. Since each τ_k is a winning strategy for II in $G_l(x_{i_{k+1}}, x_{i_k})$ we have that $\varphi_{l_k}(x_{i_{k+1}}, y_{i_{k+1}}) \leq \varphi_{l_k}(x_{i_k}, y_{i_k})$ for each k, where $s_{l_k} = (y(0), \ldots, y(k-1))$. Since $l_k \geq k$ and $\vec{\varphi}$ is very good, we have $\varphi_k(x_{i_{k+1}}, y_{i_{k+1}}) \leq \varphi_k(x_{i_k}, y_{i_k})$ for all k. Since $\vec{\varphi}$ is very good, this says that for all n that the norms $\varphi_n(x_{i_k}, y_{i_k})$ are eventually constant. Since $\vec{\varphi}$ is a semi-scale on B, this says that $(x, y) \in B$. This shows that $\vec{\psi}$ is a semi-scale on A.

The proof that $\vec{\psi}$ is a scale is very similar. To see that $\vec{\psi}$ is a scale, it suffices to show that for each $s \in \omega^{<\omega}$, II can win the game $G_s(x, x_m)$ for all large enough m. Fix such an s, and let i_0 now be a large enough integer so that the norms $\psi_s(x_m)$ are constant for all $m \geq i_0$. If now I makes move first move $y(0)$ in $G_s(x, x_{i_0})$, then II chooses an $i_1 > i_0$ such that the norms $\psi_{s^\frown y(0)}(x_m)$ are constant for $m \geq i_1$ and fixes a winning strategy τ_{i_0} in the game $G_s(x_{i_1}, x_{i_0})$ and copies the move $y(0)$ to be I's first move in this game. II then plays according to τ_{i_0} to get the first move of II in $G_s(x, x_{i_0})$. If I then plays $y(1)$ as the next move of $G_s(x, x_{i_0})$, II then chooses a x_{i_2} so that the norms $\psi_{l_1}(x_m)$ are constant for $m \geq i_2$ and picks a winning strategy τ_{i_1} for the game $G_{s^\frown(y(0),y(1))}(x_{i_2}, x_{i_1})$ and copies $y(1)$ as the first move of that game. This is illustrated in Figure 9.

If we let $z_{i_k} = s^\frown y_{i_k}$ be the real produced by following τ_{i_k} on the kth board, then we have again that for all k:

$$\varphi_{s^\frown(y(0),\ldots,y(k-1))}(x_{i_{k+1}}, z_{i_{k+1}}) \leq \varphi_{s^\frown(y(0),\ldots,y(k-1))}(x_{i_k}, z_{i_k}).$$

Since $\vec{\varphi}$ is very good, all the norms $\varphi_n(x_{i_k}, z_{i_k})$ are eventually constant, say with constant value α_n, and since $\vec{\varphi}$ is a scale we have $\varphi_n(x, z) \leq \alpha_n$. In particular $\varphi_s(x, z) \leq \varphi_s(x_{i_0}, z_{i_0})$, which shows II has won the run of the game $G_s(x, x_{i_0})$.

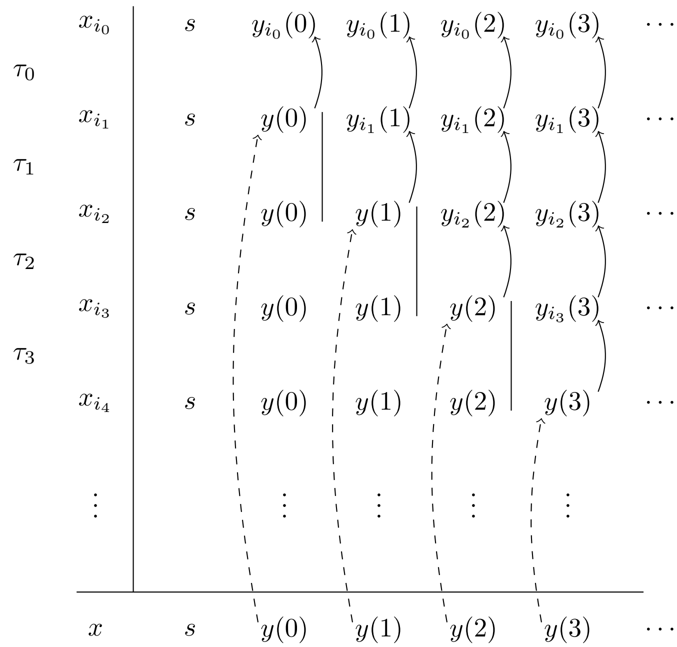

FIGURE 3. Showing $\vec{\psi}$ is a scale.

