1. Determinacy

We introduce the axiom of determinacy, which provides a powerful strengthening of ZF. Although the full axiom is inconsistent with AC, various restrictions of it are, and provide a way to extend the ZF theory of the reals presented earlier. We begin with the basic terminology. We emphasize that out background theory throughout this discussion is ZF set theory.

Let $X$ be a set. By a game on $X$ we mean a setting where two players, called I and II alternate playing elements from the set $X$ as follows:

$$
\begin{align*}
\text{I} & & x_0 & & x_2 & & x_4 & & \ldots \\
\text{II} & & x_1 & & x_3 & & x_5 & & \ldots 
\end{align*}
$$

The sequence $\vec{x} = (x_0, x_1, \ldots)$ that they jointly produce is called a run of the game. The winning condition for the game is given by a payoff set $A \subseteq X^\omega$. We say I wins the run of the game if $\vec{x} \in A$, and otherwise II wins the run. We frequently denote the game with payoff set $A$ by $G_A$.

A strategy for player I (respectively player II) is a function $\sigma$ from the sequences in $X^{<\omega}$ of even length (respectively odd length) into $X$. If $\sigma$ is a strategy for I, we say a run $\vec{x}$ is according to $\sigma$ (or I has followed $\sigma$) if for all even $n \in \omega$, $x_n = \sigma(\vec{x} \upharpoonright n)$. The definition for a strategy for II is similar, using “odd” instead of “even.” If $\sigma$ is a strategy for I, and $\vec{x} \in X^\omega$, we let $\sigma(\vec{x}) \in X^\omega$ be the run according to $\sigma$ where II has made moves $\vec{z} = (z_0, z_1, \ldots)$. Thus, $\sigma(\vec{x}) = (\sigma(0), z_0, \sigma(\sigma(0), z_0), z_1, \ldots)$.

We likewise define $\tau(\vec{x})$ when $\tau$ is a strategy for II. We let $\sigma_0(\vec{x})$ denote the sequence of moves I makes against II play of $\vec{x}$, and likewise $\tau_1(\vec{x})$ denotes II’s moves against I’s play of $\vec{x}$.

Exercise 1. Let $d$ be the standard metric on $\omega^\omega$, that is, $d(x, y) = \frac{1}{2^n}$ where $n \in \omega$ is least such that $x(n) \neq y(n)$ (and of course $d(x, y) = 0$ if $x = y$). Let $\sigma$ be a strategy for I or II in an integer game. Show that the map $x \mapsto \sigma(x)$ from $\omega^\omega$ to $\omega^\omega$ is one-to-one and Lipschitz continuous, that is, $d(\sigma(x), \sigma(y)) \leq d(x, y)$.

Exercise 2. Show that for any strategy $\sigma$ for I or II in an integer game that $\sigma[\omega^\omega] = \{\sigma(x) : x \in \omega^\omega\}$ perfect. Show that $\sigma[\omega^\omega]$ is $\Sigma^1_1$. Give an example to show $\sigma[\omega^\omega]$ is not necessarily Borel [hint: use the fact that every $\Sigma^1_1$ set $A$ in $\omega^\omega$ is the projection of a closed set in $\omega^\omega \times \omega^\omega$. It may help to assume that $A$ is dense in $\omega^\omega$.]

We say a strategy for I (respectively II) is a winning strategy if for all runs $\vec{x}$ according to $\sigma$, $\vec{x} \in A$ (respectively, $\vec{x} \notin A$). We say the game $G_A$ is determined if one of the players has a winning strategy.

Exercise 3. Show that for any game $G_A$ on any set $X$, that it cannot be the case that both players have a winning strategy.

If $X$ cannot be wellordered in ZF (e.g., $X = \mathbb{R}$), then the notion of a winning strategy is too strong. One uses instead the notion of a quasi-strategy. A quasi-strategy $\sigma$ for I is a function with domain the set of $s \in X^{<\omega}$ of even length and such that $\sigma(s)$ is a non-empty subset of $X$. The definition for II is similar. Thus, a
quasi-strategy is like a strategy except that instead of giving a single move for the player, it produces a non-empty set of possible moves. We say \( \vec{x} = (x(0), x(1), \ldots) \) is a run according to \( \sigma \) if for all even \( n \), \( x(n) \in \sigma(x \upharpoonright n) \). Likewise for II. The notion of a winning quasi-strategy is then defined in the obvious manner. We say \( A \subseteq X^\omega \) is quasi-determined if one of the players has a winning quasi-strategy. We sometimes still use the term “determined” in this case. To illustrate the difference, suppose there is a relation \( R \subseteq \omega^\omega \times \omega^\omega \) which has no uniformization (this will be the case, for example, assuming \( \text{AD} + V = L(\mathbb{R}) \)). Consider the game on \( X = \mathbb{R} \) where I plays \( x \) in the first move, and II plays \( y \) in the next move. All moves after this are irrelevant. II wins the run iff \( R(x, y) \). Then II has a winning quasi-strategy, namely \( R \) itself, but has no winning strategy as a winning strategy for II produces a uniformization for \( R \).

**Exercise 4.** We say a game on a set \( X \) is finite if there is an \( n \in \omega \) such that the payoff only depends on \( \vec{x} \upharpoonright n \). Show that any finite game on any set \( X \) is determined. [hint: Say the game is of length \( \omega \) for either I or II in integer games). We define inductively sets \( A_n \) determined. 

We define inductively sets \( A_n \) determined. 

**Lemma 1.3.** \( \text{AD} \) is inconsistent with \( \text{ZFC} \).

**Proof.** By \( \text{AC} \), let \( \{x_\alpha\}_{\alpha < \omega^\omega} \) enumerate the reals \( \omega^\omega \) (and hence all the strategies for either I or II in integer games). We define inductively sets \( A_\alpha, B_\alpha \subseteq \omega^\omega \) which are monotonically increasing (i.e., \( \alpha < \beta \rightarrow A_\alpha \subseteq A_\beta \)) and disjoint at each step (i.e., \( A_\alpha \cap B_\alpha = \emptyset \)). We also assume inductively that \( |A_\alpha|, |B_\alpha| < \omega^\omega \). We think of the reals in \( A_\alpha \) as being reals we wish to add to the set we are building, and the
reals in $B_\alpha$ as those we wish to forbid from being in our set. At stage $\alpha$, first let $A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta$ and $B_{<\alpha} = \bigcup_{\beta<\alpha} B_\beta$. Note that $A_{<\alpha} \cap B_{<\alpha} = \emptyset$, assuming the above inductive hypotheses. Consider then $x_\alpha$. If $x_\alpha$ is not a strategy for I or II, let $A_\alpha = A_{<\alpha}$, $B_\alpha = B_{<\alpha}$. Since $x \mapsto \sigma(x)$ is one-to-one, $C \doteq \{ z : \sigma(z) \in A_{<\alpha} \cup B_{<\alpha} \}$ has size $< 2^\omega$. Let $z \in \omega^\omega - C$. If $x_\alpha$ is a strategy for I, let $A_\alpha = A_{<\alpha}$, $B_\alpha = B_{<\alpha} \cup \{ x_\alpha(z) \}$. If $x_\alpha$ is a strategy for II, let $A_\alpha = A_{<\alpha} \cup \{ x_\alpha(z) \}$, $B_\alpha = B_{<\alpha}$. Since $z \notin C$, this maintains our disjointness hypothesis. Clearly $x_\alpha$ cannot be a winning strategy for I for any set disjoint from $B_\alpha$, and cannot be a winning strategy for II for any set containing $A_\alpha$. Let $A = \bigcup_{\alpha<2^\omega} A_\alpha$. So, $A \cap B = \emptyset$, where $B = \bigcup_{\alpha<2^\omega} B_\alpha$. So, no $x \in \omega^\omega$ can be a winning strategy for either I or II in $G_A$.

In view of lemma 1.3, in working with the full axiom of determinacy AD we work in the background theory $ZF + AD$. In fact, a weak form of choice, DC, is frequently also added to the background theory; we discuss this in more detail below. Actually, AD implies a weak form of choice, namely countable choice for reals. We recall the definitions.

**Definition 1.4.** For any cardinal $\kappa$, $AC_\kappa$ is the statement that for any $\kappa$-sequence $\{ A_\alpha \}_{\alpha<\kappa}$ of non-empty sets, there is a function $f$ with domain $\kappa$ such that $\forall \alpha < \kappa \, ( f(\alpha) \in A_\alpha )$. *Countable choice* is the statement $AC_\omega$. Countable choice for reals is the statement $AC_\omega$ restricted to sequences $\{ A_n \}_{n\in\omega}$ for which $A_n \subseteq \omega^\omega$ for all $n$.

**Lemma 1.5.** AD implies countable choice for reals.

**Proof.** Let $\{ A_n \}_{n\in\omega}$ be given where each $A_n \subseteq \omega^\omega$ is non-empty. Consider the game

$I$ $n$

II $x(0)$ $x(1)$ $x(2)$ …

where I plays an integer $n$ and II plays out a real $x$. II wins the run if $x \in A_n$. Clearly I cannot have a winning strategy $\sigma$ since as soon as I plays $n$, can play any $x \in A_n$ to defeat $\sigma$. A winning strategy $\tau$ for II gives a choice function.

A small variation of this argument shows the following.

**Lemma 1.6.** Assume AD. Then countable choice holds in $L(\mathbb{R})$.

**Proof.** There is a definable map $\pi$ in $L(\mathbb{R})$ from $\omega^\omega \times \text{On}$ onto $L(\mathbb{R})$. Given the sequence $\{ A_n \}_{n\in\omega}$ of non-empty sets, let $A'_n = \{ (x, \alpha) \in \omega^\omega \times \text{On} : \pi(x, \alpha) \in A_n \}$. It clearly suffices to get a choice function for the $A'_n$. For each $n$, let $\alpha_n \in \text{On}$ be least such that $\exists x \in A'_n$. Let $A''_n = \{ x \in \omega^\omega : (x, \alpha_n) \in A'_n \}$. From lemma 1.5, let $\langle x_n \rangle_{n\in\omega}$ be a choice function for the $A''_n$. Then $\pi(x_n, \alpha_n)_{n\in\omega}$ is a choice function for the $A_n$.

The axiom of dependent choice is a strengthening of countable choice.

**Definition 1.7.** DC is the following statement. Let $X$ be a set and $R \subseteq X^{\lt \omega}$ such that for all $s \in R$ there is an $x \in X$ with $s \cup x \in R$. Then there is a $\vec{x} \in X^\omega$ such that $\forall n \, \vec{x} \upharpoonright n \in R$. We let DC$_X$ be the statement of DC for sets $R \subseteq X^{\lt \omega}$.

**Exercise 5.** Show that DC implies countable choice.
In fact, AD implies that DC holds in $L(\mathbb{R})$ (Kechris). This shows that if AD is consistent, then so is AD + DC. The proof of this theorem requires more techniques.

DC is equivalent to the assertion that every ill founded tree on a set $X$ has a branch. In fact the following exercise shows a bit more.

**Exercise 6.** Show that DC is equivalent to the assertion that every relation $R$ on a set $X$ which is illfounded has an infinite decreasing sequence. Recall a relation is wellfounded if every non-empty subset of $X$ has an $R$-minimal element. [hint: If $R$ is illfounded, let $S \subseteq X^{<\omega}$ be the set of sequences $(x_0, x_1, \ldots, x_n)$ such that $x_nRx_{n-1}R \ldots Rx_1Rx_0$ (i.e., the chain is $R$-decreasing) and $\{x : xRx_n\}, R$ is ill-founded. Apply DC to $S$.]

If $X$ cannot be wellordered in ZF, then we need DC to even produce a run according to a given quasi-strategy. More precisely, we need DC$_X$ to produce such a run.

We consider next the situation for other $X$. We let AD$_X$ denote the assertion that every game on $X$ (i.e., $A \subseteq X^\omega$) is determined. So, AD = AD$_\omega$. First note the following simple fact.

**Exercise 7.** Show that if $X \subseteq X'$ then AD$_{X'} \Rightarrow$ AD$_X$.

In particular, AD$_\omega \Rightarrow$ AD$_2$. The converse is also true by the following lemma.

**Lemma 1.8.** AD$_2 \Rightarrow$ AD$_\omega$.

**Proof.** Assume AD$_2$, and let $A \subseteq \omega^\omega$. We show that $G_A$ is determined. We define an $A' \subseteq 2^\omega$ which simulates the game $G_A$, and such that whichever player has a winning strategy in $G_{A'}$ has one in $G_A$. Call a finite sequence $s \in 2^{<\omega}$ good if it is an initial segment of a sequence of the form

$$0^{2a_0} \cdot (1, 0)^{2a_1} \cdot (0, 1)^{2a_2} \cdot (1, 0)^{2a_3} \cdot \ldots$$

In other words, $s$ is good if it is a partial play of a game on $\{0, 1\}$ of the form:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0, 0, . . . , 0</td>
<td>0, 0, . . . , 0</td>
</tr>
<tr>
<td>$a_0$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0, 0, . . . , 0</td>
<td>0, 0, . . . , 0</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0, 0, . . . , 0</td>
<td>0, 0, . . . , 0</td>
</tr>
<tr>
<td>$a_1$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

We now define $G_{A'}$. The first player to play so that the sequence is not good loses. Suppose they produce $x \in 2^\omega$, and all initial segments of $x$ are good. If $x$ is eventually equal to 0, then the last player to play a 1 wins (if they both play all 0’s then II wins). Otherwise, consider the sequence $y = (a_0, a_1, \ldots) \in \omega^\omega$. Then I wins the run of $G_{A'}$ iff $y \in A$. Suppose $\sigma'$ is a winning strategy for I in $G_{A'}$ (the case for II is similar). We define a strategy $\sigma$ for I in $G_A$ as follows. To define $\sigma(\varnothing)$, have I follow $\sigma'$ in $G_{A'}$ where II plays 0’s. I must eventually play a 1 as otherwise I loses by definition. If there have been $a_0$ rounds of 0’s played before the round where I plays a 1, then we let $\sigma(\varnothing) = a_0$.

To define $\sigma(a_0, a_1)$, have II play $a_1$ more 0’s followed by a 1. Following $\sigma'$, I must respond with a 0 to all of these moves, as otherwise I loses, contradicting $\sigma'$ being a winning strategy. Have II then continue to play 0’s until I plays a 1, which I
must do as otherwise I loses by definition. This defines \( a_2 = \sigma(a_0, a_1) \) as the above diagram illustrates. Continuing in this manner defines the strategy \( \sigma \).

For any run \( x = (a_0, a_1, \ldots) \) of \( \sigma \), there is a corresponding run \( x' \) of \( \sigma' \) where \( x' \) is good and both players have played infinitely many 1's. Furthermore, \( x' \) and \( x \) correspond as in the diagram above. So, by definition, \( x' \in a' \) iff \( x \in A \). Since \( \sigma' \) is a winning strategy for I in \( G_{A'} \), \( x' \in A' \). Thus, \( x \in A \).

\[ \square \]

**Corollary 1.9.** \( \text{AD} \) is equivalent to \( \text{AD}_X \) for any countable set \( X \) with at least 2 elements.

Determinacy is not symmetrical between the two players in the sense that I having a winning strategy for a game \( G_A \) is not the same as II having a winning strategy for the complement \( G_{A'} \). However, the asymmetry is minor according to the following lemma.

**Lemma 1.10.** If \( \Gamma \) is any pointclass then \( \det(\Gamma) \Leftrightarrow \det(\bar{\Gamma}) \). In fact for any \( A \subseteq \omega^\omega \) there is a \( B \) recursively reducible to \( \omega^\omega - A \) such that I (resp. II) has a winning strategy in \( G_B \) iff II (resp. I) has a winning strategy in \( G_A \).

**Proof.** Given \( A \subseteq \omega^\omega \), let \( B = \{x: x(n) = x(n + 1)\} \), where \( x' = x(n + 1) \). If \( \sigma \) is a winning strategy for I in \( G_A \), then \( \sigma' \) is winning for II in \( G_B \), where \( \sigma'(a_0, a_1, a_2, \ldots) = \sigma(a_1, a_2, \ldots) \), that is, II ignores I's first move and then follows \( \sigma \). Also, if \( \sigma' \) is winning for II in \( G_{A'} \), then \( \sigma \) is winning for I in \( G_A \).

Similarly, if \( \tau \) is winning for II in \( G_A \) iff \( \tau' \) is winning for I in \( G_{A'} \) where \( \tau' \) makes an arbitrary first move, and then follows \( \tau \).

Thus, \( G_A \) is determined iff \( B_B \) is determined. The first statement of the lemma follows.

\[ \square \]

**Exercise 8.** Show assuming \( \text{ZFC} \) that there are two determined games \( A \) and \( B \) such that \( A \cap B \) is not determined, and likewise for unions. [hint: Let \( C \subseteq \omega^\omega \) be a non-determined game. Let \( A = \{x: x(0) \text{ is even } \lor x' \in C\} \). Let \( B = \{x: x(0) \text{ is odd } \lor x' \in C\} \). Then I wins both \( G_A \) and \( G_B \), but \( G_{A \cap B} \) is not determined.

Two natural generalizations of \( \text{AD} = \text{AD}_\omega \) suggest themselves. One is allowing \( X \) to be a larger ordinal (we must go to at least \( \omega_1 \) to have something potentially different). Another is to allow the players to play reals (since integers can be viewed as simple reals).

Regarding the ordinal case, we have the following lemma. In the proof, we will borrow from an upcoming fact, namely that \( \text{AD} \) implies the perfect set property for all sets of reals.

**Lemma 1.11 (ZF).** For any uncountable well-ordered set \( X \), the axiom \( \text{AD}_X \) is inconsistent.

**Proof.** Assuming \( \text{ZF} \), it suffices to show that \( \text{AD}_{\omega_1} \) is inconsistent (from exercise 7). Suppose \( \text{AD}_{\omega_1} \). From exercise 7 we also have \( \text{AD} \). Consider the following game on \( \omega_1 \):

\[
\begin{array}{c|ccc}
I & \alpha \\
\hline
II & x(0) & x(1) & x(2) & \cdots
\end{array}
\]
where I plays an ordinal $\alpha < \omega_1$, and plays integers $x(0), x(1), \ldots$, thereby playing out a real $x \in \omega^\omega$. II wins iff $x \in \text{WO}$ and $|x| = \alpha$. First note that I cannot have a winning strategy $\sigma$, since if $\sigma$ calls for I to play $\alpha$, II can defeat $\sigma$ by playing any $x \in \text{WO}$ with $|x| = \alpha$. So, since we are assuming this game is determined, II must have a winning strategy $\tau$. Let $A = \{\tau(\alpha) : \alpha < \omega_1\}$. Then $A$ is a wellordered subset of $\text{WO}$ of size $\omega_1$ (for each $\alpha < \omega_1$ there is a unique $x \in A$ with $|x| = \alpha$). We will show in theorem 2.2 below that under $\text{AD}$, every uncountable subset of $\omega^\omega$ contains a perfect set. So, let $P \subseteq A$ be perfect. In particular $P \subseteq A$ is closed in $\omega^\omega$. But every $\Sigma^1_1$ subset of $\text{WO}$ codes only boundedly many ordinals since $\phi(x) = |x|$ is a $\Pi^1_1$-norm on $\text{WO}$.

Although the full axiom $\text{AD}_\kappa$ for uncountable $\kappa$ is inconsistent, nevertheless ordinal games are very useful in determinacy theory. We will see later that important classes of ordinal games are determined.

When $X = \mathbb{R}$, the axiom $\text{AD}_X$ becomes a powerful strengthening of $\text{AD}$. $\text{AD}_\mathbb{R}$, as we will see later, is strictly stronger than $\text{AD}$. Its extra strength, however, does not generally become apparent until one goes beyond the model $L(\mathbb{R})$, and so will not directly concern us for a while.

The next result is one of the most basic results in determinacy theory and is used in many arguments. For $X$ a set, we topologize $X^\omega$ by giving $X$ the discrete topology and $X^\omega$ the corresponding product topology. So, a basic open set is of the form $N_s = \{\bar{x} \in X^\omega : (\bar{x} \upharpoonright \text{lh}(s)) = s\}$ where $s \in X^{<\omega}$.

**Theorem 1.12** (Gale-Stewart). Assume $\text{ZF}$. Then for any set $X$, and any open (or closed) $A \subseteq X^\omega$ the game $G_A$ is (quasi) determined.

**Proof.** Let $E$ be the set of all $s = (s(0), \ldots, s(2n - 1))$ of even length. We define a subset $W \subseteq E$ of winning positions for $I$. Let $W_0 = \{s \in e : N_s \subseteq A\}$. In general, for $\alpha \in \text{On}$, first let $W_{<\alpha} = \bigcup_{\beta < \alpha} W_\beta$. Then set

$$W_\alpha = W_{<\alpha} \cup \{s \in E : \exists x \in X \forall y \in X (s \upharpoonright x \upharpoonright y \in W_{<\alpha})\}.$$ 

Let $\theta$ be the least ordinal such that $W_\theta = W_{<\theta}$. So, $W_\alpha = W_\theta$ for all $\alpha \geq \theta$. Let $W = W_\theta$. For $s \in W$, let $|s|$ be the least ordinal $\alpha \leq \theta$ such that $s \in W_\alpha$. Note that if $s \in W$ and $|s| > 0$, then $\exists x \forall y (|s \upharpoonright x \upharpoonright y| < |s|).

Suppose first that $\varnothing \in W$. Then $I$ has a winning strategy $\sigma$ in $G_A$. Namely, let $(s(0), s(1), \ldots, s(2n - 1)) \in \Sigma$ iff for all $i < n$ either $s \upharpoonright (2i) \in W_0$ or $|s \upharpoonright 2i| > |s \upharpoonright (2i + 2)|$. From the fact mentioned at the end of the previous paragraph, $\sigma$ is a quasi-strategy for $I$. From the wellfoundedness of $\text{On}$ it follows that $I$ is a winning quasi-strategy. That is, for any $\bar{s}$ a run according to $\Sigma$, we have the ranks $|s \upharpoonright 2i|$ decrease until we reach an $i$ such that $s \upharpoonright 2i \in W_0$, and thus $\bar{s} \in A$.

Suppose next that $\varnothing \notin W$. We claim that $\Pi$ has a winning quasi-strategy $\Sigma$ in $G_A$. Namely, let $\Sigma = E - W$. Note that $\varnothing \in \Sigma$. Also, if $s \in \Sigma$ then $\forall x \exists y (s \upharpoonright x \upharpoonright y \in \sigma)$. For if $s \in \sigma$ and $\exists x \forall y (s \upharpoonright x \upharpoonright y \notin \sigma)$, then $\exists x \forall y (s \upharpoonright x \upharpoonright y \in W)$. So, $s \in W_\theta + 1 = W_\theta = W$, a contradiction. So, $\Sigma$ is a quasi-strategy for $\Pi$. If $\bar{s} \in X^\omega$ is a run according to $\Sigma$, then $\bar{s} \notin A$. For if $\bar{s} \in A$, then for some $i$ we would have $N_{\bar{s}}(2i) \subseteq A$ as $A$ is open. But then $\bar{s} \upharpoonright 2i \in W_0 \subseteq W$, a contradiction (as $\bar{s} \upharpoonright 2i \in \Sigma = E - W$).

As an immediate corollary we have that any open or closed game on an ordinal is determined.
The proof of theorem 1.12 produces for any open or closed game $G_A$ a canonical winning quasi-strategy for $G_A$. This does not use any form of choice.

An important and deep theorem of Martin says that every Borel game on any set is determined. We will give this proof later, but for now we just extend theorem 1.12 one step further.

**Theorem 1.13 (Wolff).** Assume ZF. Every $\Sigma^0_2$ game on a set $X$ is (quasi) determined.

**Proof.** Let $A \subseteq X^\omega$ be $\Sigma^0_2$, say $A = \bigcup F_n$ where each $F_n$ is closed in $X^\omega$. Let $T_n$ be the canonical tree on $X$ such that $F_n = [T_n]$. We again define a set $W$ of winning positions for $I$. Again let $E$ be the set of all $s \in X^\omega$ of even length. Let $W_0$ be the set of all $s \in E$ such that there is an $n \in \omega$ such that $I$ has a winning quasi-strategy starting at $s$ for the game $F_n$. For general $\alpha$, let $W_{<\alpha} = \bigcup_{\beta<\alpha} W_\beta$ and then define $W_\alpha$ to be $W_{<\alpha}$ together with the set of $s \in E$ such that for some $i$, $I$ has a winning quasi-strategy starting at $s$ for $A^i_{<\alpha}$, where $A^i_{<\alpha}$ is the set of $x \in X^\omega$ such that either $\forall j x \upharpoonright 2j \in T_j$ or for the least $j$ such that $x \upharpoonright 2j \notin T_j$ we have $x \upharpoonright 2j \in W_{<\alpha}$. Note that $A^i_{<\alpha}$ is a closed set, so is quasi-determined.

Define $\theta$ and $W = W_\theta = W_{<\theta}$ as in theorem 1.12. Also as before define, for $s \in W$, $|s|$ to be the least $\alpha < \theta$ such that $s \in W_\alpha$.

First suppose $\emptyset \in W$, and we define a winning quasi-strategy $\Sigma$ for $I$ in $G_A$. We give an informal description of $\Sigma$, the underlying formal definition will be apparent. $s = (s(0), s(1), \ldots, s(2n-1))$ is according to $\Sigma$ provided the following holds. Let $n_0$ be least such that $I$ has a winning quasi-strategy starting at $\emptyset$ in $A^0_{<\theta_0}$. I follows the canonical winning quasi-strategy for this closed game until an $i_0$ is reached (if it is) such that $s \upharpoonright 2i_0 \notin T_{n_0}$. If this happens, then $s \upharpoonright 2i_0 \in W_{<\theta}$. Let $\theta_0 < \theta$ be least such that $s \upharpoonright 2i_0 \in W_{\theta_0}$. Let $n_1$ be least such that $I$ has a winning quasi-strategy in $A^0_{<\theta_0}$ starting from $s \upharpoonright 2i_0$. I then follows the canonical winning quasi-strategy for this game until an $i_1 > i_0$ is reached (if it is) such that $s \upharpoonright 2i_1 \notin T_{n_1}$. Thus, $s \upharpoonright 2i_1 \in W_{<\theta_0}$. Let $\theta_1 < \theta_0$ be least such that $s \upharpoonright 2i_1 \in W_{\theta_1}$. $\Sigma$ continues in this manner. If $\tilde{s}$ is a run according to $\Sigma$, then for some $k$ we have that $\tilde{s}$ is a run starting from $\tilde{s} \upharpoonright 2k$ which stays in the tree $T_{n_k}$ (as otherwise we get an infinite decreasing sequence of $\theta_i$). Thus, $\tilde{s} \in F_{n_k}$, so $\tilde{s}$ is a win for $I$.

Suppose next that $\emptyset \notin W$. We describe a winning quasi-strategy for $II$ in $G_A$. Start with $n = 0$. Since $\emptyset \notin W_\theta$, $II$ has a canonical winning strategy for the game $A^0_{<\theta}$ starting from $\emptyset$. II follows this strategy until a least $i_0$ is reached such that $s \upharpoonright 2i_0 \notin T_0$ and $s \upharpoonright 2i_0 \notin W_{<\theta} = W_{\emptyset}$. This must happen as $II$ is winning for $A^0_{<\theta}$. Consider then $n = 1$. Since $s \upharpoonright 2i_0 \notin W_{\emptyset}$, $II$ has a winning quasi-strategy for the game $A^1_{<\theta}$ starting from $s \upharpoonright 2i_0$. II follows this canonical strategy until an $i_1$ is reached such that $s \upharpoonright 2i_1 \notin T_1$ and $s \upharpoonright 2i_1 \notin W_{<\theta} = W_{\emptyset}$. This describes $\Sigma$. If $\tilde{s}$ is a run according to $\Sigma$, then clearly $\tilde{s} \notin [T_n] = F_n$ for all $n$, and thus $II$ wins the run.

2. **Regularity Results**

In this section we show that AD implies regularity properties for sets of reals, that is, AD eliminates the pathological sets produced from AC.

First we consider the perfect set property. The basic game argument is best illustrated on the Cantor space.
Lemma 2.1 (ZF + AD). Every $A \subseteq 2^\omega$ is either countable or else contains a perfect subset.

Proof. Given $A \subseteq 2^\omega$ we consider the following game $G^*_A$:

\[
\begin{array}{c|cccc|}
\text{I} & s_0 & s_1 & s_2 & \ldots \\
\hline
\text{II} & i_0 & i_1 & i_2 & \ldots \\
\end{array}
\]

where I plays sequences $s_k \in 2^{<\omega}$ (possibly empty) and II plays $i_k \in \{0,1\}$. I wins the run iff $x = s_0 \upharpoonright i_0 \cap s_1 \upharpoonright i_1 \cap \cdots \in A$. If I has a winning strategy $\sigma$, then clearly $\sigma[2^\omega] = \{x = s_0 \upharpoonright i_0 \cap s_1 \upharpoonright i_1 \cap \ldots : (s_0, i_0, \ldots)\}$ is according to $\sigma$ is a perfect set contained in $A$. Suppose then that II has a winning strategy $\tau$. Consider $x \in A$. We say a partial run $(s_0, i_0, \ldots, s_n, i_n)$ is good for $x$ if it is according to $\tau$ and $(s_0 \upharpoonright i_0 \cap \cdots \cap s_n \upharpoonright i_n)$ is an initial segment of $x$. There must be a maximal good sequence, since if every good sequence had a good extension then there would be a run $(s_0, i_0, \ldots)$ according to $\tau$ with $x = (s_0 \upharpoonright i_0 \cap \cdots)$. This contradicts $\tau$ being winning for II. Let $s = (s_0, i_0, \ldots, s_n, i_n)$ be maximal good for $x$. Say $s_0 \cap i_0 \cap \cdots \cap s_n \cap i_n = x \upharpoonright j$. So, for every $t \in 2^{<\omega}$, $(x \upharpoonright j) \cap t \cap (s \cap t)$ is incompatible with $x$. This, however, allows us to compute $x$ from $s$ and $\tau$. Namely, $x(j) = 1 - \tau(s \cap \emptyset)$ (by $s \cap \emptyset$ we mean the partial run consisting of $s$ and the one extra move by I consisting of $s_{n+1} = \emptyset$). Then $x(j+1) = 1 - \tau(s \cap (x(j)))$, etc. Since the set of possible $s$ is countable (and $\tau$ is a single fixed real), this shows that $A$ is countable.

As a consequence we have the following.

Theorem 2.2 (ZF + AD). Let $X$ be a Polish space. Then every $A \subseteq X$ is either countable or else contains a perfect subset.

Proof. We may clearly assume $X$ is uncountable, and thus is Borel isomorphic to $2^\omega$. Let $\pi : 2^\omega \to X$ be a Borel isomorphism. If $A \subseteq X$ is uncountable, then so is $B = \pi^{-1}(A)$. Let $P \subseteq B$ be perfect. Then $\pi(P) \subseteq A$ is uncountable and Borel, and so contains a perfect subset.

We can also prove theorem 2.2 directly on an arbitrary Polish space $X$ by a game argument. We sketch this alternate proof. Let $U$ be a countable base for $X$. Consider the game $G$ played as follows.

\[
\begin{array}{c|ccc|}
\text{I} & U_0^0 & U_1^0 & U_0^1 & U_1^1 & U_0^2 & U_1^2 & \ldots \\
\hline
\text{II} & i_0 & i_1 & i_2 & \ldots \\
\end{array}
\]

For the first move, I plays a pair $U_0^0, U_1^0 \in U$ where $U_0^0 \cap U_1^0 = \emptyset$, and $\text{diam}(U_0^0) < \frac{1}{2^0}$. II plays an $i_0 \in \{0,1\}$ signifying a choice of one of these sets. At the next round, I again plays a pair $U_0^i, U_1^i$ with $U_0^i \cap U_1^i = \emptyset$, $\text{diam}(U_0^i), \text{diam}(U_1^i) < \frac{1}{2^i}$, and $U_0^0 \subseteq U_0^i, U_1^0 \subseteq U_0^i$. II plays $i_1 \in \{0,1\}$ which picks one of these sets. The play continues in this manner, and I wins the run iff $x \in A$ where $x = \bigcap_{n} U_{i_n}^n$. If I has a winning strategy $\sigma$, it is clear that following $\sigma$ against all possible plays of II
produces a perfect subset of $A$. Suppose II has a winning strategy $\tau$. Say a partial run of the game

$$(U_0^0, U_1^0), i_0, \ldots, (U_0^n, U_1^n), i_n$$

is $x$-good if it is a run according to $\tau$ where I has followed the rules above and $x \in U_i^n$. Since $\tau$ is winning for II, there is a maximal $x$-good sequence, say $s = (U_0^0, U_1^0), i_0, \ldots, (U_0^n, U_1^n), i_n$. So, for any pair $(U_0^{n+1}, U_1^{n+1})$ satisfying the requirements for I, if $i_{n+1} = \tau(s^{-\tau}(U_0^{n+1}, U_1^{n+1}))$, then $x \notin U_{i_{n+1}}$. Let $W$ be the union of all open sets $U_{i_{n+1}}$ which are attained in this manner. So, $W \subseteq U_i^n$. Also, $x \notin W$. We claim that $U_{i_{n+1}} - W$ is a singleton. To see this, suppose $x \neq y$ are both in $U_{i_{n+1}} - W$. Let $U_0^{n+1}, U_1^{n+1}$ be disjoint basic open sets of diameters $< \frac{1}{2n+1}$ whose closures are contained in $U_{i_{n+1}}^n$, and with $x \in U_0^{n+1}$, $y \in U_1^{n+1}$. Let $i_{n+1}$ be II’s response. If $i_{n+1} = 0$, then $x \notin W$, and if $i_{n+1} = 1$, then $y \notin W$, a contradiction.

The determinacy used in theorem 2.2 was local, that is, to get the perfect set property for a pointclass $\Gamma$ requires only $\text{det}(\Gamma)$. In fact, we can improve this a little using an “unfolding argument” for an existential quantifier.

