1. Polish spaces

Polish spaces are the main objects of study in analysis and descriptive set theory. Descriptive set theory is largely the study of the definable subsets of Polish spaces.

Definition 1.1. A *Polish space* X is a topological space which is separable and completely metrizable.

If we wish to specify a complete metric giving the topology we will write (X, ρ). Recall that separable means that there is a countable dense set, which for metric spaces is the same as saying the space is second countable. Also, complete metrizability is a topological property, although there will be different metrics giving the topology, some of which will be complete and some not.

Most of the familiar objects of study in analysis involve Polish spaces; the next examples record a few of them.

Examples 1.2. $\mathbb{R}_{\text{std}}$, $[0,1]_{\text{std}}$, $\mathbb{R}^n$, $[0,1]^n$ are all Polish spaces. Any G_δ subset of a Polish space is Polish according to theorem 1.3 below, so $(0,1)$ and $\mathrm{Irr} = \mathbb{R} - \mathbb{Q}$ (with the subspace topology) are also Polish. We will see below that Irr is homeomorphic to the *Baire space* ω^ω. $\mathbb{Q}$, however, is not Polish (see exercise 3 below). A countable product of Polish spaces is Polish according to exercise 1 below, so $\mathbb{R}^\omega$ (which by a theorem of Anderson is homeomorphic to ℓ^2) and the Hilbert cube $[0,1]^\omega$ are Polish. A separable Banach space is by definition Polish, so the ℓ^p, $p < \infty$, are Polish as are the $L^p([0,1])$, or $L^p(\mathbb{R})$ $(p < \infty)$. c_0 (the sequences converging to 0 with the sup norm topology) is also Polish.

Any countable discrete space is Polish, so $\mathbb{N}$, $\mathbb{Z}$ with their usual topologies are Polish.

Any compact metric space is Polish (it is helpful to recall that a metric space is compact iff it is complete with respect to every compatible metric). In particular, the Cantor space 2^ω (see below) is Polish.

If X is Polish the set $F(X)$ of all closed subsets of X carries a natural topology under which it becomes Polish called the *Beer topology*; more on this later. When X is compact this coincides with the Hausdorff metric topology on $K(X)$ (the space of compact subsets of X).

If X is a compact metric space, then $C(X)$ = the space of continuous real valued functions on X with the sup norm topology is Polish.

Exercise 1. Show that if X_n are Polish then the product space $\prod_n X_n$ (with the product topology) is also Polish. (hint: without loss of generality we may assume that the topology on X_n is given by a complete metric ρ_n which is bounded by 1. Then let $\rho = \sum_n \frac{1}{2^n} \rho_n$. Show this works.)

Exercise 2. Show that if (X, ρ) is Polish and $F \subseteq X$ is closed, then $\rho \restriction F \times F$ is a compatible complete metric on F.

Exercise 3. Show directly that $\mathbb{Q}$ (with the subspace topology) is not a Polish space (by directly we mean not quoting the Baire category theorem).

Exercise 4. Suppose ρ_n are metrics on a set X which are all bounded by 1. Show that $\rho \doteq \sum_n \frac{1}{2^n} \rho_n$ is a metric on X which gives the supremum of the ρ_n-topologies (i.e., the smallest topology containing all the ρ_n-topologies).

We first recall the following classical theorem of Alexandroff. Recall a set is G_δ if it is a countable intersection of open sets (the dual notion is F_σ, a countable union of closed sets).

Theorem 1.3 (Alexandroff)**.** *Let X be Polish. Then $Y \subseteq X$ (with the subspace topology) is Polish iff Y is a G_δ in X.*

Proof. Suppose first that Y is a G_δ, say $Y = \bigcap_n U_n$, where each U_n is open. Let ρ be a compatible complete metric on X bounded by 1. Define ρ_n on U_n by $\rho_n(x,y) = \rho(x,y) + |f_n(x) - f_n(y)|$, where $f_n(x) = \rho(x, X - U_n) = \inf_{y \in X - U_n} \rho(x,y)$. Clearly $\tau_\rho \subseteq \tau_{\rho_n}$ (since $\rho \leqslant \rho_n$). Since f_n is continuous, we also have $\tau_{\rho_n} \subseteq \tau_\rho$. So, ρ_n is a compatible metric on U_n. ρ_n is also a complete metric on U_n, for suppose $\{x_m\}$ is a ρ_n-Cauchy sequence from U_n. Since the $\rho_n(x_1, x_m)$ are bounded, so are the $f(x_m)$. So, $\exists \epsilon > 0\ \forall m\ \rho(x_m, X - U_n) \geqslant \epsilon$. Clearly $\{x_m\}$ is ρ-Cauchy, so the x_m converge to some $x \in X$. Since $\{x \colon \rho(x, X - U_n) \geqslant \epsilon\}$ is closed, $x \in U_n$. So, ρ_n is a complete compatible metric on U_n. Replacing ρ_n by $\min\{\rho_n, 1\}$, we may now assume that the ρ_n are all bounded by 1. Let now $d = \sum_n \frac{1}{2^n} \rho_n$, a metric on Y. Easily, d is a compatible metric on Y (this also follows from exercise 4). Also, d is complete on Y, for suppose $\{y_m\} \subseteq Y$ is d-Cauchy. Since $\rho_n \leqslant 2^n \cdot d$, $\{y_m\}$ is also ρ_n-Cauchy. So, there is a $y \in U_n$ such that $y_m \to y$. Since limits of sequences are unique in T_2 spaces, we must have $y \in \bigcap_n U_n$.

Suppose next that $Y \subseteq X$ is Polish in the subspace topology. Let ρ again be a compatible complete metric on X bounded by 1, and let d be a compatible complete metric on Y. For U an open set intersecting Y, define $\operatorname{osc}(d, U) = \sup\{\rho(x,y) \colon x, y \in U \cap Y\}$. For $y \in \bar{Y} = \operatorname{cl}_X(Y)$ define $\operatorname{osc}_d(y) = \inf\{\operatorname{osc}(d, U) \colon y \in U\}$. We claim that $Y = \{y \in \bar{Y} \colon \operatorname{osc}_d(y) = 0\}$. Clearly if $y \in Y$ then $\operatorname{osc}_d(y) = 0$. Suppose $y \in \bar{Y}$ and $\operatorname{osc}_d(y) = 0$. For each n choose an open set U_n about y such that $\operatorname{osc}(d, U_n) < \frac{1}{n}$ and $U_n \subseteq B_\rho(y, \frac{1}{n})$. If we let $y_n \in U_n$, then $\{y_n\}$ is d-Cauchy, so converges to some $z \in Y$. Since $U_n \subseteq B_\rho(y, \frac{1}{n})$ we also have $y_n \to z$, and so $y = z$. This proves the claim. Thus

$$\begin{aligned} Y &= \{y \in \bar{Y} \colon \operatorname{osc}_d(y) = 0\} \\ &= \bigcap_n \{y \in \bar{Y} \colon \operatorname{osc}_d(y) < \frac{1}{n}\} \end{aligned}$$

and $\{y \in \bar{Y} \colon \operatorname{osc}_d(y) < \frac{1}{n}\}$ is the intersection of the closed set $\bar{Y}$ with the open set $\bigcup\{U \colon U \cap Y \neq \varnothing \wedge \operatorname{osc}(d, U) < \frac{1}{n}\}$. Since a closed set in a metric space is a G_δ, we are done. $\square$

Let us consider in more detail two Polish spaces of particular interest, the Cantor space 2^ω and the Baire space ω^ω. For the Cantor space, we will take as our official definition the space $2^\omega = 2^{\mathbb{N}} = \prod_{\mathbb{N}} \{0,1\}$. The Baire space is $\omega^\omega = \prod_{\mathbb{N}} \omega$, where $\omega = \mathbb{N}$ carries the discrete topology. A basis for the product topology on 2^ω is the collection of sets of the form $N_s = \{x \in 2^\omega \colon x \restriction \operatorname{lh}(s) = s\}$, that is, all x which extend s, where $s \in 2^{<\omega}$ is a finite string of 0's and 1's. Similarly for the Baire space, where now $s \in \omega^{<\omega}$. We refer to $\{N_s\}$ as the standard basis for 2^ω or ω^ω. It is convenient to use the following standard metrics on 2^ω and ω^ω: $\rho(x,y) = \frac{1}{2^{i+1}}$, where i is least so that $x(i) \neq y(i)$.

From the definition of the product topology we see that a sequence $\{x_n\} \subseteq 2^\omega$ (or ω^ω) converges to x iff it converges coordinatewise, that is, $\forall i\ \exists n \forall m \geqslant n\ x_n(i) = x(i)$. Note that a function $f \colon 2^\omega \to 2^\omega$ (or from ω^ω to ω^ω) is continuous at x iff for every k there is an l so that if x, x' agree on the first l coordinates, then $f(x)$, $f(x')$ agree on the first k coordinates.

We visualize 2^ω (or ω^ω) as the set of branches or paths through the tree $2^{<\omega}$ (or $\omega^{<\omega}$). We make this precise.

Definition 1.4. A *tree* on a set X is a subset of $X^{<\omega}$ closed under initial segments, that is, if $s \in T$ and $m < \mathrm{lh}(s)$, then $s\restriction m \in T$. A *branch* or *path* through T is an $f\colon \omega \to T$ such that $\forall n\; f\restriction n \in T$. We let $[T]$ denote the set of branches through T.

The basic open sets N_s are the "cones" of branches determined by the sequence s.

Proposition 1.5. *The Cantor set 2^ω is homeomorphic to the Cantor middle thirds set $C \subseteq \mathbb{R}$.*

Proof. Recall $C = \bigcap_n F_n$ where $F_n \subseteq [0,1]$ is a disjoint union of 2^n closed intervals of length $(\frac{1}{3})^n$, obtained by successively removing the middle thirds of the previous intervals. So, $F_1 \supseteq F_2 \supseteq \dots$. Recall C is closed and nowhere dense. The disjoint intervals comprising F_n are naturally indexed by $s \in 2^n$. So, the intervals in F_1 are I_0, I_1, the intervals in F_2 are $I_{0,0}, I_{0,1} \subseteq I_0$ and $I_{1,0}, I_{1,1} \subseteq I_1$, etc. Define $f\colon 2^\omega \to C$ by $f(x) = \bigcap_n I_{x\restriction n}$. Clearly f is an onto mapping from 2^ω to C. f is one-to-one since if $x(i) \neq y(i)$ then $I_{x\restriction i+1} \cap I_{y\restriction i+1} = \varnothing$. To see that f is continuous note that if $x, y \in 2^\omega$ and $x\restriction n = y\restriction n = s$, then $f(x), f(y) \in I_s$, and so $|f(x) - f(y)| \leqslant \mathrm{diam}(I_s) = \frac{1}{3^n}$. Since f is a continuous bijection and 2^ω is compact (see below), it follows that f is also an open map and so a homeomorphism. We can also see this directly as follows. Suppose $y \in C$ and $f^{-1}(y) = x$. Let $\epsilon > 0$ and consider $B(x, \epsilon) \subseteq 2^\omega$. Take n large enough so that $\frac{1}{2^n} < \epsilon$. Let $s \in 2^n$ denote the unique sequence of length n such that $y \in I_s$. So, $x\restriction n = s$. Note that if $t \in 2^n$, $t \neq s$, then any point of I_t is at least $\frac{1}{3^n}$ away from any point of I_s. Hence, if $z \in C$ and $|y - z| \leqslant \frac{1}{3^n}$, then $f^{-1}(z)\restriction n = s$, and so $\rho(f^{-1}(y), f^{-1}(z)) \leqslant \frac{1}{2^n} < \epsilon$. So, f^{-1} is continuous. $\square$

Proposition 1.6. *The Baire space ω^ω is homeomorphic the space of irrationals Irr (with the subspace topology from $\mathbb{R}_{std}$).*

Proof. Let $\mathbb{Q} = \{q_0, q_1, q_2, \dots\}$. Let $I_0, I_1, \dots$ enumerate all of the open intervals of the form $(i, i+1)$ in $\mathbb{R}$. Suppose that I_s has been defined for all $s \in \omega^{\leqslant n}$, and each $I_s = (a_s, b_s)$ is an open interval of $\mathbb{R}$. Let $E_s \subseteq \mathbb{Q} \cap I_s$ be such that the only limit points of E_s are a_s and b_s. We may also assume that $q_n \in \bigcup_{s \in \omega^{\leqslant n}} E_s$. Let $\{I_{s^\frown i}\}_{i\in\omega}$ enumerate the subinterval of I_s determined by the points of E_s (that is, the two endpoints of each subinterval are in the set E_s). Without loss of generality we may assume that $\mathrm{diam}(E_{s^\frown i}) \leqslant \frac{1}{2^i}$ for all i. Note that $\overline{E_{s^\frown i}} \subseteq E_s$ for all s, i.

Define a map $f\colon \omega^\omega \to \mathbb{R}$ as follows. For $x \in \omega^\omega$ let $f(x) = \bigcap_n I_{x\restriction n}$. Since the I_s are strongly nested (i.e., if t extends s then $\overline{I_t} \subseteq I_s$) and $\mathrm{diam}(I_s) \leqslant \frac{1}{2^{\mathrm{lh}(s)}}$, $f(x)$ is a well-defined point in $\mathbb{R}$. We must have $f(x) \in \mathrm{Irr}$ since $f(x) \notin \bigcup_s E_s$, and $\bigcup_s E_s$ contains all of $\mathbb{Q}$. Since $\bigcup_s E_s \subseteq \mathbb{Q}$, for each $y \in \mathrm{Irr}$ there is for each n a unique sequence of length n such that $y \in I_s$. These sequences must extend each other to define an $x \in \omega^\omega$ (since if $s \perp t$ then $I_s \cap I_t = \varnothing$). By definition of f we then have $f(x) = y$. Thus, f is a bijection between ω^ω and Irr.

f is continuous since if $\rho(x, x') < \frac{1}{2^{n+1}}$, that is $x\restriction n = x'\restriction n$, then $f(x)$, $f(x')$ are in the same E_s for s of length n, and $\mathrm{diam}(E_s) \leqslant \frac{1}{2^n}$.

To see f^{-1} is continuous, let $y \in \mathrm{Irr}$ and let $x = f^{-1}(y)$. Fix an open set N_s about x, say $s = x\restriction n$. Since $f(x) = y$, $y \in I_s$. Since I_s is open in $\mathbb{R}$, let $\epsilon > 0$ be such that if $|y - y'| < \epsilon$ then $y' \in I_s$ as well. Thus, if $y' \in \mathrm{Irr}$ and $|y - y'| < \epsilon$, $f^{-1}(y')\restriction n = f^{-1}(y)\restriction n = s$. This shows f^{-1} is continuous. □

Let us study the open and closed sets in 2^ω and ω^ω a bit more, in particular we get a useful representation for the closed sets in these spaces.