So far we have shown that the ψ_n form a scale on A. If we let $\psi'_n(x) = \langle \psi_0(x), \psi_n(x) \rangle_{\text{lex}}$ for $x \in A$, then it follows that $\{\psi'_n\}$ is also a scale on A. Moreover, each of the norms ψ'_n is now a $\forall^{\omega^\omega} \exists^w w\ \mathbf{\Gamma}$ norm. For example, we have

$$x_0 \leq^*_{\psi'_n} x_1 \leftrightarrow [(x_0 <^*_{\psi_0} x_1) \vee ((x_0 \leq^*_{\psi_0} x_1) \wedge (x_1 \leq^*_{\psi_0} x_0) \wedge (x_0 \leq^*_{\psi_n} x_1))]$$

From the first periodicity theorem we have that ψ_0 is a $\forall^{\omega^\omega} \exists^{\omega^\omega}$ $\mathbf{\Gamma}$-norm on A. For $n > 0$ the norm ψ_n is not necessarily a $\forall^{\omega^\omega} \exists^{\omega^\omega}$ $\mathbf{\Gamma}$-norm on A (for example, if $x_0 \in A$ and $x_1 \notin A$, we don't necessarily have that II wins the game $G_s(x_0, x_1)$, so we can't say that $x_0 \leq^*_{\psi_n} x_1$ holds iff II wins this game as we did in the first periodicity theorem). However, in the above equation we can replace "$x_0 \leq^*_{\psi_n} x_1$" with "II wins the game $G_{s_n}(x_0, x_1)$" as at this point in the formula we have already guaranteed that both x_0, x_1 are in A.

This completes the proof of the second periodicity theorem.

Corollary 9.13 (projective determinacy)**.** *We have* scale($\mathbf{\Pi}^1_{2n+1}$), scale($\mathbf{\Sigma}^1_{2n+2}$) *for all* $n \geq 0$.

Corollary 9.14 (projective determinacy)**.** *We have* unif($\mathbf{\Pi}^1_{2n+1}$), unif($\mathbf{\Sigma}^1_{2n+2}$) *for all* $n \geq 0$.

We next give the third periodicity theorem. Although this is not needed for the propagation of scales, it is an impotant theorem in descriptive set theory, and the proof is similar to that of the second periodicity theorem. The third periodicity theorem concerns the existence of canonical/definable winning strategies for games whose payoff sets are Suslin (i.e., have scales).

Theorem 9.15 (Third Periodicity)**.** *Let $A \subseteq \omega^\omega$ and assume A admits a scale $\{\varphi_n\}$ in a pointclass $\mathbf{\Gamma}$. Assume* $\det(\mathbf{\Gamma})$*. If I has a winning strategy in the game G_A (recall I tries to get in the set A), then I has a* canonical winning strategy σ *for the game G_A which is a $\Game\mathbf{\Gamma}$ real.*

Proof. Consider $s = (a(0), a(1), \dots, a(2n-1)) \in \omega^{<\omega}$ of even length (so I's turn to move). Suppose s is a winning position for I in the game G_A. We compare the possible next moves for I by the following game. For $a, b \in \omega$ the game $G_s(a, b)$ is played as shown:

$$
G_s(a,b) \qquad
\begin{array}{lcc|cccc}
\mathrm{I} & s & b & x_1(0) & x_1(1) & x_1(2) & \cdots \\
 & & & \uparrow & \downarrow & \uparrow & \\
\mathrm{II} & s & a & x_0(0) & x_0(1) & x_0(2) & \cdots
\end{array}
$$

FIGURE 4. The position comparison game.

