

Forcing and the Continuum

1. CARDINAL ARITHMETIC

Our first goal is to show the independence of CH from ZFC. First consider the partial order for adding a single real number $x \in 2^\omega$. One way would be to take the partial order $\mathcal{P}_1 = \langle 2^{<\omega}, \leq \rangle$, that is, conditions are finite sequences of 0's and 1's, and $p \leq q$ iff p extends q . \mathcal{P}_1 can thus be viewed as the full binary tree. Alternatively, we could consider \mathcal{P}_2 which consists of all partial functions $p: \omega \rightarrow \{0, 1\}$ with finite support (that is, a finite domain), and again ordered by extension. There is clearly a dense embedding (namely inclusion) of \mathcal{P}_1 into \mathcal{P}_2 , and thus forcing with \mathcal{P}_1 is equivalent to forcing with \mathcal{P}_2 by lemma ???. It is for the moment a little more convenient to work with \mathcal{P}_2 . We will call this the Cohen forcing, and refer to a generic G for this forcing as a Cohen real. Technically a generic filter $G \subseteq \mathcal{P}_2$ is not a real, but can be identified with one. Namely, let $x(n) = i$ iff $\exists p \in G (p(n) = i)$. This defines a function as any two conditions in G are compatible, and this function has domain ω as $D_n = \{p: n \in \text{dom}(p)\}$ is dense for each $n \in \omega$. Clearly $M[G] = M[x]$, and we henceforth identify a generic with the corresponding real x . Although this produces a model $M[x]$ which satisfies $V \neq L$, and in fact $V \neq \text{HOD}$ (as we show later), adding a single real is not enough to change the value of the continuum function. To do this we must add more reals.

Let us work in a model M of (enough of) ZF, and let κ be a cardinal (i.e., $(\kappa \in \text{CARD})^M$). The natural generalization of \mathcal{P}_2 to add κ many reals would be the partial order $\text{FN}(\kappa \times \omega, 2) =$ all partial functions from $\kappa \times \omega$ to $\{0, 1\}$ with finite support. We order again by extension, that is $p \leq q$ iff $p \upharpoonright \text{dom}(q) = q$. Of course, a function $F: \kappa \times \omega \rightarrow 2$ can be identified with a κ sequence of functions $F_\alpha: \omega \rightarrow 2$, namely $F_\alpha(n) = F(\alpha, n)$. A filter G on $\text{FN}(\kappa \times \omega, 2)$ can again be identified with a partial function from $\kappa \times \omega$ to 2. If G is generic, then this function (which we will also call G) has domain $\kappa \times \omega$. This is because for each $\alpha < \kappa$, $n \in \omega$, the set $D_{\alpha, n} = \{p: (\alpha, n) \in \text{dom}(p)\}$ is dense. We summarize this discussion in the following lemma.

Lemma 1.1. *Let M be a transitive model of ZF, κ a cardinal of M , and G an M -generic for $\text{FN}(\kappa \times \omega, 2)^M$. Then G can be identified with a function $G: \kappa \times \omega \rightarrow 2$. If $\{G_\alpha\}_{\alpha < \kappa}$ is the corresponding sequence of reals, then $G_\alpha \neq G_\beta$ whenever $\alpha \neq \beta$.*

Proof. If $\alpha \neq \beta$, then $D_{\alpha, \beta} = \{p: \exists n \in \omega p(\alpha, n) \neq p(\beta, n)\}$ is dense (we mean here that both $p(\alpha, n)$, $p(\beta, n)$ are defined and not equal). \square

Note that $\text{FN}(\kappa \times \omega, 2)$ is isomorphic to $\text{FN}(\kappa, 2)$, by taking a bijection between $\kappa \times \omega$ and κ . Thus, we may work with $\text{FN}(\kappa, 2)$ which is notationally a little easier.

If G is generic for $\text{FN}(\kappa, 2)$, then from lemma 1.1 in $M[G]$ we have $2^\omega \geq \kappa$. So, if we take $\kappa = (\omega_2)^M$, then in $M[G]$ we have $2^\omega \geq (\omega_2)^M$. To get a model of $\neg\text{CH}$, we also need to know that $(\omega_2)^M = (\omega_2)^{M[G]}$. In fact, we will show that this forcing preserves all cardinals and cofinalities. There is a general principle involved which is useful in many arguments.

Definition 1.2. A partial order \mathcal{P} has the *countable chain condition*, or is said to be c.c.c. if every antichain $A \subseteq \mathcal{P}$ is countable. More generally, \mathcal{P} is said to be κ -c.c. if every antichain $A \subseteq \mathcal{P}$ has size $< \kappa$. The saturation of a partial order $\text{sat}(\mathcal{P})$, also called its cellularity, is the least κ such that \mathcal{P} has the κ -c.c.