Recall that the sets of the form N_s, for $s \in 2^{<\omega}$ (or $s \in \omega^{<\omega}$ in the case of the Baire space) form a base for the topology. Recall a topological space is said to be 0-dimensional if it has a base of clopen sets.

Proposition 1.7. *2^ω, ω^ω are 0-dimensional.*

Proof. To see that N_s is clopen, note that if $x \notin N_s$, then for some n, $t \doteq x\restriction n \perp s$. Also, $N_t \subseteq X - N_s$. Thus, $X - N_s = \bigcup\{N_t \colon t \perp s\}$ is open. □

In particular, 2^ω, ω^ω are totally disconnected (i.e., the maximal connected sets are points). We note that it is a theorem of topology that for compact T_2 spaces, 0-dimensional is the same as totally disconnected (but the two are not the same in general).

We can also say a tiny bit more about open sets in these spaces.

Lemma 1.8. *Any open set in 2^ω or ω^ω can be written as a (countable) disjoint union of basic open sets N_s.*

Proof. Let U be open in X (either 2^ω or ω^ω). Let A be the set of all $s \in 2^{<\omega}$ (or $s \in \omega^{<\omega}$) of minimal length such that $N_s \subseteq U$. If $x \in U$, then for some n, $x\restriction n \in A$ (as U is open). Thus $U = \bigcup_{s\in A} N_s$. If $s \neq t$ are both in A, then $s \perp t$ as otherwise one would not satisfy the minimality condition. Thus, $N_s \cap N_t = \varnothing$. □

Exercise 5. Show that lemma 1.8 also holds for $\mathbb{R}_{\mathrm{std}}$ (with the usual basis of open intervals), but fails for $\mathbb{R}^n$, $n \geqslant 2$.

The next lemma gives us a good picture of the closed sets in these spaces.

Lemma 1.9. *The closed sets in ω^ω (or 2^ω) are precisely those of the form $[T]$ where T is a tree on ω (or on $\{0, 1\}$ in the case of 2^ω; recall this means $T \subseteq \omega^{<\omega}$ or $T \subseteq 2^{<\omega}$). Also, we may take the tree T to be* pruned*, that is, T has no finite branches (that is, no terminal nodes in the tree).*

Proof. If T is a tree on ω, then $[T] \subseteq \omega^\omega$ is closed. To see this, suppose $x \notin [T]$. Then for some n, $x\restriction n \notin T$. Let $s = x\restriction n$. Then $N_s \subseteq \omega^\omega - [T]$, as no node extending s can be in T (by the definition of T being a tree). So, $[T]$ is closed.

Suppose next that $F \subseteq \omega^\omega$ is closed. Let T be the set of all $s \in \omega^{<\omega}$ such that there is an $x \in F$ which extends s. Clearly T is a tree. Also clearly T is pruned (if $s \in T$ the for some $x \in F$, x extends s. But then any longer initial segment of x is also in T). By definition, if $x \in F$ then $x\restriction n \in T$ for all n, so $F \subseteq [T]$. To see the other direction, suppose $x \in [T]$. So, for each n, $x\restriction n \in T$. By definition this means that there is an $x_n \in F$ with $x_n\restriction n = x\restriction n$. But then $x_n \to x$, and since F is closed, $x \in F$. So, $F = [T]$ for some pruned tree T. □

Exercise 6. Show directly that 2^ω is compact.

Exercise 7. Show that a closed set $F = [T] \subseteq \omega^\omega$ is compact iff T is *finitely splitting*, that is, every node of T has only finitely many immediate successors in T.

Exercise 8. Show that ω^ω is not σ-compact (i.e., it is not a countable union of compact sets).

We next give the Cantor–Bendixson analysis of the closed sets in a Polish space. Recall that for a set A in a topological space, A' denotes the set of limit points of A.

Definition 1.10. Let X be Polish and let $F \subseteq X$ be closed. The Cantor–Bendixson derivatives of F_α, $\alpha \in \mathrm{On}$, are defined as follows.

$$\begin{aligned} F_0 &= F \\ F_{\alpha+1} &= (F_\alpha)' \\ F_\alpha &= \bigcap_{\beta<\alpha} F_\beta \quad \text{for } \alpha \text{ limit} \end{aligned}$$

Thus, in passing from F_α to $F_{\alpha+1}$ we throw out all the points of F_α which are isolated points of F_α. Clearly the sequence of derivatives is monotonically decreasing, that is, if $\alpha < \beta$ then $F_\alpha \supseteq F_\beta$. Because the sequence is monotonically decreasing, there is a least ordinal such that $F_\alpha = F_{\alpha+1}$.

Definition 1.11. The least ordinal α such that $F_\alpha = F_{\alpha+1}$ is called the Cantor–Bendixson *rank* of the closed set F. We denote this ordinal by α_{cb}.

The next result, which is the Cantor–Bendixson analysis, shows that this is always a countable ordinal. Recall a (non-empty) set P in a topological space is said to be *perfect* if P contains no isolated points.

Exercise 9. Show that a perfect set P in a complete metric space has size $\mathfrak{c} = 2^\omega$. In fact, show that there is a continuous injection from 2^ω into P.

Theorem 1.12 (Cantor–Bendixson)**.** *Let F be a closed set in the Polish space X. Then the Cantor–Bendixson rank α_{cb} of F is countable. Furthermore, if $F_{\alpha_{cb}} = \varnothing$ then F is countable. If $F_{\alpha_{cb}} \neq \varnothing$, then $F_{\alpha_{cb}}$ is perfect (so F contains a perfect set).*

Proof. Let $U_0, U_1, \dots$ be a base for the topology. For each $x \in F - F_{\alpha_{\mathrm{cb}}}$, let $\alpha(x) < \alpha_{\mathrm{cb}}$ denote the unique ordinal such that $x \in F_{\alpha(x)} - F_{\alpha(x)+1}$. Let $n(x)$ be the least basic open set which isolates x in $F_{\alpha(x)}$, that is, $F_{\alpha(x)} \cap U_{n(x)} = \{x\}$. We claim that $x \mapsto n(x)$ is one-to-one. To see this, suppose $x \neq y$ are in $F - F_{\alpha_{\mathrm{cb}}}$ and $n(x) = n(y) = n$. If $\alpha(x) = \alpha(y)$ then we would have $\{x\} = U_n \cap F_{\alpha(x)} = U_n \cap F_{\alpha(y)} = \{y\}$, a contradiction. So, assume without loss of generality that $\alpha(x) < \alpha(y)$. Since $F_{\alpha(x)} \cap U_n = \{x\}$ and $x \notin F_{\alpha(x)+1}$, $U_n \cap F_{\alpha(x)+1} = \varnothing$, and so $U_n \cap F_{\alpha(y)} = \varnothing$, a contradiction.

If $F_{\alpha_{\mathrm{cb}}} = \varnothing$, then $\alpha(x)$, $n(x)$ are defined for all $x \in F$. Since $x \mapsto n(x)$ is one-to-one, this shows that F is countable. Suppose now $F_{\alpha_{\mathrm{cb}}} \neq \varnothing$. Since $(F_{\alpha_{\mathrm{cb}}})' = F_{\alpha_{\mathrm{cb}}}$, $F_{\alpha_{\mathrm{cb}}}$ is perfect. So, F contains the perfect set $F_{\alpha_{\mathrm{cb}}}$. $\square$

Corollary 1.13. *If F is a closed set in a Polish space X, then either F is countable or has cardinality $\mathfrak{c} = 2^\omega$.*

Corollary 1.14. *Any Polish space X is of the form $X = P \cup C$ where P is perfect (possibly empty) and C is countable.*

If the continuum hypothesis fails (that is, $\mathfrak{c} > \aleph_1$), then by AC we get a set $A \subseteq \mathbb{R}$ of size $\aleph_1$, that is $|\mathbb{N}| < |A| < \mathfrak{c}$. By corollary 1.13 such a set cannot be closed.

That is, the "pathological" set given by AC cannot be a simple set such as a closed set. A motivating principle in descriptive set theory is hinted at in this corollary: the reasonably defined subsets of a Polish space should avoid pathologies and have a structural theory.

It is instructive to interpret the proof of theorem 1.12 in the special case $X = \omega^\omega$. Given a tree T on a set X, say that a node $s \in X^{<\omega}$ is a *splitting node* if there are t_1, t_2 extending s with $t_1 \perp t_2$. Let T' denote the set of splitting nodes of T. So, suppose $F = [T] \subseteq \omega^\omega$ is closed. Let $T_0 = T$, and define by induction $T_{\alpha+1} = (T_\alpha)'$ and $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$ for α limit. Since T is countable, the least ordinal δ such that $T_{\delta+1} = T_\delta$ is countable. For each $x \in F - [T_\delta]$, let $\alpha(x)$ be largest such that $x \in [T_{\alpha(x)}]$ (this is clearly well-defined). Let $n(x)$ be least so that $x \restriction n(x) \notin T_{\alpha(x)+1}$. So, $x \restriction n(x)$ is a non-splitting node of $T_{\alpha(x)}$. Also, $N_{x \restriction n(x)} \cap [T_{\alpha(x)}] = \{x\}$. The same argument as before show that the map $x \mapsto n(x)$ is one-to one on $F - [T_\delta]$. So, if $T_\delta = \varnothing$, then F is countable. Otherwise, every node of T_δ is splitting, and so $[T_\delta]$ is perfect.

We can also run the Cantor–Bendixson analysis on an arbitrary set $C \subseteq X$. Let $C_0 = C$, $C_{\alpha+1} = C_\alpha \cap (C_\alpha)'$, and $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$ for α limit as before. As before, there is a least countable ordinal α_{cb} such that $C_{\alpha_{\mathrm{cb}}} = C_{\alpha_{\mathrm{cb}}+1}$. If $C_{\alpha_{\mathrm{cb}}} = \varnothing$, then C is countable as before. Otherwise, $C_{\alpha_{\mathrm{cb}}}$ is dense in itself. Conversely, if C contains a dense in itself set, then all derivatives of C are non-empty.

We can say a little more.

Theorem 1.15. *For $C \subseteq X$ (X Polish) the following are equivalent:*

(1) $C_{\alpha_{cb}} = \varnothing$
(2) *C is a countable G_δ.*

Proof. If C is a G_δ, then it is Polish with the subspace topology. So either $C_{\alpha_{\mathrm{cb}}} = \varnothing$ or else C contains a perfect set. The latter cannot happen as C is countable.

Suppose now that $C_{\alpha_{\mathrm{cb}}} = \varnothing$. Thus, C is countable (and hence an F_σ). It remains to show that C is a G_δ. For each $\alpha < \alpha_{\mathrm{cb}}$ and each $n \in \omega$ we define an open set U^n_α as follows. For each $x \in C_\alpha - C_{\alpha+1}$, and each n, let $B^n(x)$ be an open ball about x satisfying:

(1) $\rho(B^n(x), C_{\alpha+1}) > 0$
(2) $\mathrm{diam}(B^n(x)) < \frac{1}{2}\rho(x, C_\alpha - \{x\})$
(3) $\mathrm{diam}(B^n(x)) < \frac{1}{2^n}$

We can do this since $x \notin C_{\alpha+1}$ which is a closed set (which allows (1)) and x is not a limit point of C_α (which allows (2)). Note that if $x \neq y$ are both in $C_\alpha - C_{\alpha+1}$, then $B^n(x) \cap B^m(y) = \varnothing$ for any n, m (by (2)). Let $U^n_\alpha = \bigcup\{B^n(x) \colon x \in C_\alpha - C_{\alpha+1}\}$. Let $U^n = \bigcup_{\alpha<\alpha_{\mathrm{cb}}} U^n_\alpha$. Clearly $C_\alpha - C_{\alpha+1} \subseteq U^n_\alpha$, and hence $C \subseteq U^n$ for each n. Thus, $C \subseteq \bigcap_n U^n$. Suppose now that $x \in \bigcap_n U^n$. Let α_0 be least such that for some n we have $x \in U^n_{\alpha_0}$. Fix n_0 so that $x \in U^{n_0}_{\alpha_0}$. Let $x' \in C_{\alpha_0} - C_{\alpha_0+1}$ be the unique point such that $x \in B^{n_0}(x') \doteq B$. By (1), fix ϵ_0 such that $\rho(B, C_{\alpha_0+1}) > \epsilon_0$. Consider now any n large enough so that $\frac{1}{2^n} < \epsilon_0$. By assumption, $x \in U^n_\alpha$ for some $\alpha < \alpha_{\mathrm{cb}}$. We cannot have $\alpha > \alpha_0$ by (1) and (3). By minimality of α_0 we must then have $x \in U^n_{\alpha_0}$. By disjointness we must have $x \in B^n(x')$. By (3) we then have $x = x' \in C$. ∎

We prove another result concerning $\mathbf{\Delta}^0_2$ sets in a Polish space. The next result analyzes $\mathbf{\Delta}^0_2$ sets in a Polish space in terms of the so-called *difference hierarchy* which we now define.

Definition 1.16. Let $\{A_\alpha\}_{\alpha<\theta}$ be a sequence of sibsets of X. The difference operator is defined by

$$\mathcal{D}(\{A_\alpha\}) = \{x \colon \mu\alpha\, (x \notin A_\alpha) \text{ is odd}\}$$

whwre $\mu\alpha$ denotes "the least α." Note that there is no harm in assuming the sequence A_α is decreasing, as we can replace A_α with $\bigcap_{\beta\leqslant\alpha} A_\beta$.

We let $\mathcal{D}_\alpha(\mathbf{\Gamma})$ denote the collection of sets of the form $\mathcal{D}(\{A_\beta\}_{\beta<\alpha})$ where each $A_\beta \in \mathbf{\Gamma}$. In particular, we are interested in the classes $\mathcal{D}_\alpha(\mathbf{\Pi}^0_1)$ where $\alpha < \omega_1$.

Exercise 10. Show that if $A \in \mathcal{D}_\alpha(\mathbf{\Gamma})$, then $X - A \in \mathcal{D}_{\alpha+1}(\mathbf{\Gamma})$. [hint: put an extra copy of X at $\beta = 0$ and an extra copy of $\bigcap_{\alpha<\beta}$ at β for all limit β.] Deduce that $\mathcal{D}_\alpha(\mathbf{\Pi}^0_1) \subseteq \mathbf{\Delta}^0_2$ for all $\alpha < \omega_1$.