In the game $G_s(a, b)$, players I and II alternate moves as usual with I moving first. They play in somewhat different locations, however. I makes first move $x_0(0)$, and then II responds with move $x_1(0)$. This is illustrated by the straight arrow in Figure 4. On the next round, I first makes the move $x_1(1)$, and then II responds with $x_0(1)$ as illustrated by the arrow again. This two-round cycle of moves then repeats. We think of the moves $x_0(0)$ and $x_1(0)$ as being offensive moves made by the players; they are playing as the $\forall^\omega$ player on their opponent's board. Likewise, we likewise think of the moves $x_0(1)$ and $x_1(1)$ as defensive moves; they are playing as the $\exists^\omega$ player on their home boards (in this description, we view II as responsible for the board corresponding to $s^\frown a$. II is trying to get $z_0 = s^\frown a^\frown x_0$ to be "more in the set" A than the real $z_1 = s^\frown b^\frown x_1$ which I is responsible for). At the end of the rounds, we have reals $z_0 = s^\frown a^\frown x_0$ and $z_1 = s^\frown b^\frown x_1$. II then wins the run of the game iff $z_0 \leq^*_{\varphi_n} z_1$, where $n = |s|$.

Claim. For each s which is a winning position for I (with I to move) in the game G_A, let W_s be the set of a such that $s^\frown a$ is still a winning position for I (with now II to move). Then the relation on W_s defined by $a \leq_s b$ iff II has a winning strategy in the game $G_s(a, b)$ is a prewellordering on the set W_s.

We prove the claim shortly, but first we see how it defines a canonical winning strategy for I in G_A. Let σ be the quasistrategy for I in G which plays by always playing moves which are minimal in the prewellorderings $\leq_s$. That is, if $s = (a(0), \dots, a(2n-1))$ has followed σ so far, then $a \in \sigma(s)$ iff $a \in W_s$ is minimal in $\leq_s$. We show this is a winning quasistrategy for I.

Fix a winning strategy σ' for I in G. Let $y = (y(0), y(1), \dots)$ be a play according to σ. We must show that $y \in A$. Consider the diagram shown in Figure 5.

The moves $y_0(2k)$ on the top row come from the strategy σ'. Since $y(0) \leq_0 y_0(0)$, we can fix a winning strategy τ_0 in the game $G_\varnothing(y(0), y_0(0))$. The vertical arrows between the first and second rows of the diagram come from τ_0. The move $y(1)$ is copied as shown by the dashed arrow, and this then determines the moves $y_0(1)$, $y_0(2)$, and $y_1(2)$. Note that $(y(0), y(1), y(2))$ and $(y(0), y(1), y_1(2))$ are both in $W_{(y(0),y(1))}$. For $(y(0), y(1), y(2))$ this is because this is following σ, which by