Thus, c.c.c. is the same as ω_1 -c.c. This definition is also made for topological spaces: a topological space is κ -c.c. if every collection of pairwise disjoint open sets has size $< \kappa$. If we recall that every partial order \mathcal{P} can also be viewed as a topological space (with basic open sets of the form $N_p = \{q: q \leq p\}$), then the two definitions coincide.

Exercise 1. Show that for topological spaces 2^{nd} countable \Rightarrow separable \Rightarrow c.c.c.. Give examples to show that the implications are not reversible (hint: see the following exercise).

Exercise 2. Show that $\prod_{\alpha \in I} \mathbb{R}_{\text{std}}$ is c.c.c. (hint: see lemma ?? below).

The significance of the notion of κ -c.c. for forcing is explained in the following lemma.

Lemma 1.3. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$ a partial order which is wellorderable in M and with $(\mathcal{P}$ is κ -c.c.) M . Let G be M -generic for \mathcal{P} . Then if $A, B \in M$ and $f: A \rightarrow B$ is a function, $f \in M[G]$, then there is a function $F: A \rightarrow \mathcal{P}(B)$, $F \in M$, with $f(a) \in F(a)$ for all $a \in A$ and $(\forall a \in A |F(a)| < \kappa)^M$.*

Proof. Let $\tau \in M^{\mathcal{P}}$ and $p \Vdash (\tau \text{ is a function from } \check{A} \text{ to } \check{B})$. Since \mathcal{P} is wellorderable, working in M there is a function $H: A \rightarrow \mathcal{P}(P)$ satisfying:

- (1) $\forall a \in A H(a)$ is an antichain in P .
- (2) $\forall a \in A \forall p \in H(a) \exists b \in B (p \Vdash \tau(\check{a}) = \check{b})$.
- (3) $H(a)$ is maximal subject to (1) and (2).

Working still in M , let $F(a) = \{b \in B: \exists p \in H(a) (p \Vdash f(\check{a}) = \check{b})\}$. Since $|H(a)| < \kappa$, $|F(a)| < \kappa$ as well. If $f(a) = b \in B$, then for some $p \in P$, $p \Vdash (\tau(\check{a}) = \check{b})$. By (3), there is a $q \in H(a)$ such that $p \parallel q$. Let $r \leq p, q$. So, $r \Vdash (\tau(\check{a}) = \check{b})$. Since $q \in H(a)$, $q \Vdash (\tau(\check{a}) = \check{c})$ for some $c \in b$, and we must have $b = c$. So, $b \in H(a)$. \square

Definition 1.4. Let M be a transitive model of ZF and $\mathcal{P} \in M$. We say \mathcal{P} preserves cardinals if $(\kappa \in \text{CARD})^M \leftrightarrow (\kappa \in \text{CARD})^{M[G]}$. We say \mathcal{P} preserves cardinals $\leq \lambda$ (or $\geq \lambda$, $< \lambda$, etc.) if for all $\kappa \leq \lambda$, $(\kappa \in \text{CARD})^M \leftrightarrow (\kappa \in \text{CARD})^{M[G]}$. We say \mathcal{P} preserves cofinalities if for all $\alpha \in \text{ON} \cap M$, $(\text{cof}(\alpha))^M = (\text{cof}(\alpha))^{M[G]}$. We say \mathcal{P} preserves cofinalities $\leq \lambda$ (or $\geq \lambda$, etc.) if whenever $(\text{cof}(\alpha) \leq \lambda)^M$ then $\text{cof}(\alpha)^M = \text{cof}(\alpha)^{M[G]}$.

We note two simple facts.

Fact 1.5. If M is a transitive model of ZF and $\mathcal{P} \in M$, then \mathcal{P} preserves cofinalities iff for all $\kappa \in M$, $(\kappa \text{ is regular})^M \leftrightarrow (\kappa \text{ is regular})^{M[G]}$. Similarly, if $(\kappa \text{ is regular})^M \leftrightarrow (\kappa \text{ is regular})^{M[G]}$ for all $\kappa \leq \lambda$ (or $\kappa \geq \lambda$), then \mathcal{P} preserves cofinalities $\leq \lambda$ (or $\geq \lambda$).

Proof. We prove the first statement, the other being essentially the same. Suppose $(\text{cof}(\alpha) = \lambda)^M$, so $(\lambda \text{ is regular})^M$. Let $f: \lambda \rightarrow \alpha$, $f \in M$, be increasing and cofinal. By assumption, λ is regular in $M[G]$. From lemma ?? we have in $M[G]$ that $\text{cof}(\alpha) = \text{cof}(\lambda) = \lambda$. \square

Fact 1.6. If M is a transitive model of ZFC and $\mathcal{P} \in M$, then if \mathcal{P} preserves cofinalities then it preserves cardinals. Similarly, if \mathcal{P} preserves cofinalities $\leq \lambda$, then it preserves cardinals $\leq \lambda$. If \mathcal{P} preserves cardinals $\geq \lambda$ and $(\lambda \text{ is regular})^M$, then \mathcal{P} preserves cardinals $\geq \lambda$.