**Theorem 2.3.** Let $\Gamma$ be a pointclass and assume $\text{det}(\Gamma)$. Then for any Polish space $X$ and every $A \subseteq X$ in $\exists^\omega \Gamma$, $A$ is either countable or else contains a perfect set.

**Proof.** We give the proof in the case $X = 2^\omega$, leaving the general case to an exercise.

Let $A \subseteq 2^\omega$ be in $\exists^\omega \Gamma$. Let $B \subseteq 2^\omega \times 2^\omega$ be such that $A(x) \leftrightarrow \exists y B(x, y)$. Consider the following “unfolded perfect set game”:

$$
\begin{array}{cccc}
I & \begin{array}{cccc}
y(0) & y(1) & y(2) & \ldots \\
s_0 & s_1 & s_2 & \ldots \\
\end{array} \\
II & \begin{array}{cccc}
i_0 & i_1 & i_2 & \ldots \\
\end{array}
\end{array}
$$

where I plays sequence $s_n \in 2^{<\omega}$ and integers $y(n) \in \omega$, and II plays $i_n \in \{0, 1\}$. Let $x = s_0^\omega \sim i_0 \sim s_1^\omega \sim i_1 \sim \cdots$, and $y = (y(0), y(1), \ldots)$. Then I wins the run iff $(x, y) \in B$. The payoff set is in $\Gamma$, and so the game is determined. If I has a winning strategy, then we clearly get a perfect set contained in $A$ ( {$(\sigma(z))_0 : z \in 2^\omega$} is a perfect set contained in $A$). Suppose that II has a winning strategy $\tau$. Fix for the moment $x \in A$, and fix $y \in \omega^\omega$ such that $(x, y) \in B$. We say a partial run $s$ according to $\tau$ is $x$-good if it is of the form $s = ((y(0), s_0), i_0, \ldots, (y(n), s_n), i_n)$ where $s_0^\omega \sim i_0 \sim \cdots s_n^\omega \sim i_n = x \upharpoonright j$ is an initial segment of $x$. Since $\tau$ is winning for II, there is a maximal $x$-good sequence. Say $s = (y(0), s_0), i_0, \ldots, (y(n), s_n), i_n)$ is maximal good. Thus, if I plays $(y(n + 1), s_{n+1})$ as the next move, where $s_{n+1} = (x(j), \ldots, x(t - 1))$, and $\tau$ responds with $i_{n+1}$, then $x(t) = 1 - i_{n+1}$. This again gives an algorithm for computing $x$ from $s$ and the fixed strategy $\tau$. Since there are only countable many possible $s$, this shows $A$ is countable. □

**Exercise 9.** Give a direct game proof of theorem 2.3 on a general Polish space. [hint: modify the previously given perfect set game on a general $X$ to include the witness $y$.]

If $A$ is $\Sigma^1_1$, then there is a closed $B \subseteq X \times \omega^\omega$ such that $A(x) \leftrightarrow \exists y B(x, y)$. In this case, the unfolded perfect set game is closed, and so it is determined in ZF.
This gives another proof of the perfect set property for $\Sigma^1_1$. On the other hand, $\det(\Pi^1_1)$ suffices to show the perfect set property for $\Sigma^1_2$. We will investigate the strength of $\det(\Pi^1_1)$ later. Note that projective determinacy give the perfect set property for all projective sets.

We next consider the Baire property, which is the topological notion of regularity. Recall from definition ?? that a set $A$ in a topological space is meager if it is a countable union of nowhere dense sets (equivalently, contained in a countable union of closed nowhere dense sets). Also, $A$ has the Baire property if there is an open set $U$ such that $A \Delta U$ is meager. An easy argument (c.f. lemma ??) shows every Borel set has the Baire property, and in theorem ?? we showed that every $\Sigma^1_1$ (or $\Pi^1_1$) set has the Baire property. This last fact will also follow by a game argument below.

We again first consider the case $X = \omega^\omega$ where the basic ideas are more clear. Let $A \subseteq \omega^\omega$. We consider the game $G^*_A$ played as follows:

\begin{align*}
\text{I} & \quad s_0 & s_2 & s_4 & \ldots \\
\text{II} & s_1 & s_3 & s_5 & \ldots
\end{align*}

where each player plays $s_i \in \omega^{<\omega}$. II wins the run iff $x \in A$ where $x = s_0 \cdot s_1 \cdot \ldots$.

**Lemma 2.4.** II has a winning strategy in $G^*_A$ iff $A$ is comeager in $\omega^\omega$. I has a winning strategy in $G^*_A$ iff there is a neighborhood on which $\omega^\omega - A$ is comeager.

**Proof.** First suppose $A$ is comeager in $\omega^\omega$. Say $A \supseteq \bigcap_n D_n$ where $D_n$ is dense open. Define the strategy $\tau$ for II as follows. Let $\tau(s_0, s_1, \ldots, s_{2n})$ be the least sequence $s_{2n+1}$ such that $N_s \subseteq D_n$, where $s = s_0 \cdot s_1 \cdot \ldots \cdot s_{2n}$. This exists since $D_n$ is dense open. Clearly $\tau$ is a winning strategy for II in $G^*_A$.

Suppose next that II has a winning strategy $\tau$ in $G^*_A$. We define a sequence $D_n$ of dense open sets. Say a sequence $s \in \omega^{<\omega}$ is 1-good if it is of the form $s = s_0 \cdot \tau(s_0)$ for some $s_0$. In general, say $s$ is n-good if it is of the form $s = s_0 \cdot s_1 \cdot \ldots \cdot s_{2n-1}$ where $(s_0, s_1, \ldots, s_{2n-1})$ is a partial run according to $\tau$.

Let $M_1$ be a maximal set of pairwise incompatible 1-good elements. Let $D_1 = \cup \{N_s: s \in M_1\}$. To see that $D_1$ is dense, consider a basic open set $N_t$. Then $u = t \cdot \tau(t)$ is 1-good, so there is an $s \in M_1$ compatible with $u$. So, $N_s \cap N_t \neq \emptyset$, and so $N_s \cap N_t \neq \emptyset$. For each 1-good sequence $s$, pick a canonical partial run $(s_0, \tau(s_0))$ with $s_0 \cdot \tau(s_0) = s$.

In general, suppose an antichain $M_n$ of $n$-good sequences has been defined and $D_n = \cup \{N_s: s \in M_n\}$ is dense. Assume also that for each $s \in M_n$ we have defined a canonical partial run $(s_0, s_1, \ldots, s_{2n-1})$ according to $\tau$ with $s = s_0 \cdot \ldots \cdot s_{2n-1}$, and that for all $m < n$, $(s_0, s_1, \ldots, s_{2m-1})$ is the canonical run associated to $s_0 \cdot \ldots \cdot s_{2m-1}$. Let $M_{n+1}$ be maximal subject to being an antichain, every $s \in M_{n+1}$ is $n+1$-good, and every $s \in M_{n+1}$ is of the form $s' \cdot t \cdot \tau(s_0, s_1, \ldots, s_{2n-1}, t)$ for some $s' \in M_n$ with associated partial run $(s_0, \ldots, s_{2n-1}, t)$. Let $D_{n+1} = \cup \{N_s: s \in M_{n+1}\}$. To see that $D_{n+1}$ is dense, consider a basic open set $N_t$. Let $s' \in M_n$ be compatible with $t$ and let $(s_0, s_1, \ldots, s_{2n-1})$ be the partial run associated to $s'$ (so $s' = s_0 \cdot \ldots \cdot s_{2n-1}$). Note that $u = s' \cdot (t-s') \cdot \tau(s_0, s_1, \ldots, s_{2n-1}, t-s')$ is $n+1$-good and extends $t$ (where $t-s'$ is the sequence such that $s' \cdot (t-s') = t$).
if \( \text{lh}(s') \leq \text{lh}(t) \), and otherwise \( t - s' = \emptyset \). By maximality, there is an \( s \in M_{n+1} \) such that \( s \parallel u \). Then \( N_s \cap N_u \neq \emptyset \), and so \( N_s \cap N_t \neq \emptyset \).

If \( x \in \bigcap_n D_n \), then for each \( n \) there is an initial segment \( x \upharpoonright i_n \) of \( x \) in \( M_n \). \( x \upharpoonright i_n \) is of the form \( s_0^{-1} s_1^{-1} \cdots s_{2n-1}^{-1} \) where \( (s_0, s_1, \ldots, s_{2n-1}) \) is the canonical partial run associated to \( x \upharpoonright i_n \). Note that \( (s_0, \ldots, s_{2n-3}) \) is the canonical partial run associated to \( x \upharpoonright i_{n-1} \) since \( M_{n-1} \) is an atichain. Thus, the \( s_n \) give a run according to \( \tau \) which produces the real \( x \). Hence, \( x \in A \).

If there is a neighborhood on which \( \omega^\omega - A \) is comeager, let \( s_0 \) be such that \( \omega^\omega - A \) is comeager on \( N_{s_0} \). Have I play \( s_0 \) for the first move, and then follow a strategy for II to get into \( \omega^\omega - A \) as in the first part of the proof. Conversely, if I has a winning strategy \( \sigma \), let \( s_0 = \sigma(\emptyset) \). \( \sigma \) then gives a strategy for II in the ** game starting from \( s_0 \) to get into \( \omega^\omega - A \). The first part of the proof shows that \( \omega^\omega - A \) is comeager on \( N_{s_0} \). \( \square \)

Lemma 2.4 says that every \( A \) is either comeager or else there is a basic open set \( U \) on which it is meager. Applying this to \( \omega^\omega - A \) gives that every \( A \) is either meager or else there is a basic open set \( U \) on which it is comeager (i.e., \( U - A \) is meager).

As a corollary we have the following.

**Theorem 2.5 (ZF + AD).** Every \( A \subseteq \omega^\omega \) has the Baire property.

**Proof.** Let \( A \subseteq \omega^\omega \). Let \( \{U_i\}_{i \in \omega} \) be a maximal, pairwise disjoint collection of basic open sets such that \( A \) is comeager of \( U_i \) (i.e., \( U_i - A \) is meager). By countable additivity, \( A \) is comeager on \( U = \bigcup_i U_i \). It suffices to show that \( A - U \) is meager. If \( A - U \) is not meager, then by lemma 2.4, there is a basic open set \( V \) on which \( A - U \) is comeager. By maximality, \( V \cap U_i \neq \emptyset \) for some \( i \). Let \( W \subseteq V \cap U_i \) be basic open. Then \( \omega^\omega - U \) is comeager on \( W \), a contradiction since \( U_i \subseteq U \) is nonmeager. \( \square \)

We can also define an analog of the ** game on a general Polish space \( X \). In fact, it makes sense to define this game on a general topological space. We assume for this discussion that \( X \) has a wellorderable base \( \mathcal{U} \) (which, of course, holds if \( X \) is second countable). Given \( A \subseteq X \), \( X \) a topological space, the general game \( G_A^* \) is played as follows.

\[
\begin{array}{cccccc}
I & U_0 & U_2 & U_4 & \ldots & \\
II & U_1 & U_3 & U_5 & \ldots & \\
\end{array}
\]

where the \( U_i \) are basic open sets and \( U_i \subseteq U_{i-1} \) (the first player to violate this rule loses). II wins the run iff \( \bigcap_i U_n \subseteq A \). If \( A \) is comeager, then easily II has a winning strategy as before. Assume now II has a winning strategy \( \tau \) in \( G_A^* \). Let \( M_1 \) be a maximal pairwise disjoint collection of basic open sets \( U_1 \) such that for some \( U_0 \), \( (U_0, U_1) \) is a partial run of \( \tau \). As before, \( \cup M_1 \) is dense in \( X \). For each \( U_1 \in M_1 \), let \( s_1(U_1) = (U_0, U_1) \) be a partial run of \( \tau \) ending with \( U_1 \). We can do this since \( \mathcal{U} \) is wellordered. In general, assume \( M_{2n-1} \) has been defined and is a pairwise disjoint collection of open sets \( U_{2n-1} \), and each \( U_{2n-1} \) is the last set played in a partial run following \( \tau \). In fact, assume for each \( U_{2n-1} \in M_{2n-1} \), a canonical run \( s_{2n-1}(U_{2n-1}) = (U_0, U_1, \ldots, U_{2n-3}, U_{2n-2}, U_{2n-1}) \) according to
\( \tau \) is given, where each \( U_{2j-1} \in M_{2j-1} \) and \( s_{2n-1}(U_{2n-1}) \) extends \( s_{2n-3}(U_{2n-3}) \).
Assume also that \( \bigcup M_{2n-1} \) is dense in \( X \). Let then \( M_{2n+1} \subseteq \mathcal{U} \) be maximal subject to being an antichain and for every \( U_{2n+1} \in M_{2n+1} \), there is a partial run \( (U_0, U_1, \ldots, U_{2n-1}, U_{2n}, U_{2n+1}) \) according to \( \tau \) such that \( U_{2n-1} \in M_{2n-1} \) and \( s_{2n-1}(U_{2n-1}) = (U_0, \ldots, U_{2n-1}) \). As before, \( \bigcup M_{2n+1} \) is dense. Using the wellordering of \( \mathcal{U} \), we can easily define \( s_{2n+1} \) (the sequence of functions \( (s_n)_{n \in \omega} \) is actually being constructed from the wellordering of \( \mathcal{U} \)).

Let \( D_{2n+1} = \bigcup M_{2n+1} \). Suppose \( x \in \bigcap_n D_{2n+1} \). For each \( n \), let \( U_{2n+1} \in M_{2n+1} \) be such that \( x \in U_{2n+1} \). Let \( s_{2n+1} = s_{2n+1}(U_{2n+1}) \). The antichain property of the \( M_{2n+1} \) gives that \( s_1 \subseteq s_3 \subseteq \cdots \subseteq s_{2n+1} \) (that is, these sequences extend each other). Thus, there is a run of \( \tau \) where the odd moves are the \( U_{2n+1} \). Since \( \tau \) is winning for II, \( \bigcap_n U_{2n+1} \subseteq A \), and so \( x \in A \). Thus, \( A \) is comeager.

The same argument essentially works when I has a winning strategy, provided \( X \) satisfies a technical assumption \((\ast)\): there are \( \mathcal{U}_n \subseteq \mathcal{U} \), each \( \mathcal{U}_n \) a base, such that if \( U_n \in \mathcal{U}_n \) for all \( n \), then \( \bigcap_n U_n \) contains at most one point. For example, any metric space satisfies \((\ast)\) since we can take \( \mathcal{U}_n = \{ u \in \mathcal{U} : \text{diam}(U) < \frac{1}{n} \} \). To see this, suppose I has a winning strategy \( \sigma \). Let \( U_0 = \sigma(\emptyset) \). Its strategy \( \sigma \) gives a strategy \( \tau \) for II in the game stating from \( U_0 \) to produce a sequence \( U_0, U_1, \ldots \) such that \( \bigcap_n U_n \not\subseteq A \). From \( \tau \) we can easily get a strategy \( \tau' \) for II which is also winning for II in this game starting from \( U_0 \), and with the additional property that for every \( U_2n \) which \( \tau \) plays at stage \( n \), there is a \( U \in U_n \) such that \( U_n \subseteq U \) [after I plays \( U_{2n-1} \), two picks the least basic open set \( V \subseteq U_{2n-1} \) with \( V \in U_n \) and follows \( \tau \) against I’s last move of \( V \).] Define the sets \( D_{2n} \) as before using \( \tau' \), so the \( D_{2n} \) are open and dense in \( U_0 \). If \( x \in \bigcap_n D_{2n} \), then as before there is a run \( (U_0, U_1, \ldots) \) following \( \tau' \) such that \( \forall n \ (x \in U_{2n}) \) and \( \bigcap_n U_{2n} \not\subseteq A \). From \((\ast)\) we have \( \bigcap_n U_{2n} = \{ x \} \). Thus, \( x \not\in A \).

So, for \( X \) satisfying \((\ast)\), and \( A \subseteq X \), II has a winning strategy if \( A \) is comeager, and I has a winning strategy if \( A \) is meager on some non-empty open \( U_0 \). If \( X \) is second countable, then the game \( G_{X}^{*} \) is essentially an integer game, and thus determined. Thus, for every \( A \subseteq X \), either \( A \) is comeager or else meager in some open set \( U \). Equivalently, every \( A \subseteq X \) is either meager or else comeager in some open \( U \). This gives the following.

**Theorem 2.6 (ZF + AD).** Every subset \( A \) of a second countable space \( X \) satisfying \((\ast)\) has the Baire property.

The proof is exactly as before (theorem 2.5), using the following exercise.

**Exercise 10.** Assume ZF + AD, and let \( X \) be second countable. Show that a countable union of meager sets is meager. [hint: Let \( A_n \) be meager. A sequence of closed, nowhere sets whose union is \( A_n \) can be coded by a real. This reduces the choice needed to countable choice for reals, which follows from AD.]

**Exercise 11.** Assume ZF + AC. Let \( \{ U_{\alpha} \}_{\alpha < \kappa} \) be a pairwise disjoint collection of non-empty open sets in a topological space \( X \). Suppose \( \{ A_{\alpha} \}_{\alpha < \kappa} \) are such that each \( A_{\alpha} \subseteq U_{\alpha} \) is meager. Show that \( \bigcup_{\alpha < \kappa} A_{\alpha} \) is meager. [hint: use AC to write each \( A_{\alpha} \) as a union \( A_{\alpha} = \bigcup_n F_{n}^\alpha \), where \( F_{n}^\alpha \) is closed nowhere dense in \( X \). Show that \( \bigcup_{\alpha} F_{n}^{\alpha} \) is nowhere dense for each \( n \).]

Recall a function \( f : X \to Y \) is said to be Baire measurable if for every open \( U \subseteq Y \), \( f^{-1}(U) \) has the Baire property in \( X \).
Lemma 2.7. Let \( X, Y \) be Polish and \( F: X \to Y \) have the Baire property. Then there is a comeager \( A \subseteq X \) such that \( f \upharpoonright A \) is continuous.

Proof. Let \( \{ V_i \}_{i \in \omega} \) be a base for \( Y \). For each \( i \), let \( U_i \subseteq X \) be open such that \( M_i \equiv U_i \triangle f^{-1}(V_i) \) is meager. Let \( M = \bigcup_i M_i \), so \( M \) is meager in \( X \), and let \( A = X - M \). Then \( (f \upharpoonright A)^{-1}(V_i) = U_i \cap A \) and so \( f \upharpoonright A \) is continuous. \( \square \)

As a corollary we have the following theorem.

Theorem 2.8 (ZF + AD). Let \( f: X \to Y \), where \( X, Y \) are Polish. Then there is a comeager \( A \subseteq X \) such that \( f \upharpoonright A \) is continuous.

We next prove a general unfolding theorem for the \( \ast \ast \) game. We do this for the spaces \( X = Y = \omega^\omega \), though we can generalize to arbitrary Polish spaces. Let \( A \subseteq \omega^\omega \), \( B \subseteq \omega^\omega \times \omega^\omega \), and suppose \( \forall x \in A \exists y \in \omega^\omega B(x, y) \) (i.e., \( A \subseteq p(B) \)). We have the \( G_A^{\ast \ast} \) game defined above in which the player \( s \) play \( s_0, s_1, \ldots \) and \( II \) wins iff \( x = s_0 \upharpoonright s_1 \upharpoonright \cdots \in A \). The unfolded version of this game \( \overline{G}_A^{\ast \ast} \) is the following game:

\[
\begin{array}{cccc}
I & s_0 & s_2 & s_4 & \\
II & s_1 & s_3 & s_5 & \\
 & y(0) & y(1) & y(2)
\end{array}
\]

where \( s_i \in \omega^{<\omega} \), \( y(i) \in \omega \). \( II \) wins the run of \( \overline{G}_A^{\ast \ast} \) iff \( (x, y) \in B \).

Theorem 2.9 (\( \ast \ast \)-unfolding). Assume the determinacy of \( \overline{G}_A^{\ast \ast} \). Then Player I wins \( \overline{G}_A^{\ast \ast} \) iff I wins \( G_A^{\ast \ast} \) iff and likewise II wins \( \overline{G}_A^{\ast \ast} \) iff II wins \( G_A^{\ast \ast} \).

Proof. To simplify the following argument a bit notationally, we assume \( Y = 2^\omega \), that is, \( y(i) \in \{ 0, 1 \} \) (the case \( Y = \omega^\omega \) is similar).

Since we are assuming the determinacy of \( \overline{G}_A^{\ast \ast} \), it suffices to show that whoever wins this unfolded game also wins the game \( G_A^{\ast \ast} \). If II wins \( \overline{G}_A^{\ast \ast} \) it is clear that II wins \( G_A^{\ast \ast} \) (just by using the same strategy and ignoring the \( y(i) \) moves). So, suppose I has a winning strategy \( \sigma \) in the unfolded game. We define a strategy \( \sigma \) for I in \( G_A^{\ast \ast} \). The strategy \( \sigma \) will have the property that for any \( x \in \omega^\omega \) which is the result of a play according to \( \sigma \), for all \( y \in 2^\omega \) there is a play according to \( \sigma \) for which \( (x, y) \) is the pair of real produced. This will show that \( \sigma \) is winning for I in \( G_A^{\ast \ast} \) [If \( x \) the the result of a run by \( \sigma \), we must have \( x \notin A \) since otherwise we can fix a \( y \) with \( B(x, y) \) and \( (x, y) \) is the result of a run by \( \sigma \). This will contradict \( \sigma \) being a winning strategy for I in the unfolded game].

We define \( \sigma \) as follows. Let \( s_0 = \sigma(\emptyset) \) be I’s first move in \( G_A^{\ast \ast} \). That is, \( \sigma(\emptyset) = \sigma(\emptyset) \). Let \( s_1 \) be II’s first move in \( G_A^{\ast \ast} \). We define I’s response \( s_2 \) in two steps by defining \( u_0 = s_1 = s_2 \). Let \( u_0 = \sigma(s_0 \upharpoonright (s_1, 0)) \). Let \( u_1 = u_0 \upharpoonright \sigma(s_0, (s_1 \upharpoonright u_0, 1)) \).

Note that \( s_2 \) has the property that for either value of \( y(0) \in \{ 0, 1 \} \), there is an initial segment \( s \) of \( s_0 \upharpoonright s_1 \upharpoonright s_2 \) and a partial play according to \( \sigma \) which produces \( (s, y(0)) \).

Assume inductively that we have defined I’s responses in \( G_A^{\ast \ast} \) up through the moves \( s_0, s_1, \ldots, s_{2n} \), and II now moves \( s_{2n+1} \) in \( G_A^{\ast \ast} \). Assume inductively that \( s_0 \upharpoonright s_1 \upharpoonright \cdots \upharpoonright s_{2n} \) has the property that for every \( t = (y(0), \ldots, y(n-1)) \in 2^n \), there is an initial segment \( s_t \) of \( s_0 \upharpoonright s_1 \upharpoonright \cdots \upharpoonright s_{2n} \) and a partial play \( p = p(t) \) of the game \( \overline{G}_A^{\ast \ast} \) in which \( II \) has played \( t \) for the \( y(i) \) moves and this play has resulted in the
sequences \((s,t)\). We define \(s_{2n+2}\) in \(2^{n+1}\) steps, one for each \(t \in 2^{n+1}\). We will define \(u_0 \subseteq u_1 \subseteq \cdots \subseteq u_{2n+1}\), and then let \(s_{2n+2} = u_{2n+1}\). Let \(t_0, t_1, \ldots, t_{2n+1}\) enumerate \(2^{n+1}\). Given \(u_t\), let \(s\) be the initial segment of \(s_0 \cdots s_1 \cdots s_2\), such that there is a partial play \(p\) according to \(\sigma\) which produces \((s,t_1 \upharpoonright n)\). Let \(s_0 \cdots s_{2n} = s^\infty\). Then let \(u_{t+1} = u_t \upharpoonright (p, (v^\infty s_{2n+1} \supset u_t, t_{t+1}(n)))\). Clearly \(u_t \subseteq u_{t+1}\) and there is a partial play \(p_t\) of \(\sigma\) which extends the partial play \(p\) and which produces \((s_0 \cdots s_{2n+1} \supset u_{t+1}, t_{t+1})\).

This defines the strategy \(\sigma\) for \(I\) in \(G^*_A\). Suppose \(x\) is a run according to \(\sigma\). Say \(x = s_0 \cdots s_t \cdots\) where \((s_0, s_1, s_2, \ldots)\) is a run of \(\sigma\). Let \(y \in \omega^\omega\) be arbitrary. We define a run \(\bar{s}_0, (\bar{s}_1, y(0)), \bar{s}_2, \ldots\) according to \(\sigma\) in which \(\bar{I}\) has played \(y\) and also \(s_0 \cdots s_t \cdots = x\). Suppose \((\bar{s}_0, (\bar{s}_1, y(0)), \ldots, \bar{s}_{2n})\) has been defined, which is a partial play according to \(\sigma\) and \(s_0 \cdots s_{2n}\) is an initial segment of \(s_0 \cdots s_{2n}\). Let \(\bar{s}_0 \cdots \bar{s}_{2n} = s_0 \cdots s_{2n} \supset v\). Let \(t_{t+1} = (y(0), \ldots, y(n))\). By definition of \(s_{2n+2}\) we have that \(s_0 \cdots s_{2n+2}\) extends

\[
\bar{s}_0 \cdots \bar{s}_{2n} \supset v \supset s_{2n+1} \supset u_t \supset \sigma(\bar{s}_0, (\bar{s}_1, y(0)), \ldots, \bar{s}_{2n}, (v\supset s_{2n+1} \supset u_t, y(n)))
\]

which show we may extend the play by \(\sigma\) and continue to build up \((x, y)\).

\textbf{Exercise 12.} Show the version of the unfolding Theorem 2.9 for any Polish space \(X\). In the unfolded game, the players play basic open sets which are decreasing and \(II\) also plays the integers \(y(i)\) as before.

As a corollary of the unfolding we get the Baire property for \(\Sigma^1_1\) (and \(\Sigma^1_3\)).

\textbf{Theorem 2.10 (ZF).} Every \(\Sigma^1_1\) (or \(\Pi^1_1\)) set in a Polish space \(X\) has the Baire property.

\textbf{Proof.} Let \(A \subseteq X\) be \(\Pi^1_1\). Let \(U = \bigcup \{N_i : N_i \cap A = \text{meager}\}\) where \(\{N_i\}\) is a base for \(X\). Clearly \(U \setminus A\) is meager. We show \(A \setminus U\) is also meager. Suppose \(A \setminus U\) is non-meager. Note that \(A \setminus U\) is still \(\Pi^1_1\). Write \(x \in (A \setminus U)^c \iff \exists y \in \omega^\omega \ (x, y) \in B\) where \(B \subseteq X \times \omega^\omega\) is closed. Using this \(B\) we define the unfolded game \(G^*_B(X)\), which is a closed game for \(II\), hence determined. If \(II\) wins this game, then \(II\) also wins \(G^*_{(A \setminus U)^c}\) which gives that \((A \setminus U)^c\) is comeager, a contradiction since we are assuming \(A \setminus U\) is non-meager. If \(I\) wins the unfolded game, then \(I\) also wins \(G^*_{(A \setminus U)^c}\), and so there is a basic neighborhood \(V\) such that \((A \setminus U)\) is comeager on \(V\). By definition of \(U\) we have \(V \subseteq U\), and so \((A \setminus U) \cap V = \emptyset\), a contradiction.

Another important consequence of AD is comeager uniformization.

\textbf{Theorem 2.11 (ZF + AD).} Let \(X, Y\) be Polish and \(R \subseteq X \times Y\) be such that \(\forall x \exists y \ R(x, y)\). Then there is a comeager \(A \subseteq X\) and a function \(f : A \to Y\) such that \(\forall x \in A \ R(x, f(x))\).

\textbf{Proof.} We give the proof for the case \(X = \omega^\omega\), \(Y = 2^\omega\) leaving the general case to an exercise below. So, suppose \(R \subseteq \omega^\omega \times 2^\omega\) and \(\text{dom}(R) = \omega^\omega\). Consider the “unfolded” variation of the ** game.

\[
\begin{array}{c|c|c|c|c}
I & s_0 & s_2 & s_4 & \ldots \\
\hline
II & s_1 & s_3 & s_5 & \ldots \\
& y(0) & y(1) & y(2) & \ldots \\
\end{array}
\]
where \( s_i \in \omega^\omega \) and \( y(i) \in \{0, 1\} \). Let \( x = s_0 \upharpoonright s_1 \upharpoonright \cdots \), and let \( y = (y(0), y(1), \ldots) \). Then II wins the run iff \( R(x, y) \). Applying Theorem 2.9 to the set \( \omega^\omega \) we see that II has a winning strategy \( \tau \) in the unfolded game \( G^*_R \). We define a sequence of dense open sets \( D_i \) as in the proof of lemma 2.4. Let \( M_1 \subseteq \omega^\omega \) be a maximal antichain of 1-good sequences, where \( s \) is 1-good if it is of the form \( s = t \upharpoonright \tau_0(t) \) for some \( t \) (here \( \tau_0 \) denotes the sequence part of \( \tau \)’s response). Let \( D_1 = \bigcup \{ N_s : s \in M_1 \} \), so \( D_1 \) is dense open as in lemma 2.4. Suppose the maximal antichain \( M_i \) has been defined and consists of \( i \)-good sequences, where \( s \) is \( i \)-good if there is a partial run \( (s_0, (s_1, y(0)), \ldots, s_{2i-1}, (s_{2i-1}, y(i-1))) \) according to \( \tau \) with \( s = s_0 \upharpoonright s_1 \upharpoonright \cdots \upharpoonright s_{2i-1} \). Assume we have associated to each \( s \in M_i \) a canonical such partial run of \( \tau \).

Let \( M_{i+1} \) be maximal subject to being an antichain and for every \( s \in M_{i+1} \) there is a (unique) \( t \in M_i \) and associated partial run \( (s_0, (s_1, y(0)), \ldots, s_{2i-1}, (s_{2i-1}, y(i-1))) \) (so \( t = s_0 \upharpoonright \cdots \upharpoonright s_{2i-1} \)) such that \( s = t \upharpoonright s_{2i} \upharpoonright s_{2i+1} \) where

\[
(s_0, (s_1, y(0)), \ldots, s_{2i-1}, (s_{2i-1}, y(i-1))) \\
(s_{2i+1}, (s_{2i+1}, y(i)))
\]

is according to \( \tau \) (for some \( y(0), \ldots, y(i) \)). As in lemma 2.4, each \( M_i \) is a maximal antichain, and so \( D_i = \bigcup \{ N_s : s \in M_i \} \) is dense. If \( x \in \bigcap_i D_i \), then there is a unique run according to \( \tau \) in which \( x \) is the concatenation of the sequences played. Let \( y \) be the corresponding real resulting from the integer moves that \( \tau \) makes in this run. Since \( \tau \) is winning for II, \( R(x, y) \). Note, in fact, that if \( A = \bigcap_i D_i \), then the map \( x \mapsto y \) is continuous.

\[ \square \]

**Corollary 2.12 (ZF+AD).** Let \( R \subseteq X \times Y \) with \( X, Y \) Polish. Suppose \( \forall x \exists y R(x, y) \).
Then there is a comeager \( A \subseteq X \) and a continuous function \( f : A \to Y \) such that \( \forall x \in A \ R(x, f(x)) \).

**Exercise 13.** Prove theorem 2.11 for arbitrary Polish spaces. [hint: Play the game where I and II play basic open sets \( U_i \) with \( U_0 \supseteq U_1 \supseteq \cdots \), \( \text{diam}(U_i) < \frac{1}{2^i} \), and \( U_i \subseteq U_{i-1} \). II also plays basic open sets \( V_i \) satisfying these same requirements. Let \( \{ x \} = \bigcap_i U_i \), and \( \{ y \} = \bigcap_i V_i \). II wins the run iff \( R(x, y) \). If II has a winning strategy, get the dense open sets \( D_i \) as in the discussion after theorem 2.5. Argue that I cannot have a winning strategy as in the proof of theorem 2.11.

Let us recall the Kuratowski-Ulam theorem, which is the “Fubini” theorem for category. If \( X \) is a topological space we use the abbreviation \( \forall^* x \in X A(x) \) to abbreviate “for comeager many \( x \) in \( X \) we have \( x \in A \)”.

**Theorem 2.13 (Kuratowski-Ulam).** Let \( X, Y \) be topological spaces with \( Y \) second countable. If \( R \subseteq X \times Y \) is comeager (in the product topology on \( X \times Y \)) then \( \forall^* x \in X \forall^* y \in Y (x, y) \in R \). Furthermore, if \( A \) has the Baire property and \( X, Y \) are Baire spaces, then the reverse implication holds.

**Proof.** Suppose \( R \supseteq \bigcap_n D_n \) where \( D_n \subseteq X \times Y \) is dense open. We claim that for each dense open \( D_n \subseteq X \times Y \), there is a comeager set \( A_n \subseteq X \) such that for all \( x \in A_n \), \( (D_n)_x = \{ y : (x, y) \in D_n \} \) is dense open in \( Y \). To see this, let \( U \subseteq Y \) be basic open. Given an open set \( U \subseteq X \), the set \( U \times V \) is open in \( X \times Y \), and since \( D_n \) is dense open there is an open set \( U' \subseteq U \) and an open set \( V' \subseteq V \) such that \( U' \times V' \subseteq D_n \). This shows that \( A_n \) is dense open in \( X \), where \( A_n \) is the set of \( x \) such that there is a neighborhood \( U \) of \( x \) and a neighborhood \( V \subseteq V' \) such that \( U \times V \subseteq D_n \). So, \( A_n = \bigcap_n A_n \) is comeager in \( X \), and for all \( x \in A_n \) we have that
(D_n)_x is dense (and, of course, open). Let A = \bigcap_n A_n, so A \subseteq X is comeager. If x \in A then all sections (D_n)_x are dense open so R_x is comeager.

