3. Axioms of Set theory

Before presenting the axioms of set theory, we first make a few basic comments about the relevant first order logic. We will give a somewhat more detailed discussion later, but for now just say what we need to present the axioms. We work in the *language of set theory*. This allows only a single binary relation symbol $\in$ in the language (all first-order languages allow a symbol for equality; thus these are are the only two relation symbols allowed). There are no function or constant symbols.

A formula of set theory is, roughly speaking, any expression built up, using the standard logical operations, from these two binary relation symbols. More precisely, our language allows in addition to these two symbols the symbols: $\exists$, $\forall$, $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, (,), and an infinite list of variables $x_1, x_2, x_3, \dots$ (the reader will recall that only $\neg$, $\vee$, and $\exists$ suffice, for example). An *atomic formula* is an expression of the form $x_i \in x_j$ or $x_i \approx x_j$ (formally we should use a different symbol for the equality symbol in the formal language versus real equality). The collection of formulas (in the language of set theory) is then defined recursively by the following:

(1) An atomic formula is a formula.
(2) If ϕ, ψ are formulas, so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\neg \phi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$, $\exists x_i\, \phi$, $\forall x_i\, \phi$.

We will drop parentheses when the meaning is clear. An occurance of a variable x in the formula ϕ is *free* if it is not within the scope of an $\exists x$ or $\forall x$ quantifier. The free variables of a formula are the variable which have a free occurance in the formula. More formally, this notion is defined recursively through the following:

(1) A variable is free in an atomic formula if it occurs in the formula.
(2) A variable is free in $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\neg \phi)$, $(\phi \rightarrow \psi)$, or $(\phi \leftrightarrow \psi)$ iff it is free in ϕ or ψ.
(3) A variable is free in $\exists x_i\, \phi$, or $\forall x_i\, \phi$ if it is free in ϕ and not equal to x_i.

We frequently write $\phi(x_1, \dots, x_n)$ to indicate that $x_1, \dots, x_n$ are the free variables in ϕ. A formula is said to be a *sentence* if it has no free variables. For example, the sentence $\phi = \exists x\, \exists y\, \exists z\, (y \in x \wedge z \in x \wedge \forall w\, (w \in x \rightarrow (w = y \vee w = z)))$ asserts that there is a set with exactly two elements.

We present now the formal axioms of ZFC set theory. The axioms, for the most part, describe allowable rules of set formation. These rules encompass the methods of forming sets used commonly in mathematics. That some rules are necessary is evident from Russell's paradox: if the collection $S = \{x \colon x \notin x\}$ were a set, then we would have a contradiction as $S \in S \rightarrow S \notin S$ and $S \notin S \rightarrow S \in S$.

The axioms of set theory are all expressed in the language of set theory which has a single binary relation symbol $\in$. We now list the axioms with brief comments after them.

(**Extensionality**) $\forall x\, \forall y\, (x = y \leftrightarrow \forall z\, (z \in x \leftrightarrow z \in y))$.

This asserts two sets are equal iff they have the same elements, that is, a set is determined by its elements.

(**Pairing**) $\forall x\, \forall y \exists z\, (x \in z \wedge y \in z)$.

This say that for any sets x, y, we can find a set containing both x and y.

(**Union**) $\forall x\, \exists u\, \forall y\, \forall z (y \in x \wedge z \in y \rightarrow z \in u)$.

This says that for all sets x, a set u exists which contains the union of x (i.e., the union of all the elements of x).

(**Infinity**) $\exists x\ (\emptyset \in x \wedge \forall y\ (y \in x \rightarrow y \cup \{y\} \in x))$.

This asserts that a set x exists which contains the emptyset, and is closed under the successor function. This gives that a set containing all of the $S^n(0)$ exists, that is, an infinite ordinal exists.

(**Power Set**) $\forall x\ \exists y\ \forall z\ (z \subseteq x \rightarrow z \in y)$.

Here "$z \subseteq x$" abbreviates $\forall w\ (w \in z \rightarrow w \in x)$. This clearly asserts that a set containing the power set of x exists.