Proof. We prove by induction on the cardinals κ of M that they are cardinals of $M[G]$. For limit cardinals κ of M the inductive step is trivial (since a limit of cardinals is a cardinal). If κ is a successor cardinal of M , then since M satisfies ZFC, κ is regular in M . By assumption, κ is regular in $M[G]$, and hence is a cardinal of $M[G]$. \square

Theorem 1.7. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$ with $(\mathcal{P} \text{ is } \kappa\text{c.c.})^M$, where κ is a regular cardinal of M . Then \mathcal{P} preserves cardinals $\geq \kappa$ and preserves cofinalities $\geq \kappa$.*

Remark 1.8. Tarski's theorem 3.1 says that $\text{sat}(\mathcal{P})$ is always a regular cardinal (assuming ZFC or P is wellorderable). Thus, there is no loss of generality in assuming the κ of the lemma is a regular cardinal in M .

Proof. We first show \mathcal{P} preserves cardinals $\geq \kappa$. Let $\lambda \geq \kappa$ with $(\lambda \in \text{CARD})^M$. Suppose $(\lambda \notin \text{CARD})^{M[G]}$. Let $f: \alpha \rightarrow \lambda$ be onto, where $f \in M[G]$ and $\alpha < \lambda$. By lemma 1.3, there is a $F \in M$ with $F: \alpha \rightarrow \mathcal{P}(\lambda)$, $f(\beta) \in F(\beta)$ for all $\beta < \alpha$, and $(|F(\beta)| < \kappa)^M$ for all $\beta < \alpha$. If $\lambda = \kappa$, then F expresses, in M , κ as a union of α ($< \kappa$) union of sets each of which has size $< \kappa$, a contradiction to the regularity of κ in M . If $\lambda > \kappa$, then in M F gives a map from $\alpha \cdot \kappa$ onto λ , a contradiction as λ is a cardinal in M .

To see \mathcal{P} preserves cofinalities $\geq \kappa$, let $\lambda \geq \kappa$ with $(\lambda \text{ is a regular cardinal})^M$. If λ is not regular in $M[G]$, let $f \in M[G]$ with $f: \alpha \rightarrow \lambda$ cofinal, for some $\alpha < \lambda$. Let $F \in M$ be as in lemma 1.3. Since ρ is regular in M , each $F(\beta)$ is a bounded subset of λ . By regularity of ρ again, F has bounded range in λ , a contradiction as $f(\beta) \in F(\beta)$ and f is cofinal. \square

Corollary 1.9. *If M is a transitive model of ZF, $\mathcal{P} \in M$ and $(\mathcal{P} \text{ isc.c.})^M$, then \mathcal{P} preserves all cardinals and cofinalities.*

We now show that the Cohen poset for adding an arbitrary number of real, that is $\text{FN}(\kappa, 2)$, is c.c.c. The following combinatorial lemma embodies the heart of the proof. It is known as the Δ -system lemma.

Lemma 1.10. *Let $\alpha \in \text{ON}$ and A an uncountable collection of finite subsets of α . Then there is a "root" $r \in A^{<\omega}$ and an uncountable $B \subseteq A$ such that $r \subseteq s$ for all $s \in B$, and $\{s - r : s \in B\}$ are pairwise disjoint.*

Proof. Without loss of generality we may assume $|A| = \omega_1$ (note that A is wellorderable, so there is no problem discussing its cardinality). Replacing α with $\cup A$, we may assume $|\alpha| = \omega_1$, and by taking a bijection with ω_1 we may assume $\alpha = \omega_1$. Write each element of A in the form $s = (\alpha_0(s), \dots, \alpha_k(s))$ where $\alpha_0 < \dots < \alpha_k < \omega_1$. Thinning out A , we may assume that each $s \in A$ has size $k + 1$ for some fixed $k \in \omega$. For $l \leq k$, let

$$\beta_l = \sup\{\alpha_l : s = (\alpha_0, \dots, \alpha_l, \dots, \alpha_k) \in A\}.$$