Assume now R has the Baire property and \( \forall^* x \forall^* y R(x,y) \). If R is not comeager, then by the Baire property there is a basic open set \( U \times V \) on which \( R^c \) is comeager. By the first part we have that \( \forall^* x \in U \forall^* y \in V R^c(x,y) \). We can then choose x \in U such that \( \forall^* y \in V R^c(x,y) \) and also \( \forall y R(x,y) \) by intersecting two comeager sets. We then get y \in V such that \( R^c(x,y) \) and \( R(x,y) \), a contradiction. \( \square \)

Note that from AD and the Kuratowski-Ulam theorem we may switch comeager quantifiers, that is, \( \forall^* x \forall^* y A(x,y) \) holds if \( \forall^* y \forall^* x A(x,y) \) holds.

From these results on category we now have the following basic fact.

**Theorem 2.14 (ZF + AD).** There does not exist an uncountable wellordered set of reals.

*Proof.* Suppose \( \{x_\alpha\}_{\alpha<\omega_1} \subseteq \omega^\omega \) was a sequence of distinct reals. Let \( P \subseteq \{x_\alpha\}_{\alpha<\omega_1} \) be perfect. Since \( P \) is in bijection (in fact, homeomorphic to) to \( 2^{\omega} \), we may assume that we have a wellordering \( \prec \) of length \( \omega_1 \). Consider \( \prec \) as a subset of \( 2^{\omega} \times 2^{\omega} \). For every \( y \), \( \{x: x \prec y\} \) is countable, and therefore is meager. So, \( \forall^* y \forall^* x \neg(x \prec y) \).

Since \( \prec \) has the Baire property, by Kuratowski-Ulam we have \( (2^{\omega} \times 2^{\omega}) - \prec \) is comeager, and thus \( \forall^* x \forall^* y \neg(x \prec y) \). This is a contradiction since for any \( x \), \( \{y: y \prec x\} \) is co-countable and thus comeager. \( \square \)

**Remark 2.15.** The proof of theorem 2.14 shows in ZF that if there is a wellordering of the reals, then there is a set without the Baire property. See also remark 2.22.

**Exercise 14.** Show from AD that there does not exists an \( \omega_1 \) sequence of distinct closed or open sets in a Polish space. [remark: the same is actually true for any level of the Borel hierarchy; we’ll show this later.]

Another application of this argument gives the following important fact, the full aditivity of the category ideal.

**Theorem 2.16 (ZF + AD).** A wellordered union of meager sets is meager.

*Proof.* Suppose \( \{A_\alpha\}_{\alpha<\theta} \) is given with each \( A_\alpha \) meager, but \( A = \bigcup_{\alpha<\theta} A_\alpha \) non-meager. Let \( \rho \leq \theta \) be the least ordinal such that \( \bigcup_{\alpha<\rho} A_\alpha \) is non-meager. Since \( A \) has the Baire property, there is a basic open set \( U \) in \( X \) such that \( A \) is comeager on \( U \). Let

\[
\prec = \{(x,y) \in U \times U: \exists \alpha < \beta (x \in A_\alpha - \bigcup_{\alpha' < \alpha} A_{\alpha'} \land y \in A_\beta - \bigcup_{\beta' < \beta} A_{\beta'})\}.
\]

For all \( x \) in \( U \), \( \{y \in U: x \prec y\} \) is comeager in \( U \). So, for comeager many \( y \in U \), \( \{x \in U: x \prec y\} \) is comeager in \( U \). However, the last set is meager by the minimality of \( \rho \).

We now discuss the situation concerning measure.

**Exercise 15.** Let \( \mu \) be a Borel probability measure on \( 2^{\omega} \). Let \( A \) be a \( \mu \) measurable set. Show that for any \( \epsilon > 0 \), there are basic open sets \( N_j \) (in the usual base for \( 2^{\omega} \)) such that \( A \subseteq \bigcup_j N_j \) and \( |\mu(A) - \sum_j \mu(N_j)| < \epsilon \). [hint: Since \( \mu \) is regular, it is enough to show this for open \( A \). Write \( A \) as a countable disjoint union of basic open sets.]
Theorem 2.17. Let $\mu$ be a Borel probability measure on a Polish space $X$. Then every $A \subseteq X$ is $\mu$-measurable.

Proof. We first argue that it is enough to prove the theorem in the case $X = 2^\omega$. Let $\mu$ be a Borel measure on the Polish space $X$, and let $\pi : X \to 2^\omega$ be a Borel bijection (we assume $X$ is uncountable, or the result is trivial). Let $\nu = \pi(\mu)$. That is, $\nu$ is the measure on $2^\omega$ given by $\nu(B) = \mu(\pi^{-1}(B))$. Suppose $A \subseteq X$ is given.

Let $B = \pi[A]$. By the $2^\omega$ case of the theorem, there is a Borel set $C \subseteq 2^\omega$ such that $Z = B \bigtriangleup C$ has $\nu$ measure $0$. So, there is a Borel set $D$ with $Z \subseteq D$ and $\nu(D) = 0$.

Then, $\pi^{-1}(C)$ and $\pi^{-1}(D)$ are Borel sets in $X$ and $\mu(\pi^{-1}(D)) = 0$. Since $\pi$ is a bijection, $A \Delta \pi^{-1}(C) = \pi^{-1}(D)$. Thus, $A$ is $\mu$-measurable.

Suppose now $A \subseteq 2^\omega$, and $\mu$ is a Borel probability measure on $2^\omega$. Let $M$ be a maximal collection of Borel sets of positive measure which are pairwise almost disjoint, and almost contained in $A$ (“almost” refers to the measure $\mu$). $M$ must be countable, say $M = \{B_i\}_{i < \omega}$. Let $B = \bigcup_i B_i$. So, $\mu(B - A) = 0$ and every Borel subset of $A - B$ has $\mu$ measure $0$. It suffices to show that $A - B$ has measure $0$, as then $A = (A - B) \cup (B - (B - A))$, which is a union of a measure $0$ set and a Borel set minus a measure $0$ set.

Changing notation, let us assume that $A \subseteq 2^\omega$ and every Borel subset of $A$ has $\mu$ measure $0$. We must show that $A$ has $\mu$ measure $0$. Note that in fact every $\mu$-measurable subset of $A$ has $\mu$ measure $0$.

For every fixed $\epsilon > 0$ we consider the following “covering” game $G_\epsilon(A)$ (due to Harrington).

\[
\begin{array}{cccc}
I & x(0) & x(1) & x(2) & \ldots \\
II & a_0 & a_1 & a_2 & \ldots \\
\end{array}
\]

where I plays $x(i)\in\{0,1\}$ building a real $x\in 2^\omega$, and II plays integers $a_i \in \omega$ which we think of as coding a finite sequence $N_0, \ldots, N_k$ of basic open sets in $2^\omega$.

II must play so that $\sum_{j \leq k_i} \mu(N_j) - \epsilon$ (otherwise II loses). If II follows the rules, then II wins the run iff $(x \in A \rightarrow x \in \bigcup_i \bigcup_{j \leq k_i} N_j)$.

Suppose I had a winning strategy $\sigma$. Note that $B = \sigma[\omega^\omega] \subseteq A$, and $B \in \Sigma_1^1$. In particular, $B$ is $\mu$ measurable, and so $\mu(B) = 0$. Let $N_1, N_2, \ldots$ be a sequence of basic open sets in $2^\omega$ with $A \subseteq \bigcup_i N_j$ and such that $\sum_i \mu(N_j) < \epsilon$. Let $a_0$ code the first $k_0$ of the $N_j$, where $k_0$ is large enough so that $\sum_{j > k_0} \mu(N_j) < \frac{\epsilon}{2}$. Let $a_1$ code the sets $N_{k_0+1}, \ldots, N_{k_1-1}$, where $k_1$ is large enough so that $\sum_{j > k_1} \mu(N_j) < \frac{\epsilon}{2}$. Continue in this manner to define $a_i$, so that the sum of the $\mu(N_j)$ for $j$ coded by $a_i$ is less than $\frac{\epsilon}{2^i}$. If II plays the $a_i$, then II wins the run, a contradiction.

Suppose now that $\tau$ is a winning strategy for II. For each $i$, let $U_i$ be the union of all the $N_j^i = N_j^i(s)$ coded by some $a_i$ of the form $\tau(s)$ for $s \in 2^i$. The $\mu$ measure of $U_i$ is at most $2^i \cdot \frac{\epsilon}{2^i} = \frac{\epsilon}{2}$. Then $U = \bigcup_i U_i$ contains $A$ and $\mu(U) \leq \sum_i \frac{\epsilon}{2^i} = 2\epsilon$.

So, for every $\epsilon > 0$, $A$ can be covered by an open set of $\mu$ measure less than $\epsilon$. This shows that $A$ has $\mu$ measure $0$. \qed

Many of the sets constructed with AC give rise to sets which are non-measurable and without the Baire property. For example, consider an ultrafilter $U$ on $\omega$. We
view $\mathcal{U}$ as a subset of $2^\omega$ by identifying an $A \subseteq \omega$ with its characteristic function $\chi_A \in 2^\omega$. We then have the following.

**Theorem 2.18 (ZF).** Suppose $\mathcal{U}$ is a non-principal ultrafilter on $\omega$. Then $\mathcal{U}$ is non-measurable and does not have the Baire property (measurable here refers to the standard Bernoulli measure on $2^\omega$).

**Proof.** Suppose first that $\mathcal{U}$ has the Baire property. Note that for any $x,y \in 2^\omega$, if $\exists n \forall m \geq n \ (x(m) = y(m))$, then $x \in \mathcal{U}$ iff $y \in \mathcal{U}$ (by $x \in \mathcal{U}$ we mean $\{n: x(n) = 1\} \in \mathcal{U}$). This is because every co-finite set is in the ultrafilter. If $s \in 2^{\omega^\omega}$ is such that $\mathcal{U}$ is comeager on $N_s$, then it follows that $\mathcal{U}$ is comeager on every $N_t$ with $\text{lh}(t) = \text{lh}(s)$. This is because the natural bijection between $N_s$ and $N_t$ is a homeomorphism, and so preserves category. Thus, either $\mathcal{U}$ is meager or else comeager. Consider now the map $\pi: 2^\omega \to 2^\omega$ defined by $\pi(x)(n) = 1 - x(n)$. Clearly $\pi$ is a homeomorphism, and so $\mathcal{U}$ is meager (or comeager) iff $\pi(\mathcal{U})$ is meager (or comeager). However, $x \in \mathcal{U}$ iff $\pi(x) \notin \mathcal{U}$, and so $\mathcal{U}$ is meager (or comeager) iff $2^\omega - \mathcal{U}$ is meager (or comeager), a contradiction.

Suppose next that $\mathcal{U}$ is measurable. If $\mathcal{U}$ has positive measure, then for any $\epsilon > 0$ there is a basic open set $N_s$ such that $\mu(\mathcal{U} \cap N_s) > (1 - \epsilon)\mu(N_s)$ (Lebesgue density theorem). Again, the natural bijection between $N_s$ and $N_t$, where $\text{lh}(s) = \text{lh}(t)$ is measure preserving, and so $\mu(\mathcal{U} \cap N_t) > (1 - \epsilon)\mu(N_t)$. It follows that $\mu(\mathcal{U}) \geq 1 - \epsilon$. Since this holds for all $\epsilon > 0$, we have $\mu(\mathcal{U}) = 1$. So, either $\mu(\mathcal{U}) = 0$ or $\mu(\mathcal{U}) = 1$. Consider again the map $\pi$, and note that $\pi$ is also measure preserving. So, $\mu(\mathcal{U}) = 0$ (or $1$) iff $\mu(\pi(\mathcal{U})) = 0$ (or $1$). Since $\pi$ flips membership in $\mathcal{U}$, we have that $\mu(\mathcal{U}) = 0$ (or $1$) iff $\mu(2^\omega - \mathcal{U})) = 0$ (or $1$), a contradiction. □

As a consequence of this, we have the following.

**Theorem 2.19 (AD).** Every ultrafilter on a set $X$ is countably additive.

**Proof.** Let $\mathcal{U}$ be an ultrafilter on $X$. We may assume $\mathcal{U}$ is non-principal since principal ultrafilters are arbitrarily additive. Suppose $\mathcal{U}$ is not countable additive, say $A_n \subseteq X$ are such that $A_n \notin \mathcal{U}$ but $A = \bigcup_n A_n \in \mathcal{U}$. We may assume the $A_n$ are increasing by finite additivity of $\mathcal{U}$. Define $f: x \to \omega$ by $f(x) = \text{the least } n \text{ such that } x \in A_n$. Let $\mu = f(\mathcal{U})$, so $\mu$ is an ultrafilter on $\omega$. $\mu$ is non-principal since each $A_n \notin \mathcal{U}$. From theorem 2.18, $\mu$ does not have the Baire property as a subset of $2^\omega$. This contradicts theorem 2.5. □

**Definition 2.20.** A measure on a set $X$ is a countably aditive ultrafilter on $X$.

So, assuming AD, every ultrafilter on a set $X$ is a measure. As another excample we have the following.

**Lemma 2.21.** If there is a wellordering of the reals, then there is a set without the Baire property, and there is a set which is not measurable.

**Proof.** Suppose $\{x_\alpha\}_{\alpha < c}$ is a wellordering of $2^\omega$. suppose that every subset of $2^\omega$ has the Baire property. Let $\rho \leq c$ be least such that $A_\rho = \{x_\alpha\}_{\alpha < \rho}$ is non-meager. Let $\prec = \{(x_\alpha, x_\beta) : \alpha < \beta < \rho\}$. We have that $\forall y \in 2^\omega \forall^* x \in 2^\omega \ ((x,y) \notin \prec)$. Since $\prec$ has the Baire property, by Kuratowski-Ulam we have $\forall^* x \forall^* y \ ((x,y) \notin \prec)$. Since $A_\rho$ is non-meager, there is an $x \in A_\rho$ such that $\forall^* y \ ((x,y) \notin \prec)$. Say $x = x_\alpha$, where $\alpha < \rho$. Since $\{x_\beta : \alpha < \beta < \rho\}$ must be non-meager, there is an $x_\beta$ in this set such that $(x_\alpha, x_\beta) \notin \prec$, contradicting the definition of $\prec$. 
The argument for measure is similar, using Fubibi’s theorem instead of Kuratowski-Ulam.

**Remark 2.22.** Shelah has shown in $\text{ZF} + \text{DC}$ that if there is an uncountable wellordered set of reals, then there is a non-measurable set. The proof of this also shows that if every $\Sigma^1_3$ is measurable, then $\forall x (\omega^L[x] < \omega_1)$. Thus, the statement that every $\Sigma^1_3$ set is measurable has the consistency strength of an inaccessible cardinal. On the other hand, Shelah has shown that if $\text{ZF}$ is consistent then so is $\text{ZF} + \text{“every set of reals has the Baire property”}$. So, even the Baire property for all sets of reals doesn’t have consistency strength beyond $\text{ZF}$. This is an interesting asymmetry between measure and category.

### 3. Turing Degrees

If $x, y$ are in $\omega^\omega$ or $2^\omega$, then we say $x$ is *Turing* reducible to $y$, $x \leq_T y$ iff $x$ can be effectively computed from $y$. This can be made precise in several different ways. For example, one use the notion of computation from an oracle. In this approach, the ordinary notion of a Turing machine is modified to include an auxiliary tape (the “oracle”) which is read-only, and which the machine is allowed to scan at any time (in the current position). The action the machine takes at any step is allowed to depend on this scanned value as well as the value on the read-write tape (and the state of the machine as usual). Thus, when we say “$x$ is computed from $y$,” we mean that there is a Turing machine which when started with input $n$ on the read-write tape and $y$ on the read-only tape will terminate with the correct value of $x(n)$.

We identify subsets of $\omega$ with elements of $2^\omega$ via their characteristic functions, and we frequently pass between the two points of view. We say an $x$ in $\omega^\omega$ or $2^\omega$ (or a subset of $\omega$) is recursive if $x$ is computable from the constant 0 sequence (i.e., $x$ is just outright computable). We say $x \equiv_T y$ if $x \leq_T y$ and $y \leq_T x$. It is clear that $\equiv_T$ is an equivalence relation on the reals and $\leq_T$ is a partial order on the equivalence classes. By a *Turing degree* we mean an equivalence class $[x]_T = \{y : y \equiv_T x\}$. Clearly each Turing degree is a countable set of reals. We sometimes write 0 for the Turing equivalence class of the constant 0 real, that is, the class of all recursive reals. We write $\mathcal{D}$ for the set of Turing degrees. We frequently use $d$ to denote a Turing degree (i.e., $d = [x]_T$ for some $x \in \omega^\omega$).

By the *cone above* $x$, we mean the set $\{y : x \leq_T y\}$. This is clearly a set of Turing degrees. We have the following fundamental theorem of Martin.

**Theorem 3.1** (Martin). Every set of Turing degrees either contains or omits a cone of degrees.

**Proof.** Let $A \subseteq \mathcal{D}$ be a set of degrees. Consider the usual game $G_A$:

<table>
<thead>
<tr>
<th>I</th>
<th>$x_0$</th>
<th>$x_2$</th>
<th>$x_4$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$x_1$</td>
<td>$x_3$</td>
<td>$x_5$</td>
<td>...</td>
</tr>
</tbody>
</table>

So, I wins the run iff $x \in A$. Suppose that I has a winning strategy $\sigma$. Let $d \in \mathcal{D}$ and $\sigma \leq_T d$ (i.e., $d$ is in the cone above $\sigma$). Consider the run of the game where I
follows $\sigma$ and II plays any $x$ with $d = [x]_T$. The resulting run $\sigma(x)$ is computable from $\sigma$ and $x$, and thus computable from $x$ (since $\sigma \leq_T x$). Clearly $x$ is also computable from $\sigma(x)$ ($x = (\sigma(x))_1$). So, $d = [x]_T = [\sigma(x)]_T$. Since $\sigma$ is winning for I, $\sigma(x) \in A$, and since $A$ is a set of degrees, $d \in A$. So, $A$ contains the cone above $\sigma$. A similar argument shows that if II has a winning strategy $\tau$, then $D - A$ contains the cone above $\tau$. \hfill $\Box$

In view of theorem 3.1 we make the following definition.

**Definition 3.2.** The Martin measure on the set of turing degrees $D$ is the measure defined by cones, that is, if $A \subseteq D$ has measure one iff $A$ contains a cone of degrees.

Theorem 3.1 says that the Martin measure is in fact a measure (i.e., a countably additive ultrafilter) on $D$ (recall that from AD every ultrafilter is countably additive, although here it is obvious directly that the Martin measure is countably additive).

We can give an improvement to theorem 3.1. To state this, we make the following definition.

**Definition 3.3.** A tree $T$ on $\{0,1\}$ (or on $\omega$) is said to be pointed if for any $x \in [T]$, $T \leq_T x$.

Thus, a tree $T$ is pointed if $T$ is computable from any of its branches. We now state our improvement to theorem 3.1.

**Theorem 3.4 (ZF + AD).** Let $A \subseteq 2^\omega$ (or $\omega^\omega$). Then there is a perfect pointed tree $T$ on $\{0,1\}$ (or on $\omega$) such that $[T] \subseteq A$ or $[T] \subseteq 2^\omega - A$.

**Proof.** Suppose $A \subseteq 2^\omega$, and consider again the basic game $G_A$. Suppose I has a winning strategy $\sigma$ (the case for II is similar). For any $x = (x(0), x(1), \ldots) \in 2^\omega$, let $\sigma(x) = (y(0), x(0), y(1), x(1), \ldots)$ be the real produced when II plays $x$ and I follows $\sigma$. So, for any $x \in 2^\omega$, $\sigma(x) \in A$. Consider the set $C \subseteq 2^\omega$ of those $x$ such that for all $n$, $x(2n) = 1$ iff $\sigma(n) = 1$ (viewing $\sigma$ as an element of $2^\omega$). Clearly $\sigma[C] = [T]$ for some perfect tree $T$. So, $[T] \subseteq A$. If $z \in [T]$, then $)((z)_1)_0 = \sigma$, and so $\sigma \leq_T z$. However, $T$ is computable from $\sigma$, and so $T \leq_T z$. Thus, $T$ is pointed. \hfill $\Box$

We note that theorem 3.4 is indeed a strengthening of theorem 3.1. For suppose $A \subseteq D$, and $T$ is perfect pointed with $[T] \subseteq A$. Suppose $x \geq_T T$. Define a branch $y$ of $T$ as follows. Let $y = s_0 \cdot x(0) \cdot s_1 \cdot x(1) \cdot \ldots$, where $s_0$ is the least splitting node of $T$ (i.e., such that $s_0 \cdot 0$, $s_0 \cdot 1$ are both in $T$), $s_1$ is the least splitting node extending $s_0 \cdot x(0)$, etc. Clearly $y$ is computable from $T$ an $x$. Since $x \geq_T T$, we have $y \leq_T x$. On the other hand, $x$ is computable from $y$ and $T$. Since $T$ is pointed, $T \leq_T y$, and so $x \leq_T y$. Thus, $x \equiv_T y$. Since $y \in [T]$, $y \in A$ and thus $x \in A$ as $A \subseteq D$. So, $A$ contains the cone above $T$.

To give an application of the Martin measure we use the following fact, whose proof uses the coding lemma, which we give later (in fact, the coding lemma provides a much stronger version of the fact).

**Fact 3.5 (ZF + AD).** If there is a map from $\omega^\omega$ onto an ordinal $\lambda$, then there is a map from $\omega^\omega$ onto $\mathcal{P}(\omega^\omega)$.

We make the following important definition.

**Definition 3.6.** $\Theta$ is the supremum of lengths of the prewellorderings of $\omega^\omega$. 
Of course, it makes no difference in the definition of $\Theta$ whether we use $\omega^\omega$, $2^\omega$, or any other uncountable polish space (as they are all isomorphic). Easily $\Theta$ is a limit ordinal. Equivalently, we can say that $\Theta$ is the supremum of the ordinals $\lambda$ such that there is a surjection from $\omega^\omega$ onto $\lambda$. Assuming $\text{AC}$, where $\omega^\omega$ has a welldefined cardinality, $\Theta = (2^\omega)^+$ (so $\Theta = \omega_2$ assuming $\text{CH}$). However, the definition is meant primarily in the determinacy context, and provides an $\text{AD}$ version of the cardinality of the reals.

Assuming countable choice gives that $\text{cof}(\Theta) > \omega$ according to the following exercise.

Exercise 16. Assume $\text{ZF}+$ countable choice. Show that $\text{cof}(\Theta) > \omega$. [hint: suppose $\{\alpha_n\}_{n \in \omega}$ were a cofinal $\omega$ sequence in $\Theta$. By countable choice, let $\preceq_n$ by a prewellordering of $\omega^\omega$ of length $\alpha_n$. Glue the $\preceq_n$ together to get a prewellordering of length $\Theta$, a contradiction.]

**Theorem 3.7 ($\text{ZF} + \text{AD}$).** Every countably additive filter $F$ on an ordinal $\lambda < \Theta$ can be extended to a measure on $\lambda$.

**Proof.** Since $\lambda < \Theta$, from fact 3.5 there is a map $\pi: \omega^\omega \rightarrow \mathcal{P}(\lambda)$. For every degree $d$, define

$$A_d = \bigcap \{ \pi(x): x \in d \land \pi(x) \in F \}.$$ 

Let $f(d)$ be the least element of $A_d$, which makes sense since $F$ is countably additive. So, $f: D \rightarrow \lambda$. Let $\mu = f(\nu)$, where $\nu$ is the Martin measure on $D$. So, $\mu$ is a measure on $\lambda$. If $F \subseteq F$, then consider $x$ such that $\pi(x) = F$. If $d \in D$ with $x \leq_T d$, then $A_d \subseteq F$, and so $f(d) \in F$. So, $\forall d (f(d) \in F)$. Thus, $\mu(F) = 1$. So, $\mu$ extends the filter $F$. $\square$

**Corollary 3.8.** If $\lambda < \Theta$ and $\text{cof}(\lambda) > \omega$, then there is a measure $\mu$ on $\lambda$ such that $\mu([0, \alpha]) = 0$ for all $\alpha < \lambda$.

**Proof.** Let $F$ be the co-bounded filter on $\lambda$. Since $\text{cof}(\lambda) > \omega$, $F$ is countably additive. From theorem 3.7, there is a measure $\mu$ on $\lambda$ extending $F$. Since $\mu$ extends $F$, every co-bounded set has $\mu$ measure 1, and so every bounded in $\lambda$ set has $\mu$ measure 0. $\square$

As another application of the Martin measure, we give the following theorem of Kunen.

**Theorem 3.9 ($\text{ZF} + \text{AD}$).** Let $\lambda < \Theta$. Then the set of measures on $\lambda$ is wellorderable.

**Proof.** Again let $\pi: \omega^\omega \rightarrow \mathcal{P}(\lambda)$. Suppose $\mu_1, \mu_2$ are measures on $\lambda$. For $\mu$ a measure on $\lambda$ and $d \in D$, again let

$$A_d^\mu = \bigcap \{ \pi(x): x \in d \land \mu(\pi(x)) = 1 \}.$$

Let also $f^\mu(d)$ be the least element of $A^\mu(d)$. Define $\mu_1 < \mu_2$ iff $\forall d (f^{\mu_1}(d) < f^{\mu_2}(d))$. For any two measure $\mu_1$, $\mu_2$ we have either $\mu_1 < \mu_2$ or $\mu_2 < \mu_1$. To see this, let $A \subseteq \lambda$ such that (without loss of generality) $\mu_1(A) = 0$, $\mu_2(A) = 1$. Let $\pi(x_0) = A$, $\pi(x_1) = \lambda - A$. If $x_0, x_1 \leq_T d \in D$, then $f^{\mu_1}(d) \in A$ and $f^{\mu_2}(d) \in A$, and thus $f^{\mu_1}(d) \neq f^{\mu_2}(d)$. So, $\forall d (f^{\mu_1}(d) \neq f^{\mu_2}(d))$, and it follows that either $\mu_1 < \mu_2$ or $\mu_2 < \mu_1$. Easily $<$ is transitive and irreflexive. So, $<$ is a linear ordering of the measures on $\lambda$. To see it is wellfounded, suppose $\mu_{n+1} < \mu_n$ for all $n$. Let
$A_n \subseteq D$ be such that for all $d \in A_n$, $f^{\mu_n+1}(d) < f^{\mu_n}(d)$. By countable additivity of $\nu$, $\bigcap_n A_n \neq \emptyset$. Let $d \in \bigcap_n A_n$. Then $f^{\mu_1}(d) > f^{\mu_2}(d) > \ldots$, a contradiction. \qed

In view of theorem 3.9, we make the following definition.

**Definition 3.10 (ZF + AD).** For $\lambda < \Theta$, let $\beta(\lambda)$ denote the cardinality of the set of measures on $\lambda$.

We will give estimates for $\beta(\lambda)$ later.

4. **Wadge Degrees**

We now turn to a discussion of the basic facts concerning Wadge degrees. The objects of study now are not reals but sets of reals. It is most convenient for this discussion to work in the space $\omega^\omega$, which we do for the rest of this section. We will frequently write $A^c$ for the complement $\omega^\omega - A$.

Recall that a function $f$ from a metric space $(X, d)$ to a metric space $(Y, \rho)$ is said to be Lipschitz continuous, with Lipschitz constant $C$, if $\rho(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. In the case $X = Y = \omega^\omega$, and $d = \rho = \text{the usual metric}$: $d(x, y) = \frac{1}{2^n}$, where $n$ is least such that $x(n) \neq y(n)$ (and $d(x, y) = 0$ if $x = y$), to say that $f$ is Lipschitz continuous (with constant 1) simply means that if $x \upharpoonright n = y \upharpoonright n$, then $f(x) \upharpoonright n = f(y) \upharpoonright n$. We simply call such a function Lipschitz continuous.

Note that Lipschitz continuous functions are essentially strategies for II in integer games (a strategy for I is also Lipschitz continuous, and in fact slightly better).

**Definition 4.1.** Let $A, B \subseteq \omega^\omega$. We say $A$ is Wadge reducible to $B$, $A \leq_w B$, if there is a continuous function $f: \omega^\omega \rightarrow \omega^\omega$ such that $A = f^{-1}(B)$. We say $A$ is Lipschitz reducible to $B$, $A \leq^\ell B$, if there is a Lipschitz continuous function $f: \omega^\omega \rightarrow \omega^\omega$ such that $A = f^{-1}(B)$.

Note that to say $A = f^{-1}(B)$ means that for all $x$ we have $x \in A$ iff $f(x) \in B$. Thus $f$ reduces the question of membership in $A$ to that of membership in $B$. Trivially, if $A \leq^\ell B$ then $A \leq_w B$. Both $\leq^\ell$ and $\leq_w$ are clearly reflexive and transitive (the composition of two Lipschitz functions is Lipschitz), that is, both are partial orders. Note also that $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ is a pointclass iff $\Gamma$ is closed under Wadge reduction.

Note that $A \leq^\ell B$ iff $A^c \leq^\ell B^c$, and likewise for $\leq_w$.

AD gives us direct information about the partial order $\leq^\ell$. The following fundamental lemma, due to Wadge, is known as Wadge’s lemma.

**Lemma 4.2 (ZF + AD).** Let $A, B \subseteq \omega^\omega$. Then either $A \leq^\ell B$ or $B \leq^\ell A^c$.

**Proof.** Consider the game $G_{A,B}$ where I plays out $x \in \omega^\omega$ and II plays out $y \in \omega^\omega$, and where II wins the run iff $(x \in A \leftrightarrow y \in B)$. If II has a winning strategy for $G_{A,B}$, then a winning strategy $\tau$ gives a Lipschitz continuous function, which we also call $\tau$, from $\omega^\omega$ to $\omega^\omega$ such that $x \in A$ iff $\tau(x) \in B$. Thus, $A \leq^\ell B$. A winning strategy $\sigma$ for I also gives a (slightly better than) Lipschitz continuous function from $\omega^\omega$ to $\omega^\omega$ such that for all $x, x \in B$ iff $\sigma(x) \notin A$. Thus, $B \leq^\ell A^c$. \qed

We can also define a game $G^c_{A,B}$ corresponding to Wadge reduction. In this game, the players make integer moves but II is also allowed to pass. If II makes a terminal sequence of passes, II loses. Otherwise, the payoff is exactly as in the Lipschitz game $G_{A,B}$. It is easy to see that $A \leq_w B$ iff II has a winning strategy in the game $G_{A,B}$. 

Of course, it follows immediately that for any \( A, B \) that either \( A \leq_w B \) or \( B \leq_w A \).

So, \( \leq_f \) and \( \leq_w \) are not linear orders on \( \mathcal{P}(\omega^\omega) \), but if we amalgamate sets with their complements, then they are. We make this precise in the next definition.

**Definition 4.3.** By a Lipschitz degree we mean the equivalence class \( \{ A, A' \} \) of a set \( A \subseteq \omega^\omega \) together with its complement \( A' \) under the relation \( \{ A, A' \} \equiv_f \{ B, B' \} \) iff \( A \) is Lipschitz reducible to either \( B \) or \( B' \), and \( B \) is Lipschitz reducible to either \( A \) or \( A' \). Likewise, we define a Wadge degree to be the equivalence class of a pair \( \{ A, A' \} \) using \( \leq_w \) instead of \( \leq_f \).

Note that \( \leq_f \) is well-defined on the Lipschitz degrees, and likewise \( \leq_w \) is well-defined on the Wadge degrees. That is, \( \{ A, A' \} \leq_f \{ B, B' \} \) iff \( A \leq_f B \) or \( A \leq_f B' \).

We write \( [A]_f \) and \( [A]_w \) for the Lipschitz and Wadge degrees of a set \( A \) (that is, \( [A]_f \) is the equivalence class of \( \{ A, A' \} \)). From Wadge’s lemma it follows that \( \leq_f \), and thus also \( \leq_w \), is a linear ordering of the Lipschitz (resp. Wadge) degrees. As usual, we write \( [A]_f \leq [B]_f \) to mean \( [A]_f \leq [B]_f \) and \( [B]_f \nleq [A]_f \).

We next present the following important result of Martin and Monk which states that the Lipschitz and Wadge degrees are actually wellordered by these orders.

**Theorem 4.4 (ZF + DC + AD).** The Lipschitz degrees are wellordered under \( \leq_f \). Likewise, the Wadge degrees are wellordered by \( \leq_w \).

**Proof.** Suppose \([A_0]_f > [A_1]_f > [A_2]_f > \cdots \) For each \( n \), \( \Pi \) does not have a strategy in the Lipschitz game \( G_{A_n,A_{n+1}} \), and so I has a winning strategy \( \sigma_n \) for this game. So, \( \sigma_n(x) \in A_n \) iff \( x \notin A_{n+1} \). Likewise, \( \Pi \) does not win \( G_{A_n,A_{n+1}} \), so let \( \tau_n \) be a winning strategy for I in this game. Thus, \( \tau_n(x) \in A_n \) iff \( x \in A_{n+1} \).

So, \( \sigma_n \) “flips” membership between \( A_{n+1} \) and \( A_n \), and \( \tau_n \) preserves membership. It is important for the following argument that all of the \( \sigma_n \), \( \tau_n \) are strategies for I. For \( z \in 2^\omega \) we fill in the following diagram in such a way that for all \( n \), \( x_n = \sigma_n(x_{n+1}) \) if \( z(n) = 1 \) and \( x_n = \tau_n(x_{n+1}) \) if \( z(n) = 0 \).

