The c.u.b. Filter and Silver's Theorem

1. Ideals and Filters

We first recall the standard notions of ideal and filter.

Definition 1.1. An *ideal* on a set X is a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ of subsets of X satisfying:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

We say the ideal $\mathcal{I}$ is proper if $X \notin \mathcal{I}$ (equivalently $\mathcal{I} \neq \mathcal{P}(X)$).

We think of an ideal as a notion of smallness for the subsets of X; those subsets of X which are in $\mathcal{I}$ are the small ones.

The "dual" notion is the concept of a filter:

Definition 1.2. A *filter* on a set X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of X satisfying:

(1) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

We say the filter $\mathcal{F}$ is proper if $\emptyset \notin \mathcal{F}$ (equivalently $\mathcal{F} \neq \mathcal{P}(X)$).

Recall than an *ultrafilter* on a set X is a maximal filter. Equivalently, an ultrafilter is a filter $\mathcal{F}$ with the property that for every $A \in \mathcal{P}(X)$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$. It is a standard fact that from the axiom of choice one may extend any filter on a set X to an ultrafilter.

Exercise 1. Show that $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal iff $\mathcal{F} = \{A \colon X - A \in \mathcal{I}\}$ is a filter.

We say an ideal $\mathcal{I}$ is κ-additive if whenever $\alpha < \kappa$ and $\{A_\beta\}_{\beta<\alpha}$ is an α sequence of members of $\mathcal{I}$, then $\bigcup_{\beta<\alpha} A_\beta \in \mathcal{I}$. The dual notion would be: a filter $\mathcal{F}$ is κ-additive if whenever $\alpha < \kappa$ and $\{A_\beta\}_{\beta<\alpha}$ is an α sequence of members of $\mathcal{F}$, then $\bigcap_{\beta<\alpha} A_\beta \in \mathcal{I}$. Note that κ-additive refers to closure under *less than* κ unions (or intersections).

The notions of ideal and filter are thus interchangeable, and we will pass back and forth between the two. For $\mathcal{I}$ an ideal (or $\mathcal{F}$ a filter), we sometimes call the sets $A \in \mathcal{I}$ (or sets A such that $X - A \in \mathcal{F}$) "measure zero." We call the A such that $X - A \in \mathcal{I}$ (or $A \in \mathcal{F}$) "measure one." If neither $A \in \mathcal{I}$ nor $X - A \in \mathcal{I}$, we say A is "positive."

Exercise 2. Show that for any ideal (or filter) there is a largest $\lambda \in \mathrm{CARD}$ such that $\mathcal{I}$ is λ-additive. We call this the *additivity* of the ideal (or filter).

Exercise 3. Let κ be a cardinal and let $\mathcal{I}$ be the ideal of subsets of κ which have size $< \kappa$. Identify the additivity of this ideal.

If $\mathcal{I}$ is an ideal (or filter) on a set X, an *antichain* is a collection $\{A_\alpha\}$ of $\mathcal{I}$-positive subsets of X such that $A_\alpha \cap A_\beta \in \mathcal{I}$ for all $\alpha \neq \beta$. We say the ideal is λ-saturated if all anti-chains have size $< \lambda$. The saturation of the ideal, $\mathrm{sat}(\mathcal{I})$ is the largest λ such that $\mathcal{I}$ is λ-saturated (which is easily well-defined).

If $\mathcal{I}$ is an ideal (or $\mathcal{F}$ a filter) on a set X, and $S \subseteq X$ is positive, then define the notion of the ideal (or filter) restricted to S, which we denote by $\mathcal{I}_{|S}$ (or $\mathcal{F}_{|S}$), and defined by $\mathcal{I}_{|S} = (\mathcal{I} \cap \mathcal{P}(S)) \cup (X - S)$ (that is we declare complement of S to be in the restricted ideal, i.e, the restricted ideal "lives" on S). Equivalently, $\mathcal{F}_{|S} = \{A \cap S \colon A \in \mathcal{F}\}$.