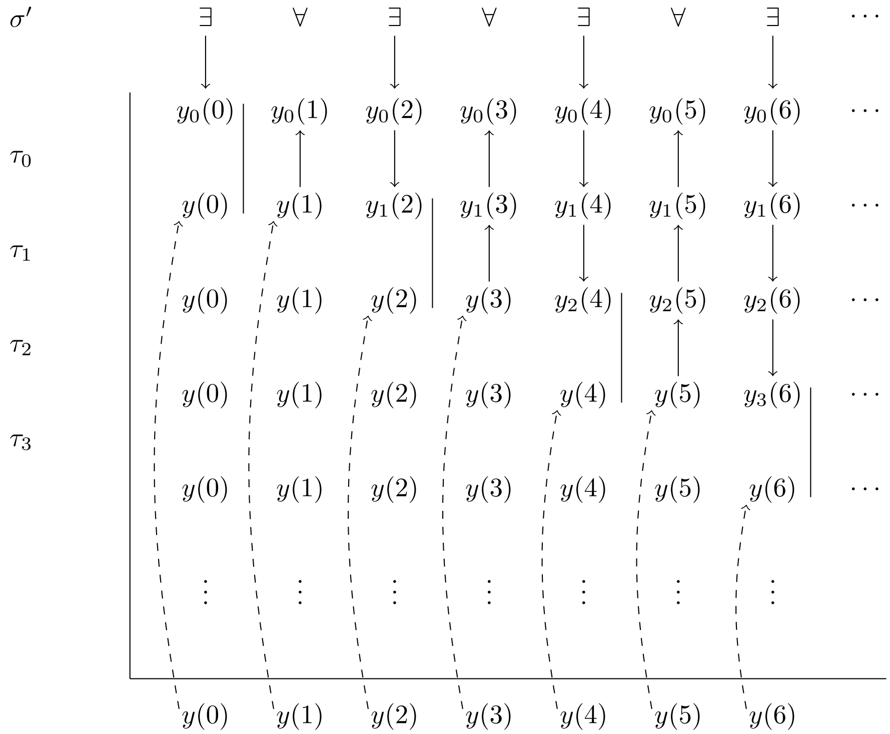

FIGURE 5. Showing the canonical strategy is winning.

definition always stays inside the corresponding W_s sets. For $(y(0), y(1), y_1(2))$ this is because the pair $s = (y_0(0), y_0(1), y_0(2))$, $t = (y(0), y(1), y_1(2))$ is according to τ_0 (if t were not a winning position for $\exists$ in the basic game $G_t(A)$, then I could defeat τ_0 starting from s, t by following a winning strategy for $\forall$ in the second board; this would give $y_1 \notin A$ which defeats τ_0). Since $y(2) \leq_{(y(0),y(1))} y_1(2)$, we can fix a winning strategy τ_1 for II in the game $G_{(y(0),y(1))}(y(2), y_1(2))$. The vertical arrows between the second and third rows of the diagram come from τ_1. Continuing, we fill-in the diagram.

Let $y_0, y_1, \dots$ be the reals produced in the diagram. Clearly $y_m \to y$. Since each τ_k is winning for II, $y_{k+1} \in A$ for all k. Also $\varphi_{2k}(y_{k+1}) \leq \varphi_{2k}(y_k)$ for all k. Since $\vec{\varphi}$ is very good, it follows that $y \in A$.

Next we prove the claim. Fix s a winning position in G_A with I to move, and let W_s again denote the set of moves a such that I can win the basic game from $s^\frown a$. Consider the relation $\leq_s$ on W_s. The transitivity of $\leq_s$ follows easily by composing strategies as in the first and second periodicity theorems.

We make the subclaim that given any sequence $a_0, a_1, \dots$ from W_s, there is a k such that $a_k \leq_s a_{k+1}$. Suppose not, and let σ_k be a winning strategy for I in the game $G_s(a_k, a_{k+1})$. Fix a winning strategy σ for I in the game $G(A)$ starting from position $s^\frown a_0$. We fill in the diagram as shown in Figure 6.

Each σ_k first makes the move $y_k(0)$. This fills in the first column of the diagram. Each σ_k then moves (viewing II's move as $y_{k+1}(0)$) $y_{k+1}(1)$. This gives the second column except for the move $y_0(1)$ which comes from the strategy σ (indicated by the

$$
\begin{array}{cc|ccccc}
\sigma & & \forall & \exists & \forall & \exists & \cdots \\
 & & & \downarrow & & \downarrow & \\
 & a_0 & y_0(0) & y_0(1) & y_0(2) & y_0(3) & \cdots \\
\sigma_0 & & & & & & \\
 & a_1 & y_1(0) & y_1(1) & y_1(2) & y_1(3) & \cdots \\
\sigma_1 & & & & & & \\
 & a_2 & y_2(0) & y_2(1) & y_2(2) & y_2(3) & \cdots \\
\sigma_2 & & & & & & \\
 & a_3 & y_3(0) & y_3(1) & y_3(2) & y_3(3) & \cdots \\
\sigma_3 & & & & & & \\
 & a_4 & y_4(0) & y_4(1) & y_4(2) & y_4(3) & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

FIGURE 6. Proving the claim.

vertical arrow from the $\exists$ to $y_0(1)$). This then repeats to fill in the entire diagram. Since σ is a winnimg strategy in $G(A)$ we have that $z_0 = s^\frown a_0{}^\frown y_0 \in A$. Since each σ_k is winning for I in $G_s(a_k, a_{k+1})$, we have that $\neg(z_k \leq^*_{|s|} z_{k+1})$ where $z_k = s^\frown a_k{}^\frown y_k$ and $z_{k+1} = s^\frown a_{k+1}{}^\frown y_{k+1}$. Since $z_k \in A$ inductively, this says $z_{k+1} \in A$. So, $z_k \in A$ for all k and also $\varphi_{|s|}(z_{k+1}) < \varphi_{|s|}(z_k)$ for all k, a contradiction.