(**Comprehension Scheme**) For each formula $\phi(x, z, w_1, \ldots, w_n)$ the axiom

$$\forall z\ \forall w_1, \ldots, w_n\ \exists y\ \forall x\ (x \in y \leftrightarrow (x \in z \wedge \phi(x, z, w_1, \ldots, w_n))).$$

This says that for any set z and parameters $w_1, \ldots, w_m$, the set $y = \{x \in z \colon \phi(x, z, w_1, \ldots, w_n)\}$ exists. We will, in fact, use this common mathematical notation for describing this set. Note that ϕ is not allowed to refer to y (i.e., y cannot occur as a free variable in ϕ), that is, we are not allowed to define y in terms of itself (this would easily give a contradiction).

(**Replacement Scheme**) For each formula $\phi(x, z, w_1, \ldots, w_n)$ the axiom

$$\forall a\ \forall w_1, \ldots w_n\ [\forall x \in a\ \exists z \phi(x, z, w_1, \ldots, w_n) \rightarrow \exists b\ \forall x \in a\ \exists z \in b\ \phi(x, z, w_1, \ldots, w_n)].$$

This says we can, for all sets a, and parameters $w_1, \ldots, w_n$ find a set b large enough to collect the range of a definable "function" (defined by ϕ) applied to the set a. Note that we are anot assuming that this "function" actually exists as a set (if it did, we wouldn't need this axiom).

The remaining axiom of ZF, the foundation axiom, is a little different from the previous ones in that it is not a set existence axiom, but rather resticts the universe of sets. The effect of this axiom is to restrict our attention to the collection of well-founded sets. Adding the axiom to ZF is, in some sense, not critical as we could always just choose to restrict our attention to these sets (a more precise discussion of this point will be given later).

(**Foundation**) $\forall x\ [(\exists y\ y \in x) \rightarrow (\exists y\ (y \in x \wedge \neg \exists z\ (z \in x \wedge z \in y)))]$.

This axiom asserts that the $\in$ relation on every set is well-founded. That is, given any non-empty set x, an $\in$-minimal element y of x can be found. Note that this is true for the ordinals by definition, without the axiom of foundation. Foundation is asserting this is true for every set.

This complete the axiom scheme ZF. The scheme ZFC consists of the ZF axioms plus the axiom of choice (AC):

(**Axiom of Choice**) $\forall x\ \exists r\ (r$ well-orders $x)$

We leave it to the reader to write out "r well-orders x" in the first order language of set theory.

We now fix some fairly standard terminology, proving along the way that the relevant sets exist. If ϕ is a formula in the language of set theory, say $\phi = \phi(x, z, w_1, \ldots, w_n)$, and $w_1, \ldots, w_n$ are sets, we frequently write

$$y = \{x \in z \colon \phi(x, z, w_1, \ldots, w_n)\}$$

to denote the set y obtained by applying the comprehension axiom to ϕ and the set z (and parameters $w_1, \ldots, w_n$; note that y is not allowed to be free in ϕ, this is not a problem as we can always change the name of the set from y to something else).

Exercise 20. Show that there is a unique empty set, that is, a unique set which has no elements.

Exercise 21. Show that for any sets $x_1, \ldots, x_n$, the set $y = \{x_1, \ldots, x_n\}$ exists. More precisely, show there is a unique set y such that $\forall z\ (z \in y \leftrightarrow (z = x_1 \vee \cdots \vee z = x_n))$.

We define the *ordered pair* $\langle x, y\rangle$ by: $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$. This is well-defined by the previous exercise.

Exercise 22. Show that if $\langle x, y\rangle = \langle x', y'\rangle$ then $x = x'$ and $y = y'$. Thus, this serves as a coding function.

For each set x, there is a unique set $\cup x$ such that $\forall z\ (z \in \cup x \leftrightarrow \exists y\ (y \in x \wedge z \in y))$. To see this, first apply the union axiom to get a set u such that $\forall y\ \forall z\ (y \in x \wedge z \in y \rightarrow z \in u)$. Then apply the comprehension axiom to u to form $\{z \in u \colon \exists y\ (y \in x \wedge z \in y)\}$. Frequently one thinks of x as a "collection" of sets in writing $\cup x$, in which case we think of $\cup x$ as the union of the sets in this collection. Note that $x \cup y = \cup\{x, y\}$.