<table>
<thead>
<tr>
<th>( A_0 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>( A_3 )</th>
<th>( A_4 )</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0(0) )</td>
<td>( x_1(0) )</td>
<td>( x_2(0) )</td>
<td>( x_3(0) )</td>
<td>( x_4(0) )</td>
<td>( \cdots )</td>
</tr>
<tr>
<td>( x_0(1) )</td>
<td>( x_1(1) )</td>
<td>( x_2(1) )</td>
<td>( x_3(1) )</td>
<td>( x_4(1) )</td>
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</tr>
<tr>
<td>( x_0 )</td>
<td>( x_1 )</td>
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<td>( x_3 )</td>
<td>( x_4 )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>

For \( n \) such that \( z(n) = 1 \), let \( x_n(0) = \sigma_n(\emptyset) \), and for \( n \) such that \( z(n) = 0 \) let \( x_n(0) = \tau_n(\emptyset) \). In a similar manner we fill in all of the \( x_n(i) \). Let \( x_0(z) \) be the value of \( x_0 \) produced as described above using \( z \in 2^\omega \) Let \( B_0 = \{ z : x_0(z) \in A_0 \} \), and \( B_1 = \{ z : x_0(z) \notin A_0 \} \). At least one of \( B_0 \), \( B_1 \) must be non-meager. Say \( N_\sigma \) be a basic open set in \( 2^\omega \) on which one of these sets is comeager. Say, \( B_0 \) is comeager on \( N_\sigma \) (the other case is similar). So, \( B_0 \) is comeager on \( N_{\sigma^{-}\emptyset} \) and on \( N_{\sigma^{-}1} \). Consider the map \( \pi \) from \( N_{\sigma^{-}0} \) to \( N_{\sigma^{-}1} \) obtained by flipping the \( \ell \text{th} \) digit.
So, $\pi$ is a homeomorphism between $N_{x^0}$ and $N_{x^1}$. Since $B_0$ is comeager in $N_{x^0}$, $\pi(B_0)$ is comeager in $N_{x^1}$. However, for any $z$, $x_0(z) \in A_0$ if $x_0(\pi(z)) \notin A_0$ (as $\tau_n$ does not flip membership but $\sigma_n$ does). Thus, $\pi(B_0 \cap N_{x^0}) = B_1 \cap N_{x^1}$. Thus, $B_1$ is also comeager in $N_{x^1}$, a contradiction (as $B_0$, $B_1$ are disjoint).

The proof also works for Wadge degrees, since if $[A_0]_w >_w [A_1]_w >_w \cdots$, then it is also the case that for each $n$ that $[A_n]_\ell >_\ell [A_{n+1}]_\ell$. \[\Box\]

Variations of the Martin-Monk method are frequently used in the abstract theory of pointclasses.

As an immediate consequence of Wadge's lemma we have the following.

**Theorem 4.5 (ZF + AD).** If $\Gamma$ is a non-selfdual pointclass, then $\Gamma \upharpoonright \omega^\omega$ has a universal set.

**Proof.** Let $A \in \Gamma - \Gamma$. If $B \in \Gamma$, then we must have $B \leq_\ell A$, as otherwise we would have $A \leq_\ell B^c$ which would give $A \in \Gamma$. Define then $U(x,y) \leftrightarrow x(y) \in A$, where we view every real $x$ as coding a Lipschitz continuous map from $\omega^\omega$ to $\omega^\omega$ (for example $x(s) = m$, for $s \in \omega^{<\omega}$, iff $x(\langle s \rangle) = m$). The map $(x,y) \mapsto x(y)$ is continuous, and so $U \in \Gamma$. Since every $B \in \Gamma$ is Lipschitz reducible to $A$, it follows that $U$ is universal for $\Gamma \upharpoonright \omega^\omega$. \[\Box\]

**Definition 4.6.** If $A \subseteq \omega^\omega$, we define the Wadge rank of $A$, $o(A)$ to be the rank of $\{A,A^c\}$ in $\leq_w$. Likewise we define the Lipschitz rank $o(A)$. If $\Gamma$ is a pointclass, we define $\omega(\Gamma) = \sup\{o(A) : A \in \Gamma\}$.

So, if $\Gamma$ is non-selfdual, then $\omega(\Gamma) = o(A)$ for any $A \in \Gamma - \Gamma$.

We develop some of the facts concerning the Lipschitz and Wadge hierarchies. Note that every Wadge degree is a union of Lipschitz degrees. The following lemma is another connection.

**Definition 4.7.** We say $\{A,A^c\}$ is a selfdual Lipschitz (or Wadge) degree if $A \equiv_\ell A^c$, otherwise we say the degree is non-selfdual.

Note that if $A \leq_\ell A^c$ then $A^c \leq_\ell A$, and so $A \equiv_\ell A^c$. The following important lemma is due to Steel.

**Lemma 4.8 (ZF + AD).** $A \equiv_w A^c$ iff $A \equiv_\ell A^c$.

**Proof.** Suppose $A \leq_w A^c$. Let $f : \omega^\omega \rightarrow \omega^\omega$ be continuous such that $x \in A$ iff $f(x) \in A^c$. Suppose $A \not\leq_\ell A^c$. Then I wins the game $G_{A,A^c}$, say by $\sigma$. For each $z \in 2^{<\omega}$, we consider the diagram as in the proof of theorem 4.4, where for $n$ such that $z(n) = 1$ we use $\sigma$ (that is $x_n = \sigma(x_{n+1})$, and if $z(n) = 0$ we use $f$ (i.e., $x_n = f(x_{n+1})$). Since $f$ is just assumed to be continuous, it is no longer the case that the diagram can always be filled in. However, we claim that for comeager many $z \in 2^{<\omega}$, the diagram can be filled in. This is because the the set of $z$ such that $x_0 \upharpoonright n,\ldots,x_n \upharpoonright n$ are defined is dense open in $2^{<\omega}$. To see this, note that for any $s \in 2^{<\omega}$, there is a $t$ extending $s$ of the form $t = \overline{s^wm}$ such that a sufficiently large initial segment of $x_n$ is defined so that $x_0 \upharpoonright n,\ldots,x_n \upharpoonright n$ are all defined. Let $C \subseteq 2^{<\omega}$ be the comeager set of $z$ such that the diagram can be completely filled in. Let $x_t(z)$ denote the corresponding reals produced. Note that $\sigma$ preserves membership in $A$ and $f$ flips membership.

The argument is now similar to that of theorem 4.4. Let $C_0 = \{z \in C : x_0(z) \in A\}$, and $C_1 = \{z \in C : x_0(z) \notin A\}$. At least one of the $C_i$ is nonmeager, and
so comeager on some neighborhood $N_s$ of $2^w$. So, $C_i$ is comeager in $N_{s^0}$ and $N_{s^{-1}}$. Let $\pi$ be the natural homeomorphism between $N_{s^0}$ and $N_{s^{-1}}$. Let $D = N_{s^0} \cap C_i \cap \pi^{-1}(C)$, so $D$ is comeager in $N_{s^{-1}}$. Thus, $\pi(D)$ is comeager in $N_{s^0}$, and $\pi(D) \subseteq C_{1-i}$, a contradiction since $C_i$ is also comeager in $N_{s^0}$ and $C_i, C_{1-i}$ are disjoint.

In view of lemma 4.8, we may speak unambiguously of a degree being selfdual or non-selfdual.

**Definition 4.9.** If $A, B \subseteq \omega^\omega$, their *join* is defined by

$$A \oplus B = \{x: (x(0) \text{ is even and } x' \in A) \lor (x(0) \text{ is odd and } x' \in B)\},$$

where $x'(n) = x(n + 1)$. Similarly, if $A_n \subseteq \omega^\omega$ are given for each $n$, their join is

$$\oplus_n A_n = \{x: x' \in A_{x(0)}\}.$$

**Lemma 4.10 (ZF + AD).** For any $A, B \subseteq \omega^\omega$, $A, B \leq_\ell A \oplus B$. Furthermore, if \{C, C\} \leq_\ell A \oplus B then \{C, C\} \leq_\ell A$ or \{C, C\} \leq_\ell B.

**Proof.** Clearly $A, B \leq_\ell A \oplus B$. For example, to see that $A \leq_\ell A \oplus B$, have II play first play a 0 in the game $G_{A,A\oplus B}$, and then copy I’s moves. Suppose now that $C \leq_\ell A \oplus B$, but $A \oplus B \not<_\ell C$. Thus, I wins the game $G_{A\oplus B,C}$ by $\sigma$. Suppose $\sigma$’s first move is even (the other case is similar). Then $\sigma$ gives a winning strategy for II in the game $G_{C,A^c}$. Thus, $C \leq_\ell A^c$. So, \{C, C\} \leq_\ell \{A, A^c\}$. So, $o(C) \leq \max\{o(A), o(B)\}$. \hfill \square

Note that lemma 4.10 does not say that $o_\ell(A \oplus B) = \max\{o_\ell(A), o_\ell(B)\}$ (it does say that $o_\ell(A \oplus B) \leq \max\{o_\ell(A), o_\ell(B)\} + 1$). We do get the following.

**Lemma 4.11.** If $A$ is non-selfdual, then $A \oplus A^c$ is the least $\ell$-degree strictly above the degree of $A$. Also, $A \oplus A^c <_w A$, so $A \oplus A^c$ is also the least Wadge degree strictly above $A$. Furthermore, $A \oplus A^c$ is selfdual.

**Proof.** First note that $A \oplus A^c <_w A$. For if $A \oplus A^c \leq_w A$, then $A^c \leq_w A$. This contradicts $A$ being non-selfdual. To see that II wins $G_{A\oplus A^c,(A\oplus A^c)^c}$, if I plays $i$ for the first move, have II play $i + 1$ (to change the parity), and then copy I’s moves. Thus, $A \oplus A^c$ is selfdual. So, $A \oplus A^c$ is selfdual and $o(A \oplus A^c) > o(A)$. Finally, suppose $\{B, B^c\} \leq_\ell A \oplus A^c$. So, I has a winning strategy $\sigma$ in $G_{A\oplus B,A^c}$. If I’s first move is even (the other case is similar), then $\sigma$ gives a strategy for II in the game $G_{B,A^c}$, and so $o(B) \leq o(A)$. \hfill \square

Corollary 4.11 identifies the next Lipschitz (in fact Wadge) degree after $\{A, A^c\}$ when $A$ is non-selfdual. We now identify the next Lipschitz degree when $A$ is selfdual.

**Lemma 4.12 (ZF + AD).** Suppose $A$ is selfdual. Then $A' = \{0^\omega: x \in A\}$ is the next Lipschitz degree above $A$. Also, $A'$ is selfdual.

**Proof.** Clearly $A \leq_\ell A'$. Suppose $A' \leq_\ell \{A, A^c\}$. Since $A$ is selfdual, we have $A' \leq_\ell A^c$. So, II has a winning strategy $\tau$ in the game $G_{A,A^c}$. From the definition of $A'$, it follows that $\tau$ give a strategy for I in the game $G_{A,A}$ (make a 0 for the first move, and then follow $\tau$ as a strategy for I). II can defeat this, however, by copying. To see that II wins $G_{A',(A')^c}$, if I plays 0, then have II play 0 and then follow a strategy for II in $G_{A,A^c}$. If I make a first move other than 0, have II play a 0, and then any sequence in $A$. Thus, $A'$ is selfdual. Finally, suppose $\{B, B^c\} <_\ell A'$.\]
Lemma 4.13. \(\text{cofinality } \omega\) schitz hierarchy. We next consider limit stages. First we consider limit stages of all \(n\) selfdual, I wins the game \(G\).

Proof. Let \(o\) we show that II wins \(G\). From the definition of join, this gives a winning strategy for II in the game \(\{A, B, C, D\} \). Thus, II has a winning strategy in \(G\) for all \(n\), then \(\oplus_n A_n \equiv_w A_0\).

Exercise 17. Show that if \(A\) is selfdual, then \(A' \equiv_w A\) (where \(A'\) is as in lemma 4.12).

Exercise 18. Show that if \(A\) is non-selfdual, then \(A' \equiv_e A\). [hint: Since \(A\) is non-selfdual, I wins the game \(G\). This gives a winning strategy for II in the game \(G\)].

Lemmas 4.11, 4.12 identify the Lipschitz degrees at successor stages in the Lipschitz hierarchy. We next consider limit stages. First we consider limit stages of cofinality \(\omega\).

Lemma 4.13. Suppose \(\{A_n\}_{n \in \omega}\) is given with \(o_l(A_n) < o_l(A_{n+1})\) for all \(n\). Then \(o_l(\oplus_n A_n) = \sup_n o_l(A_n)\). Also, \(\oplus_n A_n\) is selfdual. Furthermore, if \(A_n \equiv_w A_0\) for all \(n\), then \(\oplus_n A_n \equiv_w A_0\).

Proof. Let \(A = \oplus_n A_n\). Clearly \(A_n \leq \ell A\) for all \(n\). To see that \(A\) is selfdual, we show that II wins \(G_{A,A^\ell}\). If I makes first move \(n\), II plays any \(m\) such that \(o_l(A_n) < o_l(A_m)\). II then follows any winning strategy for II in \(G_{A_n,A_m}\). Suppose \(\{B, B^\ell\} \leq \ell A\). Since \(A \not\leq \ell B\), I has a winning strategy \(\sigma\) in \(G_{A,B}\). From the definition of join, this gives a winning strategy for II in the game \(G_{B,A^\ell}\), where \(n = \sigma(\emptyset)\). Thus, \(\{B, B^\ell\} \leq \ell \{A_n, A^\ell\}\). Therefore, \(o_l(A) = \sup_n o_l(A_n)\).

Finally, suppose \(A_n \equiv_w A_0\) for all \(n\). II has a winning strategy in \(G_{A,A^\ell}\) defined as follows. If I makes first move \(n\), II passes, and then follows a strategy for II in the game \(G_{A_n,A^\ell}w\). \(\square\)

Thus, at limit stages of cofinality \(\omega\) in the Lipschitz hierarchy, there is a selfdual degree. From lemma 4.13 it follows that after a non-selfdual degree \(\{A, A^\ell\}\), the next \(\omega_1\) Lipschitz degrees are all selfdual and all of the same Wadge degree as \(\{A, A^\ell\}\).

Now we consider stages in the Lipschitz hierarchy of uncountable cofinality.

Lemma 4.14. Suppose \(\text{cof}(\alpha) > \omega\). Then the \(A\) such that \(o_l(A) = \alpha\) is non-selfdual.

Proof. Suppose \(o(A) = \alpha\), and \(A\) is selfdual. For each \(n \in \omega\), let \(A_n = \{x: n \leq x \in A\}\). Thus, \(A = \oplus_n A_n\). Let \(\alpha_n = o(A_n)\). If \(\alpha_n < \alpha\) for all \(\alpha\), then \(\alpha_n = \sup_n o_l(A_n) < \alpha\), a contradiction. So, fix \(n\) such that \(o_l(A_n) = o_l(A)\), in particular, \(A_n\) is selfdual. Thus, II has a winning strategy in \(G_{A,A^\ell}\). If we fix I’s first move as \(n\), and then copy II’s moves, we defeat II’s winning strategy. \(\square\)

From the previous lemmas, we now have a complete picture of the Lipschitz and Wadge hierarchies.

Theorem 4.15 (ZF + AD). The non-selfdual and the selfdual Wadge degrees alternate. At limit stages of countable cofinality there is a selfdual Wadge degree which is the degree of the join of Waadge degrees from a cofinal \(\omega\) sequence. At limit stages of uncountable cofinality there is a non-selfdual Wadge degree. Every non-selfdual Lipschitz degree is non-selfdual as a Wadge degree. Every selfdual Wadge degree consists of an \(\omega_1\) block of selfdual Lipschitz degrees.
Along the way, we have also obtained some additional information about the degrees. For \( A \subseteq \omega^\omega \) and \( n \in \omega \), recall \( A_n = \{ x : n^s x \in A \} \). Also, for \( s \in \omega^{<\omega} \), let \( A_s = \{ x : s^s x \in A \} \).

**Lemma 4.16.** If \( A \) is a selfdual degree, then for every \( n \in \omega \), \( A_m <_\ell A \). Also, for every \( x \in \omega^\omega \) there is an \( n \) such that \( A_{x|n} <_w A \). If \( A \) is non-selfdual, then there is an \( x \in \omega^\omega \) such that for all \( n \), \( A_{x|n} \equiv_w A \) (thus, this property characterizes the non-selfdual degrees).

**Proof.** Clearly \( A_s \leq_\ell A \) for any \( s \) and any \( A \). If \( A \) is selfdual, then we showed in the proof of lemma 4.14 that for every \( n \) that \( A_n <_\ell A \). So, for any \( x \in \omega^\omega \), the sequence \( A_{x|0}, A_{x|1}, \ldots \) is strictly decreasing as Lipschitz degrees until we hit a non-selfdual degree \( A_{x|n} \) (which we must eventually do by wellfoundedness). We then have that \( A_{x|n} <_w A \) as otherwise \( A \) is non-selfdual.

Suppose now that \( A \) is non-selfdual. If for each \( n \) we had \( A_n <_w A \), then \( A = \bigoplus_n A_n \) would be selfdual. Namely, \( \Pi \) could win \( G^w(A, A') \) as follows: \( \Pi \) plays \( n \), \( \Pi \) passes, and then \( \Pi \) follows a strategy Wadge reducing \( A_n \) to \( A' \). So, for some \( n \) we have \( A_n \equiv_w A \), and in particular \( A_n \) is also non-selfdual. Repeating the argument gives an \( x \) such that for all \( n \), \( A_{x|n} \equiv_w A \).

\( \square \)

**Exercise 19.** Show the property of lemma 4.16 for Wadge degrees directly, that is without mentioning Lipschitz degrees. [hint: suppose \( A \) is selfdual, and suppose \( x \in \omega^\omega \) were such that for all \( n \), \( A_{x|n} \equiv_w A \). For each \( n \), fix winning strategies \( \sigma_n, \tau_n \) for \( \Pi \) in the Wadge games \( G^w(A, A_{x|n}) \) and \( G^w(A, A'_{x|n}) \). Define a sequence of integers \( k_n \) inductively so that for all \( n \), if we set \( s_n = x \upharpoonright k_n \) and for \( m < n \) let \( s_m = (x \upharpoonright k_m)^c \rho_{k_m}(s_{m+1}) \), then \( \ell(s_0) \geq n \) (for all choices of \( \rho_{k_m} \in \{ \sigma_{k_m}, \tau_{k_m} \} \)). For every \( z \in 2^n \), this defines a filling-in of a diagram producing \( x_0, x_1, \ldots \) with \( x_n \upharpoonright k_n = x \upharpoonright k_n \), and where \( x_n = \sigma_{k_n}(x_{n+1}) \) if \( x(n) = 0 \) and \( x_n = \tau_{k_n}(x_{n+1}) \) if \( x(n) = 1 \). This gives a contradiction as in theorem 4.4.]

The analysis of Wadge degrees (Theorem 4.15) gives immediately an analysis of the pointclasses.

**Theorem 4.17 (ZF + AD).** Let \( \Gamma \) be a pointclass. then one of the following cases holds.

1. There is a non-selfdual Wadge degree \( [(A, A')]_w \) with \( \Gamma = \{ B : B \leq_w A \} \) (note: this case is symmetrical between \( A \) and \( A' \)).
2. There is a selfdual Wadge degree \( [(A)]_w \) such that \( \Gamma = \{ B : B \leq_w A \} \).
3. There is a non-selfdual Wadge degree \( [(A, A')]_w \) with \( \Gamma = \{ B : B \leq_w A \} \cup \{ B : B \leq_w A' \} \).
4. There is a limit ordinal \( \alpha \) such that \( \Gamma = \{ B : o_w(B) < \alpha \} \).

In the first case, \( \Gamma \) is non-selfdual, and in the last three cases \( \Gamma \) is selfdual.

**Proof.** Consider \( I = \{ \alpha \in \text{On} : \exists A \in \Gamma \ o_w(A) = \alpha \} \). Clearly \( I \) is an ordinal (if \( A \in \Gamma \) and \( (B, B') <_w (A, A') \), then \( B, B' \in \Gamma \) by Wadge). If \( I \) is a limit ordinal (that is, there is no largest \( \alpha \) in \( I \)), then we have that \( \Gamma \) is selfdual (if \( A \in \Gamma \), let \( B \in \Gamma \) with \( (A, A') <_w (B, B') \), then \( A' \leq B \) and so \( A' \in \Gamma \)). Suppose then that there is a largest ordinal \( \alpha \) in \( I \). Let \( o_w(A) = \alpha \). If \( A \) is selfdual, then for every \( B \in \Gamma \) we have \( B \leq_w A \) and so \( \Gamma = \{ B : B \leq_w A \} \). As \( A \) is selfdual, so is \( \Gamma \) in this case. If \( (A, A') \) is a non-selfdual Wadge degree, at least one of \( A, A' \) must be in \( \Gamma \) (as \( \alpha \) is in \( I \)). If exactly one of these sets, say \( A \), is in \( \Gamma \), then for every
B ∈ Γ we must have B ≤_w A as otherwise, by Wadge, A^c ≤_w B would be in Γ. So, Γ = \{B: B ≤_w A\}. Γ is clearly non-selfdual in this case. If Γ contains both A and A^c, then Γ ⊃ {B: B ≤_w A} ∪ {B: B ≤_w A^c}. We must have equality here as otherwise α is not the largest element of I. Clearly Γ is selfdual then. □

5. Theory Of Pointclasses

We continue with the AD theory of Wadge degrees, developing an abstract theory of pointclasses. Recall the definitions of the separation and reduction properties.

**Definition 5.1.** Let Γ be a pointclass. Γ has the separation property if for all A, B ∈ Γ with A ∩ B = ∅, there is a C ∈ Δ = Γ ∩ Γ with A ⊆ C and C ∩ B = ∅.

Note that this definition is symmetrical between A and B. We say the set C (as in the definition) separates A from B. We let sep(Γ) denote the statement that Γ has the separation property.

**Definition 5.2.** Let Γ be a pointclass. Γ has the reduction property if for all A, B ∈ Γ there are A', B' ∈ Γ with A' ⊆ A, B' ⊆ B, A' ∩ B' = ∅, and A' ∪ B' = A ∪ B.

Note that (in the notation of the definition), A − B ⊆ A' and B − A ⊆ B'. the points in A ∩ B can go in either A' or B' (but only one of them). We let red(Γ) denote the reduction property for Γ.

The reduction property is a stronger property in the sense of the following.

**Fact 5.3 (ZF).** red(Γ) → sep(Γ).

*Proof.* Assume red(Γ), and let A, B ∈ Γ with A ∩ B = ∅. Let A_1 = B^c, B_1 = A^c, so A_1, B_1 ∈ Γ and A_1 ∪ B_1 = ω^ω. □

First we consider the separation property. We have the following general result of Steel and Van-Wesep.

**Theorem 5.4 (ZF+AD).** For every non-selfdual pointclass Γ, exactly one of sep(Γ) or sep(Γ̄) holds.

Steel showed [?] that for every non-selfdual Γ that either sep(Γ) or sep(Γ̄) holds. Van-wesep [?] showed that both sides of a non-selfdual class cannot have the separation property.

We first prove Steel’s theorem, which we state separately in the following.

**Theorem 5.5 (Steel).** Assume ZF + AD. For every non-selfdual pointclass Γ, either sep(Γ) or sep(Γ̄) holds.

*Proof.* Let Γ be non-selfdual, and let Δ = Γ ∩ Γ̄. We say a pair of sets A, B ⊆ ω^ω is Δ-inseparable if there is no Δ set C separating them, that is, A ⊆ C, C ∩ B = ∅ (this is symmetric in A and B).

**Lemma 5.6.** Let A_0, A_1 be sets which are Δ inseparable. Then for any pair B_0, B_1 of disjoint sets both of which in Γ or both of which are in Γ̄, there is a strategy σ for I (i.e., a Lipschitz 1/2 function) such that for all x ∈ ω^ω, (x ∈ B_0 → σ(x) ∈ A_0) and (x ∈ B_1 → σ(x) ∈ A_1).
Proof. Play the game where I plays out \( z \in \omega^\omega \) and II plays out \( x \in \omega^\omega \) and I wins the run iff
\[
(x \in B_0 \rightarrow z \in A_0) \land (x \in B_1 \rightarrow z \in A_1).
\]
If I has a winning strategy \( \sigma \), then we are done. If II has a winning strategy \( \tau \), then note that \( \forall z (\tau(z) \in B_0 \cup B_1) \). Also, if \( z \in A_0 \), then \( \tau(z) \in B_1 \) and if \( z \in A_1 \) then \( \tau(z) \in A_1 \). So, \( \{ z: \tau(z) \in B_1 \} = \omega^\omega - \{ z: \tau(z) \in B_0 \} \), and so these are both \( \Delta \) sets. This gives a separation of \( A_0 \) and \( A_1 \), a contradiction. \( \square \)

To prove theorem 5.5, suppose toward a contradiction that \( \neg \text{sep}(\Gamma) \) and \( \neg \text{sep} (\bar{\Gamma}) \).

Let \( A_0, A_1 \) be \( \Gamma \) sets which are \( \Delta \) inseparable, and let \( B_0, B_1 \) be \( \bar{\Gamma} \) sets which are \( \Delta \)-inseparable. From the lemma (applied to the \( \Delta \)-inseparable pair \( B_0, B_1 \)) there is a continuous function \( f \) such that \( (x \in A_0 \rightarrow f(x) \in B_0) \) and \( (x \in A_1 \rightarrow f(x) \in B_1) \).

Let \( C_0 = f^{-1}(B_0), C_1 = f^{-1}(B_1) \). Thus, \( C_0, C_1 \in \bar{\Gamma}, A_0 \subseteq C_0, A_1 \subseteq C_1 \), and \( C_0 \cap C_1 = \emptyset \).

From the lemma (applied to the \( \Delta \)-inseparable pair \( A_0, A_1 \)) there is a strategy \( \sigma_0 \) for I such that \( (x \in C_0 \rightarrow \sigma_0(x) \in A_1) \) and \( (x \in C_1 \rightarrow \sigma_0(x) \in A_0) \). There is also a strategy \( \sigma_1 \) for I such that \( (x \in A_0 \rightarrow \sigma_1(x) \in A_0) \) and \( (x \in C_0 \rightarrow \sigma_1(x) \in A_1) \). Finally, there is a strategy \( \sigma_2 \) for I such that \( (x \in A_1 \rightarrow \sigma_2(x) \in A_1) \), and \( (x \in C_1 \rightarrow \sigma_2(x) \in A_0) \). For any \( z \in 3^\omega \) there is a filling-in of the diagram to produce reals \( x_0, x_1, \ldots \) such that \( x_n = \rho(x_{n+1}) \) where \( \rho = \sigma_0 \) or \( \sigma_1 \) or \( \sigma_2 \) if \( z(n) = 0 \) or 1 or 2 respectively. Let \( x_0(z) \) be the value of \( x_0 \) produced for this particular \( z \). Note that all of the \( \sigma_i \) have the property that if \( x \in A_0 \cup A_1 \) then \( \sigma_i(x) \in A_0 \cup A_1 \).

We now get the usual contradiction an in theorem 4.4. Namely, let \( E_0 = \{ z: x_0(z) \in C_0 \} \), \( E_1 = \{ z: x_0(z) \in C_1 \} \), and \( E_2 = \{ z: x_0(z) \notin C_0 \cup C_1 \} \). Let \( s \in \omega^\omega \) be such that one of the \( E_i \) is comeager on \( N_s \). Suppose \( E_0 \) is comeager on \( N_s \). So, \( E_0 \) is comeager on the neighborhood determined by \( t = s^{-1}k^{-1}s \) for any \( k \).

By choosing \( k \) of the appropriate parity, we have that for almost all \( z \) in \( N_t \) that \( x_0(z) \in A_1 \subseteq C_1 \), a contradiction. The argument in the case \( E_1 \) is comeager on \( N_s \) is identical. Finally, suppose \( E_2 \) is comeager on \( N_s \). Then \( E_2 \) is also comeager on \( N_t \), for \( t = s^{-1}k^{-1}s \). However, for almost all \( z \) in \( N_t \) we have \( x_0(z) \in A_0 \cup A_1 \) from the definition of \( \sigma_1 \) (we use here the fact that if \( x \notin C_0 \), then \( \sigma_1(x) \in A_1 \), and so all further \( \sigma_i \) applied to this point stay in \( A_0 \cup A_1 \)). This is a contradiction and completes the proof of theorem 5.5. \( \square \)

We now give the other half of theorem 5.4.

**Theorem 5.7** (Van Wesep). Assume \( ZF + AD \). If \( \Gamma \) is non-selfdual, then \( \text{sep}(\Gamma), \text{sep}(\bar{\Gamma}) \) cannot both hold.

**Proof.** Suppose toward a contradiction that \( \text{sep}(\Gamma), \text{sep}(\bar{\Gamma}) \) both hold. Let \( A \in \Gamma - \bar{\Gamma} \). We again regard every real \( x \in \omega^\omega \) as coding a Lipschitz continuous function from \( \omega^\omega \) to \( \omega^\omega \). Define \( A_0(x, y) \leftrightarrow ((x)_0(y) \in A) \) and \( A_1(x, y) \leftrightarrow ((x)_1(y) \in A) \). Then \( A_0, A_1 \in \Gamma \) and form a \( \Gamma \)-universal pair. That is, for every pair \( B_0, B_1 \) of \( \Gamma \) sets, there is an \( x \) such that \( B_0 = (A_0)_x, B_1 = (A_1)_x \) (recall \( C_x \) denotes the section \( \{ y: C(x, y) \} \) of the set \( C \)).

We cannot have that \( A_0 - A_1 \) and \( A_1 - A_0 \) can be separated by a \( \Delta \) set. This is the same argument that \( \text{red}(\Gamma) \rightarrow \neg \text{sep}(\Gamma) \). Here briefly is the argument again. Suppose \( C \in \Delta \) and \( A_0 - A_1 \subseteq C, C \cap (A_1 - A_0) = \emptyset \). Then \( C \in \Delta \) and is universal for \( \Delta \) sets, a contradiction (since we then define \( D(x) \leftrightarrow \neg C(x, x) \), so \( D \in \Delta \), and for all \( x, D \neq C_x \)). Since we are assuming
sets \(\tilde{\Gamma}\), it follows that \(A_0 - A_1\) and \(A_1 - A_0\) cannot be separated by disjoint \(\tilde{\Gamma}\) sets (that is, there does not exist disjoint \(\tilde{\Gamma}\) sets \(D_1, D_2\) with \(A_0 - A_1 \subseteq D_0\) and \(A_1 - A_0 \subseteq D_1\)).

Consider the game \(G_0\) where I plays out \(x\), II plays out \(y\), and II wins the run iff
\[
(y \in A_0 \cup A_1) \\
\land ((x \in A_0 - A_1) \rightarrow (y \in A_0 - A_1)) \\
\land ((x \in A_1 - A_0) \rightarrow (y \in A_1 - A_0)).
\]

If II had a winning strategy \(\tau\), then \(D_0 = \tau^{-1}(A_0^c)\) and \(D_1 = \tau^{-1}(A_0^c)\) would be disjoint \(\tilde{\Gamma}\) sets with \(A_0 - A_1 \subseteq D_0\), \(A_1 - A_0 \subseteq D_1\), a contradiction. Let \(\sigma_0\) be a winning strategy for I in \(G_0\). Then \(y \in (A_0 - A_1) \rightarrow (\sigma_0(y) \in A_1 - A_0)\), \(y \in (A_1 - A_0) \rightarrow (\sigma_0(y) \in A_0 - A_1)\), and \(y \in (A_0 \cup A_1) \rightarrow (\sigma_0(y) \in ((A_0 - A_1) \cup (A_1 - A_0)))\).

Now apply the same argument to to the pair \(A_0^c, A_1^c\), which is a universal pair for \(\tilde{\Gamma}\). As before, we cannot have that \(A_0^c - A_1^c = A_1 - A_0\) and \(A_1^c - A_0^c = A_0 - A_1\) can be separated by disjoint \(\tilde{\Gamma}\) sets. We consider the game \(G_1\) where I plays out \(x\), II plays out \(y\), and II wins the run iff
\[
(y \in A_0^c \cup A_1^c) \\
\land ((x \in A_0 - A_1) \rightarrow (y \in A_1 - A_0)) \\
\land ((x \in A_1 - A_0) \rightarrow (y \in A_0 - A_1)).
\]

If II had a winning strategy \(\tau\), then \(D_0 = \tau^{-1}(A_1)\), \(D_0 = \tau^{-1}(A_0)\) would be disjoint \(\tilde{\Gamma}\) sets separating \(A_0 - A_1\) and \(A_1 - A_0\). Let \(\sigma_1\) be a winning strategy for I in \(G_1\). Then \(y \in (A_0 - A_1) \rightarrow (\sigma_0(y) \in A_1 - A_0)\), \(y \in (A_1 - A_0) \rightarrow (\sigma_0(y) \in A_0 - A_1)\), and \(y \in (A_0^c \cup A_1^c) \rightarrow (\sigma_0(y) \in ((A_0 - A_1) \cup (A_1 - A_0)))\).

Notice that \(\sigma_0\) flips membership between \(A_0 - A_1\) and \(A_1 - A_0\), while \(\sigma_1\) preserves membership. Also, \(\sigma_0\) maps \(A_0 \cup A_1\) into the union of these two sets, while \(\sigma_1\) maps \((A_0 \cap A_1)^c\) into these two sets.

We get the usual Martin-Monk contradiction. For every \(z \in 2^\omega\), consider the filling-in to produce \(x_0, x_1, \ldots\) where \(x_n = \sigma_0(x_{n+1})\) if \(z(n) = 0\), \(x_n = \sigma_1(x_{n+1})\) if \(z(n) = 1\). Suppose that for nonmeager many \(z\) that \(x_0(z) \notin (A_0 - A_1) \cup (A_1 - A_0)\). Say for nonmeager many \(z\) that \(x_0(z) \notin (A_0 \cup A_1)^c\) (the case where for nonmeager many \(z\) we have \(x_0(z) \in A_0 \cap A_1\) is similar). Say \(E = \{z : x_0(z) \notin A_0 \cup A_1\}\) is comeager on \(N_s\). So, \(E\) is also comeager on \(N_1\), where \(t = s^{-1}1\text{ }s\). But \(y \notin (A_0 \cup A_1)\) implies \(\sigma_1(y) \in (A_0 - A_1) \cup (A_1 - A_0)\). So, for comeager in \(N_1\) many \(z\) we have that \(x_0(z) \notin (A_0 - A_1) \cup (A_1 - A_0)\), a contradiction. So, for comeager many \(z\) we have \(x_0(z) \notin (A_0 - A_1) \cup (A_1 - A_0)\). Suppose without loss of generality that \(F = \{z : x_0(z) \in A_0 - A_1\}\) is comeager on \(N_s\). Then \(F\) is comeager on \(N_1\), where \(t = s^{-1}0^k \text{ }s\), and this is a contradiction as \(\sigma_0\) flips membership between \(A_0 - A_1\) and \(A_1 - A_0\), provided we take \(k\) of the appropriate parity.