Theorem 1.17. *A set is $\mathbf{\Delta}^0_2$ iff it is in $\mathcal{D}_\alpha(\mathbf{\Pi}^0_1)$ for some $\alpha < \omega_1$.*

Proof. One direction is Exercise 10. For the other direction, assume $A \subseteq X$ is $\mathbf{\Delta}^0_2$. We define a decreasing sequence of closed sets F_α and associated open sets U_α, V_α (for α even) as follows. Let $F_0 = X$, and for limit α let $F_\alpha = \bigcap_{\beta<\alpha} A_\beta$. Given F_α where α is even, we define $F_{\alpha+1}$, $F_{\alpha+2}$, U_α and V_α as follows. Since both of A and $A^c = X - A$ are G_δ, they cannot both be dense in the Polish space F_α (otherwise their intersection would be dense, and in particular non-empty, a contradiction). So there an open set W in X such that $W \cap F_\alpha \neq \varnothing$ and either $W \cap F \subseteq A$ or $W \cap F \subseteq A^c$. Let $U_\alpha = \cup\{W \colon W \cap F_\alpha \subseteq A\}$ and $V_\alpha = \cup\{W \colon W \cap F_\alpha \subseteq A^c\}$. Let $F_{\alpha+1} = F_\alpha - U_\alpha$ and $F_{\alpha+2} = F_\alpha - (U_\alpha \cup V_\alpha)$. Clearly as long as $F_\alpha \neq \varnothing$ we have $F_{\alpha+2} \subsetneq F_\alpha$. Let $\alpha < \omega_1$ be least such that $F_\alpha = \varnothing$. Suppose $x \in X$. Let $\beta \leqslant \alpha$ be the least ordinal such that $x \notin F_\beta$. So, β is a successor ordinal. Let $\beta' < \beta$ be the largest even ordinal less than β (all limit ordinals are even). So, $\beta = \beta' + 1$ or $\beta = \beta' + 2$. Also, $x \in F_{\beta'}$, and $x \notin F_{\beta'+2}$. If $x \in F_{\beta'} - F_{\beta'+1}$ then $x \in F_{\beta'} \cap U_{\beta'} \subseteq A$, and if $x \in F_{\beta'+1} - F_{\beta'+2}$ then $x \in F_{\beta'} \cap V_{\beta'} \subseteq A^c$. So, $A = \mathcal{D}_\alpha(\{F_\beta\})$. □

1.1. **The Borel Hierarchy.** We introduce the Borel hierarchy of sets in a Polish space X. We will study the Borel sets in more detail shortly, but for now we introduce the hierarchy and state a few facts. Although we intend to study Polish spaces, the definition makes sense in a general topological space.

Definition 1.18. The Borel sets in a topological space X is the smallest σ-algebra (i.e., closed under countable unions, intersection, complements) containing the open sets.

The next definition stratifies the Borel sets into a natural hierarchy.

Definition 1.19. Let X be a topological space. The $\mathbf{\Sigma}^0_1$ sets in X are the open sets of the space X. The $\mathbf{\Pi}^0_1$ sets are the closed sets of X. A set A is $\mathbf{\Sigma}^0_\alpha$ if it is a countable union of sets $A = \bigcup_n A_n$ with each A_n being $\mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$. A set A is $\mathbf{\Pi}^0_\alpha$ if it is a countable intersection $A = \bigcap_n A_n$ with each A_n being $\mathbf{\Sigma}^0_{\beta_n}$ for some $\beta_n < \alpha$. A set is $\mathbf{\Delta}^0_\alpha$ if it is both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$.

So, $\mathbf{\Sigma}^0_1$ is the collection of open sets, $\mathbf{\Pi}^0_1$ the closed sets, $\mathbf{\Delta}^0_1$ the clopen sets, $\mathbf{\Sigma}^0_2$ the F_σ sets, and $\mathbf{\Pi}^0_2$ the G_δ sets.

Exercise 11. Let $Y \subseteq X$ be topological spaces. Show that if $A \subseteq X$ is $\boldsymbol{\Sigma}^0_\alpha$ (or $\boldsymbol{\Pi}^0_\alpha$, etc.), then $A \cap Y$ is $\boldsymbol{\Sigma}^0_\alpha$ in the relative topology on Y.

Definition 1.20. A (boldface) pointclass $\boldsymbol{\Gamma}$ is a collection of subsets of Polish spaces which is closed under continuous preimages. That is, for each Polish space X, $\boldsymbol{\Gamma}\restriction X \subseteq \mathcal{P}(X)$, and if $f\colon X \to Y$ is continuous and $A \subseteq Y$ is in $\boldsymbol{\Gamma}$, then $f^{-1}(A) \subseteq X$ is in $\boldsymbol{\Gamma}$.

The next definition abstracts the operations used in generating the Borel hierarchy.

Definition 1.21. Let $\boldsymbol{\Gamma}$ be a pointclass. The *dual class* $\check{\boldsymbol{\Gamma}}$ is defined by $\check{\boldsymbol{\Gamma}}\restriction X = \{X - A \colon A \in \boldsymbol{\Gamma}\restriction X\}$. We let $\bigcup_\omega \boldsymbol{\Gamma}$ be the collection of those $A \subseteq X$ which can be written as a countable union $A = \bigcup_n A_n$ with each $A_n \in \boldsymbol{\Gamma}$. Likewise we define $\bigcap_\omega \boldsymbol{\Gamma}$ using countable intersections.

The next lemma gives some of the elementary properties of these sets.

Lemma 1.22. *For any topological space X, the $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$ sets are closed under continuous preimages. The $\boldsymbol{\Sigma}^0_\alpha$ sets are closed under countable unions, and the $\boldsymbol{\Pi}^0_\alpha$ sets are closed under countable intersections. The complement of a $\boldsymbol{\Sigma}^0_\alpha$ set is a $\boldsymbol{\Pi}^0_\alpha$ set and vice-versa. $\boldsymbol{\Delta}^0_\alpha$ is closed under complements. For any $\alpha \geqslant 2$, $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$ are contained in $\boldsymbol{\Delta}^0_{\alpha+1}$. If X is a metric space, then this last fact holds also for $\alpha = 1$. For $\alpha \neq 3$, $\boldsymbol{\Sigma}^0_\alpha$ is closed under finite intersections, $\boldsymbol{\Pi}^0_\alpha$ under finite unions, and $\boldsymbol{\Delta}^0_\alpha$ under finite unions and intersections. If X is a metric space than this holds also for $\alpha = 3$. If we let $\mathcal{B} = \bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha$, then $\mathcal{B}$ is the σ-algebra of Borel subsets of X.*

Proof. The fact that $\boldsymbol{\Sigma}^0_1$ is closed under countable (in fact, arbitrary) unions and $\boldsymbol{\Pi}^0_1$ is closed under countable intersections follows immediately from the definition of a topology. For $\alpha \geqslant 2$, these same closure properties for $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ follow immediately from the definition of these classes. The complement of a $\boldsymbol{\Sigma}^0_1$ set is $\boldsymbol{\Pi}^0_1$ by definition and vice-versa, and a straightforward induction shows that $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$ are likewise dual classes for all α.

For $2 \leqslant \alpha < \beta$, it is immediate from the definitions that $\boldsymbol{\Sigma}^0_\alpha \subseteq \boldsymbol{\Sigma}^0_{\alpha+1}$ and likewise $\boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Pi}^0_{\alpha+1}$. For all $\alpha < \beta$ it is immediate that $\boldsymbol{\Sigma}^0_\alpha \subseteq \boldsymbol{\Pi}^0_\beta$ and $\boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Sigma}^0_\beta$ (any $\boldsymbol{\Sigma}^0_\alpha$ set is the countable intersection of itself). So, for $2 \leqslant \alpha < \beta$ we have $\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Delta}^0_\beta$. If X is metric then $\boldsymbol{\Pi}^0_1 \subseteq \boldsymbol{\Pi}^0_2$ as well, since for any closed set F we may write $F = \bigcap_n U_n$ where $U_n = \{x \colon \rho(x, F) < \frac{1}{n}\}$ is open. By duality we also have $\boldsymbol{\Sigma}^0_1 \subseteq \boldsymbol{\Sigma}^0_2$. Hence, for metric X, $\boldsymbol{\Sigma}^0_1, \boldsymbol{\Pi}^0_1 \subseteq \boldsymbol{\Delta}^0_2$.

For $\alpha \neq 3$ we show that $\boldsymbol{\Sigma}^0_\alpha$ is closed under finite intersections, and then also follows that $\boldsymbol{\Pi}^0_\alpha$ is closed under finite unions and $\boldsymbol{\Delta}^0_\alpha$ is closed under both finite unions and intersections. $\boldsymbol{\Sigma}^0_1$ is closed under finite intersections from the definition of a topology. If $A, B \in \boldsymbol{\Sigma}^0_2$, then write $A = \bigcup F_n$, $B = \bigcup_n H_n$, where the F_n, H_n are closed. Then $A \cap B = \bigcup_{n,m}(F_n \cap H_m)$, and each $F_n \cap H_m$ is closed. Similarly, if $A, B \in \boldsymbol{\Sigma}^0_\alpha$ and $\alpha \geqslant 4$, then write $A = \bigcup_n A_n$, $B = \bigcup_n B_n$ where each A_n, B_n are $\boldsymbol{\Pi}^0_{\alpha_n}$, $\boldsymbol{\Pi}^0_{\beta_n}$ respectively, where $\alpha_n, \beta_n < \alpha$. We again have $A \cap B = \bigcup_{n,m}(A_n \cap B_m)$. In an arbitrary space X we still have $\boldsymbol{\Pi}^0_1 \subseteq \boldsymbol{\Sigma}^0_2 \subseteq \boldsymbol{\Pi}^0_3$. From this and the fact that $\boldsymbol{\Pi}^0_\gamma \subseteq \boldsymbol{\Pi}^0_\delta$ whenever $2 \leqslant \gamma \leqslant \delta$, it follows that $A_n \cap B_m \in \boldsymbol{\Pi}^0_\delta$, where $\delta = \max\{\alpha_n, \beta_m, 3\} < \alpha$. Thus, $A \cap B \in \boldsymbol{\Sigma}^0_\alpha$. When X is a metric space, the argument works also for $\alpha = 3$ since $\boldsymbol{\Pi}^0_1 \subseteq \boldsymbol{\Pi}^0_2$ also holds in this case.

The fact that $\mathcal{B}$ is closed under countable unions follows from the fact that $\mathrm{cof}(\omega_1) > \omega$ (this uses AC). It is also closed under complements since every $\boldsymbol{\Pi}^0_\alpha$ set is $\boldsymbol{\Sigma}^0_{\alpha+1}$. So, $\mathcal{B}$ is a σ-algebra, and hence contains the Borel sets. On the other hand, clearly the $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$ sets stay inside any σ-algebra containing the open sets, and so $\mathcal{B}$ is equal to the Borel sets. □

Exercise 12. Let X be the topological space $[0, \omega_2)$ with the order topology. Show that there is an open set U and a closed set F in this space such that $U \cap F$ is not $\boldsymbol{\Sigma}^0_3$. [hint: Let $H = \{\omega_1 \cdot \alpha \colon \alpha < \omega_2\}$, that is, the set of all ordinal multiples of ω_1. H is closed, and let $U = X - H$. Let F be the set of all limit ordinals below ω_2, so F is closed. Suppose $F \cap H = \bigcup_n A_n \cup \bigcup_n B_n$ where each A_n is $\boldsymbol{\Pi}^0_1$ and each B_n is $\boldsymbol{\Pi}^0_2$. Argue that each A_n, and hence $\bigcup_n A_n$ is bounded below ω_2. Thus, for any large enough copy of ω_1, its set of limit ordinals is a union of $\boldsymbol{\Pi}^0_2$ sets. One of these $\boldsymbol{\Pi}^0_2$ sets must be stationary. But a stationary $\boldsymbol{\Pi}^0_2$ set in ω_1 must contain a tail, a contradiction.]

Exercise 13. Show that in the space $X = [0, \omega_1)$ the class $\boldsymbol{\Sigma}^0_3$ is closed under finite intersections. [hint: First argue that it is enough to show that the intersection of a $\boldsymbol{\Pi}^0_1$ set and a $\boldsymbol{\Pi}^0_2$ set is $\boldsymbol{\Sigma}^0_3$. So consider $F \cap \bigcap_n U_n$, where F is closed and each U_n is open. If any of these sets is bounded, the result is easy so assume otherwise. If all of the open sets U_n contain a tail of ω_1, then so does $\bigcap_n U_n$, and the result is again easy. Without loss of generality assume $H_0 \doteq X - U_0$ is c.u.b. in ω_1. Thus, $U_0 = \bigcup_{\alpha<\omega_1} I_\alpha$, where each I_α is a countable interval of ordinals, and these intervals are pairwise disjoint. Write each $F \cap I_\alpha$ as a countable intersection of sets relatively open in I_α (and so open in X). Say, $F \cap I_\alpha = \bigcap_j V^j_\alpha$. Let $V^j = \bigcup_\alpha V^j_\alpha$. Then $\bigcap_j V^j = F$. So, $F \cap \bigcap_n U_n$ is a $\boldsymbol{\Pi}^0_2$, and hence a $\boldsymbol{\Sigma}^0_3$ set.]

In particular, in the case of interest where X is a Polish space, all of the $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$ are closed under finite unions and intersections. $\boldsymbol{\Sigma}^0_\alpha$ is closed under countable unions, $\boldsymbol{\Pi}^0_\alpha$ under countable intersections, and $\boldsymbol{\Delta}^0_\alpha$ under complements. Also, $\boldsymbol{\Sigma}^0_\alpha, \boldsymbol{\Pi}^0_\alpha \subseteq \boldsymbol{\Delta}^0_\beta$ for any $\alpha < \beta < \omega_1$. Thus, we have the following picture of the Borel hierarchy in any Polish space:

$$\begin{tikzcd}[column sep=small]
 & \mathbf{\Sigma}^0_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Sigma}^0_2 \arrow[dr, phantom, sloped, "\subseteq"] & & & & \mathbf{\Sigma}^0_\alpha \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Sigma}^0_{\alpha+1} \arrow[dr, phantom, sloped, "\subseteq"] & & \\
\mathbf{\Delta}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Delta}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Delta}^0_3 \; \cdots & \cdots & \mathbf{\Delta}^0_\alpha \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Delta}^0_{\alpha+1} \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Delta}^0_{\alpha+2} \; \cdots \\
 & \mathbf{\Pi}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \mathbf{\Pi}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] & & & & \mathbf{\Pi}^0_\alpha \arrow[ur, phantom, sloped, "\subseteq"] & & \mathbf{\Pi}^0_{\alpha+1} \arrow[ur, phantom, sloped, "\subseteq"] & &
\end{tikzcd}$$

Exercise 14. Show that the collection of Borel sets in a metric space is the smallest collection containing the open and closed sets and closed under countable increasing unions and countable decreasing intersections.