Note that $\beta_0 \leq \beta_1 \leq \dots \leq \beta_k$. If $\beta_k < \omega_1$, then every $s \in A$ is a subset of the countable ordinal β_k . This is impossible as there are only countably many finite subsets of β_k . So, let $l \leq k$ be least such that $\beta_l = \omega_1$. Inductively choose $A_1 = \{t_\gamma\}_{\gamma < \omega_1} \subseteq A$ such that if $\gamma_1 < \gamma_2$ then $\alpha_k(t_{\gamma_1}) < \alpha_l(t_{\gamma_2})$. Since $\alpha_0(s), \dots, \alpha_{l-1}(s) < \beta_{l-1} < \omega_1$ for all $s \in A_1$, there is an uncountable $B \subseteq A_1$ and $\alpha_0 < \dots < \alpha_{l-1}$ such that $(\alpha_0(s), \dots, \alpha_{l-1}(s)) = (\alpha_0 < \dots < \alpha_{l-1})$ for all $s \in B$. Then B is as desired, with the root of the " Δ -system" being $r = \{\alpha_0, \dots, \alpha_{l-1}\}$. \square

Lemma 1.11. *For any κ , $FN(\kappa, 2)$ is c.c.c.*

Proof. Suppose $\{p_\alpha\}$ were an uncountable antichain. Let $s_\alpha = \text{dom}(p_\alpha)$. From lemma 1.10 we may assume the s_α form a Δ system with root r . However, there are only finitely many possibilities for $p \upharpoonright r$ and so we may get $\alpha, \beta < \omega_1$ with $p_\alpha \upharpoonright r = p_\beta \upharpoonright r$. Then p_α, p_β are compatible, a contradiction. \square

Summarizing, we have now show the following.

Theorem 1.12. *Let M be a transitive model of ZF anf $\mathcal{P} = FN(\omega_1^M, 2)$. Let G be M -generic for \mathcal{P} . Then $(2^\omega \geq \omega_2)^{M[G]}$.*

Proof. Let $\kappa = \omega_2^M$. Then in $M[G]$ there is a κ sequence of distinct reals. From lemma 1.10 and corollary 1.9 we have that $\omega_2^M = (\omega_2)^{M[G]}$. Thus (There is an ω_2 sequence of distinct reals) $^{M[G]}$. So, $(-\text{CH})^{M[G]}$. \square

From theorem 1.12 and our previous metamathematical remarks on forcing, we now have the following.

Corollary 1.13. *ZFC does not prove CH.*

Thus, the continuum hypothesis is not provable from within ZFC set theory.

Although Gödel's model L provides a model of CH (in fact GCH), we can also use forcing to get a model of CH. This also introduces another very useful forcing poset.

Definition 1.14. $\text{coll}(\omega, \kappa)$ is the poset of all partial functions $p: \omega \rightarrow \kappa$ with finite domain. The partial functions are ordered by $p \leq q$ iff p extends q . More generally, if $\lambda < \kappa$ are cardinals with λ regular, then $\text{coll}(\lambda, \kappa)$ is the poset of partial functions $p: \lambda \rightarrow \kappa$ with $|\text{dom}(p)| < \lambda$.

Easily, in any generic extension by $\text{coll}(\lambda, \kappa)$ we will have $|\lambda| = |\kappa|$. Thus, this forcing collapses κ to have cardinality λ . We would like to show, though, that the forcing does not add any bounded subsets of λ . This brings us to another important property of partial orders.

Definition 1.15. We say a poset \mathcal{P} is $\leq \lambda$ closed (or $< \lambda$ closed) is whenever $\alpha \leq \lambda$ (respectively, $\alpha < \lambda$) and $p_0 \geq p_1 \geq \dots \geq p_\beta \geq \dots$, $\beta < \alpha$, is an α -decreasing sequence in P , then there is a $p \in P$ with $p \leq p_\beta$ for all $\beta < \alpha$. We say P is $\leq \lambda$ (or $< \lambda$) distributive if whenever $\alpha \leq \lambda$ ($\alpha < \lambda$) and $\{D_\beta\}_{\beta < \alpha}$ are dense in P , then $\{p: \forall \beta < \alpha \exists q \in D_\beta (p \leq q)\}$ is dense.

The name distributive comes from the fact if P is a complete Boolean algebra, then the definition of $\leq \lambda$ distributive is equivalent to saying P satisfies the λ -length distributive law: $\prod_{\alpha < \lambda} \sum_{\beta \in I_\alpha} a_{\alpha, \beta} = \sum_{f \in \prod I_\alpha} \prod_{\alpha < \lambda} a_{\alpha, f(\alpha)}$. [brief sketch: let b the left-hand side, and c the right-hand side. Clearly $c \leq b$. Suppose $b - c \neq 0$. For each $\alpha < \lambda$ let $D_\alpha = \{p: (\forall \beta \in I_\alpha (p \perp a_{\alpha, \beta})) \vee (\exists \beta \in I_\alpha (p \leq a_{\alpha, \beta}))\}$. Each D_α is clearly dense. By assumption, $D = \{p: \forall \alpha < \lambda \exists q \in D_\alpha (p \leq q)\}$ is dense. Let $u \in D$ with $u \leq b - c$. Then either for all $\alpha < \lambda$ we have $u \leq a_{\alpha, \beta}$ for some $\beta \in I_\alpha$, or else for some α , p is incompatible with all elements $a_{\alpha, \beta}$. In the first case, this defines a function $f \in \prod I_\alpha$ such that $u \leq a_{\alpha, f(\alpha)}$ for all $\alpha < \lambda$, and hence $u \leq \prod_{\alpha < \lambda} a_{\alpha, f(\alpha)}$. This shows $u \leq c$, a contradiction. In the second case, $u \perp \sum_{\beta \in I_\alpha} a_{\alpha, \beta}$ for some α , and thus $u \perp b$, also a contradiction.]