2. Boolean Algebras

Definition 2.1. A Boolean algebra is a set $\mathcal{B}$ with two distinguished elements 0 and 1 and two binary operations $+$, $\cdot$, and one unary operations $A \mapsto \overline{A}$. The axioms are:

(commutative laws) $a + b = b + a$, $a \cdot b = b \cdot a$.
(associative laws) $a + (b + c) = (a + b) + c$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(distributive laws) $a \cdot (b + c) = a \cdot b + a \cdot c$, $a + (b \cdot c) = (a + b) \cdot (a + c)$.
(identity laws) $a + a = a$, $a \cdot a = a$.
(de Morgan's laws) $\overline{a + b} = \overline{a} \cdot \overline{b}$, $\overline{a \cdot b} = \overline{a} + \overline{b}$.
(negation laws) $a + \overline{a} = 1$, $a \cdot \overline{a} = 0$.
(0, 1 laws) $0 + a = a$, $0 \cdot a = 0$, $1 + a = 1$, $1 \cdot a = a$.

In analogy with set operations, we sometimes write $\vee$ for $+$ and $\wedge$ for $\cdot$ in a Boolean algebra. We also sometimes write a^c for $\overline{a}$. The axioms imply all of the usual set identities.

Exercise 4. Show that in any Boolean algebra $a = \overline{\overline{a}}$. Show that $a + a \cdot b = a$ and $a \cdot (a + b) = a$. Show that $a \cdot b = a$ iff $a + b = b$ iff $a \cdot (\overline{b}) = 0$.

We write $a \leq b$ in a Boolean algebra to denote $a \cdot b = a$, or equivalently, $a + b = b$. We also write $a - b$ for $a \cdot (\overline{b})$. We have $a \leq b$ iff $\overline{b} \leq \overline{a}$.

The concepts of ideal, filter, ultrafilter generalize naturally from $\mathcal{P}(X)$ to any Boolean algebra.

Definition 2.2. An ideal on the Boolean algebra $\mathcal{B}$ is a collection $\mathcal{I} \subseteq \mathcal{B}$ satisfying:

1. If $a \in \mathcal{I}$, and $b \leq a$ then $b \in sI$.
2. If a, b are in $\mathcal{I}$, then $a + b \in \mathcal{I}$.

The ideal $\mathcal{I}$ is proper if $1 \notin \mathcal{I}$.

A filter on the Boolean algebra $\mathcal{B}$ is a collection $\mathcal{F} \subseteq \mathcal{B}$ satisfying:

1. If $a \in \mathcal{F}$, and $a \leq b$ then $b \in sF$.
2. If a, b are in $\mathcal{F}$, then $a \cdot b \in \mathcal{F}$.

The filter $\mathcal{F}$ is proper if $0 \notin \mathcal{F}$. An ultrafilter on $\mathcal{B}$ is a maximal filter.

It is straightforward to check that a filter $\mathcal{F}$ on a Boolean algebra $\mathcal{B}$ is an ultrafilter iff for ever $a \in \mathcal{B}$ either $a \in \mathcal{F}$ or $\overline{a} \in \mathcal{F}$. With AC, every filter on a Boolean algebra can be extended to an ultrafilter (the proof is the same as that for filters on $\mathcal{P}(X)$.

If X is any set, then all $\mathcal{B} \subseteq \mathcal{P}(X)$ which contains $\emptyset$, X, and is closed under finite unions, finite intersections, and complements is a Boolean algebra under the operations of union, intersection, and complement. We call such a $\mathcal{B}$ an algebra of subsets of X. Conversely, Stone's theorem says any boolean algebra is isomorphic to an algebra of subsets of some set X:

Theorem 2.3. *(ZFC) Every Boolean algebra is isomorphic to an algebra of subsets of some set X.*

Proof. Let $\mathcal{B}$ be a Boolean algebra. Let $X = \{u \colon u \text{ is an ultrafilter on } \mathcal{B}\}$. Define $\pi \colon \mathcal{B} \to \mathcal{P}(X)$ by $\pi(a) = \{u \in X \colon a \in u\}$. Let $S = \operatorname{ran}(\pi)$. We claim that π is a Boolean algebra isomorphism between $\mathcal{B}$ and the algebra of sets S. Clearly $\pi(0) = \emptyset$ and $\pi(1) = X$. That S is a field of sets and π is a homomorphism (i.e., preserves the Boolean operations) follows from the equations: $\pi(a \cdot b) = \pi(a) \cap \pi(b)$, $\pi(a + b) = \pi(a) \cup \pi(b)$, $\pi(\overline{a}) = X - \pi(a)$. For example, the first equation says $a \cdot b$ is in an ultrafilter iff a and b are. It is immediate from the definition that this in fact holds for all filters. For the second equation, note that $\pi(a), \pi(b) \subseteq \pi(a + b)$ as $a, b \leq a + b$. If $u \in \pi(a + b)$ but $u \notin \pi(a)$ and $u \notin \pi(b)$, then since u is an ultrafilter, $u \in \pi(\overline{a})$ and $u \in \pi(\overline{b})$. Since u is a filter, $u \in \pi(\overline{a} \cdot \overline{b})$, and so $u \in \pi((\overline{a} \cdot \overline{b}) \cdot (a + b)) = \pi(0) = \emptyset$. The third equation follows from the fact that any ultrafilter must contain either a or $\overline{a}$. It remains to show that π is one-to-one. Suppose $a \neq b$. Without loss of generality $a \nleq b$ (since if $a \leq b$ and $b \leq a$ then $a = a \cdot b = b$). So, $a - b \neq 0$. Let u be an ultrafilter on $\mathcal{B}$ with $a - b \in u$. Then $u \in \pi(a)$ but $u \notin \pi(b)$. $\square$