This completes the proof of theorem 5.4.

One use of the separation property is to transfer closure properties from \(\Delta\) to \(\Gamma\).

**Theorem 5.8** (Steel). **Assume ZF + AD.** Let \(\Gamma\) be non-selfdual and assume \(\text{sep}(\tilde{\Gamma})\). Then:

1. If \(\Delta\) is closed under finite (countable) unions, then \(\Gamma\) is closed under finite (resp. countable) unions.
2. If \(\Delta\) is closed under \(\exists^\omega\), then \(\Gamma\) is closed under \(\exists^\omega\).
Proof. To prove (1), let \( A, B \in \mathfrak{F} \) and assume \( A \cup B \notin \mathfrak{F} \). By Wadge, every \( \Gamma \) set is Wadge reducible to \( A \cup B \), and thus every \( \Gamma \) set can be written as the union of two \( \mathfrak{F} \) sets. Say \( A', B' \) are \( \mathfrak{F} \) sets with \( A' \cup B' \in \mathfrak{F} - \mathfrak{F} \). By sep(\( \Gamma \)), let \( C \subseteq \Delta \) and separate \( A' \) from \( (A' \cup B')^c \), and let \( D \in \Delta \) separate \( B' \) from \( (A' \cup B')^c \). Then \( A' \cup B' = C \cup D \in \Delta \), a contradiction.

To prove (2), suppose \( 3^{\omega^\omega} \Gamma \nsubseteq \Gamma \). By Wadge, \( \Gamma \subseteq 3^{\omega^\omega} \mathfrak{F} \). We will show sep(\( \Gamma \)), a contradiction. Let \( A, B \) be disjoint \( \Gamma \) sets. Let \( A(x) \leftrightarrow \exists y A'(x, y), \) and \( B(x) \leftrightarrow \exists y B'(x, y) \), where \( A', B' \) are \( \mathfrak{F} \) subset of \( \omega^\omega \times \omega^\omega \). Define \( A''(x, y, z) \leftrightarrow A'(x, y) \) and \( B''(x, y, z) \leftrightarrow B'(x, z) \). Clearly \( A'', B'' \) are disjoint \( \Gamma \) sets. Let \( A'' \subseteq D \subseteq (B'')^c \), with \( D \in \Delta \), from sep(\( \Gamma \)). Let \( E(x) \leftrightarrow \exists y \forall z D(x, y, z) \). Then \( E \in \Delta \) and \( A \subseteq E \subseteq B^c \). [Note that if \( x \in B \cap E \) then \( \exists z \forall y B''(x, y, z) \) and \( \exists y \forall z D(x, y, z) \). Fix \( z, y \) witnessing these two existential statements. Then \( B''(x, y, z) \) and \( D(x, y, z) \), a contradiction as \( B'' \) and \( D \) are disjoint.] \( \square \)

Remark 5.9. Using the coding lemma one has that if \( 3^{\omega^\omega} \Delta \nsubseteq \Delta \) and \( \alpha < \text{cof}(\omega(\Delta)) \), then \( \Delta \) is closed under \( \alpha \)-length unions. The proof of (1) then generalizes (using the coding lemma) to show that \( \mathfrak{F} \) is closed under \( \alpha \)-length unions.

An example due to Van Wesep shows that there is a non-selfdual class with neither sider having the reduction property. However, we have the following.

Theorem 5.10 (Steel, Van Wesep). Assume ZF + AD. Suppose \( \Gamma \) is non-selfdual, sep(\( \Gamma \)), and the intersection of two \( \Delta \) sets is in \( \Gamma \). Then red(\( \Gamma \)).

Proof. The proof of (1) of theorem 5.8 shows that \( \mathfrak{F} \) is closed under finite unions (so \( \Gamma \) is closed under finite intersections). Since we are assuming sep(\( \Gamma \)), we have \( \neg \text{sep}(\mathfrak{F}) \). Let \( C, D \) be a disjoint pair of \( \Gamma \) sets which are not separable by a \( \Delta \) set. Let \( A, B \) be a pair of \( \Gamma \) sets. Consider the game where \( I \) plays out \( x, II \) plays out \( y \), and \( II \) wins if \( y \in A \cup B \), \( (x \in C \rightarrow y \in (A - B)) \), and \( (x \in D \rightarrow y \in (B - A)) \). If \( II \) had a winning strategy \( \tau \), then \( \tau^{-1}(B^c) \), \( \tau^{-1}(A^c) \) would be disjoint \( \Gamma \) sets with \( C \subseteq \tau^{-1}(B^c) \), \( D \subseteq \tau^{-1}(A^c) \). From sep(\( \Gamma \)) we would then have a \( \Delta \) set separating \( C \) from \( D \), a contradiction. Let \( \sigma \) be a winning strategy for \( I \). So, \( y \in (A \cup B) \rightarrow \sigma(y) \in (C \cup D) \), \( (y \in (A - B) \rightarrow \sigma(y) \in D) \), and \( (y \in (B - A) \rightarrow \sigma(y) \in C) \). Then \( A' = \{ y \in A : \sigma(y) \in D \} \) and \( B' = \{ y \in B : \sigma(y) \in C \} \) are in \( \Gamma \) (since \( \Gamma \) is closed under intersections) and reduce \( A, B \).

Corollary 5.11. If \( \Gamma \) is non-selfdual and \( \Delta \) is closed under finite intersections (equivalently, finite unions), then \( \text{red}(\Gamma) \) or \( \text{red}(\mathfrak{F}) \).

Corollary 5.12. If \( \Gamma \) is non-selfdual, sep(\( \Gamma \)), and \( \Gamma \) is closed under finite intersections, then \( \text{red}(\Gamma) \).

In particular, if \( \Gamma \) is non-selfdual and \( \Gamma \) is closed under finite unions and intersections, then \( \text{red}(\Gamma) \) or \( \text{red}(\mathfrak{F}) \) holds.

6. The Coding Lemma

The coding lemma of Moschovakis is a basic tool in determinacy theory. It provides a choice-like principle which holds assuming AD. We first need to some abstract recursion theoretic or “lightface” notions.

Let \( \Gamma \) denote a non-selfdual pointclass. Recall that we may define a univer set \( U \subseteq \omega^\omega \times \omega^\omega \) for the \( \Gamma \) subsets of \( \omega^\omega \) by fixing a set \( A \in \mathfrak{F} - \mathfrak{F} \) and then letting

\[
U(x, y) \leftrightarrow x(y) \in A,
\]

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Let \( \Gamma \) denote a non-selfdual pointclass. Recall that we may define a univer set \( U \subseteq \omega^\omega \times \omega^\omega \) for the \( \Gamma \) subsets of \( \omega^\omega \) by fixing a set \( A \in \mathfrak{F} - \mathfrak{F} \) and then letting

\[
U(x, y) \leftrightarrow x(y) \in A,
\]
where every \( x \in \omega^\omega \) is viewed as giving a Lipschitz continuous function (strategy for \( \Pi \)) from \( \omega^\omega \) to \( \omega^\omega \).

Recall we also have recursive coding and decoding functions on \( \omega^\omega \) which we collectively denote by \((x,y) \mapsto (x,y), (x_0,x_1,\ldots) \mapsto (x_0,x_1,\ldots), z \mapsto ((z)_0,(z)_1), z \mapsto ((z)_0,(z)_1,\ldots)\), etc. Exactly which of these coding/decoding functions the notation refers to will be clear from the context.

In effective (“lightface”) descriptive set theory, the existence of continuous (even recursive) so-called \( s-m-n \) functions is a basic starting point for much of the theory. We first show that this set-up can be developed for an arbitrary non-selfdual \( \Gamma \). This is the content of the next lemma. For the lemma, we only require that \( \Gamma \) have a universal set (which under AD holds for all non-selfdual \( \Gamma \)). By a “product space” we mean the smallest collection of spaces containing \( \omega^\omega \) and closed under finite products, e.g., \( X = (\omega^\omega \times \omega^\omega \times \omega^\omega) \times (\omega^\omega \times \omega^\omega) \). The coding and decoding functions extend naturally to these product spaces.

We assume \( \Gamma \) is a pointclass for products spaces as defined above.

**Lemma 6.1 (s-m-n theorem).** Let \( \Gamma \) be a pointclass and assume \( \Gamma \) has a universal set. Then for every product space \( X \) there is a universal set \( U_X \subseteq \omega^\omega \times X \) such the following holds:

\( \text{(s-m-n property)} \) for every pair of product spaces of the form \( X = X_1 \times \ldots \times X_n, \ Y = X_1 \times \ldots \times X_n \times \ldots \times X_m \ (m > n) \) there is a continuous function \( s_{X,Y} : \omega^\omega \times X \to \omega^\omega \) such that

\[
U_Y(y,x_1,\ldots,x_n,\ldots,x_m) \leftrightarrow U_X(s_{X,Y}(y,x_1,\ldots,x_n),x_{n+1},\ldots,x_m),
\]

where \( X' = X_{n+1} \times \ldots \times X_m \).

**Proof.** Let \( U \subseteq \omega^\omega \times \omega^\omega \) be the universal \( \Gamma \) set for subseteq of \( \omega^\omega \) as mentioned above. The idea is to define all the \( U_X \) by referring to this single set \( U \). For \( X = X_1 \times X_2 \times \ldots \times X_n \) we define

\[
U_X(y,x_1,\ldots,x_n) \leftrightarrow U((y)_0,((s)_1,x_1,\ldots,x_n)).
\]

Let \( Y = X_1 \times \ldots \times X_n \times X_{n+1} \times \ldots \times X_m \). So we also have

\[
U_Y(y,x_1,\ldots,x_n,\ldots,x_m) \leftrightarrow U((y)_0,((y)_1,x_1,\ldots,x_n,\ldots,x_m))
\]

and

\[
U_{X'}(s,x_{n+1},\ldots,x_m) \leftrightarrow U((s)_0,((s)_1,x_{n+1},\ldots,x_m)).
\]

So, we take

\[
s_{X,Y}(y,x_1,\ldots,x_n) = (\epsilon, (y,x_1,\ldots,x_n))
\]

where \( \epsilon \in \omega^\omega \) is a fixed real such that

\[
U(\epsilon,(y,x_1,\ldots,x_n),x_{n+1},\ldots,x_m) \leftrightarrow U((y)_0,((y)_1,\ldots,x_m))
\]

holds for all \( y,x_1,\ldots,x_m \). This can be done by choosing \( \epsilon \) such that

\[
U(\epsilon, z) \leftrightarrow U((z)_0,0,0,0,\ldots),(z_0,0,1,\ldots),(z_{0,n},z_{1,1},\ldots),(z_{m-n}),\ldots)
\]

where in this last equation we have simplified the notation, so that \( z_{0,0} \) means \(((z)_0)_0,0,0,\ldots\), etc.

**Exercise 20.** Show that the \( s-m-n \) functions of Lemma 6.1 may be taken to be recursive. [hint: we need to show that there is a recursive function \( f : z = (y,x_1,\ldots,x_n) \mapsto \epsilon \) such that for all \( z,x_{n+1},\ldots,x_m \in \omega^\omega \) we have \( f(z)((z,x_{n+1},\ldots,x_m)) = z_{0,n}(w) \), where \( w = (z_{0,1},z_{1,1},\ldots,z_{n,n},x_{n+1},\ldots,x_m) \). It suffices to get a recursive \( f \) such that
for all \( z, x_{n+1}, \ldots, x_m, u \in \omega^\omega \) we have \((f(z))(\langle u, x_{n+1}, \ldots, x_m \rangle) = z_0,0(u)\), where \( w = \langle u_{0,1}, u_{1,1}, \ldots, u_n, x_{n+1}, \ldots, x_m \rangle \). Argue this directly, assuming our coding and decoding functions are reasonable, in particular, the first \( k \) digits of \( \langle a_0, \ldots, a_\ell \rangle \) determine the first \( k \) digits of \( a_0, \ldots, a_\ell \) for all of our coding functions.

An important consequence of the \( s-m-n \) theorem is the recursion theorem. Again, this is a classical result in recursion theory (due to Kleene) which also generalizes to arbitrary non-selfdual pointclasses. The recursion theorem says the remarkable fact that in defining a \( \Gamma \) set, we may actually use the code for the set we are defining in its definition. This can also be viewed as a fixed-point property on the set of codes for \( \Gamma \) sets.

**Theorem 6.2** (recursion theorem). Let \( \Gamma \) have a universal set. Then for every product space \( X = X_1 \times \cdots \times X_n \) and every \( \Gamma \) set \( A \subseteq \omega^\omega \times X \), there is a \( y^* \in \omega^\omega \) such that \( U_X(y^*, x_1, \ldots, x_n) \) iff \( A(y^*, x_1, \ldots, x_n) \) for all \( x_1, \ldots, x_n \).

**Proof.** Let \( X = X_1 \times \cdots \times X_n \) and fix \( A \subseteq \omega^\omega \times X \) in \( \Gamma \). Let \( s : \omega^\omega \times \omega^\omega \rightarrow \omega^\omega \) be the \( s-m-n \) function corresponding to the product spaces \( \omega^\omega \) and \( \omega^\omega \times X \). Fix \( \epsilon \) in \( \omega^\omega \) such that \( U_{\omega^\omega \times X}(\epsilon, y, x) \leftrightarrow A(s(y, y), x) \).

We thus have for all \( x, y, z \):

\[
U_X(s(\epsilon, y), x) \leftrightarrow U_{\omega^\omega \times X}(\epsilon, y, x) \leftrightarrow A(s(y, y), x).
\]

We then set \( y = \epsilon \), that is we take \( y^* = s(\epsilon, \epsilon) \). \( \square \)

We first prove a version of the coding lemma which applies to prewellorderings, and then prove a more general version which applies to wellfounded relations.

**Theorem 6.3** (AD). Let \( \Gamma \) be a nonselfdual pointclass closed under \( \exists^\omega_1 \) and \( \wedge \) (and assume \( \Gamma \supseteq \Pi^0_1 \)). Let \( \preceq \) be a prewellordering with both \( \preceq \) and the strict part \( \prec \) in \( \Gamma \). Let \( R \subseteq \text{dom}(\preceq) \times \omega^\omega \) be a relation with \( \text{dom}(R) = \text{dom}(\preceq) \). Then there is a \( \Gamma \) relation \( R' \subseteq R \) such that \( \text{dom}(R') \) meets every \([x]\) for \( x \in \text{dom}(\preceq) \) (here \([x] = \{y : y \preceq x \wedge x \preceq y\}\)).

**Proof.** Let \( U \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega \) be universal for \( \Gamma \upharpoonright \omega^\omega \times \omega^\omega \), and we assume all universal sets are good, that is, we have continuous \( s-m-n \) functions. We prove the coding lemma by induction on the length \(|\preceq|\) of the prewellordering \( \preceq \). If \(|\preceq| = \alpha + 1\) is a successor ordinal, the result follows easily by induction. Namely, given \( R \subseteq \text{dom}(\preceq) \times \omega^\omega \), apply induction to \( \preceq' \) where \( x \preceq' y \iff (x \preceq y \wedge y \preceq z) \), where \([z]\) is maximal in \( \preceq \). Note that \( \preceq' \) and the strict part \( \prec \) are in \( \Gamma \). Applying induction to \( \preceq' \) and \( S = R \cap (\text{dom}(\preceq') \times \omega^\omega) \) given an \( S' \subseteq S \in \Gamma \). Let \( R' = S' \cup \{(z, y)\}\) where \( R(z, y) \). Since by induction \( \text{dom}(S') \) meets every class below \([z]\), \( \text{dom}(R') \) meets every class of \( \preceq \). Also, \( R' \in \Gamma \) since \( S' \in \Gamma \), \( \Gamma \) is closed under \( \lor \) (since it is closed under \( \exists^\omega_1 \)), and \( \Gamma \) contains all closed sets.

So, assume that \(|\preceq| = \lambda \) is a limit. If \( \alpha \leq \lambda \) we say \( \epsilon \in \omega^\omega \) is \( \alpha \)-good if \( U_\epsilon \subseteq R \) and for all for all \( x \in \text{dom}(\preceq) \) with \(|x| < \alpha \) we have \( \text{dom}(U_\epsilon) \cap [x] \neq \emptyset \) (i.e., \( U_\epsilon \) is a choice relation below \( \alpha \)). Note that if \( \epsilon \) is \( \alpha \)-good for some \( \alpha \), then there is a maximal \( \alpha \leq \lambda \) such that \( \epsilon \) is \( \alpha \)-good which we denote by \( \alpha(\epsilon) \). So, \( \alpha(\epsilon) \) is defined if \( U_\epsilon \subseteq R \). By induction, for any \( \alpha < \lambda \) there is an \( \epsilon \) which is \( \alpha \)-good. To see this, pick \( z \in \text{dom}(\preceq) \) with \(|z| = \alpha \). Let \( \preceq' = \preceq \setminus \{(x, y) : y \prec z\} \), so \( \preceq' \) and the strict part \( \prec \) are in \( \Gamma \). Apply induction to \( \preceq' \) and \( S = R \cap \{(x, w) : x \prec z\} \) to produce \( S' \). If \( U_\epsilon = S' \), then \( \epsilon \) is \( \alpha \)-good.
Consider now the following game $G$: I and II play out reals $\epsilon, \delta$ respectively in $\omega^\omega$. II wins the run provided: $(U_\epsilon \subseteq R) \rightarrow (U_\delta \subseteq R \land \alpha(\delta) > \alpha(\epsilon))$.

Suppose first that I has a winning strategy $\sigma$ for $G$. Let $A = \sigma[\omega^\omega]$. For every $\epsilon \in A, U_\epsilon \subseteq R$. Furthermore $\{\alpha(\epsilon) : \epsilon \in A\}$ is unbounded in $\lambda$ since if $\delta$ is such that $\alpha(\delta) = \beta$, then $\alpha(\sigma(\delta)) \geq \beta$, and there are $\delta$ which are $\beta$-good for arbitrarily large $\beta < \lambda$. Define then $R'(x, y) \leftrightarrow \exists \epsilon \in A (U_\epsilon(x, y))$. $R' \in \Gamma$ from our assumed closure properties, and clearly $R'$ is $\lambda$-good so we are done.

Assume next that II has a winning strategy $\tau$ for $G$. The relation

$$S(\epsilon, z, x, w) \leftrightarrow U_\epsilon(x, w) \land (x \prec z)$$

is in $\Gamma$, and so $S(\epsilon, z, x, w) \leftrightarrow U(a, \epsilon, z, x, w) \leftrightarrow U(s(a, \epsilon, z), x, w)$ for some $a \in \omega^\omega$, and where $s$ is our continuous $s$-$m$-$n$ function. Let $f(\epsilon, z) = s(a, \epsilon, z)$, so $f$ is continuous. Note that if $z$ is in the field of $\leq$, then $U_{f(\epsilon, z)} = U_\epsilon \cap \{(x, w) : x \prec z\}$. If $z$ is not in the field of $\leq$, then $U_{f(\epsilon, z)} = \emptyset$. In particular, for any $z$ and $\epsilon$, if $U_\epsilon \subseteq R$, then $U_{f(\epsilon, z)} \subseteq R$.

From the recursion theorem, let $\epsilon$ be such that

$$U_\epsilon(x, w) \leftrightarrow \exists z (z \leq x \land x \leq z \land U(\tau(f(\epsilon, z)), x, w)).$$

We claim that $R' = U_\epsilon$ works. Note that $U_\epsilon(x, w) \rightarrow x \in \text{dom}(\leq)$. We show by induction on $|x|$ that $U_\epsilon(x, w) \rightarrow R(x, w)$. For any $z \in [x]$, by induction we have $U_{f(\epsilon, z)} = U_\epsilon \cap \{(x, w) : x \prec z\} \subseteq R$. It follows that $U_{\tau(f(\epsilon, z))} \subseteq R$ as well, and so $R(x, w)$.

Finally, we prove by induction on $|x|$ that for all $x \in \text{dom}(\leq)$ that $\text{dom}(R') \cap [x] \neq \emptyset$. Let $x \in \text{dom}(\leq)$, and by induction we may assume that $\text{dom}(R') \cap [x'] \neq \emptyset$ for all $x' \prec x$. It follows that for any $z \in [x]$ that $\text{dom}(U_{f(\epsilon, z)}) \cap [x'] \neq \emptyset$ as well. That is, $U_{f(\epsilon, z)}$ is $|x|$-good. It follows that $\tau(f(\epsilon, x))$ is $|x|$ good, that is, $U_{\tau(f(\epsilon, z))} \cap [x] \neq \emptyset$. Since the definition of $U_\epsilon$ involves unioning of all possible $z \in [x]$, it follows that $U_\epsilon \cap [x] \neq \emptyset$. 

\[\square\]

**Remark 6.4.** The hypothesis that both $\leq$ and $\prec$ are in $\Gamma$ from theorem 6.3 is more than necessary. The result holds just assuming the strict part $\prec$ is in $\Gamma$, although a different proof must be given. Theorem 6.5 below contains this stronger result, and also requires $\prec$ to only be a wellfounded relation.

**Theorem 6.5 (AD).** Let $\Gamma$ be a nonselfdual pointclass closed under $\exists^\omega \omega$ and $\land$ (and assume $\Gamma \supseteq \Pi^0_1$). Let $\prec$ be a wellfounded relation in $\Gamma$. Let $R \subseteq \text{dom}(\leq) \times \omega^\omega$ be a relation with $\text{dom}(R) = \text{dom}(\prec)$. Then there is a $\Gamma$ relation $R' \subseteq R$ such that for every $\beta < |\prec|$, $\text{dom}(R')$ meets $\{x \in \text{dom}(\prec) : |x| = \beta\}$.

**Proof.** Let $\prec_\beta$ denote $\{x \in \text{dom}(\prec) : |x| = \beta\}$. We again proceed by induction on $|\prec|$, and we may assume $|\prec|$ is a limit. Similar to before, we say $\epsilon$ is $\alpha$-good, for $\alpha \leq \lambda = |\prec|$, if $U_\epsilon \subseteq R$ and for all $\beta < \alpha$ with $\text{dom}(U_\epsilon) \cap \prec_\beta \neq \emptyset$. Also as before, if $U_\epsilon \subseteq R$, then we let $\alpha(\epsilon)$ be the largest $\alpha \leq \lambda$ such that $\epsilon$ is $\alpha$-good. As before we consider the game $G$ where I and II play out reals $\epsilon, \delta$ respectively in $\omega^\omega$ and II wins the run provided: $(U_\epsilon \subseteq R) \rightarrow (U_\delta \subseteq R \land \alpha(\delta) > \alpha(\epsilon))$. Note that by induction, considering $\prec_0 z = \{(x, y) : x \prec y \land y \prec z\}$ for $z \in \text{dom}(\prec)$, we have that there are $\epsilon$ which are $\alpha$-good for arbitrarily large $\alpha$ below $\lambda$.

Suppose first that I has a winning strategy $\sigma$ for $G$. Let $A = \sigma[\omega^\omega]$. So, for $\epsilon \in A$, $U_\epsilon \subseteq R$. Also, $\sup(\alpha(\epsilon) : \epsilon \in A) = \lambda$. Define $R'$ by $R'(x, w) \leftrightarrow \exists \epsilon \in A (U_\epsilon(x, w))$. Then $R' \subseteq R$ and $\text{dom}(R') \cap \prec_\beta \neq \emptyset$ for all $\beta < \lambda$. 


Suppose next that II has a winning strategy \( \tau \) for \( G \). Here we must argue a little differently from theorem 6.3. We attempt to define using the recursion theorem a relation \( U_\epsilon \) with \( \text{dom}(U_\epsilon) = \text{dom}(\prec) \) and such that \( U_\epsilon(x,u) \) implies \( U_u \subseteq R \) and \( \alpha(u) > |x|_\epsilon \). Consider the relation
\[
S(\epsilon, x, y, w) \leftrightarrow \exists u \ (z < x \wedge U_\epsilon(z, u) \wedge U_u(y, w)).
\]
From the s-m-n theorem, let \( s: \omega^\omega \times \omega^\omega \to \omega^\omega \) be continuous such that \( S(\epsilon, x, y, w) \leftrightarrow U(s(\epsilon, x), y, w) \). Define then, using the recursion theorem,
\[
U_\epsilon(x, u) \leftrightarrow (x \in \text{dom}(\prec)) \land u = \tau(s(\epsilon, x)).
\]
Clearly \( \text{dom}(U_\epsilon) = \text{dom}(\prec) \), and for every \( x \in \text{dom}(\prec) \) there is exactly one \( u \) such that \( U_\epsilon(x, u) \). We prove by induction on \( |x|_\epsilon \) that \( U_\epsilon(x, u) \) implies that \( U_u \subseteq R \) and \( \alpha(u) > |x|_\epsilon \). By induction, it follows that \( U_{\epsilon, x} \) is a union of relations which are \( \beta \)-good for various \( \beta \) whose supremum is at least \( |x|_\epsilon \). Thus, \( s(\epsilon, x) \) is \( |x|_\epsilon \)-good.

Finally, define \( R' \) by:
\[
R'(x, w) \leftrightarrow \exists y \ \exists u \ (y \in \text{dom}(\prec) \land U_\epsilon(y, u) \land U_u(x, w)).
\]
Clearly \( R' \subseteq R \) since \( U_\epsilon(y, u) \) implies \( U_u \subseteq R \). Also, since \( \alpha(u) > |y|_\epsilon \), it follows that \( \text{dom}(R') \cap \prec(\beta) \neq \emptyset \) for all \( \beta < \lambda \).

As a corollary it follows that if we can map the reals onto an ordinal \( \lambda \), then we can map the reals onto \( P(\lambda) \). First we note a general fact. Suppose \( A \subseteq \omega^\omega \). Define the pointclass \( \Sigma^1_1(A) \) to be collection of \( B \) which can be written in the form
\[
B(x) \leftrightarrow C(x) \lor \exists y \ (\forall n \ (y)_n \in A \land D((x, y)_n)),
\]
where \( C, D \subseteq \omega^\omega \) are \( \Sigma^1_1 \). We likewise define \( \Sigma^1_{1+(\omega^\omega)^n} \) for any \( n \), using the same formula and our recursive bijection between \( (\omega^\omega)^n \) and \( \omega^\omega \).

**Exercise 21.** Show that \( \Sigma^1_1(A) \) is a pointclass which contains \( A \) and is closed under \( \exists \omega^\omega \), \( \land \) and \( \lor \). Also, \( \Sigma^1_1(A) \) has a universal set. [hint: closure under continuous preimages is immediate. Note that \( A(x) \leftrightarrow \exists y \ (\forall n \ (y)_n \in A \land D((x, y)_n)) \) where \( D(z) \leftrightarrow \forall i, j \ ((z)_1)_i = ((z)_1)_j \land x = ((z)_1)_0 \). Thus, \( A \in \Sigma^1_1(A) \). To see closure under \( \land \) notice that \( \exists y \ (\forall n \ (w)_n \in A \land D((x, y)_n)) \lor \exists z \ (\forall n \ (z)_n \in A \land D'(((x, z)_n))) \leftrightarrow \exists w \ (\forall n \ (w)_n \in A \land \exists u \ (D((x, u)) \lor D'((x, v)))) \). To see closure under \( \lor \), note that \( C(x) \lor \exists y \ (\forall n \ (y)_n \in A \land D((x, y)_n)) \leftrightarrow \exists y \ (\forall n \ (y)_n \in A \land D((x, y)_n) \lor C(x)) \). Also, \( \exists y \ (\forall n \ (y)_n \in A \land D((x, y)_n)) \land \exists z \ (\forall n \ (z)_n \in A \land D'((x, z)_n)) \leftrightarrow \exists w \ (\forall n \ (w)_n \in A \land \exists u \ (D((x, u)) \land D'((x, v))) \land \forall j \ ((w)_j = (w)_j, (v)_j = (w)_j, (v)_j)) \). To see closure under \( \exists \omega^\omega \), suppose that \( B'(x) \leftrightarrow \exists z \ B((x, z)) \leftrightarrow \exists z \ [C((x, z)) \lor \exists y \ (\forall n \ (y)_n \in A \land D(((x, z), y)_n)) \]. Finally, if \( U \subseteq \omega^\omega \times \omega^\omega \) is universal for \( \Sigma^1_1 \ | \ \omega^\omega \), then define \( V(\epsilon, x) \leftrightarrow U((\epsilon)_0, x) \lor \exists y \ (\forall n \ (y)_n \in A \land U((\epsilon)_1, (x, y))) \). Then \( V \) is universal for \( \Sigma^1_1 \ | \ \omega^\omega \).

**Definition 6.6.** The ordinal \( \Theta \) is the supremum of the lengths of the prewellorderings of \( \omega^\omega \).

The ordinal \( \Theta \) plays a role in \( \text{AD} \) models similar to that of the continuum in \( \text{ZFC} \) models. Note that for any \( \alpha < \Theta \) there is a map from \( \omega^\omega \) onto \( \alpha \) (since there is a prewellordering of length \( \alpha \)). With \( \text{ZFC} \) we clearly have \( \Theta = (2^\omega)^+ \). Note that \( \Theta \) is a limit ordinal.

**Corollary 6.7 (AD).** If \( \lambda < \Theta \), then there is a map \( f \) from \( \omega^\omega \) onto \( P(\lambda) \).
Proof. Suppose \( \lambda < \Theta \), and let \( \leq \) be a prewellordering of \( \omega^\omega \) of length \( \lambda \). Let \( \Gamma \) be a nonselfdual pointclass closed under \( \exists \omega^\omega \), \( \land \) and with \( \leq, \in \in \Gamma \). Let \( U \subseteq \omega^\omega \times \omega^\omega \) be universal for \( \Gamma \setminus \omega^\omega \). If \( A \subseteq \lambda \), then the coding lemma applied to the characteristic function of \( A \) show that the relation \( R(x, y) \leftrightarrow (x \in \text{dom}(\leq) \land ((x | \in A \land y = \bar{1}) \lor (x | \notin A \land y = 0)) \) is in \( \Gamma \). In particular, the code set of \( A, C_A = \{ x \in \text{dom}(\leq) : |x| \in A \} \) is in \( \Gamma \). Define \( f(\epsilon) = A \) if \( U_{\epsilon} \) is the code set for \( A \), and \( f(\epsilon) = \emptyset \) otherwise. Then, \( f \) is onto \( \mathcal{P}(\lambda) \). \( \square \)

**Corollary 6.8 (AD).** \( \Theta \) is a limit cardinal.

Proof. Suppose \( \alpha < \Theta \) but \( \Theta \leq \alpha^+ \). Since \( \alpha < \theta \), there is a prewellordering of length \( \alpha \), and hence a map from \( \omega^\omega \) to \( \alpha \). From corollary 6.7 there is a map \( \pi \) from \( \omega^\omega \) onto \( \mathcal{P}(\alpha) \). Since there is map from \( \mathcal{P}(\alpha) \) onto \( \alpha^+ \) (send \( A \subseteq \alpha \) to \( |A| \) if \( A \) is a wellordering, and to \( 0 \) otherwise), we have that there is a map from \( \omega^\omega \) onto \( \alpha^+ \). Since the supremum defining \( \Theta \) is not attained, we have \( \alpha^+ < \Theta \). \( \square \)

**Exercise 22.** Show that countable choice implies that \( \text{cof}(\Theta) > \omega \).

**Theorem 6.9 (AD).** \( \Theta = \sup\{|A|_w : A \subseteq \omega^\omega\} \) is the supremum of the wadge ranks of sets of reals.

Proof. If \( A \subseteq \omega^\omega \), then there is a map \( \pi \) from \( \omega^\omega \) onto \( |A|_w \), namely, \( \pi(x) = |\pi^{-1}(A)|_w \), where \( x \in \omega^\omega \) codes the strategy \( \pi \) as before. Thus, \( \Theta \geq \sup\{|A|_w : A \subseteq \omega^\omega\} \).

For the other direction, fix a prewellordering \( \leq \) of \( \omega^\omega \). There is a function \( F : \mathcal{P}(\omega^\omega) \to \mathcal{P}(\omega^\omega) \) such that for all \( A \subseteq \omega^\omega, F(A) > A \). Namely, let \( F(A) = A' \) be defined by \( A'(x) \leftrightarrow (x(0) = 0 \land \tau_{\omega'}(x) \notin A) \lor (x(0) = 0 \land \tau_{\omega'}(x) \in A) \), where as before \( x'(n) = x(n+1) \) and \( \tau_y \) is the continuous function from \( \omega^\omega \) to \( \omega^\omega \) coded by \( y \) (in some fixed coding scheme of the continuous functions). By construction, \( A' \) is not Wadge reducible to either \( A \) or \( A' \), so has strictly higher Wadge degree [if say \( \tau_y \) reduced \( A' \) to \( A \) then \( 0^\omega y \in A \) if \( \tau_y(0^\omega y) \in A \), however by definition \( 0^\omega y \notin A' \) if \( \tau_y(0^\omega y) \notin A \)].

Now define by induction on \( \alpha < |\leq| \) a set \( A_\alpha \) by \( A_0 = \emptyset \), and for \( \alpha > 0 \):

\[
A_\alpha = F(\{ (x, y) : (x \in \text{dom}(\leq)) \land (|x| \leq \alpha) \land (y \in A_{|x| \leq \alpha}) \}).
\]

A straightforward induction on \( \alpha \) shows that \( |A_\alpha|_w \geq \alpha \). \( \square \)

Using the recursion theorem we can prove an effective strengthening of Theorem 6.9.