It is easy to see that $\leq \lambda$ closed (or $< \lambda$ closed) implies $\leq \lambda$ (or $< \lambda$) distributive. To see this, let D_β , $\beta < \alpha$ be dense. Let $p \in P$. Define inductively for $\beta \leq \alpha$ an element p_β such that $p \geq p_0 \geq p_1 \geq \dots \geq p_\beta$ where $p_\beta \in D_\beta$. We can do this as P is $\leq \lambda$ closed, say. Then $p_\alpha \leq p$ and p_α extends elements in each of the D_β .

Exercise 3. Show that $\text{id } \mathcal{P}$ is $\leq \lambda$ distributive and D_α , $\alpha < \lambda$ are all dense below $p \in P$, then any generic filter containing p contains a $q \in G$ such that $\forall \alpha < \lambda \exists r \in D_\alpha (q \leq r)$.

In practice, most of the posets we see which are λ -distributive are actually λ -closed, but the proof of the main fact goes through for the notion of λ -distributive. This is expressed in the following lemma.

Lemma 1.16. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$ be $\leq \lambda$ distributive in M . Let G be M generic for \mathcal{P} . If $f: \lambda \rightarrow M$ is a function in $M[G]$, then $f \in M$.*

Proof. Let $f = \tau_G$. Let $q \Vdash (\tau \text{ is a function with domain } \check{\lambda})$, with $q \in G$. For each $\alpha < \lambda$, let $D_\alpha = \{p: \exists x \in M (p \Vdash \tau(\check{\alpha}) = \check{x})\}$. Clearly each D_α is dense below q . Thus there is a $r \in G$ which extends elements of each D_α . So, $\forall \alpha < \lambda \exists! x \in M (r \Vdash \tau(\check{\alpha}) = \check{x})$. This shows f is definable in M and so is in M (by replacement in $M[G]$ we may assume the given f has range in a set in M). \square

We have the following fact whose proof is immediate.

Lemma 1.17. *If λ is regular and $\lambda < \kappa$, then $\text{coll}(\lambda, \kappa)$ is $< \lambda$ -closed.*

Theorem 1.18. *Let M be a transitive model of ZFC, and $\mathcal{P} = (\text{coll}(\omega_1, 2^\omega))^M$. Then for any G which is M -generic over \mathcal{P} , $(2^\omega = \omega_1)^{M[G]}$.*

Proof. Since G gives a function from ω_1^M onto $(2^\omega)^M$, we have $(|(2^\omega)^M| \leq |\omega_1|^M \leq \omega_1)^{M[G]}$. Since \mathcal{P} is $< \omega_1$ closed in M , we have that $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$. So, $(2^\omega \leq \omega_1)^{M[G]}$. \square

Corollary 1.19. *ZFC, if it is consistent, cannot prove $\neg\text{CH}$.*

We now know that CH is independent of ZFC; that is, ZFC set theory can neither prove CH nor prove $\neg\text{CH}$.

2. MORE ON THE CONTINUUM FUNCTION

We will show a little later that starting with a model M of ZFC we may force over M to get a model of ZFC + GCH. Furthermore, starting with a model of ZFC + GCH we may then force to get a model of ZFC where the continuum function takes any values on the regular cardinals consistent with monotonicity (if $\kappa < \lambda$ then $2^\kappa \leq 2^\lambda$), Cantor's theorem ($2^\kappa > \kappa$) and König's theorem ($\text{cof}(2^\kappa) > \kappa$). In this section, we prove a warm-up version.

First we show that we may force to get the GCH up to any ω_n .