Definition 2.4. A Boolean algebra $\mathcal{B}$ is *complete* is for every $A \subseteq \mathcal{B}$, a least upper bound, l.u.b.(A) for the elements of A under $\leq$ exists. Equivalently, for every A the greatest lower bound g.l.b.(A) exists. We also write $\Sigma(A)$, $\sup(A)$ for l.u.b.(A) and $\Pi(A)$, $\inf(A)$ for g.l.b.(A). A Boolean algebra is said to be κ-complete if $\Sigma(A)$, $\Pi(A)$ exists for all A of size $< \kappa$.

For example, $\mathcal{P}(X)$ is a complete Boolean algebra. On the other hand, $\mathcal{P}(\omega)/\text{FIN}$ is not complete, where FIN denotes the ideal of finite subsets of ω. To see this, let A_n be disjoint infinite subsets of ω whose union is ω. Then $\{[A_n]\}_{n \in \omega}$ does not have a least upper bound.

The notions of κ-additive and κ-saturated generalize from ideals on a set X (i.e., the Boolean algebra $\mathcal{P}(X)$) to arbitrary Boolean algebras:

Definition 2.5. An ideal $\mathcal{I}$ on a κ-complete Boolean algebra $\mathcal{B}$ is said to be κ-additive if $\Sigma(A) \in \mathcal{I}$ whenever $A \subseteq \mathcal{I}$ and $|A| < \kappa$. A Boolean algebra is κ-saturated if every antichain in $\mathcal{B}$ has size $< \kappa$. $\operatorname{sat}(\mathcal{B})$ is the largest κ such that $\mathcal{B}$ is κ-saturated.

Thus, an ideall $\mathcal{I}$ on κ is λ-saturated iff the Boolean algebra $\mathcal{P}(\kappa)/\mathcal{I}$ is λ-saturated.

We will be mainly interested in complete Boolean algebras. For complete Boolean algebras it is a theorem that $\operatorname{sat}(\mathcal{B})$ is a regular cardinal.

3. The c.u.b. Filter

We introduce now a specific filter of basic importance called the c.u.b. filter (the corresponding ideal is called the non-stationary ideal).

Definition 3.1. If $A \subseteq \text{ON}$, then the *closure* of A, $\overline{A}$, is the set of all $\alpha \in \text{ON}$ such that $\forall \beta < \alpha\ \exists \gamma\ (\beta < \gamma \leq \alpha \wedge \gamma \in A)$. We cay A is *closed* if $A = \overline{A}$.

It is easy to see that $\overline{A}$ consists of A together with the ordinals α which are limit points of A, that is, α is a limit ordinal and A is unbounded in α. Thus, A is closed iff it contains all its limit points.

The topological terminology is justified. We can put a topology on the ordinals (order topology) by defining the basic open sets to be of the form $(\alpha, \beta) = \{\gamma \colon \alpha <$

$\gamma < \beta\}$ (together with $\{0\}$ since 0 is the least element in the ordering of ordinals). It is easily checked that is a base for a topology (in fact, this is true for any linear ordering on any set). A neighborhood base at α consists of sets of the form $(\beta, \alpha]$, where $\beta < \alpha$. In this topology, the closure operation defined above is just topological closure in the order topology.