It follows that $\leq_s$ is reflexive on W_s as otherwise the sequence $a, a, a, \dots$ would violate the above subclaim. Likewise, if $a, b \in W_s$ than we must have $a \leq_s b$ or $b \leq_s a$ as otherwise $a, b, a, b, a, b, \dots$ violates the subclaim. The subclaim also immediately shows that $\leq_s$ is wellfounded on W_s. This proves the claim, that is, shows that $\leq_s$ is a prewellordering on W_s.

So far we have shown that the canonical quasistrategy for I in G_A is a winning quasistrategy. We next consider the complexity of this strategy. Let $\leq^*_s$, $<^*_s$ be the relations associated to the prewellordering $\leq_s$ on W_s.

Let $G'_s(a, b)$ be the game, similar to $G_s(a, b)$, played as shown in Figure 7. Here I makes the first move $x_1(0)$, and then II responds with $x_0(0)$ (these are their "offensive" moves on their opponent's boards; here we view I's home board as the one corresponding to $s^\frown a$ as shown). I then makes the move $x_0(1)$ followed by II's move of $x_1(1)$ (these are their "defensive" moves). I wins the run of the game iff $z_0 <^*_s z_1$ where $z_0 = s^\frown a^\frown x_0$ and $z_1 = s^\frown b^\frown x_1$.

Claim. For all a, b, we have $a \leq^*_s b$ iff II has a winning strategy in the game $G_s(a, b)$. Also, $a <^*_s b$ iff I has a winning strategy in $G'_s(a, b)$.

To prove the claim, first suppose that II has a winning strategy in $G_s(a, b)$. We must have that $a \in W_s$ (i.e., I can win $G(A)$ starting from $s^\frown a$) as otherwise II can make the moves $x_0(0), x_0(2), x_0(4), \dots$ in $G_s(a, b)$ according to a winning strategy for $\forall$ in $G(A)$. This would result in a play of $G_s(a, b)$ where $z_0 \notin A$, where

$$G'_s(a,b)\qquad \begin{array}{ccc|cccc} \mathrm{II} & s & b & x_1(0) & x_1(1) & x_1(2) & \cdots \\ & & & \downarrow & \uparrow & \downarrow & \\ \mathrm{I} & s & a & x_0(0) & x_0(1) & x_0(2) & \cdots \end{array}$$

FIGURE 7. The game $G'_s(a,b)$.

$z_0 = s^\frown a^\frown x_0$. Thus, $\neg(z_0 \leq^*_s z_1)$, contradicting that II has won $G_s(a,b)$. So, $a \in W_s$. If $b \notin W_s$ then $a \leq^*_s b$ holds by definition. If $b \in W_s$ as well, then by definition we have $a \leq_s b$, and so $a \leq^*_s b$ holds.

Next suppose that $a \leq^*_s b$. By definition we have $a \in W_s$. If $b \notin W_s$ then II can win $G_s(a,b)$ as follows: II plays (makes the moves $x_1(2k)$) in the top board (the $s^\frown b$ board) to ensure $z_1 \notin A$, which is possible since $b \notin W_s$. II plays (makes the moves $x_0(2k+1)$) in the bottom board to ensure $z_0 \in A$, which is possible since $a \in W_s$. We then have $z_0 \leq^*_{\varphi_{|s|}} z_1$. If $b \in W_s$ as well, then by definition $a \leq_s b$, that is, II wins $G_s(a,b)$.