Exercise 15. Show that the collection of Borel sets in a metric space is the smallest collection containing the open and closed sets and closed under countable disjoint unions and countable decreasing intersections.

Exercise 16. Show that the collection of Borel sets in a metric space is the smallest collection containing the open and closed sets and closed under countable disjoint unions and complements.

Exercise 17. Show that if $\mathcal{C} \subseteq \mathcal{P}(X)$ is a collection of subsets of a set X, and $\mathcal{C}$ is closed under complements and countable disjoint unions, then $\mathcal{C}$ is closed under countable increasing unions and countable decreasing intersections.

Exercise 18. Show that in $\mathbb{R}_{\text{std}}$, every Borel set is in the smallest collection containing the open intervals (a, b) and closed under complements and countable disjoint unions. Remark: This is also true for $\mathbb{R}^n_{\text{std}}$ by a result of [1].

We next introduce a notion of complexity for functions between Polish spaces.

Definition 1.23. Let $\mathbf{\Gamma}$ be a pointclass, and X, Y Polish spaces. We say a function $f \colon X \to Y$ is $\mathbf{\Gamma}$-measurable if for every open set $U \subseteq Y$, $f^{-1}(U) \in \mathbf{\Gamma} \restriction X$. We simply say f is Borel to mean Borel measurable.

Thus, continuity is equivalent to $\mathbf{\Sigma}^0_1$-measurability. We will see later that Borel measurability is equivalent to saying that the graph of f is a Borel set in $X \times Y$.

Exercise 19. Show that a composition of two Borel functions is a Borel function. Show that a composition of a $\mathbf{\Sigma}^0_n$ and a $\mathbf{\Sigma}^0_m$ measurable function is $\mathbf{\Sigma}^0_{n+m-1}$-measurable.

We next establish several "transfer" theorems which allow us to transfer results from one Polish space to another. In particular, we show that all uncountable Polish spaces are Borel isomorphic in a strong sense. We will also establish several results of independent interest along the way. The first result says in some sense that ω^ω is universal among the Polish spaces.

Lemma 1.24. *For any Polish space X there is a continuous surjection $\pi \colon \omega^\omega \to X$.*

Proof. Fix a countable base $\mathcal{B} = \{U_0, U_1, \dots\}$ for X. For $x \in \omega^\omega$, define a sequence of basic open sets in X as follows. Let $V_0^x = U_{x(0)}$. In general, let $V_{n+1}^x = U_{x(n+1)}$ if $\overline{U}_{x(n+1)} \subseteq V_n^x$ and $\operatorname{diam}(U_{x(n+1)}) < \frac{1}{2^n}$. Otherwise let $V_{n+1}^x = U_m$ where m is least such that $\overline{U}_m \subseteq V_n^x$ and $\operatorname{diam}(U_m) < \frac{1}{2^n}$. Let $\pi(x) = \bigcap_n V_n^x$, which is clearly a well-defined point in X.

To see π is onto, let $y \in X$. Get a sequence of basic open sets U_{i_0}, $U_{i_1}, \dots$ with $U_{i_n} \supseteq \overline{U}_{i_{n+1}}$ and $\operatorname{diam}(U_{i_n}) < \frac{1}{2^{n-1}}$ for all n. Then if $x = (i_0, i_1, \dots) \in \omega^\omega$, $\pi(x) = y$.

To see π is continuous, suppose $\pi(x) \in U$, where U is open in X. Let $\epsilon > 0$ be such that $B_\rho(\pi(x), \epsilon) \subseteq U$, and fix n large enough so that $\frac{1}{2^{n-2}} < \epsilon$. Then if $x \restriction n = x' \restriction n$, $\pi(x)$ and $\pi(x')$ lie in a set of diameter less than $\frac{1}{2^{n-2}}$, and so $\rho(x, x') \leqslant \frac{1}{2^{n-2}} < \epsilon$ implies $\pi(x') \in U$. $\square$

We restate exercise 9 in the following lemma.

Lemma 1.25. *If X is an uncountable Polish space, then there is an embedding from 2^ω into X.*

The next lemma strengthens lemma 1.24 (it is a strengthening since every closed subset of ω^ω is a retract of ω^ω).

Exercise 20. Show that every closed subset $F \subseteq \omega^\omega$ is a retract of ω^ω, and likewise for every closed subset of 2^ω (recall this means that there is a continuous function $f \colon \omega^\omega \to F$ such that $f \restriction F$ is the identity).

Lemma 1.26. *Let X be Polish space. Then there is a closed $F \subseteq \omega^\omega$ and a continuous bijection $\pi \colon F \to X$. Furthermore, the inverse map π^{-1} is $\mathbf{\Sigma}^0_2$-measurable.*

Proof. Fix a countable base $\mathcal{B} = \{U_0, U_1, \dots\}$ for X. For each $x \in X$ we define a sequence of basic open sets $U_{i_0(x)}$, $U_{i_1(x)}, \dots$ as follows. Let $i_0(x)$ be least such that $x \in U_{i_0(x)}$ and $\operatorname{diam}(U_{i_0}(x)) < 1$. In general, let $i_{n+1}(x)$ be the least integer such that $x \in U_{i_{n+1}(x)}$, $\operatorname{diam}(U_{i_{n+1}(x)}) < \frac{1}{2^{n+1}}$, and $\overline{U}_{i_{n+1}(x)} \subseteq U_{i_n(x)}$. For $x \in X$ let $f(x) = (i_0(x), i_1(x), \dots) \in \omega^\omega$. Clearly, f is one-to-one from X into ω^ω.

We claim that $F \doteq \operatorname{ran}(f) \subseteq \omega^\omega$ is closed. For suppose $z_n = f(x_n)$, and $z_n \to z$ in ω^ω. So, for any k we have that for all large enough n that $z_n(k) = z(k)$. So, for all large enough n, m, we have that x_n, x_m lie in a basic open set (i.e., $U_{z(k)}$) of diameter $< \frac{1}{2^k}$. So, $\{x_n\}$ is a Cauchy sequence in X, and hence converges to some point $x \in X$. We show that $f(x) = z$, which shows $z \in \operatorname{ran}(f)$. Suppose we have $f(x) \restriction i = z \restriction i$, and we show $(f(x))(i) = z(i)$. For all large enough n we have that

$$x_n \in U_{z(i+1)} \subseteq \overline{U}_{z(i+1)} \subseteq U_{z(i)}.$$

So, $x \in \overline{U}_{z(i+1)} \subseteq U_{z(i)}$. On the other hand, for any $j < z(i)$ such that $\overline{U}_j \subseteq U_{z(i-1)}$ and $\operatorname{diam}(U_j) < \frac{1}{2^i}$ we have for all large enough n that $x_n \notin U_j$. Since $x_n \to x$, we have $x \notin U_j$. From the definition of $f(x)$ we now have that $(f(x))(n) = z(n)$. So, $f(x) = z$.

Note that for $z = f(x) \in F$, that $x = \bigcap_n U_{z(n)}$. Thus, $\pi \doteq f^{-1} \colon F \to X$ is given by $\pi(z) = \bigcap_n U_{z(n)}$. Since for any $z \in F$, $\operatorname{diam}(U_{z(n)}) < \frac{1}{2^n}$ it follows that π is continuous.

Finally, we show that $\pi^{-1} = f$ is $\mathbf{\Sigma}^0_2$-measurable. So, let $s \in \omega^{<\omega}$ and consider the basic open set $F \cap N_s$ of F, which we may assume is non-empty. This says that $U_{z(0)} \supseteq \overline{U}_{z(1)} \supseteq U_{z(1)} \supseteq \overline{U}_{z(2)} \supseteq \cdots \supseteq U_{z(\mathrm{lh}(s))}$ and $\mathrm{diam}(U_{z(i)}) < \frac{1}{2^i}$ for all $i < \mathrm{lh}(s)$. We must compute $f^{-1}(N_s)$. Now

$$f^{-1}(N_s) = \{x \in X : x \in U_{z(\mathrm{lh}(s))} \wedge \forall i < \mathrm{lh}(s)\ \forall j \in I_i\ z \notin U_j\},$$

where I_i is the finite set of $j < z(i)$ satisfying $\overline{U}_j \subseteq U_{z(i-1)}$ and $\mathrm{diam}(U_j) < \frac{1}{2^i}$. So, $f^{-1}(N_s)$ is the intersection of an open and a closed set, so is in $\boldsymbol{\Delta}^0_2$. □

We next head toward the Borel isomorphism of two uncountable Polish spaces. First we need the following technical lemma.

Lemma 1.27. *Let $F \subseteq \omega^\omega$ be perfect, and let $C \subseteq F$ be countable and dense in F. Then $F - C$ is homeomorphic to ω^ω.*

Proof. Let $F = [T]$, where T is a pruned tree on ω. Let $C = \{c_0, c_1, \dots\}$. Since F is perfect, for every node $s \in T$ there is a splitting in T below s. We define a map f from $\omega^{<\omega}$ to T as follows. To begin, let $d_\varnothing = c_0 \in C \subseteq F$ (so $d_0 \in [T]$). Let $A_\varnothing = \{s \in T : s \perp d_\varnothing \wedge s\restriction(\mathrm{lh}(s) - 1)||d_\varnothing\}$ (recall $s||t$ means s and t are compatible, $s \perp t$ means they are incompatible). $A_\varnothing$ is infinite since every node of T eventually splits (for every k, there are two incompatible nodes s_k, t_k extending $c_0\restriction k$. At least one of these, say s_k, must be incompatible with $d_\varnothing$. Now go a larger l such that $s_k(l) \neq d_\varnothing(l)$ and repeat the argument. Continuing we get an ω sequence in A_0). Let $s_0, s_1, \dots$ enumerate A_0. Set $f(i_0) = s_{i_0}$. Now repeat this process below each of the nodes s_i. That is, for each s_i pick a $d_i \in F$ extending s_i (which is possible as C is dense). Also, if c_1 extends s_i, then let $d_i = c_1$. Let $A_i = \{s \in T : (s \text{ extends } s_i) \wedge (s \perp d_i) \wedge (s_i\restriction(\mathrm{lh}(s_i) - 1))||d_i\}$. As before, each A_i is infinite. Let $A_i = \{s_{i,0}, s_{i,1}, \dots\}$. Then set $f(i_0, i_1) = s_{i_0,i_1}$. Continuing we define the map f.

The map f naturally gives rise to a map $\pi : \omega^\omega \to F$, namely, $\pi(z) = x$ where $x\restriction n = s_{z(0),z(1),\dots,z(n-1)}$. Clearly, $\pi(z) \in F - C$ since every $f(s)$ for s of length i is incompatible with c_i. π is onto $F - C$, for suppose $x \in F - C$. Since $x \neq d_\varnothing = c_0$, there is a least n_0 such that $x(n_0) \neq d_\varnothing(n_0)$. Then $x\restriction(n_0 + 1) = s_{i_0}$ for some i_0. Since $x \neq d_{i_0}$, there is an $n_1 > n_0$ such that $x(n_1) \neq d_{i_0}(n_1)$. So, for some i_1, $x\restriction(n_1 + 1) = s_{i_0,i_1}$. Continuing, we define $z = (i_0, i_1, \dots)$ such that $\pi(z) = x$.

It is clear from the construction that both π and π^{-1} are continuous. □

Exercise 21. Give an example of a perfect Polish space X and a countable dense set $C \subseteq X$ such that $X - C$ is not homeomorphic to ω^ω. [hint: try $X = \mathbb{R}^2_{\mathrm{std}}$.]

Exercise 22. Show that if $C \subseteq \mathbb{R}_{\mathrm{std}}$ is countable dense then $\mathbb{R} - C$ is homeomorphic to ω^ω. [hint: follow the proof that $\mathbb{R} - \mathbb{Q}$ is homeomorphic to ω^ω.]

Now we are ready for the isomorphism result.

Theorem 1.28. *Let X be an uncountable Polish space. Then there is a bijection $\pi : \omega^\omega \to X$ such that both π and π^{-1} are $\boldsymbol{\Delta}^0_3$-measurable.*

Proof. From lemma 1.26, let $f : F \to X$ be a bijection where $F \subseteq \omega^\omega$ is closed, f is continuous, and f^{-1} is $\boldsymbol{\Delta}^0_2$-measurable. From the Cantor–Bendixson analysis write $F = P \cup C_1$, a disjoint union, where P is perfect and C_1 is countable. From lemma 1.27, let $C_2 \subseteq P$ be countable such that $P - C_2$ is homeomorphic to ω^ω. Also, let $D \subseteq \omega^\omega$ be countable dense, so by lemma 1.27 we also have $\omega^\omega - D \approx \omega^\omega$. Let $g : \omega^\omega - D \to F - C$ be a homeomorphism, where $C = C_1 \cup C_2$. Extend g

to a bijection $h\colon \omega^\omega \to F$ by taking an arbitrary bijection between the countable infinite sets D and C. Let $\pi = f \circ h$. Clearly, π is a bijection between ω^ω and X.

Let $U \subseteq X$ be open. So, $V = f^{-1}(U)$ is open in F. Also,

$$V = f^{-1}(U) = (V \cap (F - C)) \cup (V \cap C).$$

Now, $h^{-1}(V) = g^{-1}(V \cap (F - C)) \cup h^{-1}(V \cap C)$. Since g is a homeomorphism, $g^{-1}(V \cap (F - C))$ is open in $\omega^\omega - D$, and thus is the intersection of an open set and a co-countable set in ω^ω. Thus, $g^{-1}(V \cap (F - C))$ is $\mathbf{\Pi}^0_2$ in ω^ω. Also, $h^{-1}(V \cap C)$ is countable. So, $h^{-1}(f^{-1}(U))$ is the union of $\mathbf{\Pi}^0_2$ set and a countable set, so is $\mathbf{\Delta}^0_3$. So, π is $\mathbf{\Delta}^0_3$-measurable.

Suppose now $U \subseteq \omega^\omega$ is open. Then

$$h(U) = g(U - D) \cup h(U \cap D).$$

Since g is a homeomorphism, $g(U - D)$ is open in $F - C$. So, $g(U - D) = V - C$, where V is open in F. So, $h(U) = (V - C) \cup E$, where E is countable. Since $f(V)$ is $\mathbf{\Sigma}^0_2$, $f(h(U))$ is of the form $(S - E_1) \cup E_2$, where $S \in \mathbf{\Sigma}^0_2$ and E_1, E_2 are countable. So, $\pi(U) \in \mathbf{\Delta}^0_3$. This shows π^{-1} is $\mathbf{\Delta}^0_3$ measurable. □

Remark 1.29. The proof actually shows that there is a countable set $C \subseteq \omega^\omega$ such that $\pi \restriction (\omega^\omega - C)$ is continuous.