Definition 3.2. Let α be a limit ordinal. We say $C \subseteq \alpha$ is c.u.b. is C is closed and unbounded in α. If $\mathrm{cof}(\alpha) > \omega$, then the c.u.b. filter on α, $\mathrm{Cub}(\alpha)$ is defined to be the collection of subsets of α which contain a c.u.b. set. The corresponding ideal is denoted $\mathrm{NS}(\alpha)$; the ideal of non-stationary subsets of α (terminology explained below).

Exercise 5. Show that if $\mathrm{cof}(\alpha) = \omega$ then $\mathrm{Cub}(\alpha)$ is not a filter.

We let $\mathcal{P}(\kappa)/\mathrm{NS}$ denote the set of equivalence classes $[A]$, for $A \in \mathcal{P}(\kappa)$, under the equivalence relation $A \sim B$ iff $A \triangle B \in \mathrm{NS}(\kappa)$. In fact, for any ideal $\mathcal{I}$ on κ we may consider $\mathcal{P}(\kappa)/\mathcal{I}$. This forms a Boolean algebra.

Lemma 3.3. *Suppose* $\mathrm{cof}(\alpha) > \omega$. *Then Cub($\alpha$) is a filter, and is* $\mathrm{cof}(\alpha)$*-additive. In fact, the intersection of* $< \mathrm{cof}(\alpha)$ *many c.u.b. subsets of* α *is c.u.b..*

Proof. By definition if $A \in \mathrm{Cub}(\alpha)$ and $B \supseteq A$ then $B \in \mathrm{Cub}(\alpha)$. Suppose $\delta < \mathrm{cof}(\alpha)$ and $\{A_\beta\}_{\beta<\delta}$ is a sequence of sets in $\mathrm{Cub}(\alpha)$. Using AC, we may assume that all of the A_β are actually c.u.b. subsets of α, and show their intersection is c.u.b.. Clearly $\bigcap_{\beta<\delta} A_\beta$ is closed. We must show it is unbounded in α. For each $\beta < \delta$, let $f_\beta \colon \alpha \to \alpha$ be given by $f_\beta(\gamma) =$ the least element of A_β which is $> \gamma$. Fix $\eta < \alpha$. Let $\eta_0 = \eta$, and let $\eta_{n+1} = \sup_{\beta<\delta} f_\beta(\eta_n)$. Note that $\eta_{n+1} < \alpha$ as $\mathrm{cof}(\alpha) > \delta$. Since also $\mathrm{cof}(\alpha) > \omega$, $\eta_\omega = \sup_n(\eta_n) < \alpha$. Each A_β is unbounded in η_ω (as there is a point of A_β between η_n and η_{n+1} for any n), and thus $\eta_\omega \in A_\beta$ for all $\beta < \delta$. $\square$

In discussing the c.u.b. filter $\mathrm{Cub}(\alpha)$, there is actually no loss of generality is assuming α is a regular cardinal. For assume $\mathrm{cof}(\alpha) = \kappa$. Let $\{\gamma_\eta\}_{\eta<\kappa}$ be a continuous, increasing, cofinal sequence in α. By continuous we mean that for η limit that $\gamma_\eta = \sup_{\eta'<\eta} \gamma_{\eta'}$. Let $C = \{\gamma_\eta \colon \eta < \kappa\}$. Then C is c.u.b. in α. The map $A \mapsto A' = \{\gamma_\eta \colon \eta \in A\}$ is a bijection between $\mathcal{P}(\kappa)$ and $\mathcal{P}(C)$ which preserves the notion of c.u.b. since the γ_η are continuous. Thus we have an isomorphism between $\mathcal{P}(\kappa)/\mathrm{Cub}$ and $\mathcal{P}(C)/\mathrm{Cub}$. Finally, the map $A \mapsto A \cap C$ is a Boolean algebra isomprphism between $\mathcal{P}(\alpha)/\mathrm{Cub}$ and $\mathcal{P}(C)/\mathrm{Cub}$ (the map is one-to-one since C is c.u.b.). Thus, $\mathcal{P}(\kappa)/\mathrm{Cub} \cong \mathcal{P}(\alpha)/\mathrm{Cub}$ as Boolean algebras. Thus, as far as discussions concerning the c.u.b. filter are concerned, we may replace α by the set C of size $\kappa = \mathrm{cof}(\alpha)$.

Definition 3.4. Let κ be a cardinal and $A_\alpha \subseteq \kappa$ for $\alpha < \kappa$. The *diagonal intersection* of the A_α is defined by $\triangledown A_\alpha = \{\beta < \kappa \colon \forall \alpha < \beta\ (\beta \in A_\alpha\}$. The *diagonal union* is defined by $\triangle A_\alpha = \{\beta < \kappa \colon \exists \alpha < \beta\ (\beta \in A_\alpha)\}$.