For each non-empty set x there is also a unique set $\cap x$ such that $\forall z\ (z \in \cap x \leftrightarrow \forall y\ (y \in x \rightarrow z \in y))$. Again, $x \cap y = \cap\{x, y\}$.

Exercise 23. Show $\cap x$ exists and is uniquely determined.

For sets x, y, we let $x - y$ denote $\{z \in x \colon z \notin y\}$, which clearly exists by comprehension.

For sets A, B we define their cartesian product $A \times B$ to be the set $\{\langle x, y\rangle \colon x \in A \wedge y \in B\}$. This exists by applying the replacement axiom twice, first to show that for each $y \in B$ that $A \times \{y\}$ exists, and then to show $A \times B$ exists (using also the comprehension axiom). Details are in the text.

We define a *relation* to be a set all of whose elements are ordered pairs. If R is a relation, we define $\mathrm{dom}(R) = \{x \colon \exists y\ (\langle x, y\rangle \in R)\}$. Likewise, we define $\mathrm{ran}(R) = \{y \colon : \exists x\ (\langle x, y\rangle \in R)\}$.

Exercise 24. Show that for any relation R that $\mathrm{dom}(R)$, $\mathrm{ran}(R)$ are indeed sets.

If R is a relation, the inverse relation R^{-1} is defined by $R^{-1} = \{\langle x, y\rangle \colon \langle y, x\rangle \in R\}$. We define the notions of the relation R being a function, a one-to-one function, an onto function, a bijection, etc., in the usual way. We employ the usual notion $f \colon A \to B$ to denote f is a function, $\mathrm{dom}(f) = A$, and $\mathrm{ran}(f) \subseteq B$. We say f is a *partial* function from A to B if we only require $\mathrm{dom}(f) \subseteq A$.

Having officially now defined the notion of (binary) relation, the formal definitions of partial order, linear order, well-order, etc., can now be given. They are, of course, the same as those given informally earlier. We can now say that working within ZF set theory we have given a precise definition of the ordinals.

We wish sometimes to refer to collections of sets which are not necessarily sets themselves, in fact ZF will prove that these collections are not sets. Consider first the collection of all sets, which we might write as $\{x \colon x = x\}$. Note that this is not a legitimate application of the comprehension axiom, and so not guananteed to be a set. In fact, the Russell paradox argument given earlier is now a proof that this is not a set. More precisely, ZF proves that $\neg\exists x\ \forall y\ (y \in x)$.

Exercise 25. Go back over the proof of Russell's paradox and give a precise proof that ZF proves the above statement.

How then are we to deal with such collections which are not sets, and does the word "collection" here really mean? We answer these questions with the notion of a *class*. Formally, a class just means a formula ϕ in the language of set theory. ϕ may have free variables which we regard as set parameters. Intuitively, we are thinking of ϕ as representing the collection of sets which satisfy ϕ. For example we let V (the universe of all sets) be the class $\phi = (x \approx x)$. Thus when we write "$x \in V$" we officially mean $x = x$ (which of course is always true). Another example is the class ON of all ordinals. This corresponds to the formula $\phi(x)$ which asserts that x is a transitive set well-ordered by the $\in$ relation. Again, when we say "$x \in \mathrm{ON}$", this abbreviates $\phi(x)$. Be aware that the notion of a class is an abbreviation employed in the meta-theory, not a definition within ZF set theory. Thus a statement such as "for every class C we have …" denotes a schema of metatheorems, not a theorem (or even statement) in the language of set theory. A person who objects to this can either make the above translations and not consider statements such as "for every class C …," or else formalize the metatheory (e.g., we might consider the metatheory to be a model of Peano arithmetic, and view formulas as coded by integers). Note that the well-foundedness principle for ordinals extends to classes. That is, if $C \subseteq \mathrm{ON}$ is a non-empty class, then C contains a least element (again, this is a schema in the metatheory, not a formal statement in ZF). For any particular class C, this can be proved exactly as the proof before for sets.