The next results provide a topological characterization of the spaces 2^ω, ω^ω.

Theorem 1.30. 2^ω *is the unique up to homeomorphism space which is compact, metrizable, perfect, and* 0*-dimensional.*

Proof. 2^ω clearly has all the stated properties. Suppose X is a 0-dimensional compact metric space (recall a compact metric space is complete, so X is Polish). To begin, let $U_0, U_1, \dots, U_k$ be cover of X by disjoint clopen sets of diameter $< \frac{1}{2}$. We may do this as X is compact and 0-dimensional. Since X is also perfect, by further splitting these clopen sets we may assume that k is a power of 2, say $k = 2^{n_0}$.

Now repeat the argument to write each U_i as a finite disjoint union of clopen sets $U_{i,j}$, $j < k(i)$, with $\mathrm{diam}(U_{i,j}) < \frac{1}{4}$, and $\overline{U}_{i,j} \subseteq U_i$. Again, we can do this as each U_i is compact and X is 0-dimensional. By further splitting, we may assume that for some $n_1 > n_0$ that $k(i) = 2^{n_1}$ for all i.

Continuing, we define the clopen sets U_s for all $s \in \omega^{<\omega}$ with $s(0) < 2^{n_0}$, $s(1) < 2^{n_1}, \dots, s(\mathrm{lh}(s) - 1) < 2^{n_{\mathrm{lh}(s)-1}}$. Now define $\pi\colon 2^\omega \to X$ as follows. Let $x \in 2^\omega$. For each j, let $x \restriction 2^{n_j}$ be the i_j^{th} binary sequence of length n_j. Then set $\pi(x) = \bigcap_j U_{i_0, i_1, \dots, i_j}$.

Because the U_s are strongly nested and their diameters go to 0, $\pi(x)$ is well-defined. By the disjointness of the U_s for s of a given length, the map π is one-to-one. Since these U_s also cover, π is onto. The proofs that π and π^{-1} are continuous is straightforward. □

Theorem 1.31. ω^ω *is the unique up to homeomorphism space which is perfect, Polish,* 0*-dimensional, and the closure of every open set is not compact.*

Proof. It is clear that ω^ω has all the stated properties. Assume now X has the stated properties. Let $\mathcal{U}$ be an open cover of X with no finite subcover. Let $\mathcal{B}$ be all the basic clopen sets B of X such that $\mathrm{diam}(B) < \frac{1}{2}$ and for some $U \in \mathcal{U}$, $B \subseteq U$. Clearly, $\mathcal{B}$ is a cover of X with no finite subcover. Let $\mathcal{B} = \{B_0, B_1, \dots\}$. Define $U_0 = B_0$, $U_1 = B_1 - B_0$, $U_n = B_n - (B_0 \cup \cdots \cup B_{n-1})$. By reindexing, we may assume that all of the U_i are non-empty (using the fact that no finite subset of $\mathcal{B}$ covers X). So, the U_i are a pairwise disjoint collection of clopen sets of diameter $< \frac{1}{2}$ which cover X.

Now repeat the argument to write each U_i as a countably infinite disjoint union of clopen sets $U_{i,j}$, $j \in \omega$, with $\operatorname{diam}(U_{i,j}) < \frac{1}{4}$, and $\overline{U}_{i,j} \subseteq U_i$. We use here the fact that each clopen set U_i is not compact (by hypothesis). Define $\pi \colon \omega^\omega \to X$ by $\pi(x) = \bigcap_j U_{x \restriction j}$. This is easily a bijection and it is again straightforward to check that π and π^{-1} are continuous. $\square$

Of course, other Polish spaces of interest have topological characterizations as well. For example, $[0,1]$ is the unique up to homeomorphism space which is compact metrizable, connected, and has exactly two non-cut points. The focus in descriptive set theory, however, is not so much at the topological level, but at a somewhat higher level. According to theorem 1.28, at a little past the topological level all the uncountable Polish spaces appear the same.

One of the advantages of using ω^ω (or 2^ω) is that $\omega^\omega \times \omega^\omega \approx \omega^\omega$, and moreover $\prod_{n \in \omega} \omega^\omega \approx \omega^\omega$. This means that there are continuous coding and decoding maps from $(\omega^\omega)^n$ or $(\omega^\omega)^\omega$ into ω^ω. This simple fact is nevertheless of crucial importance in many arguments. Let us be specific about these maps. Fix a bijection $(n,m) \mapsto \langle n,m \rangle \in \omega$ from $\omega \times \omega$ to ω. We let $n \mapsto ((n)_0, (n)_1)$ denote the inverse of this map. Define $\pi \colon (\omega^\omega)^\omega \to \omega^\omega$ by $\pi(x_0, x_1, \dots) = y$ where $y(n) = x_{(n)_0}((n)_1)$. Extending the notation used for integers, we frequently write $(x_0, x_1, \dots) \mapsto \langle x_0, x_1, \dots \rangle \in \omega^\omega$ for this map. The decoding map (the inverse of π) is given by $x \mapsto ((x)_0, (x)_1, \dots)$ where $(x)_i(j) = x(\langle i,j \rangle)$. When there is no danger of confusion we will drop the extra parentheses, and just write $x \mapsto (x_0, x_1, \dots)$ for the decoding map. We can similarly get bijections from $(\omega^\omega)^n$ to ω^ω (starting with a bijection between $\omega \times n$ and ω for example). With a slight abuse of notation we will use the same notation for all of these maps (e.g., $(x,y) \mapsto \langle x,y \rangle$ and $x \mapsto (x_0, x_1)$ for the coding and decoding maps between $\omega^\omega \times \omega^\omega$ and ω^ω).

In an entirely similar manner we also define the coding and decoding maps between products of 2^ω and 2^ω, and we continue to use the same notation.

Exercise 23. Show that the coding and decoding maps in all cases are continuous.

We have analyzed the open and closed sets in Polish spaces. We push a little further and analyze the $\mathbf{\Delta}^0_2$ sets. For this we need to introduce the *difference hierarchy*. Recall that an ordinal is odd (even) if it is of the form $\lambda + k$ where λ is limit and $k \in \omega$, and k is odd (even).

Definition 1.32. Let $\mathbf{\Gamma}$ be a pointclass and $\alpha \in \mathrm{On}$. The α-$\mathbf{\Gamma}$ difference sets, denoted $\mathcal{D}_\alpha(\mathbf{\Gamma})$, are those sets A for which there is an α length sequence $\{A_\beta\}_{\beta<\alpha}$, with each $A_\beta \in \mathbf{\Gamma}$, such that $A = \mathcal{D}(\{A_\beta\})$, where $x \in \mathcal{D}(\{A_\beta\})$ iff the least β such that $x \notin A_\beta$ is odd (we regard A_α as being the empty set).

The next theorem analyzes the $\mathbf{\Delta}^0_2$ sets.

Theorem 1.33. *A set A in a Polish space X is $\mathbf{\Delta}^0_2$ iff it is α-$\mathbf{\Pi}^0_1$ for some countable α.*

Proof. If A is α-$\mathbf{\Pi}^0_1$, then both A and $X - A$ are easily $\mathbf{\Sigma}^0_2$. For example

$$A = \bigcup_{\beta<\alpha, \beta \text{ odd}} \Big(\bigcap_{\gamma<\beta} A_\gamma \cap (X - A_\beta) \Big).$$

The quantity in parentheses is the intersection of a closed and an open set, so is $\mathbf{\Delta}^0_2$. So, $A \in \mathbf{\Sigma}^0_2$. A similar computation shows $X - A \in \mathbf{\Sigma}^0_2$, so $A \in \mathbf{\Delta}^0_2$.

Suppose now that $A \in \mathbf{\Delta}^0_2$. Let $B = X - A$, so both A, B are $\mathbf{\Pi}^0_2$. Note that for any closed set $F \subseteq X$ we cannot have that both $A \cap F$ and $B \cap F$ are dense in F, as the intersection of two dense $\mathbf{\Pi}^0_2$ sets in the Polish space F is non-empty (Baire category). So for any closed F, there is a basic open set U such that $F \cap U \subseteq A$ or $F \cap U \subseteq B$ (and $F \cap U \neq \emptyset$).

We define a decreasing sequence F_β of closed sets along with open sets A_β and B_β. Let $\mathcal{B}$ be a base for X. Let $F_0 = X$. Let $A_0 = \bigcup\{U \in \mathcal{B} \colon U \subseteq A\}$, and likewise $B_0 = \bigcup\{U \in \mathcal{B} \colon U \subseteq B\}$. At least one of A_0, B_0 is non-empty. Let $F_1 = F_0 - A_0$ and $F_2 = F - (A_0 \cup B_0)$. In general, for β limit define $F_\beta = \bigcap_{\gamma<\beta} F_\gamma$. Otherwise, given F_β let $A_\beta = \bigcup\{U \in \mathcal{B} \colon F_\beta \cap U \subseteq A\}$, and $B_\beta = \bigcup\{U \in \mathcal{B} \colon F_\beta \cap U \subseteq B\}$. Note that $A_\beta \cap B_\beta \cap F_\beta = \emptyset$. Again, assuming $F_\beta \neq \emptyset$, at least one of A_β, B_β is non-empty. Let $F_{\beta+1} = F_\beta - A_\beta$, $F_{\beta+2} = F_\beta - (A_\beta \cup B_\beta)$.

For some least countable ordinal δ we have that $F_\delta = \emptyset$. Fix $x \in X$, and let α be least such that $x \notin F_\alpha$. We claim that $x \in A$ iff α is odd. Since we took intersections at limit ordinals, it make sense to let β be the largest even ordinal such that $x \in F_\beta$. So either $x \in F_\beta - F_{\beta+1} = F_\beta \cap A_\beta$ or $x \in F_{\beta_1} - F_{\beta+2} = F_\beta \cap B_\beta$. In the first case we have $x \in A$ and $\alpha = \beta + 1$ is odd. In the second case we have $x \notin A$ and $\alpha = \beta + 2$ is even. This shows that A is δ-$\mathbf{\Pi}^0_1$. $\square$

We will extend theorem 1.33 later to higher levels of the Borel hierarchy.

We next present a useful notational shorthand for describing sets and set operations in Polish spaces, called *logical notation*. The idea is to use the terminology of first order logic in describing sets and set operations. Although only a notational change, it turns out to be very useful, especially since it is intrinsically tied to the Borel and projective hierarchies (to be defined shortly). If X is a Polish space and $P \subseteq X$, we regard P as a property or unary relation of the set X. Thus, instead of "$x \in P$" we write $P(x)$, and instead of "$x \notin P$" we write $\neg P(x)$. Similarly, we write $P(x) \vee Q(x)$ in place of $x \in (P \cup Q)$ and $P(x) \wedge Q(x)$ in place of $x \in (P \cap Q)$. Countable unions and intersections become correspond to existential and universal quantification over the natural numbers respectively. To be specific, suppose $P = \bigcup_n P_n$. Define $Q \subseteq X \times \omega$ by $Q(x, n) \leftrightarrow (x \in P_n)$. Then $P(x) \leftrightarrow \exists n\ Q(x, n)$. More generally, we make the following definition.

Definition 1.34. Let $\mathbf{\Gamma}$ be a pointclass, and X, Y Polish spaces. The pointclass $\exists^Y \mathbf{\Gamma}$ is defined by: $A \subseteq X$ is in $\exists^Y \mathbf{\Gamma}$ if there is a $B \in X \times Y$, $B \in \mathbf{\Gamma}$, such that $A(x) \leftrightarrow \exists y \in Y\ B(x, y)$. Likewise $A \in \forall^Y \mathbf{\Gamma}$ if there is a $B \in X \times Y$, $B \in \mathbf{\Gamma}$, such that $A(x) \leftrightarrow \forall y \in Y\ B(x, y)$.

Note that $A \in \exists^Y \mathbf{\Gamma}$ iff A is the projection onto X of a $\mathbf{\Gamma}$ set in $X \times Y$. Thus, in non-logical notation we would describe $\exists^Y \mathbf{\Gamma}$ as the collection of projections of $\mathbf{\Gamma}$ sets (from sets in $X \times Y$).

The next exercise makes precise our comments above about countable unions and intersections.

Exercise 24. Let $\mathbf{\Gamma}$ be a (boldface) pointclass. Show that any set in $\exists^\omega \mathbf{\Gamma}$ is a countable union of sets in $\mathbf{\Gamma}$. Show that the converse is not true in general (hint: consider $\mathbf{\Gamma} = \bigcup_n \mathbf{\Sigma}^0_n$). Show that the converse also holds provided the *countable join* of sets in $\mathbf{\Gamma}$ is in $\mathbf{\Gamma}$. Given sets $P_n \subseteq X$, their join is the set $P(x, n) \leftrightarrow (x \in P_n)$.

According to the next exercise, in quantifying over Polish spaces we only need to consider the cases $Y = \omega$ and $Y = \omega^\omega$.

Exercise 25. Let $\mathbf{\Gamma}$ be a pointclass and X, Y Polish spaces. Show that if $A \subseteq X$ is in $\exists^Y \mathbf{\Gamma}$, then $A \in \exists^{\omega^\omega} \mathbf{\Gamma}$. (hint: consider a continuous map $\pi \colon X \times \omega^\omega \xrightarrow{\text{onto}} X \times Y$).

The next definition is of fundamental importance in the theory of pointclasses.

Definition 1.35. Let $\mathbf{\Gamma}$ be a pointclass, and X, Z be Polish spaces. We say $U \subseteq Z \times X$ is *universal* for $\mathbf{\Gamma}\restriction X$ if $U \in \mathbf{\Gamma}$ and for every $A \in \mathbf{\Gamma}\restriction X$, A occurs as a section of U. That is, $\exists z \in Z\ (A = U_z)$, where $U_z \doteq \{x \in X \colon (z, x) \in U\}$ is the z-section of U.

The next lemma provides universal sets at the bottom level. These will then propagate to higher pointclasses.

Lemma 1.36. *For any Polish space X there is a set $U \subseteq 2^\omega \times X$ which is universal for $\mathbf{\Sigma}^0_1\restriction X$.*

Proof. Fix a base $\mathcal{B} = \{U_0, U_1, \dots\}$ for X. Define

$$(z, x) \in U \leftrightarrow \exists n \in \omega\ (z(n) = 1 \wedge x \in U_n).$$

Clearly, every open set in X is a section of U. Also, $U \subseteq 2^\omega \times X$ is open. To see this, suppose $(z, x) \in U$. Fix n so that $z(n) = 1$ and $x \in U_n$. Then $V \doteq \{z \in 2^\omega \colon z(n) = 1\}$ is a basic open set in 2^ω and $(z, x) \in V \times U_n \subseteq U$. $\square$

The next lemma propagates universal sets under complements, countable unions, and countable intersections.