Lemma 2.1. *Let M be a countable transitive model of ZFC. Then for every $n \in \omega$ there is a generic extension $M[G]$ such that $(\forall k \leq n \ 2^{\omega_k} = \omega_{k+1})^{M[G]}$.*

Proof. Let M be a countable transitive model of ZFC. Let $\mathcal{P}_0 = (\text{coll}(2^\omega, \omega_1))^M$. So, $\mathcal{P}_0 \in M$ and is $< \omega_1$ closed in M . Let G_0 be M generic for \mathcal{P}_0 . Let $M_1 = M[G_0]$. Then $(2^\omega = \omega_1)^{M_1}$. Let $\mathcal{P}_1 = (\text{coll}(2^{\omega_1}, \omega_2))^{M_1}$. So, \mathcal{P}_1 is $< (\omega_2)^{M_1}$ closed in M_1 . Let G_1 be M_1 generic for \mathcal{P}_1 . Let $M_2 = M_1[G_1]$. Since forcing with \mathcal{P}_1 does

not add any new subsets of ω , we have $(2^\omega)^{M_1} = (2^\omega)^{M_2}$ and $(\omega_1)^{M_1} = (\omega_1)^{M_2}$. Thus, we still have $(2^\omega = \omega_1)^{M_2}$. We clearly have $(|2^{\omega_1}|^{M_1} \leq |(\omega_2)^{M_1}| \leq \omega_2)^{M_2}$, and thus $((2^{\omega_1})^{M_1} \leq \omega_2)^{M_2}$. Since forcing with \mathcal{P}_1 does not add any new subsets of ω_1 , we also have $(2^{\omega_1})^{M_1} = (2^{\omega_1})^{M_2}$. We now have $(2^{\omega_1} \leq \omega_2)^{M_2}$, and so $(2^\omega = \omega_1 \wedge 2^{\omega_1} = \omega_2)^{M_2}$. Continuing in this manner we finish. \square

The partial orders $\text{FN}(\kappa, 2)$ and $\text{coll}(\kappa, \lambda)$ are both special cases of the following.

Definition 2.2. $\text{FN}(\kappa, \lambda, \rho)$ is the partial order of partial functions $p: \kappa \rightarrow \lambda$ with $|\text{dom}(p)| < \rho$. We define $p \leq q$ iff p extends q .

Thus, $\text{FN}(\kappa, 2) = \text{FN}(\kappa, 2, \omega)$, and $\text{coll}(\kappa, \lambda) = \text{coll}(\kappa, \lambda, \kappa)$.

The following generalizes lemma 1.11.

Lemma 2.3. $\text{FN}(\kappa, \lambda, \rho)$ is $(\lambda^{<\rho})^+$ c.c.

Proof. Suppose $A \subseteq \text{FN}(\kappa, \lambda, \rho)$ is an antichain of size $(\lambda^{<\rho})^+$. Let $\delta = (\lambda^{<\rho})^+$, so δ is regular. Note that $\delta > \rho$. Let $A = \{p_\alpha\}_{\alpha < \delta}$, and let $d_\alpha = \text{dom}(p_\alpha)$. Without loss of generality we may assume that $\kappa = \delta$ (there are at most $\rho \cdot \delta = \delta$ elements of $\bigcup_{\alpha < \delta} d_\alpha$). Since δ is regular and $\delta > \rho$, there is a $\tau < \rho$ such that for δ many α we have $\text{o.t.}(d_\alpha) = \tau$. So, we may assume $\text{o.t.}(d_\alpha) = \tau$ for all $\alpha < \delta$. For $\beta < \tau$ let $\eta(\beta) = \sup\{d_\alpha(\beta) : \alpha < \delta\}$, where $d_\alpha(\beta)$ denotes the β^{th} element of d_α . For some $\beta < \tau$ we must have $\eta(\beta) = \delta$, for otherwise, since $\delta > \tau$ is regular, we would have that $\sup_{\beta < \tau} \eta(\beta) < \delta$. This would say that all of the d_α are subsets of some $\delta' < \delta$. Hence, $|\{d_\alpha\}| \leq (\delta')^\tau$. But, $|\delta'| \leq \lambda^{<\rho}$, so $(\delta')^\tau \leq \lambda^{<\rho \cdot \tau} = \lambda^{<\rho} < \delta$. Hence $|A| \leq \lambda^{<\rho} \cdot \lambda^\tau < \delta$, a contradiction.

Let $\beta_0 < \tau$ be least such that $\eta(\beta_0) = \delta$. Since δ is regular, it is straightforward using the definition of β_0 to thin $\{d_\alpha\}$ to a subsequence of size δ such that if $\alpha_1 < \alpha_2$ then $\sup(d_{\alpha_1}) < d_{\alpha_2}(\beta_0)$. We assume that the d_α now have this property. Let $\eta = \sup\{\eta(\beta) : \beta < \beta_0\}$. By regularity of δ , $\eta < \delta$. We have $d_\alpha \restriction \beta_0 \subseteq \eta$. However there are then at most $\eta^\tau < \delta$ (as argued above) many possibilities for d_α . So, we may assume the d_α are equal to some fixed d . But then there are only $\lambda^\tau < \delta$ many possibilities for $p_\alpha \restriction d$. Thus we may assume all the p_α agree on d . But any two such p_α are compatible, a contradiction. \square

To compute the cardinality of the subsets of κ in $M[G]$, we need to have a reasonable representation for the subsets of κ in $M[G]$. The following lemma, which does that, is a refinement of the argument used to show the power set axiom in $M[G]$.