Definition 3.5. A filter $\mathcal{F}$ (or ideal $\mathcal{I}$) is said to be *normal* if whenever A_α, $\alpha < \kappa$, are in $\mathcal{F}$ (or $\mathcal{I}$), then $\triangledown A_\alpha \in \mathcal{F}$ (resp. $\triangle A_\alpha \in \mathcal{I}$).

Lemma 3.6. *For every regular cardinal* κ*, the filter Cub(κ) is normal.*

Proof. Assume $A_\alpha \in \mathrm{c.u.b.}(\kappa)$ for all $\alpha < \kappa$. Using AC, let $C_\alpha \subseteq A_\alpha$ be c.u.b.. It suffices to show that $\triangledown C_\alpha$ is c.u.b. in κ. The diagonal intersection is easily closed, we show it is also unbounded. For each $\alpha < \kappa$, let $f_\alpha \colon \kappa \to \kappa$ be given by $f_\alpha(\eta) =$ least element of C_α greater than η. Let $\eta_0 < \kappa$. Define $\eta_{n+1} = \sup_{\alpha<\eta_n} f_\alpha(\eta_n)$. Let $\eta_\omega = \sup_n \eta_n$. Note that if $\alpha < \eta_\omega$, then for all n such that $\eta_n > \alpha$, there is a point of C_α between η_n and η_{n+1}, and hence $\eta_\omega \in C_\alpha$. Thus, $\eta_\omega \in \nabla A_\alpha$. □

An immediate but important consequence of this lemma is Fodor's theorem. To state it, we introduce the important notion of stationarity.

Definition 3.7. Let κ be a regular cardinal. Then $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every c.u.b. $C \subseteq \kappa$.

Note that S being stationary is just saying that S is positive with respect to the Cub filter on κ. That is, S is not in the corresponding ideal (which is why we called this ideal the non-statioary ideal).

Theorem 3.8. *(Fodor's Theorem) Let κ be a regular cardinal, $S \subseteq \kappa$ be stationary, and $f \colon S \to \kappa$ be* pressing down*, that is, $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary set $S' \subseteq S$ on which f is constant.*

Proof. If not, then for all $\alpha \in S$ there is a set $A_\alpha \in \mathrm{Cub}(\kappa)$ such that $f(\beta) \neq \alpha$ for all $\beta \in A_\alpha \cap S$. From lemma 3.6, $\nabla A_\alpha \in \mathrm{Cub}(\kappa)$ (for $\alpha \notin S$ we may take $A_\alpha = \kappa$), and thus there is some $\beta \in (\nabla A_\alpha) \cap S$ as S is stationary. Then $f(\alpha) < \alpha$ and so $\alpha \in A_{f(\alpha)} \cap S$, a contradiction to the definition of $A_{f(\alpha)}$. □

Exercise 6. Let κ be regular and $f_\alpha \colon \kappa \to \kappa$ for all $\alpha < \kappa$. Show that $C = \{\beta < \kappa \colon \forall \alpha < \beta\ (\beta \text{ is closed under } f_\alpha)\}$ is c.u.b. in κ.

If $\lambda < \kappa$ are regular cardinals, then $S^\kappa_\lambda = \{\alpha < \kappa \colon \mathrm{cof}(\alpha) = \lambda\}$ is stationary in κ. For example, for $\kappa = \aleph_2$ this gives two disjoint stationary subsets of $\aleph_2$, namely S_ω and S_{ω_1}. We will show now more generally that any stationary subset $S \subseteq \kappa$ of a regular cardinal κ can be split into κ many disjoint stationary subsets. For successor κ this is due to Ulam, and for limit κ to Solovay.

We consider first the successor case and prove a slightly more general result.