Lemma 1.37. *Let $\mathbf{\Gamma}$ be a pointclass, and $U \subseteq Z \times X$ a universal set for $\mathbf{\Gamma}\restriction X$. then $U' \doteq Z \times X - U$ is universal for $\check{\mathbf{\Gamma}}\restriction X$.*

Suppose $\mathbf{\Gamma}_n$ are pointclasses and $U_n \subseteq 2^\omega \times X$ (or $\omega^\omega \times X$) is universal for $\mathbf{\Gamma}_n\restriction X$. Define $U(z, x) \leftrightarrow \exists n \in \omega\ U_n((z)_n, x)$. Then U is universal for $\bigcup_\omega \mathbf{\Gamma}_n\restriction X$. Similarly, $U(z, x) \leftrightarrow \forall n \in \omega\ U_n((z)_n, x)$ is universal for $\bigcap_\omega \mathbf{\Gamma}_n\restriction X$.

Proof. The first claim is easily checked. Suppose now $\mathbf{\Gamma}_n$, U_n are as above and define $U(z, x) \leftrightarrow \exists n \in \omega\ U_n((z)_n, x)$. If $A \in \bigcup_\omega \mathbf{\Gamma}_n$, write $A = \bigcup_n A_n$ with $A_n \in \mathbf{\Gamma}_n$. Fix for each n a z_n such that $\forall x\ (A_n(x) \leftrightarrow U_n(z_n, x))$. Let $z = \langle z_0, z_1, \dots \rangle$. Clearly,

$$U(z, x) \leftrightarrow \exists n \in \omega\ U_n((z)_n, x) \leftrightarrow \exists n \in \omega\ U_n(z_n, x) \leftrightarrow \exists n\ (x \in A_n).$$

Also, $U = \bigcup_n C_n$, where $C_n(z, x) \leftrightarrow U_n((z)_n, x)$. Since $z \mapsto (z)_n$ is continuous and $\mathbf{\Gamma}_n$ is a pointclass, $C_n \in \mathbf{\Gamma}_n$. The argument for $\bigcap_\omega \mathbf{\Gamma}_n$ is similar. $\square$

As a corollary of lemma 1.36 and 1.37 we have the following theorem.

Theorem 1.38. *Let X be a Polish space. Then for any $\alpha < \omega_1$ there is a universal set $U \subseteq 2^\omega \times X$ for $\mathbf{\Sigma}^0_\alpha\restriction X$ and likewise a universal set for $\mathbf{\Pi}^0_\alpha\restriction X$.*

An important consequence of having universal sets is non-selfduality, according to the next result.

Theorem 1.39. *Suppose $\mathbf{\Gamma}$ is a pointclass having a universal set $U \subseteq X \times X$ (for $\mathbf{\Gamma}\restriction X$). Then $U \notin \check{\mathbf{\Gamma}}$. In particular, $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ (i.e., $\mathbf{\Gamma}$ is not self-dual).*

Proof. Suppose $U \subseteq X \times X$ is in $\mathbf{\Gamma}$ and is universal for $\mathbf{\Gamma}\restriction X$. Suppose $U \in \check{\mathbf{\Gamma}}$. Define $A(x) \leftrightarrow \neg U(x, x)$. Since $x \mapsto (x, x)$ is continuous, $A \in \mathbf{\Gamma}$. But then $A = U_z$ for some fixed $z \in X$. Then, $A(z) \leftrightarrow U(z, z) \leftrightarrow \neg A(z)$, a contradiction. $\square$

The last proof was just the usual Cantor diagonal argument.

As a corollary we have the following theorem, which says that in any uncountable Polish space there is no collapsing in the Borel hierarchy.

Theorem 1.40. *Let X be an uncountable Polish space. Then for every $\alpha < \omega_1$ there is a set $A \in \mathbf{\Sigma}^0_\alpha - \mathbf{\Pi}^0_\alpha$. There is also for each $\alpha > 1$ a set in $\mathbf{\Delta}^0_\alpha - \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta$.*

Proof. Let X be an uncountable Polish space. The claim is clear for $\alpha = 1$ so assume $\alpha > 1$. (If every open set were closed, every point, and hence every subset of X would be open, giving X the discrete topology. This contradicts X being uncountable and separable.) From lemma 1.25 there is subspace $C \subseteq X$ homeomorphic to 2^ω. Since 2^ω is compact, C is closed in X. From theorems 1.38 and 1.39 there is an $A \subseteq C$ which is $\mathbf{\Sigma}^0_\alpha$ but not $\mathbf{\Pi}^0_\alpha$ in the subspace topology on C. Thus in X, A is the intersection of a $\mathbf{\Sigma}^0_\alpha$ and a closed set, and hence is $\mathbf{\Sigma}^0_\alpha$ (as $\alpha > 1$). If A were $\mathbf{\Pi}^0_\alpha$ in X, then $A = A \cap C$ would also be $\mathbf{\Pi}^0_\alpha$ in the relative topology on C, a contradiction. This proves the first claim.

For the second claim, suppose now $\alpha \geqslant 2$. As above, it again suffices to show there is an $A \in 2^\omega$ which is $\mathbf{\Delta}^0_\alpha$ but not in $\mathbf{\Sigma}^0_\beta$ for any $\beta < \alpha$. If α is a successor, say $\alpha = \beta + 1$, then let $A \subseteq 2^\omega$ with $A \in \mathbf{\Sigma}^0_\beta - \mathbf{\Pi}^0_\beta$, and let $B \subseteq 2^\omega$ with $B \in \mathbf{\Pi}^0_\beta - \mathbf{\Sigma}^0_\beta$. Let then $C \subseteq 2^\omega$ be the join of A and B, that is,

$$C(x) \leftrightarrow (x(0) = 0 \wedge x' \in A) \vee (x(0) = 1 \wedge x' \in B),$$

where $x'(i) = x(i + 1)$ for all i. Clearly, $C \in \mathbf{\Delta}^0_\alpha$ (by closure of $\mathbf{\Delta}^0_\alpha$ under finite unions, intersections). Also, $A \notin \mathbf{\Sigma}^0_\beta$, as then $B = \{x \colon 1^\frown x \in C\}$ would be in $\mathbf{\Sigma}^0_\beta$. □

Thus, in the picture of the Borel hierarchy shown earlier, all of the containments shown are proper (assuming X is uncountable Polish).

The next two exercises analyze the Borel sets in an ordinal space $[0, \alpha)$.

Exercise 26. Let $\alpha \in \mathrm{On}$ with $\mathrm{cof}(\alpha) > \omega$. Show that if $B \subseteq [0, \alpha)$ is Borel, and $C \subseteq [0, \alpha)$ is c.u.b. (closed and unbounded), then there is a c.u.b. $D \subseteq C$ such that B either contains or omits a tail of D.

Exercise 27. Show that in any ordinal space $X = [0, \alpha)$, every Borel set is $\mathbf{\Delta}^0_4$. Show that in $X = [0, \omega_1)$ every Borel set is $\mathbf{\Delta}^0_3$. [hint: It is enough to show it is $\mathbf{\Sigma}^0_4$. Show this by induction on α. For α successor or limit with $\mathrm{cof}(\alpha) = \omega$, the result follows easily by induction. If $\mathrm{cof}(\alpha) > \omega$, let $C \subseteq \alpha$ be c.u.b. in α of order-type $\mathrm{cof}(\alpha)$. Let $B \subseteq [0, \alpha)$ be Borel. By the previous exercise B contains or omits a tail of a c.u.b. $D \subseteq C$. Say $D = \{i_\gamma\}_{\gamma<\mathrm{cof}(\alpha)}$ is an increasing, continuous enumeration of D. Consider each of the open sets $I_\gamma = (i_\gamma, i_{\gamma+1})$. By induction $B \cap I_\gamma$ is $\mathbf{\Sigma}^0_4$ in I_γ. Argue that this gives that $B \cap \bigcup_\gamma I_\gamma = B \cap (X - D)$ is $\mathbf{\Sigma}^0_4$ in $X - D$, and thus is $\mathbf{\Sigma}^0_4$ in X. If B contains a tail of D (the other case is easier), then $B \cap D = F \cup B'$ where F is closed in X and $B' \subseteq [0, \beta)$ is Borel, for some $\beta < \alpha$. By induction B' is $\mathbf{\Sigma}^0_4$ in $[0, \beta)$, which easily gives that B' is $\mathbf{\Sigma}^0_4$ in X (recall here exercise 13). So, B is $\mathbf{\Sigma}^0_4$ in X.]

One technique for proving theorems about Borel sets is to change the topology to make them clopen. The following theorem says that this is possible.

Theorem 1.41. *Let (X, τ) be Polish, and $A \subseteq X$ a $\mathbf{\Sigma}^0_\alpha$ set. Then there is a finer Polish topology $\tau' \supseteq \tau$ such that A is clopen in τ'. Furthermore, if A is open in τ', then A is $\mathbf{\Sigma}^0_{\alpha+1}$ is τ.*

Moreover, if $\{A_n\}_{n\in\omega}$ is a sequence of sets with $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$, then there is a Polish topology $\tau' \supseteq \tau$ in which all of the A_n are clopen and such that every open set in τ' is $\mathbf{\Sigma}^0_\alpha$ in τ, where $\alpha = \sup_n(\alpha_n + 1)$.

Proof. We prove this by induction on α. Let ρ be a complete metric on X giving the topology τ, and we may assume $\rho \leqslant 1$. Suppose first $A \in \mathbf{\Sigma}^0_1$. Define

$$\rho'(x,y) = \begin{cases} \min\{1, \rho(x,y) + |f(x) - f(y)|\} & \text{if } x, y \in A \\ 1 & \text{if } x \in A,\ y \notin A \text{ or vice-versa} \\ \rho(x,y) & \text{if } x, y \in X - A, \end{cases}$$

where for $x \in A$, $f(x) = \frac{1}{\rho(x, X-A)}$. It is straightforward to show that ρ' is a metric on X. Also, $\rho \leqslant \rho'$ so $\tau \subseteq \tau' \doteq$ the topology given by ρ'. To see ρ' is complete, suppose $\{x_n\}$ is ρ'-Cauchy. So, $\{x_n\}$ is ρ-Cauchy. So, for some $x \in X$, $x_n \to x$. We also clearly have that for large enough n that $x_n \in A$, or for large enough n that $x_n \notin A$. In the latter case we have $x \in X - A$, and $\rho'(x, x_n) = \rho(x, x_n) \to 0$. In the first case we have (as in the proof of Alexandroff's theorem) that there is an $\epsilon > 0$ such that $\forall n\ \rho(x_n, X - A) \geqslant \epsilon$. Since $\{z : \rho(z, X - A) \geqslant \epsilon\}$ is closed, $\rho(x, X - A) \geqslant \epsilon$. Since f is continuous on A, this shows that $\rho'(x_n, x) \to 0$. This shows ρ' is complete. τ' is also second countable, since if $\mathcal{B}$ is a countable base for τ, it is easy to see that $\mathcal{B} \cup \{B \cap (X - A)\}$ is a base for τ'. So, $\tau' \supseteq \tau$ and τ' is also a Polish topology on X. Next observe that $X - A$ is also open in τ' since if $x \in X - A$ and $\epsilon < 1$, then $B_{\rho'}(x, \epsilon) \subseteq X - A$. So, A is clopen in τ'. Finally, suppose U is open in τ'. As we observed above, U is union of sets each of which is either in $\mathcal{B}$, the base for τ, or else of the form $B \cap (X - A)$ for $B \in \mathcal{B}$. So, U is countable union of sets which are $\mathbf{\Delta}^0_2$ in τ, and hence is $\mathbf{\Sigma}^0_2$ in τ.

Suppose now $A \in \mathbf{\Sigma}^0_\alpha$, and the theorem holds for all $\beta < \alpha$. Write $A = \bigcup_n A_n$ where $A_n \in \mathbf{\Pi}^0_{\alpha_n}$, and $\alpha_n < \alpha$. Let ρ_n be a complete, separable metric (i.e., the resulting topology τ_n is separable) on X such that A_n is clopen in τ_n, and every τ_n open set is $\mathbf{\Sigma}^0_{\alpha_n+1}$ in τ. Without loss of generality we may assume $\rho_n \leqslant 1$ for each n. Define

$$\rho'(x,y) = \sum_n \frac{1}{2^n} \rho_n(x,y).$$

It is straightforward to check (as in Alexandroff's theorem) that ρ' is a complete metric on X. Clearly, $\tau \subseteq \tau'$ again. Also, τ' is at least as fine as each τ_n (the topology given by ρ_n), so each A_n is clopen in τ'. Thus, A is open in τ'. Also, $\tau' = \sup_n \tau_n$, so if $\mathcal{B}_n$ is a countable base for τ_n, then $\{B_1 \cap \cdots \cap B_k : B_1 \in \mathcal{B}_1, \dots, B_k \in \mathcal{B}_k\}$ is a base for τ'. So, τ' is separable. This observation also shows that any open set U in τ' is a countable union of sets each of which is $\mathbf{\Sigma}^0_{\alpha_n+1}$ for some n. If $\alpha = \beta + 1$, then U is a countable union of sets each of which is $\mathbf{\Sigma}^0_\alpha$ in τ, and hence U is $\mathbf{\Sigma}^0_\alpha$ in τ. Likewise, if α is limit we get U is $\mathbf{\Sigma}^0_\alpha$ in τ. By the first part of the proof, we may then enlarge τ' to τ'', also a Polish topology making A clopen in τ''. Also, every set open in τ'' is $\mathbf{\Sigma}^0_2$ in τ'. Thus, every open set in τ'' is $\mathbf{\Sigma}^0_2$ in τ' and thus a countable union of sets which are $\mathbf{\Pi}^0_\alpha$ in τ. So, every open set in τ'' is $\mathbf{\Sigma}^0_{\alpha+1}$ in τ.

Given a sequence $\{A_n\}_{n\in\omega}$ with each $A_n \in \boldsymbol{\Sigma}^0_{\alpha_n}$, let for each n, ρ_n give a Polish topology $\tau_n \supseteq \tau$ which makes A_n clopen and such that every open set in τ_n is $\boldsymbol{\Sigma}^0_{\alpha_n+1}$ in τ. We may assume $\rho_n \leqslant 1$ for all n. Let $\rho' = \sum_n \frac{1}{2^n}\rho_n$. As above, ρ' generates a Polish topology $\tau' \supseteq \tau_n$. Thus, all of the A_n are clopen in τ'. Also as before, a base for τ' consists of sets of the form $B_1 \cap \cdots \cap B_k$ where B_i is open in τ_i. Each such set lies in $\boldsymbol{\Sigma}^0_\alpha$ where $\alpha = \sup_n(\alpha_n + 1)$. Hence, every open set in τ' is $\boldsymbol{\Sigma}^0_\alpha$ as well. □

Theorem 1.41 has many applications in the study of Borel sets. The next result gives one.