Exercise 26. (Burali-Forti Paradox) Show that the class ON is proper, that is, is not a set. More precisely, show that it is a theorem of ZF that $\neg\exists x\ \forall y\ (y \in \mathrm{ON} \to y \in x)$. (hint: use theorem 2.6 and lemma 1.8.)

We can now give a precise definition, within ZF, of the natural numbers, and of the notion of "finite."

Definition 3.1. n is a natural number iff ("n is an ordinal" $\wedge\ \forall m \leq n\ (m = 0 \vee \exists\alpha\ m = S(\alpha))$). We say a set is *finite* if it can be put in bijection with a natural number.

Note that the above definition defines the natural numbers as a class. Indeed, without the axiom of infinity it can be a proper class. However, with the axiom of infinity the set of natural numbers exists. To see this, let x be a set as given by the infinity axiom, that is, $0 \in x$ and $\forall y \in x\ (S(y) \in x)$. We claim that x contains all of the natural numbers. Suppose not. Let C be the class of natural numbers which are not in x. Then C has a least element, say m. Clearly $m \neq 0$ as otherwise $m \in x$. By definition of natural number, m is a successor, say $m = S(k)$. But then $k \in x$ by minimality of m, and thus $m = S(k) \in x$ by the property of x. This shows that a set containing the natural numbers exists, and by compehension the set of natural numbers exists. We denote this set by ω.

4. Transfinite Recursion

Definitions by transfinite recursion occur frequently in mathematics and in set theory in particular. We prove a theorem which says that such definitions are legitimate. One common situation is in defining a function by recursion on an ordinal (or equivalently, a well-ordered set). However other situations also arise. First, we sometimes want to use more general well-founded relations. For example, we may wish to use the $\in$ relation on a set x (which is well-founded by the foundation axiom). Secondly, we often wish to do a transfinite recursion on a proper class such as ON or V itself. We wish to show that in all cases this is legitimate.

At the risk of slight redundancy, we first prove the theorem for sets, that is, where the domain of the function we are trying to define is a set. Then we prove the version for classes, which is really the same proof if we add the appropriate words.

Theorem 4.1. *(Set Transfinite Recursion) Let $(A, \prec)$ be a set with a well-founded relation. Let $F \colon V \times A \to V$ be a class function. That is, $F(x, a, y)$ is a formula in set theory such that ZF proves that $\forall x\ \forall a \in A\ \exists! y\ F(x, a, y)$. Then there is a unique function $G \colon A \to V$ (G is a set) such that $\forall a \in A\ [G(a) = F(G \restriction pr(a), a)]$.*

Note: in this theorem, $\mathrm{pr}(a)$ denotes the set of predecessors $\{b \in A \colon b \prec a\}$. Also, note that we are not assuming $\prec$ is transitive, just well-founded.

Note that theorem 4.1 is still a schema in the meta-theory as it talks about a class function F. Of course, F could be a set as well.

Exercise 27. Formulate a version of the transfinite recursion theorem which can be expressed as a statement in ZF. You might want to read the proof of theorem 4.1 first to suggest what hypotheses to put on the set function F.

Proof. For $a \in A$ define $\mathrm{cl}(a) \subseteq A$ by $b \in \mathrm{cl}(A)$ iff $\exists f\ \exists n$ (f is a function $\wedge\, n \in \omega \wedge \mathrm{dom}(f) = n \wedge f(0) = a \wedge f(n-1) = b \wedge \forall i < n\ (i > 0 \to f(i) \prec f(i-1))$). $\mathrm{cl}(a)$ is a set by comprehension. $a \in \mathrm{cl}(a) \subseteq A$, and $\mathrm{cl}(a)$ is the "downward" closure of a under $\prec$. In particular, if $b \prec a$ then $\mathrm{cl}(b) \subseteq \mathrm{cl}(a)$.