**Theorem 6.10 (AD).** Let \( \leq \) be a prewellordering of \( \omega^\omega \), and let \( \Gamma \) be the pointclass (see exercise 21) \( \Gamma = \Sigma^1_1(\leq \otimes \leq^c \otimes (\text{dom}(\leq))^c) \). Then there is a sequence \( \{A_\alpha : \alpha < |\leq| \} \) of sets in \( \Delta = \Gamma \cap \Gamma \) which is strictly increasing in Wadge degree.

Proof. Let \( \Gamma = \Sigma^1_1(\leq \otimes \leq^c \otimes (\text{dom}(\leq))^c) \), so \( \Gamma \) is non-selfdual and closed under \( \exists \omega^\omega, \land \) (and so also \( \lor \)). Also, \( \leq \in \Delta \). Let \( < \) denote the strict part of \( \leq \), that is, \( x < y \) iff \( x \leq y \land \neg(y \leq x) \). Note that \( \leq, <, \text{dom}(\leq) \) are also in \( \Delta \). Note that \( \Delta \) is closed under \( \land, \lor, \neg \). Let \( U \) denote the universal \( \Gamma \) sets with continuous \( s-m-n \) functions. The idea is to define a \( \Gamma \) relation \( U_\epsilon \) with domain \( \text{dom}(\leq) \) (which will actually be a function) such that \( U_\epsilon(x, w) \) implies that \( w_0, w_1 \) code \( \Gamma \) sets which are complements (so they jointly code a \( \Delta \) set) which is the “jump” (as in the proof of Theorem 6.9) of the \( \Delta \) set \( B(y, z) \) of those \( (y, z) \) such that \( y < x \) and \( z \) in the \( \Delta \) set coded by \( U_\epsilon(y) \).
Consider the relations $S_1, S_2$ defined by:

\[ S_1(\epsilon, x, y, z) \iff (x \in \text{dom}(\leq) \land (y < x) \land \exists w(U(\epsilon, y, w) \land U(w_0, z))) \]

\[ S_2(\epsilon, x, y, z) \iff (x \notin \text{dom}(\leq) \lor [(y < x) \lor \exists w(U(\epsilon, y, w) \land U(w_1, z))]) \]

From the closure properties of $\Gamma$ we see that $S_1, S_2 \in \Gamma$. Let $S_1(\epsilon, x, y, z) \iff U(a_1, \epsilon, x, y, z) \iff U(s(a_1, \epsilon, x), y, z)$, and let $s_1(\epsilon, x) = s(a_1, \epsilon, x)$. Similarly define $s_2$ using $S_2$ (and some real $a_2$). Note that if $x \in \text{dom}(\leq)$, and we assume $\epsilon$ is such that for all $y < x$ there is a unique $w$ such that $U(\epsilon, y, w)$, and this $w$ codes a $\Delta$ set (i.e., $U_{w_0} = (U_{w_1})^\circ$), then for this fixed $\epsilon$ and $x$, $S_2(\epsilon, x, y, z)$ iff $\neg S_1(\epsilon, x, y, z)$ for all $y, z$. Roughly speaking, $(S_1)_{\epsilon, x}$ is the “join” of the $\Delta$ sets corresponding to $y < x$.

Define

\[ T_1(w, u) \iff [(u(0) = 0 \land U(w_1, \tau_u(u'))) \lor (u(0) = 1 \land U(w_0, \tau_u(u')))] \]

\[ T_2(w, u) \iff [(u(0) \notin \{0, 1\}) \lor (u(0) = 0 \land U(w_0, \tau_u(u')) \lor (u(0) = 1 \land U(w_1, \tau_u(u')))] \]

Clearly $T_1, T_2 \in \Gamma$. Let $T_1(w, u) \iff U(b_1, w, u) \iff U(s(b_1, w), u)$. Let $t_1(w) = s(b_1, w)$. Similarly define $t_2(w)$ using $T_2$ (and some real $b_2$). Note that if $w$ codes a $\Delta$ set, then $t_1(w)$ codes the jump of this $\Delta$ set and $t_2(w)$ codes the complement of the jump. So, $(t_1(w), t_2(w))$ is a $\Delta$ code for the jump.

Finally, using the recursion theorem let $\epsilon$ be such that

\[ U(\epsilon, x, w) \iff [(x \in \text{dom}(\leq)) \land w = (t_1((s_1(\epsilon, x), s_2(\epsilon, x)), t_2((s_1(\epsilon, x), s_2(\epsilon, x))))]. \]

Clearly $U \in \Gamma$, dom($U$) = dom($\leq$), and $U_\epsilon$ is a function. By induction on $|x|_\leq$ we have that if $U(\epsilon, x, w)$ then $w$ codes a $\Delta$ set $A_x$ (that is $U_{w_0} = (U_{w_1})^\circ$, which we take to be $A_x$), and $|A_x| = |x|_\leq$. For $\alpha < |\leq|$ we can then let $A_\alpha = \{z: |z_0|_\leq = \alpha \land z_1 \in A_{\alpha_0}\}.

\[ \square \]

As another application of the coding lemma we prove the following.

**Theorem 6.11.** Suppose $\Gamma$ is a nonselfdual pointclass closed under $\forall^\omega \omega$, $\land$, $\lor$. Suppose also $\text{pwo}(\Gamma)$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal set for $\Gamma \restriction \omega^\omega$. Let $\phi$ be a $\Gamma$-norm on $U$. Then $|\phi| = \delta(\Gamma) = \text{the supremum of the lengths of the } \Delta \text{ prewellorderings = the supremum of the lengths of the } \Gamma \text{ wellfounded relations.}$

**Proof.** Let $\phi$ be a $\Gamma$-norm on $U$. All the initial segments of the prewellordering are in $\Delta$ since $\pi$ is a $\Gamma$-norm. Thus, $|\phi| \leq \delta(\Gamma)$. It suffices therefore to show that any $\Gamma$ wellfounded relation $\prec$ has length less than $|\phi|$. From the recursion theorem let $\epsilon$ be such that

\[ U_\epsilon(x) \iff \forall y (y \prec x \rightarrow (\epsilon, y) <^*_{\phi}(\epsilon, x)), \]

where $<^*$ is the norm relation corresponding to $\phi$ (recall for a norm $\psi$ on a set $A$ that $x <^\psi y$ iff $(x \in A \land (y \notin A \lor \psi(x) < \psi(y)))$). $U_\epsilon$ is attempting to define an embedding from $\prec$ into $|\phi|$ by $x \rightarrow |(\epsilon, x)|$.

We show by induction on $|x|_\prec$ that $U_\epsilon(x)$ and that for all $y < x$ that $\phi(y, x) < \phi(\epsilon, x)$. Let $x$ be least such that the stated property fails. So, for all $y < x$ we have $U_\epsilon(y)$. If $\neg U_\epsilon(x)$, then by definition of $<^*_\phi$ we have that for all $y < x$ that $y <^*_{\phi} x$. From the definition of $U_\epsilon$ it now follows that $U_\epsilon(x)$, a contradiction. So, $U_\epsilon(x)$. From the definition of $U_\epsilon$ it now follows that for all $y < x$ that $\phi(y, x) < \phi(\epsilon, x)$. Thus, $x \rightarrow \phi(\epsilon, x)$ is orderpreserving from $\prec$ to $|\phi|$, so $|\prec| \leq |\phi|$. We must in fact
have $| \prec | < | \phi |$ as given any $\hat{\Gamma}$ wellfounded relation $\prec$ we can easily get another $\hat{\Gamma}$ wellfounded relation $\prec'$ with $| \prec' | > | \prec |$. \hfill \Box

As another application we have the following.

**Theorem 6.12.** Let $\Gamma$ be a nonselfdual pointclass closed under $\exists \omega^\omega$, $\land$. Then $\delta = \sup \{ \text{lengths of the } \Gamma \text{ wellfounded relations} \}$ is a regular cardinal.

**Proof.** Suppose $\rho = \text{cof}(\delta) < \delta$. Let $f: \rho \to \delta$ be cofinal. Let $\prec$ be a $\Gamma$ wellfounded relation of length $\rho$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\Gamma$ set. Apply the coding lemma to the relation $R(x, y) \iff (x \in \text{dom}(\prec)) \land (U_y \text{ is wellfounded } \land |U_y| \geq f(|x|_\prec))$. Let $R' \subseteq R$ with $R' \in \Gamma$ be from the coding lemma, so $\text{dom}(R') \cap | \prec | \prec \beta \neq \emptyset$ for all $\beta < \rho$ (recall $| \prec | \beta = \{ z : |z|_\prec = \beta \}$). Let $A(y) \iff \exists x (x \in \text{dom}(\prec) \land R'(x, y))$, so $A \in \Gamma$ and consists of codes of $\Gamma$ wellfounded relations whose lengths are unbounded in $\delta$. Define then $(y, z) \ll (y', z') \iff (y = y' \land y \in A) \land (U_y(z, z'))$.

Then $\ll$ is a wellfounded relation in $\Gamma$, and $| \ll | = \delta$. \hfill \Box

7. The Prewellordering Property

The prewellordering property is one the important structural properties of pointclasses. It is also an important ingredient in the scale property. The prewellordering property for $\Gamma$ says that every set in $\Gamma$ can be written as an increasing union of sets in $\Delta$, in an effective manner. We recall the precise definition. There are two almost equivalent definitions of the notion of a $\Gamma$-norm. We take as our official definition the slightly stronger version.

**Definition 7.1 (\(\Gamma\)-norm).** Let $\Gamma$ be a pointclass, and $A \subseteq X$. We say a norm $\phi$ on $A$ is a $\Gamma$-norm if the relations $\prec^*, \preceq^*$ are in $\Gamma$, where

$$
\begin{align*}
\prec^* x & \iff x \in A \land (y \notin A \lor (y \in A \land \phi(x) < \phi(y))) \\
\preceq^* x & \iff x \in A \land (y \notin A \lor (y \in A \land \phi(x) \leq \phi(y)))
\end{align*}
$$

If $A \in \Gamma$, and $\phi$ is a $\Gamma$-norm on $A$, then we can consider the components of $A$ with respect to the norm. Say (without loss of generality) that $\phi$ is regular, so $\phi: A \to \lambda$ is onto. For $\alpha < \lambda$, let $A_\alpha = A_\alpha^\phi = \{ x \in A : \phi(x) = \alpha \}$, and let $A_{\leq \alpha} = \{ x \in A : \phi(x) \leq \alpha \}$. If $x \in A$ is such that $\phi(x) = \alpha$, then we have $A_{\leq \alpha} = \{ y : y \preceq^* x = \{ y : -(x \prec^* y) \}$

which shows that $A_{\leq \alpha} \in \Delta$. Thus $A = \bigcup_{\alpha < \lambda} A_{\leq \alpha}$ writes $A$ as an increasing union of $\Delta$ sets, the length of the union being the length of the norm $\phi$.

A similar argument shows that each set $A_{\leq \alpha} = \{ x \in a : \phi(x) < \alpha \}$ is in $\Delta$. So we can also write $A = \bigcup_{\alpha < \lambda} A_{< \alpha}$. In practice, $\lambda$ will always be a limit ordinal, so we can write $A = \bigcup_{\alpha < \lambda} A_{< \alpha}$. Written this way, the sets are continuous with respect to union at limit ordinals.

**Definition 7.2.** A pointclass $\Gamma$ has the prewellordering property (written $\text{pwo}(\Gamma)$) if every $A \in \Gamma$ admits a $\Gamma$-norm.

so, if $\text{pwo}(\Gamma)$ holds, then every $\Gamma$ (or $\hat{\Gamma}$) set admits a representation as an increasing union (or intersection) of simpler, anmely $\Delta$, sets.

We will restrict our attention now to Levy pointclasses.

**Definition 7.3.** $\Gamma$ is a Levy pointclass if it is non-selfdual and either $\exists \omega^\omega \Gamma \subseteq \Gamma$ or $\forall \omega^\omega \Gamma \subseteq \Gamma$. 

So, a Levy pointclass is non-selfdual and is closed under either existential or universal quantification over the reals (possibly closed under both). The first few Levy pointclasses are the levels of the projective hierarchy, the $\Sigma^1_n, \Pi^1_n$ sets. The Levy classes are the important pointclasses in descriptive set theory. Although there are certainly pointclasses between the Levy classes, they are of less significance.

Our goal in this section completely classify the prewellordering property among the Levy pointclasses. We will see that for every Levy pointclass $\Gamma$, either $\text{pwo}(\Gamma)$ or $\text{pwo}(\Gamma)$.

We first prove two transfer results which allow us to propagate the prewellordering property up a projective-like hierarchy, that is, by applications of real quantifiers. This is the simplest case of the so-called periodicity theorems of Moschovakis and Martin.

First we do the easy case of transfer by $\exists\omega^\omega$, and then we give the first periodicity theorem, which gives the transfer by $\forall\omega^\omega$. The transfer under $\exists\omega^\omega$ just uses the inf.

**Theorem 7.4 (ZF).** Let $\Gamma$ be a pointclass and assume $\text{pwo}(\Gamma)$. Then every set in $\exists\omega^\omega \Gamma$ admits a $\exists\omega^\omega \forall\omega^\omega \Gamma$-norm.

**Proof.** Let $A(x) \leftrightarrow \exists y \in \omega^\omega B(x, y)$, where $B \in \Gamma$. Let $\phi$ be a $\Gamma$-norm on $B$, with norm relations $<^*_\phi$, $\leq^*_\phi$. Define the norm $\psi$ on $A$ by:

$$\psi(x) = \min\{\phi(x, y) : (x, y) \in B\}.$$  

Let $<^*_\psi$, $\leq^*_\psi$ be the corresponding norm relations on $A$. We have $x_1 <^*_\psi x_2$ if $\exists y_1 \forall y_2 (x_1, y_1) <^*_\phi (x_2, y_2)$. Likewise, $x_1 \leq^*_\psi x_2$ iff $\exists y_1 \forall y_2 (x_1, y_1) \leq^*_\phi (x_2, y_2)$. □

**Exercise 23.** Show that for any non-selfdual pointclass $\Gamma$ that $\exists\omega^\omega \Gamma$ admits a $\exists\omega^\omega \forall\omega^\omega \Gamma$-norm. Deduce that $\text{pwo}(\Sigma^\alpha_n)$ holds for all $\alpha \geq 2$ for general Polish $X$, and for all $\alpha \geq 1$ if $X = \omega^\omega$.

We next do the transfer under $\forall\omega^\omega$. Here we use the “fake sup” which is defined using a game.

**Theorem 7.5 (\Delta-determinacy).** Let $\Gamma$ be a non-selfdual pointclass, and assume $\text{pwo}(\Gamma)$. Then every set in $\forall\omega^\omega \Gamma$ admits a $\forall\omega^\omega \exists\omega^\omega \Gamma$-norm.

**Proof.** Let $A(x) \leftrightarrow \forall y \in \omega^\omega B(x, y)$, and let $\phi$ be a $\Gamma$-norm on $B$ with corresponding norm relations $<^*_\phi$, $\leq^*_\phi$. We define a relation $\leq_A$ on $A$ as follows. Let $x_1, x_2 \in A$. Consider the following game $G_A(x_1, x_2)$:

\[
\begin{array}{ccccccc}
 x_1 & I & y_1(0) & y_1(1) & y_1(2) & \ldots \\
 x_2 & II & y_2(0) & y_2(1) & y_2(2) & \ldots \\
\end{array}
\]

II wins the run iff $\phi(x_1, y_1) \leq \phi(x_2, y_2)$. Note that this makes sense as $B(x_1, y_1), B(x_2, y_2)$ since $x_1, x_2 \in A$. Note also the game $G_A$ is determined as payoff set is $\{(y_0, y_1) : (x_0, y_0) \leq^*_\phi (x_1, y_1)\} = \{(y_0, y_1) : \neg(x_1, y_1) <^*_\phi (x_1, y_1)\}$ (using here that $x_0, x_1 \in A$).

**Claim.** $\leq_A$ is reflexive.

**Proof.** Have II copy I’s moves. □
Claim. $\leq_A$ is transitive.

Proof. Suppose $x_0, x_1, x_2 \in A$ and $x_0 \leq_A x_1, x_1 \leq_A x_2$. We must show that $x_0 \leq_A x_2$, that is, II wins the game $G_A(x_0, x_2)$. To do that, we fix strategies $\tau_0, \tau_1$ for II in $G_A(x_0, x_1)$ and $G_A(x_1, x_2)$, and then move as shown in Figure 7.

Claim. $\leq_A$ is wellfounded, that is, $\forall x_0, x_1 \in A \ (x_0 \leq_A x_1 \vee x_1 \leq_A x_0)$.

Proof. Let $x_0, x_1 \in A$. If II wins $G_A(x_0, x_1)$ then $x_0 \leq_A x_1$ by definition. So, suppose I wins $G_A(x_0, x_1)$. Note that since $x_0, x_1$ are in $A$, the payoff condition for I, namely $\neg((x_0, y_0) \leq_A^* (x_1, y_1))$, is equivalent to saying $(x_1, y_1) <^\phi_A (x_0, y_0)$. However, if I wins this game, then II can clearly win the same game (use the same strategy but just ignore the last move of the opponent). So, $x_1 \leq_A x_0$. \hfill $\square$

At this point we have shown that $\leq_A$ is a prelinear order on the set $A$. We next show that it is wellfounded, so that it is actually a prewellorder (and thus gives a norm on $A$).

Claim. $\leq_A$ is wellfounded, that is, there does not exist a sequence $x_0 >_A x_1 >_A x_2 \cdots$ with all $x_i \in A$ (recall $x <_A y$ means $x \leq_A y$ and $\neg(y \leq_A x)$).

Proof. Suppose $x_i \in A$, and $x_0 >_A x_1 >_A x_2 \cdots$. For each $i$ we have that $\neg(x_i \leq_A x_{i+1})$, so I wins the game $G_A(x_i, x_{i+1})$ for each $i$. Fix (countable choice for reals) winning strategies $\sigma_i$ for I in $G_A(x_i, x_{i+1})$. We play these strategies against each other as shown in Figure 7.

Since $x_i \in A$, all of the $(x_i, y_i)$ are in $B$. So, since $\neg((x_i, y_i) \leq_A^* (x_{i+1}, y_{i+1}))$ we have that $(x_i, y_i) >^\phi_A (x_{i+1}, y_{i+1})$ for all $i$, a contradiction. \hfill $\square$

At this point we know that $\leq_A$ is a prewellordering on the set $A$. Let $\psi: A \to \text{On}$ be the corresponding norm on $A$. We must show that the corresponding norm relations $\leq_A^*, <^\phi_A$ are in $\forall\omega^\omega \exists\omega^\omega \Gamma$. 

\begin{figure}
\begin{align*}
G_A(x_0, x_1) & : x_0 \quad y_0(0) \quad y_0(1) \quad y_0(2) \\
G_A(x_1, x_2) & : x_1 \quad y_1(0) \quad y_1(1) \quad y_1(2) \\
G_A(x_2, x_3) & : x_2 \quad y_2(0) \quad y_2(1) \quad y_2(2) \\
G_A(x_3, x_4) & : x_3 \quad y_3(0) \quad y_3(1) \quad y_3(2)
\end{align*}
\caption{Showing transitivity}
\end{figure}

\begin{figure}
\begin{align*}
G_A(x_0, x_1) & : x_0 \quad y_0(0) \quad y_0(1) \quad y_0(2) \\
G_A(x_1, x_2) & : x_1 \quad y_1(0) \quad y_1(1) \quad y_1(2) \\
G_A(x_2, x_3) & : x_2 \quad y_2(0) \quad y_2(1) \quad y_2(2) \\
G_A(x_3, x_4) & : x_3 \quad y_3(0) \quad y_3(1) \quad y_3(2)
\end{align*}
\caption{Showing wellfoundedness}
\end{figure}
Consider first \( \leq^*_\psi \). We claim that \( x_0 \leq^*_\psi x_1 \) iff II wins the game \( G_A(x_0, x_1) \) (recall that the payoff condition for this game is \( (x_0, y_0) \leq^*_\psi (x_1, y_1) \)). Suppose first that \( x_0 \leq^*_\psi x_1 \). In particular, \( x_0 \in A \). If \( x_1 \in A \) as well, then \( x_0 \leq_A x_1 \), and so by definition we have that II wins \( G_A(x_0, x_1) \). If \( x_1 \notin A \), Then II can win \( G_A(x_0, x_1) \) by playing any real \( y_1 \) such that \( (x_1, y_1) \notin B \). Suppose next that II wins the game \( G_A(x_0, x_1) \). We must have that \( x_0 \in A \) as otherwise I could win this game by playing a real \( y_0 \) such that \( (x_0, y_0) \notin B \). If \( x_1 \notin A \) then by definition we have \( x_0 \leq^*_\psi x_1 \). If \( x_1 \in A \), then since II wins the game we have by definition that \( x_0 \leq_A x_1 \), that is \( \psi(x_0) \leq \psi(x_1) \). So, \( x_0 \leq^*_\psi x_1 \) holds in this case as well.

So, \( x_0 \leq^*_\psi x_1 \) iff II has a winning strategy in the game \( G_A(x_0, x_1) \). This last condition is easily a \( \forall^\omega \exists^\omega \Gamma \) condition.

Consider next the relation \( x_0 <^*_\psi x_1 \). We claim that \( x_0 <^*_\psi x_1 \) iff I wins the game \( \Gamma_A'(x_0, x_1) \), which is the game where I plays the \( y_1(i) \), II plays the \( y_0(i) \), and the payoff condition for I to win is \( (x_0, y_0) <^*_\psi (x_1, y_1) \). This will again show that \( <^*_\psi \) is a \( \forall^\omega \exists^\omega \Gamma \) relation.

Suppose first that \( x_0 <^*_\psi x_1 \). In particular, \( x_0 \in A \). If \( x_1 \notin A \), I wins \( \Gamma_A' \) by playing any \( y_1 \) such that \( (x_1, y_1) \notin B \). If \( x_1 \in A \) as well, then \( \neg(\psi(x_1) \leq \psi(x_0)) \), so I wins \( G_A(x_1, x_0) \). Since both \( x_0, x_1 \) are in \( A \), this gives a strategy for I in \( \Gamma_A'(x_0, x_1) \).

Suppose next that I wins \( \Gamma_A'(x_0, x_1) \), say by \( \sigma \). We must have \( x_0 \in A \) as otherwise II could win this game by playing any \( y_0 \) such that \( (x_0, y_0) \notin B \). If \( x_1 \notin A \) then we are done, so assume \( x_1 \in A \) also. II can clearly win the game \( G_A(x_0, x_1) \) (by ignoring the last move of I), so \( \psi(x_0) \leq \psi(x_1) \). We must show strict inequality. Suppose \( \psi(x_1) \leq \psi(x_0) \). This says that II can win \( G_A(x_1, x_0) \), say by \( \tau \). Playing \( \sigma \) and \( \tau \) against each other gives a contradiction.

This completes the proof of Theorem 7.5, the first periodicity theorem.

\( \square \)

**Corollary 7.6 (PD).** The pointclasses \( \Pi^1_{2n+1}, \Sigma^1_{2n+2} \) have the prewellordering property for all \( n \geq 0 \).

Our next goal is to classify the pointclasses having the prewellordering property among the Levy pointclasses. We will see that for Levy classes \( \Gamma \), either \( \text{pwo}(\Gamma) \) or \( \text{pwo}(\Gamma) \).

First we state a useful general fact.

**Definition 7.7.** A **projective algebra** is a pointclass \( \Lambda \) closed under \( \exists^\omega \), \( \neg \), \( \land \), \( \lor \) (and so also \( \forall^\omega \)).

**Exercise 24.** Show that a projective algebra \( \Lambda \) does not have a largest Wadge degree in it. [hint: if it did, it would either have successor Wadge rank and so be a join of a non-selfdual degree immediately below it, or else have limit rank of cofinality \( \omega \). In the first case \( \Lambda \) would not be closed under finite unions, and in the second case would not be closed under \( \exists^\omega \).]

**Lemma 7.8.** Let \( \Lambda \) be a projective algebra. Then \( o(\Lambda) = \sup\{ |A|_\omega : A \in \Lambda \} \) is equal to \( \delta(\Lambda) = \sup\{ |\bar{A}| \leq |A| : \bar{A} \in \Lambda \text{ is a prewellordering} \} \).

**Proof.** If \( \alpha < o(\Lambda) \), then let \( A \in \Lambda \) with \( |A|_\omega = \alpha \). The initial segment of \( A \) in the Wadge hierarchy then determines a prewellordering of length \( \alpha \) which is in \( \Lambda \) from
suffices to show that $R$ holds if one of the following cases holds:

- For $\alpha$ let $x$ be a reduction of $A'$ to $A$, then considering $x = 1^{-}x'$ gives a contraction, and if $f_{x'}$ were a reduction of $A' \to A''$, the considering $x = 0^{-}x'$ gives a contradiction.

Now we define by induction on $\alpha < \|x\|$ a set $A_{\alpha}$. Let $A_0 = \emptyset$. Let $A_{\alpha+1} = (A_{\alpha})'$. For $\alpha$ limit let $A_{\alpha}(x) \leftrightarrow (\forall i \in \alpha)(x_{i+1} \leq y_{i})$. A straightforward induction shows that the $A_{\alpha}$ are strictly increasing in Wadge degree.

Finally, we show that each $A_{\alpha} \in \Lambda$. Let $R(x,y) \leftrightarrow x \in \text{dom}(\leq) \land y \in A_{\|x\|}$. It suffices to show that $R \in \Lambda$. Consider the relation $W(x,y,z,w,j)$ where $i,j \in \{0,1\}$, and $x,y,z,w \in \omega^{\omega}$. The intention is that $i = 1$ and $(z,w,j)$ witnesses that $R(x,y)$, or $i = 0$ and $(z,w,j)$ witnesses that $\neg R(x,y)$. We define $W(x,y,z,w,j)$ to hold if one of the following cases holds:

1. $i = 1$, $x$ is an immediate successor of $z$ in $\leq$, and either $y(0) > 0$, $w = f_{y''}(y)$, and $j = 0$, or $y(0) = 0$, $w = f_{y''}(y)$, and $j = 1$.
2. $i = 0$, and either $x \notin \text{dom}(\leq)$ or $x$ is an immediate successor of $z$ in $\leq$, and either $y(0) > 0$, $w = f_{y''}(y)$, and $j = 1$, or $y(0) = 0$, $w = f_{y''}(y)$, and $j = 0$.
3. $i = 1$, $x$ has limit rank in $\leq$, $(y)_{0} < x$, $(y)_{0} = z$, $w = (y)_{1}$, and $j = 1$.
4. $i = 0$, and either $x \notin \text{dom}(\leq)$ or $x$ has limit rank in $\leq$ and the following:
   - $(y)_{0} < x$, or $(z = (y)_{0} \land w = (y)_{1} \land j = 0$).
5. $i = 0$, $x \notin \text{dom}(\leq)$, or $|x|_{\leq} = 0$.

$W$ is easily in $\Lambda$. We then have that

$$R(x,y) \leftrightarrow \exists z,w,r \in \omega^{\omega} ((z)_{0} = x \land (w)_{0} = y \land r(0) = 1$$

$$\land \forall i (W((z)_{i},(w)_{i},r(i),(z)_{i+1},(w)_{i+1},r(i+1))).$$

\[\square\]

**Lemma 7.9** (Martin). Let $\Gamma$ be non-selfdual, closed under $\forall \omega^{\omega}$, $\land$, $\lor$, and assume $\text{pwo}(\Gamma)$. Let $\delta = \delta(\Gamma) = \sup\{|x|_{\leq} : x \in \Delta \text{ is a prewordering}\}$. Then $\Delta$ is closed under $< \delta$ unions and intersections.

**Proof.** Assume not, and let $\rho < \delta$ be the least ordinal such that there is a $\rho$ increasing sequence $\{A_{\alpha}\}_{\alpha < \rho}$ of sets in $\Delta$ with $A = \bigcup_{\alpha < \delta} A_{\alpha} \notin \Delta$. As $\rho < \delta$, the coding lemma implies that $A \in \hat{\Gamma}$ (which is closed under $\exists \omega^{\omega}$, $\land$). By Wadge, every set if $\hat{\Gamma}$ can be written as a $\rho$ union of $\Delta$ sets. Given a set $B \in \hat{\Gamma} - \Gamma$, let $B = \bigcup_{\alpha < \delta} B_{\alpha}$ where $B_{\alpha} \in \Delta$, and we may assume the sequence in increasing by minimality of $\rho$. Let $\phi$ be the norm on $B$ given by $\phi(x) = \mu \alpha (x \in B_{\alpha})$. The norm relations $\leq_{\phi}, <_{\phi}$ are both in $\hat{\Gamma}$ as they can be written as $\rho$ unions of $\Delta$ sets. This shows $\text{pwo}(\Gamma)$, a contradiction to $\text{pwo}(\Gamma)$ (as both $\Gamma$, $\hat{\Gamma}$ now have the reduction property, and so both sides have the separation property).

\[\square\]
We first show that every Levy pointclass falls into a projective-like hierarchy, which can be one of four types. We then analyze the prewellordering property within each of these four possible types.

Fix a Levy pointclass $\Gamma$.

**Definition 7.10.** Let

$$\Lambda = \Lambda(\Gamma) = \bigcup \{ \Delta : (\Delta \in \Gamma) \land \exists^\omega \Delta \subseteq \Delta \land (\neg \Delta \subseteq \Delta) \land (\cap \Delta \subseteq \Delta) \}$$

So, $\Lambda \subseteq \Gamma$ and $\Lambda$ is a projective algebra, that is, $\Lambda$ is closed under $\exists^\omega$, $\neg$, and finite unions and intersections (and so also $\forall^\omega$). $\Lambda(\Gamma)$ is the largest projective algebra contained in $\Gamma$.

We show that $\Lambda(\Gamma)$ is the base of a projective-like hierarchy containing $\Gamma$. Let $\lambda = o(\Lambda) = \sup \{|A|_w : A \in \Lambda\}$ be the supremum of the Wadge ranks of the sets in $\Lambda$ (we also call $\lambda$ the Wadge ordinal of $\Lambda$). We consider the following cases.

**Case 1:** $\text{cof}(\lambda) = \omega$.

Let $A_n$ be such that $|A_n|_w = \alpha_n < \lambda$ and $\sup \alpha_n = \lambda$. We also assume that $|A_n|_w < |A_{n+1}|_w$. Let $A = \oplus_n A_n$ be the countable join of the $A_n$. Recall that $|A|_w = \sup_n |A_n|_w = \lambda$. Also, $A$ is selfdual. Let $\Sigma_0 = \bigcup \Lambda$ be the pointclass of sets which are countable unions of sets in $\Lambda$. Note that $A \in \Sigma_0$ from the definition of the join.

$\Sigma_0$ is closed under countable unions by definition. Also $\Sigma_0$ is closed under $\exists^\omega$.

For suppose $B(x) \leftrightarrow \exists y C(x,y)$ with $C \in \Sigma_0$. Say $C = \bigcup_n C_n$ with $C_n \in \Lambda$. Then

$$B(x) \leftrightarrow \exists y C(x,y) \leftrightarrow \exists y \exists n C_n(x,y) \leftrightarrow \exists n \exists y C_n(x,y)$$

so $B = \bigcup_n B_n$ where $B_n(x) \leftrightarrow \exists y C_n(x,y)$ is in $\Lambda$. note also that $\Sigma_0$ is closed under finite intersections (since $\Lambda$ is).

Finally, $\Sigma_0$ is nonselfdual. To see this, note that we may assume that each of the $A_n$ is nonselfdual (recall the selfdual and nonselfdual Wadge degrees alternate).

Let $U_n$ be universal for $\Gamma_n = \{ B : B \leq^w A_n \}$. Let $U(x,y) \leftrightarrow \exists n U_n((x)_n,y)$. Then easily $U$ is in $\Sigma_0$ and is universal for $\Sigma_0$.

So, $\Sigma_0$ is a Levy class containing $\Lambda$, which is closed under $\exists^\omega$, $\land$, $\lor$. Let $\Pi_0 = \Sigma_0$ be the dual class, so $\Pi_0$ is selfdual and closed under $\forall^\omega$ (and countable intersections), $\land$, $\lor$. Note that any Levy class which contains $\Lambda$ must contain either $\Sigma_0$ or $\Pi_0$, since any nonselfdual class which is closed under $\exists^\omega$ is closed under countable unions.

Clearly $\Sigma_0$ is not closed under countable intersections (as it is nonselfdual, so doesn’t contain $\Pi_0$) so not closed under $\forall^\omega$, so certainly not closed under $\forall^\omega$. We thus the projective-like hierarchy starting from $\Sigma_0$, $\Pi_0$ in the usual way: $\Sigma_{n+1} = \exists^\omega \Pi_n$, $\Pi_{n+1} = \Sigma_{n+1} = \forall^\omega \Sigma_n$. for $n \geq 1$, the $\Sigma_n$, $\Pi_n$ are closed under countable unions and intersections, just as for the projective hierarchy. We must have $\Gamma \subseteq \bigcup_n \Sigma_n$, as otherwise $\bigcup_n \Sigma_n$ would be a projective algebra contained in $\Gamma$ which is larger that $\Lambda$. An easy argument using that $\Gamma$ is a Levy class now shows that $\Gamma = \Sigma_n$ or $\Gamma = \Pi_n$ for some $n$ [Let $n - 1$ be largest so that $\Gamma$ contain both $\Sigma_{n-1}$ and $\Pi_{n-1}$. As $\Gamma$ is a Levy class, it must contain either $\Sigma_n$ or $\Pi_n$. If, say, $\Sigma_n \subseteq \Gamma$ but $\Pi_n \not\subseteq \Gamma$, then by Wadge $\Gamma = \Sigma_n$].

We now place the prewellordering property within the $\Sigma_n$, $\Pi_n$ classes.
Claim. \( \text{pwo}(\Sigma_0) \).