Corollary 1.42. *Every Borel set B in a Polish space X is either countable or contains a perfect set (hence has size $\mathfrak{c}$).*

Proof. Let B be Borel in the Polish space (X, τ). Let $\tau' \supseteq \tau$ be Polish such that B is clopen in τ', so B itself is a Polish space in the subspace topology from τ'. But every Polish space is either countable or contains a perfect set (in the τ' topology). In the later case, B in fact contains a homeomorphic copy of 2^ω, so B contains a set C which is compact and perfect in the τ' topology. So, C is compact in the τ topology, hence closed in τ. Since no point of C is isolated in the τ' topology, the same holds for the coarser topology τ. □

We will extend corollary 1.42 to higher level sets later.

As another application of the method of changing topologies we prove the following generalization of theorem 1.33. Recall from definition 1.32 the difference operator $\mathcal{D}$.

Theorem 1.43. *For every countable ordinal α, a set is $\boldsymbol{\Delta}^0_\alpha$ iff there is a sequence $\{A_\gamma\}_{\gamma<\delta}$ with $\delta < \omega_1$ and each A_γ in $\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta$ such that $A = \mathcal{D}(\{A_\gamma\})$.*

Proof. An easy computation as in theorem 1.33 shows that if each $A_\gamma \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta$, then $\mathcal{D}(\{A_\gamma\}) \in \boldsymbol{\Delta}^0_\alpha$. Suppose now that $A \in \boldsymbol{\Delta}^0_\alpha$. Consider first the case where α is a successor, say $\alpha = \alpha' + 1$. Write $A = \bigcup_n A_n$, $B \doteq X - A = \bigcup_n B_n$, where each A_n, $B_n \in \boldsymbol{\Pi}^0_{\alpha'}$. Write also for each n, $A_n = \bigcap_m A_{n,m}$, $B_n = \bigcap_m B_{n,m}$, where each $A_{n,m}$, $B_{n,m}$ is in $\boldsymbol{\Pi}^0_\eta$ for some $\eta < \alpha'$. From theorem 1.41 there is a Polish topology $\tau' \supseteq \tau$ making all of the $A_{n,m}$, $B_{n,m}$ clopen, and such that every τ' open set is $\boldsymbol{\Sigma}^0_{\alpha'}$. A and B are both $\boldsymbol{\Sigma}^0_2$ in τ' and so A is $\boldsymbol{\Delta}^0_2$ in τ'. From theorem 1.33 there is a $\delta < \omega_1$ and a sequence of τ' closed sets $\{A_\gamma\}_{\gamma<\delta}$ such that $A = \mathcal{D}(\{A_\gamma\})$. Each A_γ is $\boldsymbol{\Pi}^0_{\alpha'}$, so we are done in this case.

If α is a limit, the result is easier, and we may in fact take the sequence of sets to have length ω. To see this, write again $A = \bigcup_n A_n$, $B \doteq X - A = \bigcup_n B_n$ where each A_n, $B_n \in \bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta$. Let $C_0 = X$, $C_1 = X - A_0$, and $C_2 = X - (A_0 \cup B_0)$. In general, if C_{2k} is defined, let $C_{2k+1} = C_{2k} - A_k$, $C_{2k+2} = C_{2k} - (A_k \cup B_k)$. Each of the C_n lies in $\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta$, and $A = \mathcal{D}(\{C_n\})$. □

We next introduce a few general properties of pointclasses. When $\boldsymbol{\Gamma}$ is a (boldface) pointclass, we let $\boldsymbol{\Delta}(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$, so $\boldsymbol{\Delta}(\boldsymbol{\Gamma})$ is a selfdual pointclass. When there is no danger of confusion, we simply write $\boldsymbol{\Delta}$.

Definition 1.44. We say a pointclass $\boldsymbol{\Gamma}$ has the *separation* property if whenever A, B are disjoint $\boldsymbol{\Gamma}$ sets, then they can be separated by a $\boldsymbol{\Delta}$ set, that is, there is a $C \in \boldsymbol{\Delta}$ such that $A \subseteq C$ and $B \cap C = \varnothing$.

Definition 1.45. We say a pointclass $\mathbf{\Gamma}$ has the *reduction* property if whenever A, B are $\mathbf{\Gamma}$ sets, then there are $\mathbf{\Gamma}$ sets $A' \subseteq A$, $B' \subseteq B$ with $A' \cup B' = A \cup B$ and $A' \cap B' = \varnothing$.

We write $\mathrm{sep}(\mathbf{\Gamma})$ and $\mathrm{red}(\mathbf{\Gamma})$ for the reduction and separation properties of $\mathbf{\Gamma}$ respectively.

The reduction property is the stronger of the two properties in the following sense.

Lemma 1.46. *For any pointclass* $\mathbf{\Gamma}$*,* $\mathrm{red}(\mathbf{\Gamma}) \to \mathrm{sep}(\check{\mathbf{\Gamma}})$*.*

Proof. Assume $\mathrm{red}(\mathbf{\Gamma})$, and let A, B be disjoint sets in $\check{\mathbf{\Gamma}}$. Consider $X - A$, $X - B$. These are $\mathbf{\Gamma}$ sets with $(X - A) \cup (X - B) = X$. Let $C \subseteq X - A$, $D \subseteq X - B$ be disjoint $\mathbf{\Gamma}$ sets with $C \cup D = (X - A) \cup (X - B) = X$. Thus, $D = X - C$ and so $C, D \in \mathbf{\Delta}$. Then $A \subseteq D \subseteq (X - B)$. □

The concepts of prewellordering and the prewellordering property play a central role in the subject. Recall that a binary relation $\prec$ is wellfounded if for every non-empty $A \subseteq X$ there is a $\prec$-minimal element of A (i.e., $\exists a \in A\ \forall b \in A\ \neg(b \prec a)$).

Definition 1.47. A prewellordering $\preceq$ on a set X is a binary relation which is reflexive ($x \preceq x$), connected ($x \preceq y$ or $y \preceq x$), transitive ($x \preceq y$ and $y \preceq z$ implies $x \preceq z$), and such that the strict part $\prec$ is wellfounded. The strict part is defined by: $x \prec y$ iff $x \preceq y$ and $\neg(y \preceq x)$.

Given a prewellordering $\preceq$ on X, this defines a corresponding equivalence relation $x \equiv y$ iff $x \preceq y$ and $y \preceq x$. The prewellordering $\preceq$ can thus be identified with a wellordering of these equivalence classes (More precisely, set $[x] < [y]$ iff $x \prec y$. This is easily welldefined.) Prewellorderings on a set $A \subseteq X$ can also be identified with *norms* on the set A:

Definition 1.48. A *norm* on a set $A \subseteq X$ is a map $\phi\colon A \to \mathrm{On}$. We say ϕ is *regular* if ϕ is onto an ordinal.

Given a prewellordering $\preceq$ on A, we have the corresponding norm $\phi(x) = |x|_{\prec}$, defined for $x \in A$. Conversely, given the norm ϕ on A, we have the prewellordering of A given by $x \preceq y$ iff $\phi(x) \leqslant \phi(y)$.

Note that if $\prec$ is a prewellordering on a set A, and $\preceq$ lies in a pointclass $\mathbf{\Gamma}$, then $\preceq$ gives a way of writing A as an increasing union of $\mathbf{\Gamma}$ sets. Namely, for each $x \in A$, $\{y\colon y \preceq x\}$ is in $\mathbf{\Gamma}$. It is useful to strengthen this requirement, which gives us the notion of a $\mathbf{\Gamma}$-norm.

Definition 1.49 ($\mathbf{\Gamma}$-norm)**.** Let $\mathbf{\Gamma}$ be a pointclass, and $A \subseteq X$. We say a norm ϕ on A is a $\mathbf{\Gamma}$-norm if the relations $<^*$, $\leqslant^*$ are in $\mathbf{\Gamma}$, where

$$x <^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) < \phi(y)))$$
$$x \leqslant^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \phi(x) \leqslant \phi(y)))$$

Note that if ϕ is a $\mathbf{\Gamma}$-norm on A, then this writes A as an increasing union of sets which are in $\mathbf{\Delta}$, that is, for any $y \in A$, $\{x \in A\colon \phi(x) \leqslant \phi(y)\} = \{x\colon x \leqslant^* y\} = \{x\colon \neg(y <^* x\}$. So, a $\mathbf{\Gamma}$-norm is a way of writing A as an increasing union of sets which are "uniformly" in $\mathbf{\Delta}$.

Note that if ϕ is a $\boldsymbol{\Gamma}$-norm on A, then its regularization ϕ' (i.e., the transitive collapse of ϕ) has the same norm relations, and so also is a $\boldsymbol{\Gamma}$-norm. Thus, there is generally no harm in assuming the norm is regular.

There is another definition of $\boldsymbol{\Gamma}$-norm frequently used, which we give in the next definition.

Definition 1.50 (Alternate definition of $\boldsymbol{\Gamma}$-norm)**.** Let $\boldsymbol{\Gamma}$ be a pointclass. We say ϕ is a $\boldsymbol{\Gamma}$-norm on A if there are relations $\leqslant_{\boldsymbol{\Gamma}} \in \boldsymbol{\Gamma}$ and $\leqslant_{\check{\boldsymbol{\Gamma}}} \in \check{\boldsymbol{\Gamma}}$ such that for all $y \in A$ and all $x \in X$,

$$(x \in A \wedge \phi(x) \leqslant \phi(y)) \leftrightarrow x \leqslant_{\boldsymbol{\Gamma}} y \leftrightarrow x \leqslant_{\check{\boldsymbol{\Gamma}}} y.$$

The next lemma shows that in most cases the two definitions of $\boldsymbol{\Gamma}$-norm are equivalent.

Lemma 1.51. *For any pointclass $\boldsymbol{\Gamma}$, if ϕ is a $\boldsymbol{\Gamma}$-norm according to definition 1.49, then ϕ is a $\boldsymbol{\Gamma}$-norm according to definition 1.50. If $\boldsymbol{\Gamma}$ is closed under $\wedge$, $\vee$, then definition 1.50 implies definition 1.49 as well.*

Proof. Suppose ϕ satisfies definition 1.49. Define $x \leqslant_{\boldsymbol{\Gamma}} y$ iff $x \leqslant^* y$, and $x \leqslant_{\check{\boldsymbol{\Gamma}}} y$ iff $\neg(y <^* x)$. Clearly, $\leqslant_{\boldsymbol{\Gamma}} \in \boldsymbol{\Gamma}$ and $\leqslant_{\check{\boldsymbol{\Gamma}}} \in \check{\boldsymbol{\Gamma}}$. If $y \in A$, then clearly $\{x \colon x \leqslant_{\boldsymbol{\Gamma}} y\} = \{x \in A \colon \phi(x) \leqslant \phi(y)\}$. Also, since $y \in A$ we have that

$$\begin{aligned}\{x \colon x \leqslant_{\check{\boldsymbol{\Gamma}}} y)\} &= \{x \colon \neg(y <^* x)\} \\ &= \{x \colon \neg(y \in A \wedge (x \notin A \vee \phi(y) < \phi(x)))\} \\ &= \{x \colon x \in A \wedge \phi(x) \leqslant \phi(y)\}.\end{aligned}$$

Suppose next that ϕ is a $\boldsymbol{\Gamma}$-norm according to definition 1.50, and assume now $\boldsymbol{\Gamma}$ is closed under $\wedge$, $\vee$. Then $x <^* y$ iff $x \in A \wedge (y \notin A \vee \phi(x) < \phi(y))$ iff $(x \in A \wedge \neg(y \leqslant_{\check{\boldsymbol{\Gamma}}} x))$. The last equivalence follows since for $x \in A$, $y \leqslant_{\check{\boldsymbol{\Gamma}}} x$ iff $y \in A \wedge \phi(y) \leqslant \phi(x)$. This shows $<^* \in \boldsymbol{\Gamma}$. Also $x \leqslant^* y$ iff $x \in A \wedge (y \notin A \vee \phi(x) \leqslant \phi(y))$ iff $x \in A \wedge \neg(y \leqslant_{\check{\boldsymbol{\Gamma}}} x) \vee (x \leqslant_{\boldsymbol{\Gamma}} y \wedge y \leqslant_{\boldsymbol{\Gamma}} x)$. This shows $\leqslant^* \in \boldsymbol{\Gamma}$. $\square$

Definition 1.52. We say a pointclass $\boldsymbol{\Gamma}$ has the prewellordering property if every $A \in \boldsymbol{\Gamma}$ admits a $\boldsymbol{\Gamma}$-norm.

We write $\mathrm{pwo}(\boldsymbol{\Gamma})$ to say $\boldsymbol{\Gamma}$ has the prewellordering property.

Lemma 1.53. *Suppose* $\mathrm{pwo}(\boldsymbol{\Gamma})$ *and $\boldsymbol{\Gamma}$ is closed under $\vee$ and intersections with clopen sets. Then* $\mathrm{red}(\boldsymbol{\Gamma})$.

Proof. Let A, $B \in \boldsymbol{\Gamma}$. Define $C \subseteq X \times \omega$ by $C(x,n) \leftrightarrow (n = 0 \wedge x \in A) \vee (n = 1 \wedge x \in B)$. From our closure assumptions, $C \in \boldsymbol{\Gamma}$. Let ϕ be a $\boldsymbol{\Gamma}$-norm on C, and $\leqslant^*$, $<^*$ the corresponding relations. Define $A'(x) \leftrightarrow (x,0) <^* (x,1)$ and $B'(x) \leftrightarrow (x,1) \leqslant^* (x,0)$. Then A', B' are in $\boldsymbol{\Gamma}$, $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \varnothing$, and $A' \cup B' = A \cup B$. $\square$

Theorem 1.54. *For all $\alpha > 1$,* $\mathrm{pwo}(\boldsymbol{\Sigma}^0_\alpha)$*. If X is* 0*-dimensional, then also* $\mathrm{pwo}(\boldsymbol{\Sigma}^0_1)$*.*

Proof. Let $A \subseteq X$ be $\boldsymbol{\Sigma}^0_\alpha$, say $A = \bigcup_n A_n$ where each $A_n \in \boldsymbol{\Pi}^0_{\beta_n}$, $\beta_n < \alpha$. Let ϕ be the corresponding norm on A, that is, $\phi(x) =$ the least n so that $x \in A_n$. Then $x <^* y$ iff $\exists n\ (x \in A_n \wedge \forall m < n\ (x \notin A_m) \wedge y \notin A_n)$. The expression inside parentheses defines a $\boldsymbol{\Delta}^0_\alpha$ set, and so $<^*$ is $\boldsymbol{\Sigma}^0_\alpha$. A similar computation shows $\leqslant^* \in \boldsymbol{\Sigma}^0_\alpha$. If X is 0-dimensional, and $A \in \boldsymbol{\Sigma}^0_1$, write A as an increasing union of clopen sets, and a similar computation shows the corresponding norm relations $\leqslant^*$, $<^*$ are both $\mathbf{\Sigma}^0_1$. □

Corollary 1.55. *For all $\alpha > 1$, $\mathrm{red}(\mathbf{\Sigma}^0_\alpha)$ and $\mathrm{sep}(\mathbf{\Pi}^0_\alpha)$. If X is 0-dimensional, this holds also for $\alpha = 1$.*

The prewellordering property, and a stronger property called the scale property (defined later) are both closely related to the notion of uniformizations. A uniformization of $A \subseteq X \times Y$ is essentially a choice function for the relation A.