Theorem 3.9. *(Ulam) Let κ be a successor cardinal and $\mathcal{I}$ a κ-additive ideal on κ containing all the singletons. Then there is a κ size family of pairwise disjoint $\mathcal{I}$-positive subsets of κ.*

Proof. Let $\kappa = \lambda^+$. For each $\rho < \kappa$ let $f_\rho \colon \lambda \to \kappa$ be a bijection. For each $\alpha < \lambda$ and $\beta < \kappa$ let $X^\alpha_\beta = \{\rho > \beta \colon f_\rho(\alpha) = \beta\}$. For each $\beta < \kappa$ there is an $\alpha(\beta) < \lambda$ such that $X^{\alpha(\beta)}_\beta \notin \mathcal{I}$ since $\mathcal{I}$ is κ-additive and $\bigcup_{\alpha<\lambda} X^\alpha_\beta = \kappa - (\beta+1)$, which is not in $\mathcal{I}$. For some $\alpha_0 < \lambda$ we must have $|\{\beta \colon \alpha(\beta) = \alpha_0\}| = \kappa$. If $S = \{\beta \colon \alpha(\beta) = \alpha_0\}$, then for $\beta_1 \neq \beta_2 \in S$ we have $X^{\alpha_0}_{\beta_1} \cap X^{\alpha_0}_{\beta_2} = \emptyset$. □

Corollary 3.10. *If κ is a successor cardinal and $S \subseteq \kappa$ is stationary, then S can be split into κ many paiwise disjoint stationary subsets.*

Proof. Consider $\mathcal{I}_{|S}$, the non-stationary ideal restricted to S. This is a κ-additive, proper ideal containing all the singletons. From theorem 3.9, let A_α, $\alpha < \kappa$, be a κ sequence of pairwise disjoint $\mathcal{I}$-positive subsets of κ. Then $A'_\alpha = A_\alpha \cap S$ form a κ-sequence of pairwise disjoint $\mathcal{I}$-positive subsets of S. We can enlarge one, if necessary, so they union to S. □

If κ is a regular cardinal, and $S \subseteq \kappa$ is stationary, we define the set of *thin points* $\tilde{s} \subseteq S$ by $\alpha \in \tilde{S}$ iff $S \cap \alpha$ is not stationary in α.

Lemma 3.11. *Let κ be regular and $S \subseteq \kappa$ be stationary and consist of limit ordinals. Then $\tilde{S}$ is stationary.*

Proof. Let $C \subseteq \kappa$ be c.u.b.. Let α be the least limit point of C which is in S (which exists as C' is also c.u.b.). If $\mathrm{cof}(\alpha) > \omega$, then $C' \cap \alpha$ is c.u.b. in α and is disjoint from S, and so $\alpha \in S'$. If $\mathrm{cof}(\alpha) = \omega$, then there is an ω sequence of successor ordinals cofinal in α, which also gives a c.u.b. subset of α missing S. □

We now prove the limit case of theorem 3.10. Actually, the proof (due to Solovay) works for both limit and successor cardinals, and provides a different proof of theorem 3.9.

Theorem 3.12. *Let κ be a regular cardinal. Then every stationary $S \subseteq \kappa$ can be split into κ many pairwise disjoint stationary subsets.*

Proof. Let κ be regular, and $S \subseteq \kappa$ be stationary. Without loss of generality we may assume S consists of limit ordinals. Let $\tilde{S}$ be the thin points of S. For each $\alpha \in \tilde{S}$, let η^α_ξ, $\xi < \mathrm{cof}(\alpha)$, be an increasing continuous sequence with supremum α which misses S. We claim that there is ξ such that for all $\delta < \kappa$ the set $\{\alpha \in \tilde{S} \colon \eta^\alpha_\xi > \delta\}$ is stationary. If not, then for each ξ there is a $\rho(\xi) < \kappa$ and a c.u.b. set C_ξ such that for all $\alpha \in \tilde{S} \cap C_\xi$ we have $\eta^\alpha_\xi < \rho(\xi)$ (if η^α_ξ is defined). Let $C = \nabla C_\xi$, and let $D \subseteq C$ be c.u.b. and closed under the function $\xi \mapsto \rho(\xi)$. Since $\tilde{S}$ is stationary, let $\alpha < \beta$ be two elements of $\tilde{S} \cap D$. Then for each $\xi \in \mathrm{cof}(\beta) \cap \alpha$ we have $\eta^\beta_\xi < \alpha$. This shows that $\mathrm{cof}(\beta) \geq \alpha$ and that η^β_α is defined and equal to α (since the sequence η^β_ξ is continuous). By definition of the η^β_ξ, this shows $\alpha \notin \tilde{S}$, a contradiction.