Now we consider the relation $<^*_s$. First assume that I has a winning strategy in the game $G'_s(a,b)$ (we refer to Figure 7). We must have $a \in W_s$ as otherwise II can make the moves $x_0(2k)$ according to a strategy to ensure $z_0 \notin A$. This will give $\neg(z_0 <^*_{\varphi_{|s|}} z_1)$, contradicting I winning the game. So, $a \in W_s$. If $b \notin W_s$ then $a <^*_s b$ holds by definition. If $b \in W_s$ as well, then we must show that $a \leq_s b$ and $\neg(b \leq_s a)$ that is, II wins $G_s(a,b)$ and II doesn't win $G_s(b,a)$. A winning strategy σ for I in $G'_s(a,b)$ immediately gives a strategy for II in $G_s(a,b)$, as II just uses the same strategy σ, but ignores the initial moves of I each round of the game. If II also won $G_s(b,a)$ by τ, then we have a contradiction by playing σ and τ against each other. Inspection of Figures 4, 7 (where in Figure 4 we switch a and b now) shows that this makes sense. We then have $z_0 <^*_{\varphi_{|s|}} z_1$ as σ is winning, and $z_1 \leq^*_{\varphi_{|s|}} z_0$ as τ is winning, a contradiction. So, $a <^*_s b$ holds in all cases.

Finally, assume $a <^*_s b$. In particular, $a \in W_s$. We must show that I wins $G'_s(a,b)$. If $b \notin W_s$ then I can win $G'_s(a,b)$ by making the moves $x_1(2k)$ (of Figure 7) according to a strategy to ensure $z_1 \notin A$, and making the moves $x_0(2k+1)$ according to a strategy to ensure $z_0 \in A$. So, assume $b \in W_s$ as well. So, $\psi_s(a) < \psi_s(b)$. Assume towards a contradiction that II wins $G'_s(a,b)$ by τ. Since $\neg(\psi_s(b) \leq \psi_s(a))$, we have that I wins $G_s(b,a)$, say by σ. Fix also a strategy σ' for I in the game $G_{s^\frown a}(A)$. We then fill in the diagram as shown in Figure 8.
In the first column, the moves $y_{2k+1}(0)$ are first made by σ. The moves $y_{2k}(0)$ are then made by τ is response. For the second column, all the moves $y_{2k}(1)$ are made by σ except for $y_0(1)$ which is made by σ'. The $y_{2k+1}(1)$ are then all made by τ in response. This cycle of filling in the moves then repeats. Let $z_{2k} = s^\frown a^\frown y_{2k}$ and $z_{2k+1} = s^\frown b^\frown y_{2k+1}$. As σ' is winning for $G_{s^\frown a}(A)$ we have $z_0 \in A$. It then follows that all of the z_i are in A and $\varphi_{|s|}(z_0) \geq \varphi_{|s|}(z_1) > \varphi_{|s|}(z_2) \geq \varphi_{|s|}(z_3) \cdots$, a contradiction.

It follows immediately from the claim that all of the norms ψ_s on W_s are $\Game\Gamma$ norms. An easy computation now shows that the best winning strategy σ for $G(A)$ is a $\Game\Gamma$-real, where we define

$$\sigma(s) = a \leftrightarrow [(\forall b\ a \leq^*_s b) \wedge \forall b < a\ (a <^* b)]$$

$$
\begin{array}{cc|ccccc}
\sigma' & & \forall & \exists & \forall & \exists & \cdots \\
 & & & \downarrow & & \downarrow & \\
 & a & y_0(0) & y_0(1) & y_0(2) & y_0(3) & \cdots \\
\tau & & \uparrow & \downarrow & \uparrow & \downarrow & \\
 & b & y_1(0) & y_1(1) & y_1(2) & y_1(3) & \cdots \\
\sigma & & & & & & \\
 & a & y_2(0) & y_2(1) & y_2(2) & y_2(3) & \cdots \\
\tau & & \uparrow & \downarrow & \uparrow & \downarrow & \\
 & b & y_3(0) & y_3(1) & y_3(2) & y_3(3) & \cdots \\
\sigma & & & & & & \\
 & a & y_4(0) & y_4(1) & y_4(2) & y_4(3) & \cdots \\
 & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

FIGURE 8. Showing I wins $G'_s(a,b)$ from $a <^*_s b$.

As $\Game\Gamma$ is easily closed under $\wedge, \vee$, we have that σ is a $\Game\Gamma$ strategy.

This completes the proof of the third periodicity theorem. $\square$