For $a \in A$, say a function g is an a-approximation if $\mathrm{dom}(g) = \mathrm{cl}(a)$ and g satisfies the recursive definition on $\mathrm{cl}(a)$, that is, for all $b \in \mathrm{cl}(a)$ we have $g(b) = F(g \restriction \mathrm{pr}(b), b)$. First we claim that for all $a \in A$ if g_1, g_2 are a approximations then $g_1 = g_2$. If not, let a be $\prec$ minimal such that this fails, and let g_1, g_2 witness the failure. For $b \prec a$, $g_1 \restriction \mathrm{cl}(b)$, $g_2 \restriction \mathrm{cl}(b)$ are both b-approximations and so by minimality $g_1(b) = g_2(b)$. So, $g_1 \restriction \mathrm{pr}(a) = g_2 \restriction \mathrm{pr}(a)$. But then $g_1(a) = F(g_1 \restriction \mathrm{pr}(a), a) = F(g_2 \restriction \mathrm{pr}(a), a) = g_2(a)$, a contradiction.

Next we claim that for any $a \in A$ an a-approximation exists. Again, assume this fails and let a be $\prec$ minimal such that an a-approximation does not exist. So for all $b \prec a$, a unique b-approximation exists. Let g' be the union of all the b-approximations for all $b \prec a$. This exists by the replacement axiom applied to the formula $\phi(b, x) =$"x is a b-approximation." g' is a function. To see this we must show that if $b_1, b_2 \in A$, g_1, g_2 are b_1, b_2 approximations respectively, and $c \in \mathrm{cl}(b_1) \cap \mathrm{cl}(b_2)$ then $g_1(c) = g_2(c)$. This follows since $g_1 \restriction \mathrm{cl}(c)$, $g_2 \restriction \mathrm{cl}(c)$ are both $\mathrm{cl}(c)$-approximations and hence are equal by the previous paragraph. In particular $g_1(c) = g_2(c)$. Let $g = g' \cup \{\langle a, F(g' \restriction \mathrm{pr}(a), a)\rangle\}$. Then g is a function with domain $\mathrm{cl}(a)$ and is an a-approximation.

Finally, define G to be the union of all the a-approximations for $a \in A$. This is a set by the replacement axiom as before. By the existence and uniqueness claims, G is a function with domain A, and clearly G satisfies the recursive definition. □

We now consider the class version of the transfinite recursion theorem. Suppose now A is a class and $\prec$ is a class relation on A which is well-founded (meaning every non-empty set which a subset of A has a $\prec$ least element). Suppose as above that $F\colon V \times A \to V$ is a class function. For the recursive condition on G to even make sense, we must require that for $a \in A$ that $\mathrm{pr}(a)$ is a set. We claim that this is enough.

Theorem 4.2. *(Class Transfinite Recursion Theorem) Let $(A, \prec)$ be a class A together with a class relation $\prec$ on A which is well-founded. Assume that for every $a \in A$ that $pr(a)$ is a set. Let $F\colon V \times A \to V$ be a class function. Then there is a unique class function $G\colon A \to V$ such that $\forall a \in A\ [G(a) = F(G \restriction pr(a), a)]$.*

Proof. By assumption for all $a \in A$ we have that $\mathrm{pr}(a)$ is a set. We claim that for all $a \in A$ that $\mathrm{cl}(a)$ is a set. To see this, first define for each natural number n the class $\mathrm{cl}_n(a)$ which is defined exactly as $\mathrm{cl}(a)$ except that we require that the domain of the function f used in the definition be n. We claim that for each n that $\mathrm{cl}_n(a)$ is a set. That is, we claim that $\forall a \in A\, \forall n \in \omega\, \exists z\, \forall b\, (b \in z \leftrightarrow \exists f (f$ is a function $\wedge \mathrm{dom}(f) = n \wedge f(0) = a \wedge f(n-1) = b \wedge \forall i < n-1\ (f(i+1) \prec f(i))))$. If this claim fails, then we may fix $a \in A$ such that the claim fails for some $n \in \omega$, and then fix $n \in \omega$ minimal such that the claim fails for this integer and a. By minimality, $\mathrm{cl}_{n-1}(a)$ exists. Note that $\mathrm{cl}_n(a)$ is $\mathrm{cl}_{n-1}(a)$ together with the $b \prec x$ for $x \in \mathrm{cl}_{n-1}(a)$. For each $x \in \mathrm{cl}_{n-1}(a)$ the set of $b \prec x$ is a set by assumption. Replacement then gives that $\mathrm{cl}_n(a)$ is a set. The relation $R(n, z) \leftrightarrow z = \mathrm{cl}_n(a)$ is definable, and thus replacement applied to the set ω gives that $\mathrm{cl}(a)$ exists.