Lemma 2.4. Let M be a transitive model of ZF, and $\mathcal{P} \in M$. Let $\tau \in M^{\mathcal{P}}$. Let $\mu \in M^{\mathcal{P}}$. Then there is a nice name (for a subset of τ) $\sigma \in M^{\mathcal{P}}$ such that $\mathbb{1} \Vdash (\mu \subseteq \tau \rightarrow \mu = \sigma)$. By a nice name we mean a name of the form $\bigcup_{\pi \in \text{dom}(\tau)} \pi \times A_\pi$, where $A_\pi \subseteq P$ is an antichain.

Proof. Let $\sigma = \{\langle \pi, p \rangle : \pi \in \text{dom}(\tau) \wedge (p \Vdash \pi \in \mu)\}$. Let G be M generic for \mathcal{P} . We must show that $(\mu_G \subseteq \tau_G \rightarrow \mu_G = \sigma_G)$. So, assume $\mu_G \subseteq \tau_G$. If $z \in \sigma_G$, then $z = \pi_G$ where $p \Vdash \pi \in \mu$ and $p \in G$. So, $z = \pi_G \in \mu_G$. For the other direction, assume $z \in \mu_G$. Since $\mu_G \subseteq \tau_G$, $z = \pi_G$ for some $\langle \pi, p \rangle \in \tau$ with $p \in G$. Let $q \leq p$ with $q \Vdash \pi \in \mu$. Then $\langle \pi, q \rangle \in \sigma$ and $q \in G$, so $z = \pi_G \in \sigma_G$. \square

In particular, every subset of an ordinal α in $M[G]$ has a name of the form $\bigcup_{\beta < \alpha} (\check{\beta} \times A_\beta)$, where the A_β are antichains.

Lemma 2.5. *Let M be a transitive model of ZFC and $\mathcal{P} \in M$ be λ -c.c. Let G be M generic for \mathcal{P} , and κ a cardinal of $M[G]$. Then in $M[G]$ we have $2^\kappa \leq [(|P|^{<\lambda})^\kappa]^M$.*

Proof. Since \mathcal{P} is λ -c.c., there are at most $(|P|^{<\lambda})^M$ many antichains of \mathcal{P} which lie in M . Thus, there are at most $[(|P|^{<\lambda})^\kappa]^M$ many nice names for a subset of κ which lie in M . In $M[G]$, there is a map from this set of nice names onto $\mathcal{P}(\kappa)$ by lemma 2.4. \square

We now show that starting with a model of GCH, we may force to get the continuum function to take arbitrary values on finitely many regular cardinals, subject to monotonicity and Cantor's and König's theorems.

Lemma 2.6. *Let M be a countable transitive model of ZFC + GCH. Let $\kappa_1 < \dots < \kappa_n$ be regular in M . Let $\lambda_1 \leq \dots \leq \lambda_n$ be cardinals of M with $\text{cof}(\lambda_i) > \kappa_i$. Then there is a forcing extension $M[G]$ of M which preserves all the cardinals and cofinalities of M and such that $(2^{\kappa_1} = \lambda_1 \wedge \dots \wedge 2^{\kappa_n} = \lambda_n)^{M[G]}$.*

Proof. Let $M_0 = M$. We construct a sequence of forcing extensions $M_0 \subseteq M_1 \subseteq \dots \subseteq M_n$, each of which will preserve cardinals and cofinalities. Assume inductively that M_i has been defined and

- (1) M and M_i have the same functions $f: \kappa \rightarrow \text{ON}$, for all $\kappa < \kappa_{n-i+1}$.
- (2) $(\forall \kappa < \kappa_{n-i+1} (2^\kappa = \kappa^+))^{M_i}$.
- (3) $(2^{\kappa_{n-i+1}} = \lambda_{n-i+1} \wedge \dots \wedge 2^{\kappa_n} = \lambda_n)^{M_i}$.