Proof. Let \( A \in \Sigma_0 \), say \( A = \bigcup_n A_n \), an increasing union with \( A_n \in \Lambda \). For \( x \in A \) let \( \varphi(x) = \mu_n \{ x \in A_n \} \). The relations \( <_{\varphi}, \leq_{\varphi} \) are easily both countable unions of sets in \( \Lambda \), hence in \( \Sigma_0 \). \( \square \)

From first periodicity it now follows that \( \text{pwo}(\Sigma_{2n}, \text{pwo}(\Pi_{2n+1})) \) for all \( n \geq 0 \). So, we either have \( \text{pwo}(\Gamma) \) or \( \text{pwo}(\bar{\Gamma}) \).

This ends Case 1.

In the remaining cases we have \( \text{cof}(\lambda) > \omega \). Recall in this case that there is a nonselfdual pointclass \( \Gamma' \) of Wadge rank \( \lambda \). So, \( \Delta' = \Gamma' \cap \bar{\Gamma}' = \Lambda \). Also, exactly one of \( \Gamma' \), \( \bar{\Gamma}' \) has the separation property, so choose \( \Gamma' \) so that \( \text{sep}(\bar{\Gamma}') \). From Theorem 5.8 it follows that \( \Gamma' \) is closed under \( \forall^{\omega\omega} \).

We refer to \( \Gamma' \) as the Steel pointclass associated to \( \Lambda \). We emphasize that \( \Gamma' \), in general, is not closed under both \( \wedge \) and \( \vee \). Of course, \( \Gamma' \), being nonseldual and closed under \( \forall^{\omega\omega} \) is closed under countable intersections, and likewise \( \bar{\Gamma}' \) is closed under \( \exists^{\omega\omega} \) and so countable unions.

Before considering further cases, we prove some results about the Steel pointclass \( \Gamma' \) in general. We will assume that \( \Lambda \) is not closed under arbitrary length wellordered unions (equivalently intersections). This follows immediately from the hypothesis that there is no proper (i.e., \( \not\in \mathcal{P}(\omega) \)) pointclass closed under complements and arbitrary length wellordered unions and intersection. This, in turn, follows from \( \text{AD}^+ \), as one of the axioms of this theory asserts that every set is \( \omega \)-Borel.

Lemma 7.11. \( \rho = \text{cof}(\lambda) \) is the least ordinal such that \( \Lambda \) is not closed under wellordered unions of length \( \rho \).

Proof. Since \( \Lambda \) is a proper pointclass, fix a \( \Lambda \) prewellordering \( \preceq \in \Lambda \) with \( |\preceq| = \rho \).

Let \( \delta \) be the least ordinal such that there is a \( \delta \) increasing sequence of sets in \( \Lambda \), say \( \{ A_\alpha \}_{\alpha < \delta} \) with \( A = \bigcup_{\alpha < \delta} A_\alpha \) not in \( \Lambda \). Easily \( \delta \) is a regular cardinal. Suppose first that \( \rho < \delta \). Since \( \rho = \text{cof}(\lambda) \) and \( \delta \) is regular, we may thin the sequence \( A_n \) to a \( \delta \) length subsequence where all the \( A_n \) now have Wadge degree \( \leq \lambda' < \lambda \). Assume for the moment that \( \delta < \lambda \). We may then fix a nonselfdual pointclass \( \Gamma' \subseteq \Lambda \) closed under \( \exists^{\omega\omega}, \wedge, \vee \), with all the \( A_n \in \Gamma' \) and such that there is a \( \delta \) length prewellordering in \( \Gamma' \). The coding lemma then gives that \( A = \bigcup_{\alpha < \delta} A_\alpha \in \Gamma' \), a contradiction. If \( \delta > \lambda \), then from the minimality of \( \delta \) we have that there is a \( \lambda \) prewellordering in \( \Lambda \), a contradiction. If \( \delta = \lambda \), then \( \lambda \) is regular, so \( \rho = \delta = \lambda \), and we are done.

Thus, \( \delta \leq \rho \). Suppose that \( \delta < \rho \leq \lambda \). As \( \rho = \text{cof}(\lambda) \), there is a \( \lambda' < \lambda \) such that all the \( A_n \) have Wadge rank \( < \lambda' \). As \( \delta < \lambda \), the coding lemma again shows that \( A \in \Lambda \), a contradiction. \( \square \)

We next see how to generate the Steel pointclass from \( \Lambda \). Let \( \Lambda \) be a projective algebra with \( \rho = \text{cof}(\lambda) > \omega \) (where \( \lambda = o(\Lambda) \)), and let \( \Gamma' \) be the Steel pointclass (so \( \Gamma' = \lambda, \text{sep}(\Gamma) \)). Recall \( \Gamma' \) is closed under \( \forall^{\omega\omega} \). From Lemma 7.11 we have that \( \bigcup_{\rho} \Lambda \supseteq \Lambda \). We cannot have \( \bigcup_{\rho} \Lambda = \bar{\Gamma} \) as then Martin’s argument would show \( \text{pwo}(\bar{\Gamma}) \) (c.f. Lemma 7.9), and since \( \bar{\Gamma} \) would also be closed under \( \forall, \wedge \) (as \( \bigcup_{\rho} \Lambda \) is) we would then have \( \text{red}(\bar{\Gamma}) \) which contradicts \( \text{sep}(\bar{\Gamma}) \). So, by Wadge we have \( \bigcup_{\rho} \Lambda \supseteq \Gamma \). But since \( \Lambda \) is closed under \( \exists^{\omega\omega} \), we also have that \( \bigcup_{\rho} \Lambda \) is closed under \( \exists^{\omega\omega} \), and so we have \( \bigcup_{\rho} \Lambda \supseteq \exists^{\omega\omega} \Gamma \).
Let $\Sigma'$ be the pointclass of $\Sigma_1^1$-bounded $\rho$ length unions of $\Lambda$ sets. By this we mean $A$ can be written as $A = \bigcup_{\alpha < \rho} A_\alpha$, where $A_\alpha \in \Lambda$, is an increasing union, and if $S \subseteq A$ is $\Sigma_1^1$ then $\exists \alpha < \rho \,(A \in A_\alpha)$.

Claim. $\Gamma' = \Gamma$.

Proof. First note that $\Gamma'$ is closed under $\forall^\omega$. To see this, let $A(x) \leftrightarrow \forall y \, B(x,y)$ where $B \in \Gamma'$. Let $B = \bigcup_{\alpha < \rho} B_\alpha$ be an increasing $\Sigma_1^1$ union of $\Lambda$ sets. If $A(x)$, then there is an $\alpha < \rho$ such that $\forall y \, B_\alpha(x,y)$ as $\{(x,y) : y \in \omega\}$ is a $\Sigma_1^1$ set. So, $A = \bigcup_{\alpha < \delta} A_\alpha$ where $A_\alpha(x) \leftrightarrow \forall y \, B_\alpha(x,y)$. Each $A_\alpha \in \Lambda$ by the closure of $\Lambda$. Also, $\{A_\alpha\}$ forms a $\Sigma_1^1$-bounded union, since if $S \subseteq \omega$ is $\Sigma_1^1$, then so is $S' = \{(x,y) : x \in S\}$.

Let $A \in \Gamma$. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal $\Sigma_1^1$ set. Let $B = \{x : U_x \subseteq A\}$. So, $B(x) \leftrightarrow \forall y \, (U(x,y) \rightarrow A(y))$, which shows $B \in \Gamma$ provided we know that $\Gamma$ is closed under disjunctions with $\Pi_1^1$ sets. If $P \in \Pi_1^1$, say $P(x) \leftrightarrow \forall y \, U(x,y)$ where $U$ is open. If $C \in \Gamma$ then then $x \in P \cup C$ iff $\forall y \,(U(x,y) \vee C(x))$ and since $\Gamma$ is closed under $\forall^\omega$, it suffices to know that $\Gamma$ is closed under disjunction with open sets. To see this, let $C \in \Gamma$ and $U$ be open. We define a strategy $\tau$ Wadge reducing $C \cup U$ to $C$. $\tau$ will copy $I$’s moves as long as $I$ has played so far an $s \in \omega^{<\omega}$ such that $N_s \cap (C \cup U)$ is not Borel (if $C \cup U$ is Borel, the result is immediate).

Suppose $I$ plays $s \triangleleft a_n$ which is the first position to violate this condition. Since $N_s \cap (C \cup U)$ is not Borel, it follows that $N_s \cap C$ is not Borel. Thus $N_s \cap C$ is Wadge above $N_s \triangleleft a_n \cap (C \cup U)$. In this case, $\tau$ then switches to a strategy which reduces $N_s \triangleleft a_n \cap (C \cup U)$ to $N_s \cap C$. $\tau$ is easily a continuous reduction of $C \cup U$ to $C$.

So, $B \in \Gamma \subseteq \bigcup_\rho \Lambda$. Write $B = \bigcup_{\alpha < \rho} B_\alpha$ with each $B_\alpha \in \Lambda$. Define $A_\alpha(y) \leftrightarrow \exists x \in B_\alpha(U(x,y))$. Each $A_\alpha$ is in $\Lambda$, and clearly $A = \bigcup_{\alpha < \rho} A_\alpha$ (each singleton is a $\Sigma_1^1$ set). Also, $\{A_\alpha\}_{\alpha < \rho}$ is $\Sigma_1^1$-bounded [If $S \subseteq A$ is $\Sigma_1^1$, then $S = U_x$ for some $x$, and $x \in B$. So, $x \in B_\alpha$ for some $\alpha$, and thus $S \subseteq A_\alpha$]. This shows $\Gamma \subseteq \Gamma'$.

We claim that $\bigcup_\rho \Lambda \subseteq \exists^\omega \Gamma$. If $\rho < \lambda$, this follows immediately from the coding lemma (as there is a $\Lambda \subseteq \exists^\omega \Gamma$ prewellordering of length $\rho$ in this case; note here that $\exists^\omega \Gamma$ is closed under $\land$, $\exists^\omega$). So assume $\rho = \lambda$, so $\lambda$ is regular. It suffices by the coding lemma to show that there is a $\lambda$ length wellfounded relation in $\exists^\omega \Gamma$. We show, in fact, there is a $\Gamma$ wellfounded relation of length $\lambda$. Fix $A \in \Gamma - \Gamma$ and write $A = \bigcup_{\alpha < \lambda} A_\alpha$, an increasing $\Sigma_1^1$ union of sets in $\Lambda$. Play the game where $I$ plays out $x$, $\Pi$ plays out $y$ and $II$ wins if

$$(x \in A \rightarrow (\sigma(y)_0(A) = (\sigma(y)_1(A))' \text{ is a prewellordering of length } \geq |x|),$$

where $|x| = \mu \alpha \,(x \in A_\alpha)$. In this game $I$ is playing out an element of $A$ and $II$ is playing a code of a $\Lambda$ prewellordering of length at least $|x|$. $II$ wins this game by $\Sigma_1^1$ boundedness of the $A_\alpha$. Let $\tau$ be a winning strategy for $II$. Define then

$$(x_0, y_0) \prec (x_1, y_1) \leftrightarrow (x_0 = x_1 \in A) \land (y_0 \prec_{\tau(x)} y_1).$$

Here $\prec_{\tau(x)}$ is the $\Lambda$ wellfounded relation coded by $\tau(x)$, that is, $\prec_{\tau(x)} = (\sigma(\tau(x))_0)^{-1}(A)$. Thus, $\prec$ is the “join” of the wellfounded relation $\prec_{\tau(x)}$ for $x \in A$. Since each $\prec_{\tau(x)}$ is wellfounded, so is $\prec$. Clearly $|\prec| \geq \lambda$ (in fact $= \lambda$). Finally, as $\Gamma$ is closed under $\land$ we see that $\ll \in \Gamma$. This completes the proof that $\bigcup_\rho \Lambda \subseteq \exists^\omega \Gamma$. 

To summarize, we know that \( \Gamma \subseteq \Gamma' \subseteq \bigcup_\rho A \subseteq \exists^\omega \Gamma \). If \( \Gamma \) is closed under \( \exists^\omega \) (and so closed under both quantifiers, then clearly \( \Gamma = \Gamma' \). If \( \Gamma \subsetneq \exists^\omega \Gamma \), then as \( \Gamma' \) is closed under \( \forall^\omega \) we must have \( \Gamma' = \Gamma \). For if \( \Gamma' \) is properly larger than \( \Gamma \), then by Wadge it must contain \( \Gamma \), and so must contain \( \forall^\omega \Gamma \), which is the dual class of \( \exists^\omega \Gamma \). This contradicts \( \Gamma' \subsetneq \exists^\omega \Gamma \) and the fact that \( \exists^\omega \Gamma \) is nonselfdual (since \( \Gamma \) has a universal set, so does \( \exists^\omega \Gamma \)). This completes the proof that \( \Gamma = \Gamma' \). \( \square \)

To summarize, we have shown that the Steel pointclass corresponding to a projective algebra \( \Lambda \) (defined when \( \text{cof}(\lambda) > \omega \), where \( \lambda = o(\Lambda) \)) can be described as the collection of \( \Sigma^1_1 \)-bounded, \( \rho \)-length increasing unions of sets in \( \Lambda \), where \( \rho = \text{cof}(\lambda) \) is the least ordinal such that \( \Lambda \) is not closed under \( \rho \)-length unions.

**Theorem 7.12.** Let \( \Lambda \) be a projective algebra with \( \lambda = \text{cof}(o(\Lambda)) \), and assume \( \text{cof}(\lambda) > \omega \). Let \( \Gamma \) be the Steel pointclass corresponding to \( \Lambda \) (so \( |\Gamma|_w = \lambda \) and \( \text{sep}(\Gamma) \)). Then \( \text{pwo}(\Gamma) \).

**Proof.** Let \( \rho = \text{cof}(\lambda) \). Let \( A \in \Gamma - \bar{\Gamma} \), and write \( A = \bigcup_{\alpha<\rho} A_\alpha \), an increasing, \( \Sigma^1_1 \)-bounded union with each \( A_\alpha \in \Lambda \). Let \( \varphi \) be the norm on \( \Lambda \) given by \( \varphi(x) = \mu_{\alpha} \) \( (x \in A_\alpha) \). The norm relation \( <^\varphi_{\alpha} \) can be written as \( <^\varphi_{\alpha} = \bigcup_{\lambda<\rho} B_\alpha \) where \( B_\alpha(x,y) \leftrightarrow x \in A_\alpha \land y \notin A_\alpha \). Clearly \( B_\alpha \in \Lambda \), and we see that the sequence \( \{B_\alpha\}_{\alpha<\rho} \) is \( \Sigma^1_1 \)-bounded. To see this, suppose \( S \subseteq \omega^\omega \times \omega^\omega \) is \( \Sigma^1_1 \) and \( S \subseteq <^\varphi_{\alpha} \). In particular, if \( S(x,y) \) then \( x \in A_\alpha \). As the \( \{A_\alpha\}_{\alpha<\rho} \) is \( \Sigma^1_1 \)-bounded there is an \( \alpha_0 < \rho \) such that \( S(x,y) \) implies \( x \in A_{\alpha_0} \). If \( x <^\varphi_{\alpha} y \) we then have that there is an \( \alpha \leq \alpha_0 \) such that \( x \in A_\alpha \land y \notin A_\alpha \), and so \( (x,y) \in B_\alpha \) for some \( \alpha < \alpha_0 \). Thus, \( <^\varphi_{\alpha} \) is a \( \Gamma \) relation. A similar computation shows that \( \leq^\varphi_{\alpha} \) is a \( \Gamma \) relation, so \( \varphi \) is a \( \Gamma \)-norm. \( \square \)

It is worth recording another fact about \( \Gamma \)-prewellorderings.

**Lemma 7.13.** Let \( \Lambda, \Gamma \) be as in Theorem 7.12. Then there is a \( \Gamma \)-prewellordering of length \( \lambda = o(\Lambda) \). Furthermore, all strict initial segments of this prewellordering are in \( \Lambda \) (in particular, there is a \( \lambda \) increasing sequence of \( \Lambda \) sets).

**Proof.** Let \( \rho = \text{cof}(\lambda) \). Assume first that \( \rho < \lambda \). Let \( \{A_\alpha\}_{\alpha<\rho} \) be a \( \Sigma^1_1 \)-bounded increasing sequence of sets in \( \Lambda \) with \( A = \bigcup_{\alpha<\rho} A_\alpha \in \Gamma - \bar{\Gamma} \). We must have that \( \{|A_\alpha|_w : \alpha < \rho\} \) is cofinal in \( \lambda \) (as otherwise the coding lemma shows that \( A \in \Lambda \)). Let \( f(\alpha) = |A_\alpha|_w \). By thinning the sequence, we may assume that \( f \) is strictly increasing and discontinuous. Also, we may assume that for each \( \alpha < \rho \) there is a prewellordering of length \( f(\alpha) \) which is Wadge reducible to any set of Wadge degree \( f(\alpha+1) \). We may also assume that there is a prewellordering of length \( \rho \) in \( \Gamma_0 \). Let \( \Gamma_\alpha = \Sigma^1_1(A_{\alpha+1}) \), and let \( U_\alpha \) be a universal set for \( \Gamma_\alpha \) (this is defined uniformly from \( A_{\alpha+1} \), see exercise 21). So, \( \Gamma_\alpha \) is nonselfdual, closed under \( \exists^\omega \), \( \land \), and has a prewellordering of length \( f(\alpha) \).

Fix for the moment an \( \alpha < \rho \). By the coding lemma applied to \( \Gamma_\alpha \), there is a \( \Gamma_\alpha \) set \( E \) such that

1. Every \( z \in E \) codes, relative to \( U_\alpha \) a set \( B_z \) of Wadge degree \( g(z) \in [\sup_{\beta<\alpha} f(\beta), f(\alpha)] \). Here \( B_z = \sigma^{-1}_{(z)}(U_\alpha) = (\sigma^{-1}_{(z)}(U_\alpha))^C \).
2. For every \( \gamma \in [\sup_{\beta<\alpha} f(\beta), f(\alpha)] \) there is an \( z \in E \) with \( g(z) = \gamma \).

By the coding lemma again, there is a \( D \subseteq \omega^\omega \times \omega^\omega \in \Gamma_0 \subseteq \Lambda \) (using now \( \rho < \lambda \)) such that
(1) \( \text{dom}(D) \subseteq A \times \omega^\omega \).
(2) For all \( \alpha < \rho \) there is an \( (x, y) \in D \) with \( x \in A_\alpha - \bigcup_{\beta < \alpha} A_\beta \).
(3) If \( D(x, y) \) and \( x \in A_\alpha - \bigcup_{\beta < \alpha} A_\beta \), then \( y \) codes, relative to \( U_\alpha \), a \( \Gamma_\alpha \) set \( E_y \) satisfying (1) and (2) above (for the set \( E \)) and this \( \alpha \).

For \( \alpha < \rho \) define \( E_\alpha \) by:

\[
z \in E_\alpha \iff \exists x \exists y \left(D(x, y) \land x \in A_\alpha - \bigcup_{\beta < \alpha} A_\beta \land z \in E_y \right)
\]

where "\( z \in E_y \)" means \( \sigma_\beta(z) \in U_\alpha \). So, each \( E_\alpha \) is a \( \Gamma_\alpha \) set of codes (relative to \( U_\alpha \)) for sets of Wadge degrees in \( \{ \sup_{\beta < \alpha} f(\beta), f(\alpha) \} \) (and every ordinal in this interval is coded by a real in \( E_\alpha \)).

We now define an increasing sequence \( \{ C_\gamma \}_{\gamma < \lambda} \) of sets in \( \Lambda \) as follows. Fix \( \gamma < \lambda \), and let \( \alpha \) be such that \( \gamma \in \{ \sup_{\beta < \alpha} f(\beta), f(\alpha) \} \). Let

\[
(x, z) \in C_\gamma \iff \exists \alpha_0 \leq \alpha \left((x \in A_{\alpha_0} - \bigcup_{\beta < \alpha_0} A_\beta) \land z \in E_{\alpha_0} \land ((\alpha_0 < \alpha)
\]

\( \lor (\sigma_{\alpha_0}^{-1}(U_{\alpha_0}) = (\sigma_{\alpha_0}^{-1}(U_{\alpha_0}))^c \text{ has Wadge degree} \leq \gamma) \)\]

As \( \Gamma_{\alpha+1} \) is closed under \( f(\alpha) \) unions (by the coding lemma), we easily have that each \( C_\gamma \) is in \( \Gamma_{\alpha+1} \). The \( C_\gamma \) clearly form a \( \lambda \)-length increasing sequence. Let \( \psi \) be the norm corresponding the sequence \( \{ C_\gamma \}_{\gamma < \lambda} \), so \( \psi \) has length \( \lambda \). We may write

\[
\psi = \bigcup_{\alpha < \rho} \bigcup_{\gamma < f(\alpha)} (C_\gamma \times (\omega^\omega - C_\gamma))
\]

This is a \( \rho \) union of sets, say \( F_\alpha \), each of which is easily in \( \Lambda \) (since for \( \gamma < f(\alpha) \), \( C_\gamma \) has Wadge degree \( < f(\alpha + 1) \)). Also, the sequence \( \{ F_\alpha \}_{\alpha < \rho} \) is a \( \Sigma^1_1 \)-bounded union, since if \( S \subseteq \bigcup_{\alpha < \rho} F_\alpha \) then \( \{ x : \exists y (x, y) \in S \} \) is a \( \Sigma^1_1 \) subset of \( A \), and hence is contained in some fixed \( A_\alpha \). This completes the proof in the case \( \rho < \lambda \).

The case \( \rho = \lambda \) is immediate, since in this case we may write a set \( A \in \Gamma - \Gamma \) as an \( \lambda \) increasing, \( \Sigma^1_1 \)-bounded, union of \( \Lambda \) sets, and this gives a \( \Gamma \) prewellordering of length \( \lambda \).

**Case 2:** \( \text{cof}(\lambda) > \omega \) and \( \Gamma \) is not closed under \( \lor \).

First we note that this case includes the case where \( \lambda \) is singular (and \( \text{cof}(\lambda) > \omega \)).

To see this, let \( \rho = \text{cof}(\lambda) \), so \( \omega < \rho < \lambda \). Let \( A = \bigcup_{\alpha < \rho} A_\alpha \) where \( A \in \Gamma - \Gamma \) and each \( A_\alpha \in \Lambda \). Let \( \Gamma_0 \) be a pointclass closed under \( \exists^\omega \), \( \land \) such that there is a \( \Gamma_0 \) prewellordering of length \( \rho \). By the coding lemma there is a \( \Gamma_0 \) set \( B \) such that each \( x \in B \) is a \( \Gamma \) code for one of the \( A_\alpha \) and each \( A_\alpha \) is coded by some \( x \in B \). We then have

\[
y \in A \iff \exists x \left([x \in B] \land \sigma_x(y) \in A^c \right)
\]

and since \( \Gamma \) is closed under \( \exists^\omega \), this shows that \( \Gamma \) is not closed under intersection with \( \Gamma_0 \) sets.

Since \( \Gamma \) is not closed under \( \lor \), it is not closed under \( \forall^\omega \). Let \( \Pi_1 = \Gamma \) be the Steel pointclass, and define the \( \Sigma_n, \Pi_n \) classes for \( n \geq 1 \) as usual. Note that these classes are all distinct, for example \( \Sigma_n \) is not closed under \( \forall^\omega \) as otherwise, since \( \Pi_1 = \forall^\omega \Sigma_{n-1} \), we would have that \( \Sigma_n \supseteq \Pi_n \) and so \( \Sigma_n = \Pi_n \), a contradiction as all the \( \Sigma_n \) are nondual.
We have pwo(Π₁), and by periodicity we have pwo(Π₂n+1), pwo(Σ₂n+2) for all n ≥ 0. Note that for n ≥ 2, Σₙ, Πₙ are closed under countable unions and intersections.

**Case 3**: cof(λ) > ω, and Γ is closed under ∨ but not ∃ν⁺.

As we remarked above, in this case λ is regular. As in Case 2, let Π₁ = Γ and define the Σₙ, Πₙ as usual. Since by assumption Π₁ is not closed under ∃ν⁺, we again have that all the Σₙ, Πₙ are distinct. As in Case 2, by periodicity we again have pwo(Π₂n+1), pwo(Σ₂n+2) for all n ≥ 0.

**Case 4**: cof(λ) > ω and Γ is closed under ∃ν⁺.

In this case Γ is closed under ∀ν⁺, ∃ν⁺, and hence also countable unions and intersections. As we noted in Case 2, we must have λ is regular in this case. We have pwo(Γ), but in this case we cannot propagate from Γ by periodicity as Γ is closed under quantifiers.

Let Π₁ = Γ ∧ Γ, Σ₁ = Π₁ and define the Σₙ, Πₙ as usual. Note that Π₁ is closed under ∀ν⁺ and countable intersections, but not ∨ (as otherwise it would contain Σ₁, which it does not as it is nonselfdual, as it easily has a universal set). In particular Π₁ is not closed under ∃ν⁺, and as before we have that all the Σₙ, Πₙ are distinct. Thus, we have a projective-like hierarchy defined over Π₁.

We show pwo(Π₁). For this, we use the following representation of Π₁ sets.

**Claim.** Π₁ is the collection of Σ₁₁-bounded increasing unions of Π sets of length λ.

**Proof.** Let Π₁ be the collection of Σ₁₁-bounded increasing unions of Π sets of length λ. Suppose A ∈ Π₁, say A = B ∩ C where B ∈ Γ, C ∈ Π. Write B = ⋃α<λ Bα, an increasing, Σ₁₁-bounded union of A sets. Then A = ⋃α<λ(Bα ∩ C). This is easily a Σ₁₁-bounded union of Π sets (note here Π is closed under intersections with Λ sets, in fact, Γ and Π are closed under unions and intersections). So, Π₁ ⊆ Π₁.

Since Π is closed under ∀ν⁺, the argument of Claim 7 shows that Π₁ is closed under ∀ν⁺. On the other hand, since there is a Γ preorder of length λ, the coding lemma shows that ⋃λ Π ⊆ ∃ν⁺(Γ ∧ Γ) = Σ₂. Since Π₁ ⊆ Π ⊆ Σ₂ and Π₁ is closed under ∀ν⁺, by Wadge it follows that Π₁ = Π₁.

Let A ∈ Π₁, and as above write A = B ∩ C = ⋃α<λ(Bα ∩ C), where C ∈ Π and the Bα form a Σ₁₁-bounded union of Δ sets. Let Aα = Bα ∩ C, so A = ⋃α<λ Aα is a Σ₁₁-bounded union of Π sets. Let φ be the norm on A associated to this union, that is, φ(x) = μx (x ∈ Aα). We show that φ is a Π₁, norm. Write ων⁺ − C = ⋃β<λ Cβ, a Σ₁₁-bounded union of Δ sets. Note that ων⁺ − Aα = ⋃β<λ(Cβ ∪ Bα). Note also that the Cβ form a Γ, bounded union. This is because the norm ψ associated to the Cβ is a Γ, norm (as shown above) and since Γ is closed under ∨, the usual boundedness argument shows every Γ subset of ων⁺ − C is bounded in the ψ norm. From this it follows that each sequence {Cβ ∪ Bα}β<λ is actually a Γ, bounded union (since if S ⊆ ⋃β(Cβ ∪ Bα) = C∞ ∪ Bα is in Γ, then S − Bα ⊆ C∞ is in Γ).

We then have

\[ x <^* y \iff \exists \beta < \lambda \exists \alpha \leq \beta [(x \in Aα) \land (y \in Cβ ∪ Bα)]. \]

For a fixed β < λ, the rest of the above equation after the ∃β quantifier defines a Γ set (it is a β-union of Γ sets, and Γ is closed under < λ unions by the coding lemma). So, <^* = ⋃β Eβ, where Eβ ∈ Γ. Moreover, this is a Σ₁₁-bounded union. To see this, let S ⊆ <^* be Σ₁₁. Then S₁ = \{x : \exists y \ (x, y) \in S\} ∈ Σ₁₁ and S₁ ⊆ A,
and so there is an $\alpha_0 < \lambda$ such that $S_1 \subseteq A_{\alpha_0}$. For $\alpha \leq \alpha_0$, let $S_\alpha = \{(x,y) \in S : x \in A_\alpha - \bigcup_{\gamma < \alpha} A_\gamma\} = \{(x,y) \in S : x \in C \cap (B_\alpha - \bigcup_{\gamma < \alpha} B_\gamma)\}$ which shows that $S_\alpha \in \hat{\Gamma}$. By $\hat{\Gamma}$-boundedness of the $\{B_\alpha \cup C_\beta\}_{\beta < \lambda}$ we have that there is a $\beta(\alpha) < \lambda$ such that for all $(x,y) \in S_\alpha$, $y \in (B_\alpha \cup \bigcup_{\beta < \beta(\alpha)} C_\beta)$. Since $\lambda$ is regular, $\beta_0 = \sup_{\alpha < \alpha_0} \beta(\alpha) < \lambda$. We then have that $S \subseteq E_{\beta_0}$. This shows that $\prec_\varphi$ is in $\Pi_1$.

A similar computation shows that $\forall \psi \in \Theta$. To see this, let $y \in (x,y) \in S \subseteq E_{\beta_0}$ such that shows $\prec_\varphi$ is in $\Pi_1$.

This completes the proof of $\text{pwo}(\Pi_1)$. By periodicity we then have $\text{pwo}(\Pi_{2n+1})$, $\text{pwo}(\Sigma_{2n+2})$ for all $n \geq 0$.

\section{Wellordered Unions}

In this section we investigate wellordered unions from a pointclass $\Gamma$, the main closure theorem is the following.

**Theorem 8.1.** Let $\Gamma$ be nonselfdual, closed under $\exists^{\omega^\omega}$, and assume $\text{pwo}(\Gamma)$. Then $\Gamma$ is closed under wellordered unions.

Before starting the proof, we note that there is no harm in adding the assumption that $\Gamma$ is also closed under $\land$. For inspecting the hierarchy analysis for the Levy pointclasses of §7 we see that in all cases if $\Gamma$ is nonselfdual and closed under $\exists^{\omega^\omega}$ and $\text{pwo}(\Gamma)$, then $\Gamma$ is also closed under $\land$. For if $\Lambda = \Lambda(\Gamma)$ is of type I, then $\Gamma$ must be of the form $\Sigma_2n$ for some $n \geq 0$, and all these classes are closed under $\land$.

If $\Lambda$ is of types II or III, then $\Gamma$ must be of the form $\Sigma_{2n}$ for $n \geq 1$, since in some cases the Steel class $\Gamma = \Pi_1$ is not closed under $\exists^{\omega^\omega}$. All of these classes are closed under $\land$, in fact, closed under countable unions and intersections (if $\Gamma' = \exists^{\omega^\omega} \Gamma$ where $\Gamma$ is nonselfdual and closed under $\forall^\omega$, then $\Gamma'$ is closed under countable unions and intersections by the same argument for $\Sigma_1^1$). If $\Lambda$ is of type IV (that is, the Steel class is closed under both quantifiers), then either $\Gamma$ is the Steel class (since we are assuming $\text{pwo}(\Gamma)$, $\Gamma$ must be the Steel class, not the dual class), or $\Gamma$ is of the form $\Sigma_{2n}$ for $n \geq 1$, where recall $\Sigma_2 = \exists^{\omega^\omega} (\Gamma \land \hat{\Gamma})$ (since $\text{pwo}(\Pi_1)$ where $\Pi_1 = \Gamma \land \hat{\Gamma}$). All of these classes are closed countable unions and intersections by the above comment.

So, in all cases, a nonselfdual $\Gamma$ closed under $\exists^{\omega^\omega}$ with $\text{pwo}(\Gamma)$ is also closed under $\land$ (of course, $\Gamma$ is also closed under $\lor$ as it is nonselfdual and closed under $\exists^\omega$). Note also we showed above that $\Gamma$ is actually closed under countable unions and intersections in all cases except one: $\Lambda$ is of type I and $\Gamma = \Sigma_0 = \bigcup_{\alpha} \Lambda$.

The proof of Theorem 8.1 breaks into two cases depending on whether $\Gamma$ is closed under $\forall^\omega$ or not. In both cases we are also assuming $\Gamma$ is nonselfdual, $\Gamma$ is closed under $\exists^{\omega^\omega}$, and $\text{pwo}(\Gamma)$, and by the above comments also closed under $\land$, $\lor$.

\textbf{Case 1: $\Gamma$ is closed under $\forall^\omega$.}

In this case (the type IV case), $\Gamma$ is closed under $\exists^{\omega^\omega}$, $\forall^\omega$, and so also countable unions and intersections. So, $\hat{\Gamma}$ is also closed under these operations. Going by contradiction, let $\rho$ be the least ordinal such that there is a $\rho$-union of $\Gamma$ sets which is not in $\hat{\Gamma}$. Easily, $\rho$ is a regular cardinal. Let $\delta = \sup\{|\prec\| : \prec \in \hat{\Gamma} \text{ is a wellfounded relation}\}$. $\hat{\Gamma}$ is closed under $< \delta$ length unions by the coding lemma, so $\delta \leq \rho$. Since we are assuming $\bigcup_{\rho} \Gamma \not\subseteq \hat{\Gamma}$ we have $\hat{\Gamma} \subseteq \bigcup_{\rho} \Gamma$ by Wadge. We have, in fact, that every $\Gamma$ set can be written as a $\Sigma_1^1$-bounded $\rho$-union of sets in $\Gamma$. To see this, let $B \in \hat{\Gamma}$, and let $U$ be a universal $\Sigma_1^1$ set. Let $C(x) \iff \forall y (U_x(y) \rightarrow B(y))$, so $C \in \hat{\Gamma}$ by the closure properites of $\hat{\Gamma}$. Write $C = \bigcup_{\alpha < \rho} C_\alpha$,
an increasing union with $C_\alpha \in \Gamma$. Then let $B_\alpha(y) \leftrightarrow \exists x [(x \in C_\alpha) \land (y \in U_x)]$. Then $B = \bigcup_{\alpha < \rho} B_\alpha$, each $B_\alpha \in \Gamma$, and the $B_\alpha$ form a $\Sigma^1_1$-bounded union.