For $a \in A$ we now define the notion of an a-approximation exactly as before. We prove as before that for all $a \in A$ that a-approximations exist and are unique; note that the use of replacement in construction the function g' in the previous proof is still o.k. since $\mathrm{pr}(a)$ is a set. Finally, G is defined as before to be the union of the a-approximations, but this is now a class function. That is, $G(a) = z \leftrightarrow \exists g\ (g$ is an a-approximation $\wedge\ g(a) = z)$. □

We finish this section with a note on the axiom of choice. AC has several equivalent formulations, each of which is useful. Let AC_1 be the statement as in our official definition, that is, $\mathrm{AC}_1 = \forall x$ ("x can be well-ordered"). Let AC_2 be the statement that for every non-empty relation R there is a function F with $\mathrm{dom}(F) = \mathrm{dom}(R)$ such that $\forall x \in \mathrm{dom}(R)\ ((x, F(x)) \in R$.

Theorem 4.3. *(ZF) $AC_1 \leftrightarrow AC_2$.*

Proof. We work in ZF set theory. First assume AC_1 and we show AC_2. Let R be a non-empty relation. Let $D = \mathrm{dom}(R)$ and $S = \mathrm{ran}(R)$. Let $\prec$ well-order S. Let $\phi(x, y)$ be the formula (with parameters R and $\prec$) $(x, y) \in R \wedge \forall z \prec y\ (\neg (x, z) \in R)$. By comprehension, the set $F = \{(x, y) \in R \colon \phi(x, y)\}$ exists. Clearly F is a function with domain D and is as required.

Assume next AC_2, and let X be a set. We must show that X can be well-ordered. Let R be the relation $R(A, y) \leftrightarrow A \subseteq X \wedge y \in A$. R exists since $\mathcal{P}(X)$ exists and thus so does $\mathcal{P}(X) \times X$, and hence R exists by comprehension (note that replacement is being used in getting the existence of the cartesian product). We may assume that $\emptyset \notin X$, for if it is then we well-order $X - \{\emptyset\}$ and then well-order X by adding $\emptyset$ at the end of the well-order. By AC_2, let $G\colon (\mathcal{P}(X) - \{\emptyset\}) \to X$ be such that $G(A) \in A$ for for all non-empty $A \subseteq X$. We now define by transfinite recursion on the well-founded (set like) class $(\mathrm{ON}, \in)$ a class function $F\colon \mathrm{ON} \to V$. Namely, let F be such that for all $\alpha \in \mathrm{ON}$, $F(\alpha) = G(\mathrm{ran}(F \restriction \alpha))$ if $X - \mathrm{ran}(F \restriction \alpha) \neq \emptyset$, and otherwise $F(\alpha) = \emptyset$. F exists by the transfinite recursion theorem. An immediate induction shows that $F(\alpha) \in X$ or $F(\alpha) = \emptyset$ for all $\alpha \in \mathrm{ON}$. Also, the set of α such that $F(\alpha) \in X$ forms an initial segment of the ordinals. We claim that the class A of ordinals α for which $F(\alpha) \in X$ is a set. To see this, first note by an immediate induction that F is one-to-one on A. Also, $\mathrm{ran}(F) \cap X$ is a set by comprehension, call this set B. So $\forall x \in B\ \exists! \alpha \in \mathrm{ON}\ (F(\alpha) = x)$. By replacement and the fact that F is one-to-one on A there is a set which contains all the $\alpha \in \mathrm{ON}$ such that $F(\alpha) \in X$, and thus by comprehension A is a set (thus A is an ordinal). Next we claim that F is onto, that is, $B = X$. Suppose not, and let α_0 be the least ordinal not in A. Thus, $A = \alpha_0$ actually. Since $F(\alpha_0) = \emptyset$, we must have $\mathrm{ran}(F \restriction \alpha_0) = X$ from the definition of F. Thus F is onto. Now we easily define a well-order of X by: $x_1 \prec x_2 \leftrightarrow \exists \alpha_1 < \alpha_2\ (F(\alpha_1) = x_1 \wedge F(\alpha_2) = x_2)$. (note: F is actually a set since A is, but we don't need this.) □