Note that (1) implies (2) given that cardinals are preserved from M to M_i . Let $\mathcal{P} = \text{FN}(\kappa_{n-i} \times \lambda_{n-i}, 2, \kappa_{n-i})^{M_i} \cong \text{FN}(\lambda_{n-i}, 2, \kappa_{n-i})^{M_i}$. Thus, in M_i , \mathcal{P} is $(2^{<\kappa_{n-i}})^+ = \kappa_{n-i}^+$ -c.c. Since \mathcal{P} is $< \kappa_{n-i}$ closed, it preserves all cardinals and cofinalities $\leq \kappa_{n-i}$. Since \mathcal{P} is κ_{n-i}^+ -c.c. it preserves all cardinals and cofinalities $> \kappa_{n-i}$. Thus, \mathcal{P} preserves all cardinals and cofinalities. Since \mathcal{P} is again $< \kappa_{n-i}$ closed, it adds no new functions $f: \kappa \rightarrow \text{ON}$ for $\kappa < \kappa_{n-i}$. Thus, (1) and (2) hold for M_{i+1} . We clearly have $(2^{\kappa_{n-i}} \geq \lambda_{n-i})^{M_{i+1}}$. By (1), $(\lambda_{n-i}^{\kappa_{n-i}})^{M_i} = (\lambda_{n-i}^{\kappa_{n-i}})^M$. Since $(\text{cof}(\lambda_{n-i}) > \kappa_{n-i})^M$ it follows from the GCH in M that $(\lambda_{n-i}^{\kappa_{n-i}} = \lambda_{n-i})^M$. Thus, $(\lambda_{n-i}^{\kappa_{n-i}} = \lambda_{n-i})^{M_i}$. In M_i , \mathcal{P} has cardinality at most $\sum_{\kappa < \kappa_{n-i}} \lambda_{n-i}^\kappa = \lambda_{n-i}$. From lemma 2.5 $(2^{\kappa_{n-i}} \leq (\lambda_{n-i}^{\kappa_{n-i}})^{\kappa_{n-i}} = \lambda_{n-i})^{M_{i+1}}$. \square

As a corollary of lemmas 2.6 and 2.1, it follows that starting with a countable transitive model M of ZFC, we may get a forcing extension $M[G]$ in which the continuum function moves the first n many infinite cardinals to any other ω_m 's, subject to monotonicity and Cantor's theorem (König's theorem is irrelevant as all of the ω_m are regular). For example:

Corollary 2.7. *If ZFC is consistent, then so is ZFC + $2^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_8 \wedge 2^{\omega_2} = \omega_8 \wedge 2^{\omega_7} = \omega_9$.*

Exercise 4. Compute the continuum function in the model $M[G]$ of lemma 2.6.

3. TARSKI'S THEOREM

We prove the following theorem, valid for any partial order.

Theorem 3.1. (ZFC) *Let $\mathcal{P} = \langle P, \leq \rangle$ be a partial order. Then $\text{sat}(\mathcal{P})$ is a regular cardinal.*

Proof. Let $\kappa = \text{sat}(P)$, so every antichain of P has size $< \kappa$, but for every $\lambda < \kappa$ there is an antichain of P of size λ . Suppose $\text{cof}(\kappa) = \rho < \kappa$. Let $B = \{p: \forall q \leq p \text{ (sat}(q) = \kappa)\}$, where $\text{sat}(q) = \sup\{|A|: A \subseteq N_q \text{ is an antichain}\}$ (recall $N_q = \{r: r \leq q\}$). Suppose first that $B \neq \emptyset$, and let $p \in B$. Let $A \subseteq N_p$ be an antichain of size $|A| = \rho$. For each $q \in A$, let $A_q \subseteq N_q$ be an antichain of size $f(q) < \kappa$, where $f: A \rightarrow \kappa$ is cofinal (we may do this since $p \in B$). Then $\bigcup_{q \in A} A_q$ is an antichain of P of size κ , a contradiction.

Suppose next that $B = \emptyset$. Let $D = \{p: \text{sat}(p) < \kappa\}$. Thus, D is dense. Let $A \subseteq P$ be maximal subject to A being an antichain and $A \subseteq D$. Since D is dense, A is easily a maximal antichain, that is, for every $q \in P$ there is a $p \in A$ such that $q \parallel p$. By assumption, $|A| < \kappa$. Also, we must have $\sup\{\text{sat}(p): p \in A\} = \kappa$. This is because for any regular $\lambda < \kappa$ with $\lambda > |A|$ there is an antichain of P of size λ , and there must be a single $p \in A$ such that λ many elements of this antichain are compatible with p (this then gives an antichain of size λ in N_p). We may now construct $\{p_\alpha\}_{\alpha < \rho} \subseteq A$ such that $\text{sat}(p_\alpha) > \sup\{\text{sat}(p_\beta): \beta < \alpha\}$ and $\sup_{\alpha < \rho} \text{sat}(p_\alpha) = \kappa$. For each $\alpha < \rho$, let $A_\alpha \subseteq N_{p_\alpha}$ be an antichain of size $\geq \sup_{\beta < \alpha} \text{sat}(p_\beta)$. Then $\bigcup_{\alpha < \rho} A_\alpha$ is an antichain of P of size κ , a contradiction. \square