Fix now an increasing $\Sigma^1_1$-bounded union $A = \bigcup_{\alpha < \rho} A_\alpha$ with each $A_\alpha \in \Gamma$ and $A \in \Gamma - \Gamma$. For $x \in A$ let $| x | = \mu_\alpha (x \in A_\alpha)$. Consider the game where I plays out $x$, II plays out $y,z$, and II wins if

$$(x \in A \rightarrow ((y \in A) \land (| x | < | y |) \land A_{|y|} = V_z)),$$

where $V$ denotes a universal $\Gamma$ set. By $\Sigma^1_1$-boundedness, II has a winning strategy $\tau$. Consider the relation $\prec$ defined by

$$x_0 \prec x_1 \Leftrightarrow (x_0, x_1 \in A) \land (\tau(x_1) \notin V_{\tau(x_0)})$$

which is clearly a $\Gamma$ relation. Note that for $x_0, x_1 \in A$ we have $x_0 \prec x_1$ iff $\varphi(x_0) < \varphi(x_1)$ where $\varphi(x) = |(\tau(x))_0|$. Thus, $\prec$ is a prewellordering. Since $\rho$ is regular, it follows that $\prec$ has length at least $\rho$ [By induction on $\alpha < \rho$ there is a $\beta(\alpha) < \rho$ such that every $x \in A$ with $| x | \geq \beta(\alpha)$ has $\prec$-rank $\geq \alpha$. At limit stages we use the regularity of $\rho$, and at successor stages the result easily follows.] So, $\prec$ is a $\Gamma$ prewellordering on length $\rho$. This shows that $\rho < \delta$, a contradiction.

Case 2: $\Gamma$ is not closed under $\forall^\omega$, but closed under $\exists^\omega$.

Going by contradiction again, let $\rho$ be the least ordinal such that there is a $\rho$-union of $\Gamma$ sets which is not in $\Gamma$. Again, $\rho$ is a regular cardinal. Let $\Gamma' = \bigcup_{\alpha < \rho} \Gamma$. By minimality of $\rho$, any $\rho$-length union of $\Gamma$ sets has the same union as a $\rho$-length increasing union. We are assuming $\Gamma' \notin \Gamma$, and so by Wadge, $\Gamma \subsetneq \Gamma'$. Since $\Gamma'$ is clearly closed under $\exists^\omega$, we have $\exists^\omega \Gamma \subseteq \Gamma'$. Let

$$\delta_1 = \sup \{|\prec| : \prec \in \Gamma \text{ is a wellfounded relation}\},$$

and let $\delta_2 = \sup \{|\prec| : \prec \in \exists^\omega \Gamma \text{ is a wellfounded relation}\}$. From the coding lemma, both $\delta_1$ and $\delta_2$ are regular cardinals. Also, $\delta_1 < \delta_2$ since we may put all the $\Gamma$ wellfounded relations together into a single $\Gamma \cap \Gamma \subseteq \exists^\omega \Gamma$ relation (this uses the closure of $\Gamma$ under $\forall^\omega$).

Suppose first that $\rho < \delta_2$. By periodicity we have $\pwo(\forall^\omega \Gamma)$, and it follows by Theorem 7.9 that $\Delta_1 = \exists^\omega \Gamma \cap \forall^\omega \Gamma$ is closed under $\delta_2$ unions and intersections. This would give $\bigcup_{\rho} \Gamma \subseteq \Delta_1$, contradicting $\exists^\omega \Gamma \subseteq \bigcup_{\rho} \Gamma$.

Suppose next that $\rho \geq \delta_2$. Let $\prec$ be a $\exists^\omega \Gamma$ wellfounded relation of rank $> \delta_1$. Write $\prec = \bigcup_{\alpha < \rho} A_\alpha$ an increasing unions with $A_\alpha \in \Gamma$. For $x \in \dom(\prec)$, let $| x |_{| \alpha \rangle} = | x |_{A_\alpha}$ be the rank of $x$ in the wellfounded relation $A_\alpha$. So, $| x |_{| \alpha \rangle} < \delta_1$. Since $\rho$ is regular, there is an $\alpha(x) < \delta_1$ such that for all $\alpha \geq \alpha(x)$, $| x |_{| \alpha \rangle} = | x |_{| \alpha \rangle}$. The map $x \mapsto | x |_{| \alpha \rangle}$ is an order-preserving map from $\prec$ to $\delta_1$, a contradiction.

This completes the proof of Theorem 8.1. \hfill $\Box$

Case 3: $\Gamma$ is not closed under $\forall^\omega$.

Inspecting the hierarchy analysis we see that there is a type I projective algebra $\Lambda$, that is $\cof(\lambda) = \omega$ such that $\Gamma = \Sigma_0 = \bigcup_\omega \Lambda$. Recall that in this case $\Sigma_0$ is closed under $\land, \lor, \exists^\omega$, and we have $\pwo(\Sigma_0)$ and also $\Sigma_0$ is not closed under $\forall^\omega$, so this case does occur.

9. Scales

For the basic definition and facts about semi-scales, Suslin representations, and scales, see the notes on Polish spaces. We carry on from that discussion.
For $T$ a tree on $\omega \times \kappa$ ($\kappa \in \text{On}$) with $A = p[T]$, let $\vec{\varphi}^T$ be the corresponding semi-scale on $A$, so each norm $\varphi^T_n$ maps into $\alpha$. Similarly, given a semi-scale $\vec{\varphi}$ on the set $A$, let $T^\varphi$ denote the tree of $\vec{\varphi}$ (we drop the vector notation when it causes no confusion). So, $T^\varphi$ is a tree with $A = p[T]$.

In general the maps $\varphi \mapsto T(\varphi) = T^\varphi$ and $T \mapsto \varphi(T) = \varphi^T$ are not inverses of each other.

**Fact 9.1.** If $\vec{\varphi}$ is a scale on $A$, then $\varphi = \varphi(T(\varphi))$.

*Proof.* Let $X \in A$, and let $\vec{\psi}$ denote the semi-scale $\varphi(T(\varphi))$. Then $\psi_n(x)$ is the $n$th digit of the leftmost branch of $T(\varphi)_x$. Clearly $\varphi_0(x),\varphi_1(x),\ldots$ is a branch through $T(\varphi)_x$, and since $\vec{\psi}$ is a scale, it is the (true) leftmost branch of $T(\varphi)_x$. So, $\psi_n(x) = \varphi_n(x)$. $\square$

**Fact 9.2.** If $T$ is a tree, then $T(\varphi(T)) \subseteq T$.

*Proof.* If $(s,\vec{\alpha}) \in T(\varphi(T))$, then there is an $x$ extending $s$ in $A = p[T]$ such that $\varphi_n(T)(x) = \alpha(n)$ for all $n < \ln(s)$. So, for all such $n$ we have that the leftmost branch of $T_x$ begins with $\vec{\alpha} \upharpoonright n$. In particular, $(s,\vec{\alpha}) \in T$. $\square$

From the last fact we have the following.

**Fact 9.3.** For any tree $T$ we have $\varphi(T(\varphi(T))) = \varphi(T)$.

*Proof.* We have that $T(\varphi(T)) \subseteq T$, and both of these trees project to the same set $A = p[T]$. To see that $\varphi(T(\varphi(T))) = \varphi(T)$ it is enough to observe that for any $x \in A$ that the leftmost branch of $T(\varphi(T))_x$ is the leftmost branch of $T_x$. For this it is enough to see that the leftmost branch of $T_x$ is a branch through $T(\varphi(T))_x$. If $f$ is the leftmost branch of $T_x$, then for all $n$, $(x \upharpoonright n,f \upharpoonright n)$ is in $T(\varphi(T))$ since we can use $x$ as the witness (as $x$ extends $s$ and its leftmost branch $f$ starts out with $f \upharpoonright n$). This shows $\varphi(T(\varphi(T))) = \varphi(T)$. $\square$

**Corollary 9.4.** The maps $\Sigma: \varphi \mapsto \varphi(T(\varphi))$ and $\Pi: T \mapsto T(\varphi(T))$ are idempotent (that is, $\Sigma^2 = \Sigma$, $\Pi^2 = \Pi$).

*Proof.* Since $\varphi(T(\varphi(T))) = \varphi(T)$ holds for any tree $T$, we have in particular that $\varphi(T(\varphi(T))) = \varphi(T(\varphi(T)))$, which says that $\Sigma^2 = \Sigma$. Similarly, we can “apply $T$” to both sides of the equation to get $T(\varphi(T(\varphi(T)))) = T(\varphi(T))$, which says that $\Pi^2 = \Pi$. $\square$

The scale property and basic facts about it were discussed in the other set of notes, we just recall here without proof some of these facts.

**Definition 9.5.** A $\Gamma$-scale on a set $A \subseteq \omega^\omega$ is a scale $\{\varphi_n\}_{n \in \omega}$ on $A$ such that each norm $\varphi_n$ is a $\Gamma$-norm. Likewise we define $\Gamma$-semi-scale, $\Gamma$-good scale, etc. We say $\Gamma$ has the scale property if every $\Gamma$ set $A \subseteq \omega^\omega$ has a $\Gamma$-scale.

We write $\text{scale}(\Gamma)$ to say that $\Gamma$ has the scale property. Clearly $\text{scale}(\Gamma)$ implies $\text{pwo}(\Gamma)$.

**Fact 9.6.** If $\Gamma$ is closed under $\land$, $\lor$, and $\text{scale}(\Gamma)$, then every $A \in \Gamma$ admits a $\Gamma$-excellent scale.

Recall that if $\text{pwo}(\Gamma)$ and $\Gamma$ is closed under $\forall$, then $\Gamma$ has the number uniformization property. We state next the corresponding result for the full uniformization property.
Fact 9.7. If scale($\Gamma$) and $\Gamma$ is closed under $\forall^\omega$, $\land$, $\lor$, then every $R \subseteq \omega^2 \times \omega^2$ in $\Gamma$ has a $\Gamma$ uniformization.

Proof. From Fact 9.6 we may assume $\{\varphi_n\}$ is an excellent scale on $R$. As in Lemma 2.28 of the other notes, this gives a uniformization of $R$ which is also in $\Gamma$ using the closure of $\Gamma$ under $\forall^\omega$.

Fact 9.8. If $\Gamma$ is a pointclass and unif($\Gamma$), then unif($\exists^\omega \Gamma$).

Also (from the other notes) we have the following fundamental fact.

Fact 9.9 ($\text{ZF}$). $\Pi^1_1$ has the scale property.

Corollary 9.10 ($\text{ZF}$). $\Pi^1_1$ and $\Sigma^1_2$ have the uniformization property.

We now begin the analysis of the scale property assuming AD. The arguments will be similar in some respects to the analysis of the prewelling order property given before, but new ingredients arise.

First we give the periodicity theorems which propagate the scale property under quantifiers, similar to what the first periodicity theorem did for norms. As with the prewelling order property, the propagation by the $\exists^\omega$ quantifier is easy and done in ZF (see Lemma 2.48 in the other notes).

Fact 9.11. Let $\Gamma$ be a pointclass closed under $\forall^\omega$, $\land$, $\lor$ and assume scale($\Gamma$). Then scale($\exists^\omega \Gamma$).

Next we have the second periodicity theorem which transfers the scale property by the $\forall^\omega$ quantifier.

Theorem 9.12 (Second Periodicity). Let $\Gamma$ be closed under $\exists^\omega$, $\land$, $\lor$ and assume $\Delta$ determinacy. If scale($\Gamma$) then scale($\forall^\omega \Gamma$). More generally (without assuming the closure properties of $\Gamma$), if $B \subseteq \omega^2 \times \omega^2$ admits a very good scale in $\Gamma$, then $A = \forall^\omega B$ admits a scale in $\forall^\omega \exists^\omega \Gamma$.

Let $A \in \forall^\omega \Gamma$, say $A(x) \leftrightarrow \forall y B(x, y)$ where $B \in \Gamma$. Let $\{\varphi_n\}$ be a scale on $B$. We define norms $\{\psi_n\}$ on $A$ as follows. Let $\{s_n\}$ enumerate $\omega^{\omega_1}$ with all $s_n$ preceding any of its proper extensions (so $s_0 = \emptyset$). To define the norm $\psi_n$ we consider, for $x_0, x_1 \in A$, the following game:

\[
\begin{array}{c|ccc}
    s_n & y_0(0) & y_0(1) & y_0(2) & \cdots \\
G_n(x_0, x_1) & \downarrow & \downarrow & \downarrow & \cdots \\
    s_n & y_1(0) & y_1(1) & y_1(2) & \cdots \\
\end{array}
\]

where I and II play out $y_0$ and $y_1$ respectively and II wins the run iff $(x_0, s_n \upharpoonright y_0) \leq^*_\varphi (x_1, s_n \upharpoonright y_1)$. Since $x_0, x_1$ are in $A$, this game is in $\Delta$ and hence determined. We define $x_0 \leq x_1$ iff II has a winning strategy in $G_n(x_0, x_1)$. Exactly as in the first periodicity theorem, $\leq_n$ is a reflexive, transitive, connected, wellfounded relation on $A$, that is, it defines a prewelling order on $A$. Let $\psi_n$ be the corresponding norm on $A$.

We next show that $\{\psi_n\}$ is a scale on $A$. Suppose $\{x_n\} \subseteq A$, $X_m \to x$, and for each $n$ the norms $\psi_n(x_m)$ are eventually equal to $\lambda_n$. First we show that $x \in A$. Fix $y \in \omega^\omega$, and we show $B(x, y)$. Let $i_0 < i_1 < \cdots$ be a sequence such that
∀m ≥ i_k(ψ_l(x_m)) = λ_l, where s_l = (y(0),...,y(k − 1)) (for k = 0 we stabilize ψ_0, that is, l = 0). For each k, let τ_k be a winning strategy for II in the game G_l(x_{i_k+1}, x_{i_k}), with m as above. Using the τ_k we fill in the following diagram.

<table>
<thead>
<tr>
<th></th>
<th>y_{i_0}(0)</th>
<th>y_{i_0}(1)</th>
<th>y_{i_0}(2)</th>
<th>y_{i_0}(3)</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>\tau_0</td>
<td>y(0)</td>
<td>y_{i_1}(1)</td>
<td>y_{i_1}(2)</td>
<td>y_{i_1}(3)</td>
<td>\cdots</td>
</tr>
<tr>
<td>\tau_1</td>
<td>y(0)</td>
<td>y(1)</td>
<td>y_{i_2}(2)</td>
<td>y_{i_2}(3)</td>
<td>\cdots</td>
</tr>
<tr>
<td>\tau_2</td>
<td>y(0)</td>
<td>y(1)</td>
<td>y(2)</td>
<td>y_{i_3}(3)</td>
<td>\cdots</td>
</tr>
<tr>
<td>\tau_3</td>
<td>y(0)</td>
<td>y(1)</td>
<td>y(2)</td>
<td>y(3)</td>
<td>\cdots</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

This produces reals y_{i_k} which converge to the real y. Since each τ_k is a winning strategy for II in G_l(x_{i_k+1}, x_{i_k}) we have that ϕ_{i_k}(x_{i_k+1}, y_{i_k+1}) ≤ ϕ_{i_k}(x_{i_k}, y_{i_k}) for each k, where s_k = (y(0),...,y(k − 1)). Since l_k ≥ k and ϕ is very good, we have ϕ_k(x_{i_k+1}, y_{i_k+1}) ≤ ϕ_k(x_{i_k}, y_{i_k}) for all k. Since ϕ is very good, this says that for all n that the norms ϕ_n(x_{i_k}, y_{i_k}) are eventually constant. Since ϕ is a semi-scale on B, this says that (x, y) ∈ B. This shows that ϕ is a semi-scale on A.

The proof that ϕ is a scale is very similar. To see that ϕ is a scale, it suffices to show that for each s ∈ ω^{<ω}, II can win the game G_n(x, x_m) for all large enough m. Fix such an s, and let i_0 now be a large enough integer so that the norms ψ_s(x_m) are constant for all m ≥ i_0. If now I makes move first move y(0) in G_n(x, x_{i_0}), then II chooses an i_1 > i_0 such that the norms ψ_{i_1}(x_m) are constant for m ≥ i_1 and fixes a winning strategy τ_{i_0} in the game G_n(x_{i_1}, x_{i_0}) and copies the move y(0) to be I’s first move in this game. II then plays according to τ_{i_0} to get the first move of II in G_n(x, x_{i_0}). If I then plays y(1) as the next move of G_n(x, x_{i_0}), II then chooses a x_{i_2} so that the norms ψ_{i_2}(x_m) are constant for m ≥ i_2 and picks a winning strategy τ_{i_1} for the game G_n(ψ_{i_2}(y(0), y(1)), x_{i_2}, x_{i_1}) and copies y(1) as the first move of that game. This is illustrated in Figure 9.

If we let z_{i_k} = s^{−1}y_{i_k} be the real produced by following τ_{i_k} on the kth board, then we have again that for all k:

ϕ_{s^{−1}(y(0),...,y(k−1))}(x_{i_k+1}, z_{i_k+1}) ≤ ϕ_{s^{−1}(y(0),...,y(k−1))}(x_{i_k}, z_{i_k}).

Since ϕ is very good, all the norms ϕ_n(x_{i_k}, z_{i_k}) are eventually constant, say with constant value α_n, and since ϕ is a scale we have ϕ_n(x, z) ≤ α_n. In particular ϕ_n(x, z) ≤ ϕ_n(x_{i_0}, z_{i_0}), which shows II has won the run of the game G_n(x, x_{i_0}).
So far we have shown that the $\psi_n$ form a scale on $A$. If we let $\psi'_n(x) = \langle \psi_0(x), \psi_n(x) \rangle_{\text{box}}$ for $x \in A$, then it follows that $\{\psi'_n\}$ is also a scale on $A$. Moreover, each of the norms $\psi'_n$ is now a $\forall \omega \exists \omega \Gamma$ norm. For example, we have

$$x_0 \leq_{\psi'_n} x_1 \leftrightarrow [(x_0 <_{\psi_0} x_1) \lor ((x_0 \leq_{\psi_0} x_1) \land (x_1 \leq_{\psi_0} x_0) \land (x_0 \leq_{\psi_n} x_1))]$$

From the first periodicity theorem we have that $\psi_0$ is a $\forall \omega \exists \omega \Gamma$-norm on $A$. For $n > 0$ the norm $\psi_n$ is not necessarily a $\forall \omega \exists \omega \Gamma$-norm on $A$ (for example, if $x_0 \in A$ and $x_1 \notin A$, we don’t necessarily have that $\Pi$ wins the game $G_n(x_0, x_1)$, so we can’t say that $x_0 \leq_{\psi_n} x_1$ holds iff $\Pi$ wins this game as we did in the first periodicity theorem). However, in the above equation we can replace “$x_0 \leq_{\psi_n} x_1$” with “$\Pi$ wins the game $G_{n+1}(x_0, x_1)$” as at this point in the formula we have already guaranteed that both $x_0, x_1$ are in $A$.

This completes the proof of the second periodicity theorem.

**Corollary 9.13** (projective determinacy). We have $\text{scale}(\Pi^1_{2n+1}), \text{scale}(\Sigma^1_{2n+2})$ for all $n \geq 0$.

**Corollary 9.14** (projective determinacy). We have $\text{unif}(\Pi^1_{2n+1}), \text{unif}(\Sigma^1_{2n+2})$ for all $n \geq 0$.

We next give the third periodicity theorem. Although this is not needed for the propagation of scales, it is an importat theorem in descriptive set theory, and the proof is similar to that of the second periodicity theorem. The third periodicity theorem concerns the existence of canonical/definable winning strategies for games whose payoff sets are Suslin (i.e., have scales).
Theorem 9.15 (Third Periodicity). Let \( A \subseteq \omega^\omega \) and assume \( A \) admits a scale \( \{ \varphi_n \} \) in a pointclass \( \Gamma \). Assume \( \det(\Gamma) \). If \( I \) has a winning strategy in the game \( G_A \) (recall \( I \) tries to get in the set \( A \)), then \( I \) has a canonical winning strategy \( \sigma \) for the game \( G_A \) which is a \( \exists \Gamma \) real.

Proof. Consider \( s = (a(0), a(1), \ldots, a(2n - 1)) \in \omega^{<\omega} \) of even length (so \( I \)'s turn to move). Suppose \( s \) is a winning position for \( I \) in the game \( G_A \). We compare the possible next moves for \( I \) by the following game. For \( a, b \in \omega \) the game \( G_s(a, b) \) is played as shown:

\[
\begin{array}{c|ccc}
I & s & b & x_1(0) & x_1(1) & x_1(2) & \cdots \\
\hline
\text{II} & s & a & x_0(0) & x_0(1) & x_0(2) & \cdots \\
\end{array}
\]

**Figure 4.** The position comparison game.

In the game \( G_s(a, b) \), players \( I \) and \( \text{II} \) alternate moves as usual with \( I \) moving first. They play in somewhat different locations, however. \( I \) makes first move \( x_0(0) \), and then \( \text{II} \) responds with move \( x_1(0) \). This is illustrated by the straight arrow in Figure 4. On the next round, \( I \) first makes the move \( x_1(1) \), and then \( \text{II} \) responds with \( x_0(1) \) as illustrated by the arrow again. This two-round cycle of moves then repeats. We think of the moves \( x_0(0) \) and \( x_1(0) \) as being offensive moves made by the players; they are playing as the \( V^\omega \) player on their opponent's board. Likewise, we likewise think of the moves \( x_0(1) \) and \( x_1(1) \) as defensive moves; they are playing as the \( \exists^\omega \) player on their home boards (in this description, we view \( \text{II} \) as responsible for the board corresponding to \( s^\omega \cdot a \). \( \text{II} \) is trying to get \( z_0 = s^\omega \cdot a \cdot x_0 \) to be "more in the set" \( A \) than the real \( z_1 = s^\omega \cdot b \cdot x_1 \) which \( I \) is responsible for). At the end of the rounds, we have reals \( z_0 = s^\omega \cdot a \cdot x_0 \) and \( z_1 = s^\omega \cdot b \cdot x_1 \). \( \text{II} \) then wins the run of the game if \( z_0 \lesssim_{\varphi_n} z_1 \), where \( n = |s| \).

**Claim.** For each \( s \) which is a winning position for \( I \) (with \( I \) to move) in the game \( G_A \), let \( W_s \) be the set of \( a \) such that \( s^\omega \cdot a \) is still a winning position for \( I \) (with now \( \text{II} \) to move). Then the relation on \( W_s \) defined by \( a \lesssim_s b \) iff \( \text{II} \) has a winning strategy in the game \( G_s(a, b) \) is a prewellordering on the set \( W_s \).

We prove the claim shortly, but first we see how it defines a canonical winning strategy for \( I \) in \( G_A \). Let \( \sigma \) be the quasistrategy for \( I \) in \( G \) which plays by always playing moves which are minimal in the prewellorderings \( \lesssim_s \). That is, if \( s = (a(0), \ldots, a(2n - 1)) \) has followed \( \sigma \) so far, then \( a \in \sigma(s) \) iff \( a \in W_s \) is minimal in \( \lesssim_s \). We show this is a winning quasistrategy for \( I \).

Fix a winning strategy \( \sigma' \) for \( I \) in \( G \). Let \( y = (y(0), y(1), \ldots) \) be a play according to \( \sigma \). We must show that \( y \in A \). Consider the diagram shown in Figure 5.

The moves \( y_0(2k) \) on the top row come from the strategy \( \sigma' \). Since \( y(0) \leq_0 y_0(0) \), we can fix a winning strategy \( \tau_0 \) in the game \( G_{\sigma'}(y(0), y_0(0)) \). The vertical arrows between the first and second rows of the diagram come from \( \tau_0 \). The move \( y(1) \) is copied as shown by the dashed arrow, and this then determines the moves \( y_0(1), y_0(2), \) and \( y_1(2) \). Note that \( (y(0), y(1), y(2)) \) and \( (y(0), y(1), y_1(2)) \) are both in \( W_{(y(0), y(1))} \). For \( (y(0), y(1), y(2)) \) this is because this is following \( \sigma \), which by
Figure 5. Showing the canonical strategy is winning.

Definition always stays inside the corresponding \( W_s \) sets. For \((y(0), y(1), y (2))\) this is because the pair \( s = (y(0), y(0)(1), y(0)(2))\), \( t = (y(0), y(1), y(2)(2))\) is according to \( \tau_0 \) (if \( t \) were not a winning position for \( \exists \) in the basic game \( G_t(A) \), then I could defeat \( \tau_0 \) starting from \( s, t \) by following a winning strategy for \( \forall \) in the second board; this would give \( y_1 \notin A \) which defeats \( \tau_0 \)). Since \( y(2) \leq (y(0), y(1)) \) \( y(2) \), we can fix a winning strategy \( \tau_1 \) for II in the game \( G_{(y(0), y(1))}(y(2), y(1)(2)) \). The vertical arrows between the second and third rows of the diagram come from \( \tau_1 \). Continuing, we fill-in the diagram.

Let \( y_0, y_1, \ldots \) be the reals produced in the diagram. Clearly \( y_m \to y \). Since each \( \tau_k \) is winning for II, \( y_{k+1} \in A \) for all \( k \). Also \( \varphi_{2k}(y_{k+1}) \leq \varphi_{2k}(y_k) \) for all \( k \). Since \( \varphi \) is very good, it follows that \( y \in A \).

Next we prove the claim. Fix \( s \) a winning position in \( G_A \) with I to move, and let \( W_s \) again denote the set of moves \( a \) such that I can win the basic game from \( s \uparrow a \). Consider the relation \( \leq_s \) on \( W_s \). The transitivity of \( \leq_s \) follows easily by composing strategies as in the first and second periodicity theorems.

We make the subclaim that given any sequence \( a_0, a_1, \ldots \) from \( W_s \), there is a \( k \) such that \( a_k \leq_s a_{k+1} \). Suppose not, and let \( \sigma_k \) be a winning strategy for I in the game \( G_s(a_k, a_{k+1}) \). Fix a winning strategy \( \sigma \) for I in the game \( G(A) \) starting from position \( s \uparrow a_0 \). We fill in the diagram as shown in Figure 6.

Each \( \sigma_k \) first makes the move \( y_k(0) \). This fills in the first column of the diagram. Each \( \sigma_k \) then moves (viewing II’s move as \( y_{k+1}(0) \) \( y_{k+1}(1) \)). This gives the second column except for the move \( y(0)(1) \) which comes from the strategy \( \sigma \) (indicated by the
relations associated to the prewellordering \( \leq \) on \( x \), \( I \) makes the first move \( G \). The vertical arrow from the \( \exists \) to \( y_0(1) \) is a winning strategy in \( G(A) \) having \( z_0 = s^\sim a_0 \sim y_0 \in A \). Since each \( \sigma_k \) is winning for \( I \) in \( G_s(a_k, a_k+1) \), we have that \( \neg(z_k \leq^*_s z_{k+1}) \) where \( z_k = s^\sim a_k \sim y_k \) and \( z_{k+1} = s^\sim a_{k+1} \sim y_{k+1} \). Since \( z_k \in A \) inductively, this says \( z_{k+1} \in A \). So, \( z_k \in A \) for all \( k \) and also \( \varphi_{\mid a}(z_{k+1}) < \varphi_{\mid a}(z_k) \) for all \( k \), a contradiction.

It follows that \( \leq_s \) is reflexive on \( W_s \) as otherwise the sequence \( a, a, a, \ldots \) would violate the above subclaim. Likewise, if \( a, b \in W_s \) then we must have \( a \leq_s b \) or \( b \leq_s a \) as otherwise \( a, b, a, b, a, b, \ldots \) violates the subclaim. The subclaim also immediately shows that \( \leq_s \) is wellfounded on \( W_s \). This proves the claim, that is, shows that \( \leq_s \) is a prewellordering on \( W_s \).

So far we have shown that the canonical quasistrategy for \( I \) in \( G_A \) is a winning quasistrategy. We next consider the complexity of this strategy. Let \( \leq^*_s, \prec^*_s \) be the relations associated to the prewellordering \( \leq_s \) on \( W_s \).

Let \( G'_s(a, b) \) be the game, similar to \( G_s(a, b) \), played as shown in Figure 7. Here \( I \) makes the first move \( x_1(0) \), and then \( II \) responds with \( x_0(0) \) (these are their “offensive” moves on their opponent’s boards; here we view \( I \)’s home board as the one corresponding to \( s^\sim a \) as shown). I then makes the move \( x_0(1) \) followed by \( II \)’s move of \( x_1(1) \) (these are their “defensive” moves). I wins the run of the game iff \( z_0 <^*_s z_1 \) where \( z_0 = s^\sim a^\sim x_0 \) and \( z_1 = s^\sim b^\sim x_1 \).

Claim. For all \( a, b \), we have \( a \leq^*_s b \) iff \( II \) has a winning strategy in the game \( G_s(a, b) \). Also, \( a <^*_s b \) iff \( I \) has a winning strategy in \( G'_s(a, b) \).

To prove the claim, first suppose that \( II \) has a winning strategy in \( G_s(a, b) \). We must have that \( a \in W_s \) (i.e., I can win \( G(A) \) starting from \( s^\sim a \)) as otherwise \( II \) can make the moves \( x_0(0), x_0(2), x_0(4), \ldots \) in \( G_s(a, b) \) according to a winning strategy for \( \forall \) in \( G(A) \). This would result in a play of \( G_s(a, b) \) where \( z_0 \notin A \), where
Thus, \( \neg(z_0 \leq^* \tau_1) \), contradicting that II has won \( G_\tau(a,b) \). So, if \( a \in W_\tau \) then \( a \leq^* b \) holds by definition. If \( b \in W_\tau \) as well, then by definition we have \( a \leq b \), and so \( a \leq^* b \) holds.

Next suppose that \( a \leq^* b \). By definition we have \( a \in W_\tau \). If \( b \notin W_\tau \) then II can win \( G_\tau(a,b) \) as follows: II plays (makes the moves \( x_1(2k) \)) in the top board (the \( s^*b \) board) to ensure \( z_1 \notin A \), which is possible since \( b \notin W_\tau \). II plays (makes the moves \( x_0(2k+1) \)) in the bottom board to ensure \( z_0 \in A \), which is possible since \( a \in W_\tau \). We then have \( z_0 \leq^* z_1 \). If \( b \in W_\tau \) as well, then by definition \( a \leq b \), that is, II wins \( G_\tau(a,b) \).

Now we consider the relation \( <^* \). First assume that I has a winning strategy in the game \( G'_\tau(a,b) \) (we refer to Figure 7). We must have \( a \in W_\tau \) as otherwise II can make the moves \( x_0(2k) \) according to a strategy to ensure \( z_0 \notin A \). This will give \( \neg(z_0 <^*_\tau z_1) \), contradicting I winning the game. So, \( a \in W_\tau \). If \( b \notin W_\tau \) then \( a <^* b \) holds by definition. If \( b \in W_\tau \) as well, then we must show that \( a \leq b \) and \( \neg(b \leq a) \) that is, II wins \( G_\tau(a,b) \) and II doesn’t win \( G_\tau(b,a) \). A winning strategy \( \sigma \) for I in \( G'_\tau(a,b) \) immediately gives a strategy for II in \( G_\tau(a,b) \), as II just uses the same strategy \( \sigma \), but ignores the initial moves of I each round of the game. If II also won \( G_\tau(b,a) \) by \( \tau \), then we have a contradiction by playing \( \sigma \) and \( \tau \) against each other. Inspection of Figures 4, 7 (where in Figure 4 we switch \( a \) and \( b \) now) shows that this makes sense. We then have \( z_0 <^*_\tau z_1 \) as \( \sigma \) is winning, and \( z_1 \leq^* z_0 \) as \( \tau \) is winning, a contradiction. So, \( a <^* b \) holds in all cases.

Finally, assume \( a <^* b \). In particular, \( a \in W_\tau \). We must show that I wins \( G'_\tau(a,b) \). If \( b \notin W_\tau \) then I can win \( G'_\tau(a,b) \) by making the moves \( x_1(2k) \) (of Figure 7) according to a strategy to ensure \( z_1 \notin A \), and making the moves \( x_0(2k+1) \) according to a strategy to ensure \( z_0 \in A \). So, \( a \in W_\tau \). Assume towards a contradiction that II wins \( G'_\tau(a,b) \) by \( \tau \). Since \( \neg(z_1 \leq^* z_2) \), we have that I wins \( G_\tau(a,b) \), say by \( \sigma \). Fix also a strategy \( \sigma' \) for I in the game \( G_\tau(a,b) \). We then fill in the diagram as shown in Figure 8.

In the first column, the moves \( y_{2k+1}(0) \) are then made by \( \sigma \). The moves \( y_{2k}(0) \) are then made by \( \tau \) is response. For the second column, all the moves \( y_{2k}(1) \) are made by \( \sigma \) except for \( y_0(1) \) which is made by \( \sigma' \). The \( y_{2k+1}(1) \) are then made all by \( \tau \) in response. This cycle of filling in the moves then repeats. Let \( z_{2k} = s^* a \sim y_{2k} \) and \( z_{2k+1} = s^* b \sim y_{2k+1} \). As \( \sigma' \) is winning for \( G_\tau(a,b) \) we have \( z_0 \in A \). It then follows that all of the \( z_i \) are in \( A \) and \( \varphi_s(z_0) \geq \varphi_s(z_1) \geq \varphi_s(z_2) \geq \varphi_s(z_3) \cdots \), a contradiction.

It follows immediately from the claim that all of the norms \( \psi_\tau \) on \( W_\tau \) are \( \exists \Gamma \) norms. An easy computation now shows that the best winning strategy \( \sigma \) for \( G(A) \) is a \( \exists \Gamma \)-real, where we define

\[
\sigma(s) = a \leftrightarrow [(\forall b \ a \leq^* b) \land \forall b < a \ (a <^* b)]
\]
As $\mathcal{O}\Gamma$ is easily closed under $\land, \lor$, we have that $\sigma$ is a $\mathcal{O}\Gamma$ strategy.

This completes the proof of the third periodicity theorem.  \hfill $\square$