Exercise 28. Explain what is wrong with the following "proof" in ZF that an ill-founded relation $\prec$ on a set X must have an infinite decreasing chain: let $S \subseteq X$ be non-empty with no $\prec$ minimal element. Define by transfinite recursion on ω a function $f\colon \omega \to S$ as follows. Let $f(n)$ be some element of S such that $f(n) \prec f(n-1)$, which exists since $f(n-1)$ is not $\prec$-minimal. Then $\{f(0), f(1), \dots\}$ is a $\prec$-decreasing chain.

5. The Rank Hierarchy

Working in ZF we now define and develop the basic properties of the "rank hierarchy," which the basic picture of the universe of sets.

Definition 5.1. For $\alpha \in \mathrm{ON}$ the set V_α is defined by transfinite recursion as follows.

(1) $V_0 = \emptyset$.
(2) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
(3) For α limit, $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$.

Lemma 5.2. *For all $\alpha \in$ ON, V_α is transitive. If $\alpha < \beta \in$ ON, then $V_\alpha \subseteq V_\beta$.*

Proof. We first show transitivity. This is trivial for $\alpha = 0$ and for α a limit since a union of transitive sets is transitive. Suppose $x \in y \in V_{\alpha+1}$. Then $y \subseteq V_\alpha$, so $x \in V_\alpha$. By induction V_α is transitive, so $x \subseteq V_\alpha$, and so $x \in V_{\alpha+1}$.

We show that $V_\alpha \subseteq V_\beta$ by induction on β for all $\beta \geq \alpha$. For $\beta = \alpha$ this is trivial. For β limit this follow immediately from induction. So assume $V_\alpha \subseteq V_\beta$ and we show $V_\alpha \subseteq V_{\beta+1}$. It is enough to show that $V_\beta \subseteq V_{\beta+1}$, and this follows from the fact that V_β is transitive (i.e., an element of V_β is also a subset of V_β). □

The above definition did not use the axiom of foundation, that is, it is given in $\mathrm{ZF}^- = \mathrm{ZF}-$Foundation. Using foundation, we can show that this rank hierarchy exhaust the universe of sets.

Definition 5.3. If x is a set, define the transitive closure of x, tr cl(x), as follows. First, for $n \in \omega$ define $\bigcup^n(x)$ recursively by: $\bigcup^0(x) = x$, $\bigcup^{n+1}(x) = \bigcup(\bigcup^n(x))$. Set tr cl$(x) = \bigcup_n(\bigcup^n(x))$.

Exercise 29. Write out a direct ZF definition of tr cl(x) not using the transfinite recursion theorem (i.e., "unravel" the recursive definition).

Clearly tr cl(x) is the smallest transitive set containing x.

Lemma 5.4. *(ZF) For every x there is an $\alpha \in ON$ such that $x \in V_\alpha$.*

Proof. Suppose not, and fix x not in any V_α. Let $y = \text{tr cl}(\{x\})$. Let $z \in y$ be $\in$-minimal such that $z \notin \bigcup_{\alpha \in \text{ON}} V_\alpha$. This exists by foundation (note that $x \in y$). Since y is transitive, $z \subseteq y$, and by minimality of z we have $\forall w \in z\ \exists \alpha\ (w \in V_\alpha)$. By replacement, there is an $\alpha \in \text{ON}$ such that $z \subseteq V_\alpha$, and hence $z \in V_{\alpha+1}$, a contradiction. □

Thus, we may now write $V = \bigcup_{\alpha \in \text{ON}} V_\alpha$.

Definition 5.5. The *rank* of a set x is the least $\alpha \in \text{ON}$ such that $x \subseteq V_\alpha$.

Equivalently, $\text{rank}(x) = \alpha$ iff $V_{\alpha+1}$ is the least element of the V-hierarchy in which x appears (since $x \subseteq V_\alpha$ iff $x \in V_{\alpha+1}$).

Exercise 30. Show that $V_\alpha = \{x \colon \text{rank}(x) < \alpha\}$.