Fix now ξ as in the claim. Note that $\alpha \mapsto \eta^\alpha_\xi$ is pressing down. For each $\gamma < \kappa$, by the claim and Fodor's theorem there is a $\tau(\gamma) > \gamma$ and a stationary set $S_\gamma \subseteq \tilde{S}$ such that $\eta^\alpha_\xi = \tau(\gamma)$ for all $\alpha \in S_\gamma$. Since κ is regular, there is a κ size set $A \subseteq \kappa$ such that $\tau(\alpha) \neq \tau(\beta)$ for $\alpha \neq \beta \in A$. Then the sets $S_\delta = \{\alpha \in \tilde{S} \colon \eta^\alpha_\xi = \tau(\delta)\}$, for $\delta \in A$, are pairwise disjoint and stationary. □

4. Silver's Theorem

We prove a theorem of Silver which shows a significant restriction on the continuum function at singular cardinals of uncountable cofinality.

Theorem 4.1. *Let κ be a singular cardinal of uncountable cofinality. If the GCH hols below κ (i.e., $\forall \lambda < \kappa$ $(2^\lambda = \lambda^+)$), then it holds at κ as well.*

Proof. Let κ_α, $\alpha < \mathrm{cof}(\kappa)$ be an increasing, continuous sequence of cardinals cofinal in κ. For $A \subseteq \kappa$ consider the function f_A with domain $\mathrm{cof}(\kappa)$ where $F_A(\alpha) = A \cap \kappa_\alpha$. Since $2^{\kappa_\alpha} = \kappa_{\alpha+1}$, we may identify f_A with a function satisfying $f(\alpha) \in \kappa_{\alpha+1}$. Consider the collection $F = \{f_A \colon A \subseteq \kappa\}$ of all such functions. Note that this forms an *almost disjoint* family of functions, that is, if $A \neq B$ then $\exists \alpha < \mathrm{cof}(\kappa)$ $\forall \beta > \alpha$ $(f_A(\beta) \neq f_B(\beta))$.

Let $g \colon \mathrm{cof}(\kappa) \to \kappa$ with $g(\alpha) < \kappa_{\alpha+1}$ for all α. Let F_g denote those $f \in F$ such that $\{\alpha < \mathrm{cof}(\kappa) \colon f(\alpha) \leq g(\alpha)\}$ is stationary. We claim that for any such g, $|F_g| \leq \kappa$. To see this, let $\pi_\alpha \colon g(\alpha) + 1 \to \kappa_\alpha$ be a bijection. If $f \in F_g$ then there is a stationary set $S_f \subseteq \mathrm{cof}(\kappa)$ and an ordinal $\delta_f < \kappa$ such that for all $\alpha \in S_f$, $\pi_\alpha(f(\alpha)) < \delta_f$ (by Fodor's theorem). Let $h_f \colon S_f \to \delta_f$ be the function $h_f(\alpha) = \pi_\alpha(f(\alpha))$. The map $f \mapsto (S_f, \delta_f, h_f)$ is one-to-one on F since (S_f, δ_f, h_f)

determines f on S, which determines $f \in F$ as F is an almost disjoint family. There are at most $2^{\mathrm{cof}(\kappa)} < \kappa$ choices for S_f, κ many choices for δ_f, and $\sup_{\delta<\kappa} \delta^{\mathrm{cof}(\kappa)} < \kappa$ many choices for h_f. Thus, $|F_g| \leq \kappa$.

We now show that $|F| \leq \kappa^+$. We define a sequence $f_\alpha \in F$ recursively so that for $\beta < \alpha$ we have $\{\xi\colon f_\beta(\xi) < f_\alpha(\xi)\}$ is stationary. Assume f_β has been defined for $\beta < \alpha$. If for every $f \in F$ there is a $\beta < \alpha$ such that $\{\xi\colon f(\xi) \leq f_\beta(\xi)\}$ is stationary, then stop the construction. Otherwise, let $f_\alpha \in F$ be such that $\forall\beta < \alpha\ \{\xi\colon f_\beta(\xi) < f_\alpha(\xi)\} \in \mathrm{Cub}(\mathrm{cof}(\kappa))$. In particular, for all $\beta < \alpha$, $\{\xi\colon f_\beta(\xi) < f_\alpha(\xi)\}$ is stationary. f_{κ^+} cannot be defined by the claim. Thus, we end with a collection $\{f_\alpha\}_{\alpha<\lambda}$, where $\lambda \leq \kappa^+$. Every $f \in F$ is then in some F_{f_α}, and from the claim it follows that $|F| \leq \kappa \cdot \lambda \leq \kappa^+$. $\square$