Lemma 5.6. *For any set x, $\textit{rank}(x) = \sup\{\textit{rank}(y) + 1 \colon y \in x\}$.*

Proof. Let $\alpha = \sup\{\text{rank}(y) + 1 \colon y \in x\}$. Since for every $y \in x$ we have $y \in V_{\text{rank}(y)+1} \subseteq V_\alpha$, we have $x \subseteq V_\alpha$, so $\text{rank}(x) \leq \alpha$. On the other hand, if $\beta < \alpha$, then there is a $y \in x$ such that $\text{rank}(y) + 1 > \beta$, and hence $y \notin V_\beta$. Thus, x is not a subset of V_β so $\text{rank}(x) > \beta$. □

The previous lemma is not only useful for computing ranks, but it suggests another way to define rank. Namely, we could define rank(x) by transfinite recursion on the class V with the $\in$-relation by the recursion $\text{rank}(x) = \sup\{\text{rank}(y) + 1 \colon y \in x\}$. One advantage of this is that the transfinite recursion theorem does use the power set axiom, and thus neither does this definition of rank(x). Another advantage is that this computes rank(x) is a purely "local" manner. To see this, we first define the notion of rank for an arbitrary well-founded relation.

Definition 5.7. Let A be a class and $\prec$ a set-like well-founded relation on A. For $a \in A$, define by transfinite recursion $\text{rank}(a) = \sup\{\text{rank}(b) \colon b \in A \wedge b \prec a\}$.

Thus, definition 5.7 generalizes the computation of lemma 5.6 to arbitrary well-founded relations.

Lemma 5.8. *For any set x, $\textit{rank}(x)$ is the rank of the $\in$ relation on the set tr cl(x).*

Proof. For $z \in \text{tr cl}(x)$, let $|z|$ denote the rank of z in the $\in$ relation on tr cl(x). We show by induction on $z \in \text{tr cl}(x)$ that $|z| = \text{rank}(z)$. We have $|z| = \sup\{|w|+1 \colon w \in z \wedge w \in \text{tr cl}(z)\} = \sup\{\text{rank}(w)+1 \colon w \in z \wedge w \in \text{tr cl}(z)\} = \sup\{\text{rank}(w)+1 \colon w \in z\} = \text{rank}(z)$. The second equality uses induction, and the third the fact that tr cl(x) is transitive. □

Exercise 31. Show that every "real" $x \in \omega^\omega$ has rank ω. Show that the set of reals has rank $\omega + 1$, and the collection of Borel subsets of the reals has rank $\omega + 2$.

Exercise 32. Show that every function $f\colon \mathbb{R} \to \mathbb{R}$ has rank $\omega + 3$, but can be coded as a set of rank $\omega + 1$.

Exercise 33. Show that for every ordinal α, $\mathrm{rank}(\alpha) = \alpha$.

Exercise 34. Show that for any set x, $\mathrm{rank}(\mathcal{P}(x)) = \mathrm{rank}(x)+1$. Show that for any x, y that $\mathrm{rank}(x \cup y) = \max\{\mathrm{rank}(x), \mathrm{rank}(y)\}$ and $\mathrm{rank}(x \cap y) \leq \min\{\mathrm{rank}(x), \mathrm{rank}(y)\}$.

Lemma 5.9. *Let A be a set and $\prec$ a well-founded relation on A. Then $\tilde{A} = \{|a|\colon a \in A\} \in ON$.*

Proof. We must show that if $\alpha = |a|$ and $\beta < \alpha$, then $\beta = |b|$ for some $b \in A$. Suppose $\beta < \alpha$ is least so that this fails. Let $\gamma < \beta$ be least such that $\gamma \in \tilde{A}$, say $\gamma = |c|$. Then $|c| = \sup\{|d| + 1\colon d \in A \wedge d \prec c\} \leq \beta$, since each $|d| < \beta$. □

From this we obtain the following.

Lemma 5.10. *Let A be a set of cardinality κ and $\prec$ a well-founded relation on A. Then $|A| < \kappa^+$.*

Proof. If $|A| \geq \kappa^+$, then from the previous lemma every $\alpha < \kappa^+$ would be the rank of some $a \in A$. This would give a one-to-one mapping of κ^+ into A, a contradiction. □