Definition 1.56. Let $A \subseteq X \times Y$. A *uniformization* A' of A is a set $A' \subseteq A$ with $\mathrm{dom}(A') = \mathrm{dom}(A)$ and such that $\forall x \in \mathrm{dom}(A)\ \exists! y\ A'(x, y)$. We say a pointclass $\mathbf{\Gamma}$ has the *uniformization property* if every $A \subseteq X \times Y$ in $\mathbf{\Gamma}$ has a uniformization A' in $\mathbf{\Gamma}$. We say $\mathbf{\Gamma}$ has the *number uniformization property* if every $A \subseteq X \times \omega$ in $\mathbf{\Gamma}$ has a uniformization A' in $\mathbf{\Gamma}$.

The prewellordering property is related to the number uniformization property, and the stronger scale property is related to the (full) uniformization property. The following lemma makes this first connection.

Lemma 1.57. *If $\mathbf{\Gamma}$ is closed under $\forall^\omega$ and has the prewellordering property, then $\mathbf{\Gamma}$ has the number uniformization property.*

Proof. Let $A \subseteq X \times \omega$ be in $\mathbf{\Gamma}$. Define

$$A'(x, n) \leftrightarrow \forall m\ ((x, n) \leqslant^* (x, m)) \wedge \forall m\ (m \neq n \to (x, n) <^* (n, m)).$$

From closure of $\mathbf{\Gamma}$ under $\forall^\omega$ we see that $A' \in \mathbf{\Gamma}$. Clearly, A' is a uniformization of A. □

The hypothesis of lemma 1.57 does not apply to the pointclasses $\mathbf{\Sigma}^0_\alpha$, but we see below that (for $\alpha > 1$) they nevertheless have the number uniformization property.

Number uniformization can also be thought of as a kind of generalized reduction property, according to the next definition.

Definition 1.58. We say $\mathbf{\Gamma}$ has the ω-reduction property if for every sequence $\{A_n\}_{n\in\omega}$ with each $A_n \in \mathbf{\Gamma}$, there are sets $B_n \subseteq A_n$ with $B_n \in \mathbf{\Gamma}$ satisfying $B_n \cap B_m = \varnothing$ if $n \neq m$ and $\bigcup_n B_n = \bigcup_n A_n$.

We say $\mathbf{\Gamma}$ has the ω-separation property if for each sequence of pairwise disjoint sets $\{A_n\}_{n\in\omega}$ with each $A_n \in \mathbf{\Gamma}$, there is a sequence $\{C_n\}_{n\in\omega}$ of $\mathbf{\Delta}$ sets with $A_n \subseteq C_n$ for all n, and $\bigcap_n C_n = \varnothing$.

Lemma 1.59. *If $\mathbf{\Gamma}$ is a pointclass closed under ω-joins, then $\mathbf{\Gamma}$ has the number uniformization property iff $\mathbf{\Gamma}$ has the ω-reduction property.*

Proof. First suppose $\mathbf{\Gamma}$ has the ω-reduction property. Let $A \subseteq X \times \omega$ be in $\mathbf{\Gamma}$. Define $A_n \subseteq X$ by $A_n(x) \leftrightarrow A(x, n)$. Since $x \mapsto (x, n)$ is continuous, each A_n is in $\mathbf{\Gamma}$. Let $\{B_n\}$ be $\mathbf{\Gamma}$ sets reducing the A_n. Define $A'(x, n) \leftrightarrow x \in B_n$. Since $\mathbf{\Gamma}$ is closed under ω-joins, $A' \in \mathbf{\Gamma}$. Since $\bigcup_n B_n = \bigcup_n A_n$, $\mathrm{dom}(A') = \mathrm{dom}(A)$. Since the B_n are pairwise disjoint, A' is the graph of a function.

Next suppose $\mathbf{\Gamma}$ has the number uniformization property. Let $\{A_n\}_{n\in\omega}$ be a sequence of $\mathbf{\Gamma}$ sets. Define $A \subseteq X \times \omega$ by $A(x, n) \leftrightarrow x \in A_n$. Since $\mathbf{\Gamma}$ is closed under ω-joins, $A \in \mathbf{\Gamma}$. Let $A' \subseteq A$ be a uniformization of A. Define B_n by $B_n(x) \leftrightarrow (x, n) \in A'$. So, each B_n is in $\mathbf{\Gamma}$. Since A' is a uniformization of A, the B_n are pairwise disjoint and union to $\bigcup_n A_n$. Clearly, $B_n \subseteq A_n$. □

Exercise 28. Show that if $\mathbf{\Gamma}$ has the ω-reduction property and is closed under countable unions then $\check{\mathbf{\Gamma}}$ has the ω-separation property. [hint: Given pairwise disjoint $A_n \in \check{\mathbf{\Gamma}}$, apply ω-reduction to the sequence $B_n = X - A_n$.]

Exercise 29. Show that if $\mathbf{\Gamma}$ has the separation property and is closed under countable unions and finite intersections, then $\mathbf{\Gamma}$ has the ω-separation property. [hint: Given pairwise disjoint sets $A_n \in \mathbf{\Gamma}$, apply first separation to A_1 and $\bigcup_{n>1} A_n$. Continue inductively.]

Next we verify that for $\alpha > 1$, each $\mathbf{\Sigma}^0_\alpha$ class has the number uniformization property.

Theorem 1.60. *For all $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha$ has the number uniformization property.*

Proof. Suppose $A \subseteq X \times \omega$ is in $\mathbf{\Sigma}^0_\alpha$. Write $A = \bigcup_n A_n$, where each $A_n \in \mathbf{\Pi}^0_{\beta_n}$ for $\beta_n < \alpha$. Define

$$A'(x,n) \leftrightarrow \exists m\ ((x,n) \in A_m \wedge \forall k < \langle n,m\rangle\ (x,(k)_0) \notin A_{(k)_1}).$$

The expression inside the parentheses defines a $\mathbf{\Delta}^0_\alpha$ set, and thus $A' \in \mathbf{\Sigma}^0_\alpha$. Clearly, A' is a uniformization of A. □

The following theorem summarizes our pointclass discussions within the Borel hierarchy.

Theorem 1.61. *For any uncountable Polish space X, all of the containments in the Borel hierarchy are proper. For all $\alpha > 1$, each $\mathbf{\Sigma}^0_\alpha \restriction X$ class has the prewellordering, reduction, and number uniformization properties, and each of the $\mathbf{\Pi}^0_\alpha \restriction X$ classes has the separation property. If X is 0-dimensional, this holds also for $\alpha = 1$.*

As an application of these techniques we next present the Lebesgue–Hausdorff theorem analyzing Borel functions between Polish spaces. This theorem connects the notions of $\mathbf{\Gamma}$-measurability defined earlier and the notion of a function having a certain Baire class, which we define below.

First, we get another representation of the $\mathbf{\Delta}^0_\alpha$ sets. Recall the notions of lim sup and lim inf for a sequence of sets:

Definition 1.62. Let $\{A_n\}_{n\in\omega}$ be a sequence of subsets of a set X. Then

$$\limsup_n A_n = \bigcap_n \bigcup_{m \geqslant n} A_m$$
$$\liminf_n A_n = \bigcup_n \bigcap_{m \geqslant n} A_m$$

If $\limsup_n A_n = \liminf_n A_n$, we say that $\lim_n A_n$ exists and set

$$\lim_n A_n = \limsup_n A_n = \liminf_n A_n.$$

Thus, $x \in \limsup_n A_n$ iff x lies in infinitely many of the A_m, and $x \in \liminf_n A_n$ iff x lies in a tail of the A_m. Clearly, $\liminf_n A_n \subseteq \limsup_n A_n$. Saying that $\lim_n A_n$ exists is equivalent to saying that for every x there is either a tail of m for which $x \in A_m$, or a tail of m for which $x \notin A_m$. In this case, $\lim_n A_n$ is the set of x which are eventually in the A_m.

Lemma 1.63. *Let X be Polish, and $\alpha > 2$ a countable ordinal. If $\alpha = \beta + 1$ is a successor, then $A \subseteq X$ is $\mathbf{\Delta}^0_\alpha$ iff there is a sequence $\{A_n\}_{n\in\omega}$ of sets, with each $A_n \in \mathbf{\Delta}^0_\beta$ for such that $A = \lim A_n$. If α is a limit, then $A \subseteq X$ is $\mathbf{\Delta}^0_{\alpha+1}$ iff there is a sequence $\{A_n\}_{n\in\omega}$ of sets, with each $A_n \in \mathbf{\Delta}^0_{\beta_b}$ for some $\beta_n < \alpha$ such that $A = \lim A_n$. So, in either case every $\mathbf{\Delta}^0_\alpha$ set is a limit of sets in $\bigcup_{\beta<\alpha} \mathbf{\Delta}^0_\beta$. If X is 0-dimensional, this holds also for $\alpha = 2$.*

Proof. Suppose first $\alpha = \beta+1$ is a successor. If $A = \lim_{n\to\infty} A_n$ with each $A_n \in \mathbf{\Delta}^0_\beta$, then $x \in A \leftrightarrow \exists n\ \forall m \geqslant n\ x \in A_m \leftrightarrow \forall n\ \exists m \geqslant n\ x \in A_m$. This first equivalence shows A is $\mathbf{\Sigma}^0_\alpha$, and second shows A is $\mathbf{\Pi}^0_\alpha$. Suppose next that $A \in \mathbf{\Delta}^0_\alpha$. Since both A and $B \doteq X - A$ are $\mathbf{\Sigma}^0_\alpha$, write $A = \bigcup_n A_n$, $B = \bigcup_n B_n$ with $A_n, B_n \in \mathbf{\Pi}^0_\beta$. Clearly, $A_n \cap B_n = \varnothing$, and by $\mathrm{sep}(\mathbf{\Pi}^0_\beta)$ let $C_n \in \mathbf{\Delta}^0_\beta$ with $A_n \subseteq C_n$, $C_n \cap B_n = \varnothing$. If $x \in A$, then x is eventually in the A_n, and hence eventually in the C_n. If $x \notin A$, then x is eventually in the B_n, and hence eventually not in the C_n. So, $A = \lim_n C_n$. If $\alpha = 2$ the argument is similar, using $\mathrm{sep}(\mathbf{\Pi}^0_1)$.

Suppose next α is a limit. If $A = \lim_n A_n$ where each $A_n \in \bigcup_{\beta<\alpha} \mathbf{\Delta}^0_\beta$, then a computation similar to the one above shows that $A \in \mathbf{\Delta}^0_{\alpha+1}$. Suppose that $A \in \mathbf{\Delta}^0_{\alpha+1}$. Again let $B = X - A$ and write $A = \bigcup_n A_n$, $B = \bigcup_n B_n$, increasing unions with $A_n, B_n \in \mathbf{\Pi}^0_\alpha$. Write each A_n, B_n as a decreasing intersection $A_n = \bigcap_m A^m_n$, $B_n = \bigcap_m B^m_n$ where each A^m_n, B^m_n are in $\bigcup_{\beta<\alpha} \mathbf{\Delta}^0_\beta$. Define

$$\begin{aligned} x \in C_n \leftrightarrow \exists n' \leqslant n\ [&\forall m \leqslant n\ x \in A^m_{n'} \wedge \exists m \leqslant n\ x \notin B^m_{n'} \\ &\wedge \forall n'' < n'\ \exists m \leqslant n\ (x \notin A^m_{n''} \wedge x \notin B^m_{n''})] \end{aligned}$$

Each C_n is clearly a Boolean combination of sets in $\bigcup_{\beta<\alpha} \mathbf{\Delta}^0_\beta$, and hence $C_n \in \bigcup_{\beta<\alpha} \mathbf{\Delta}^0_\beta$. We claim that $A = \lim_n C_n$. To see this, suppose first that $x \in A$. Let n_0 be least such that $x \in A_{n_0}$, and so $x \in A^m_{n_0}$ for all m. Also, $x \notin B_{n_0}$. Let $n_1 \geqslant n_0$ be such that $x \notin B^{n_1}_{n_0}$. For all $n < n_0$ we have $x \notin A_n$ and $x \notin B_n$, so may take an $n_2 \geqslant n_1$ large enough so that for all $n < n_0$, $x \notin (A^{n_2}_n \cup B^{n_2}_n)$. Then, $x \in C_n$ for all $n \geqslant n_2$ (using n_0 as a witness to the first existential quantifier in the definition of C_n). Suppose next that $x \notin A$. Let n_0 be least such that $x \in B_{n_0}$. So, for all m, $x \in B^m_{n_0}$. Let $n_1 \geqslant n_0$ be such that $x \notin A^{n_1}_{n_0}$, and let $n_2 \geqslant n_1$ be large enough so that for all $n < n_0$, $x \notin (A^{n_2}_n \cup B^{n_2}_n)$. Then for all $n \geqslant n_2$, $x \notin C_n$. For suppose $n' \leqslant n$ witnessed the existential statement in the definition of C_n. We cannot have $n' < n_0$ as the first conjunct in the square brackets would fail from the choice of n_2. We cannot have $n' = n_0$ as the second conjunct would then fail as $x \in B^m_{n_0}$ for all m. We also cannot have $n' > n_0$ as then the third conjunct would fail since $x \in B^m_{n_0}$ for all m. This shows the claim and completes the proof. □

We now introduce the Baire hierarchy of functions. The classical case is for functions $f\colon \mathbb{R}_{\mathrm{std}} \to \mathbb{R}_{\mathrm{std}}$. In this case, a function is said to be of Baire class 1 if it is a pointwise limit of continuous functions. In general, it said to be of Baire class α if it is a pointwise limit of functions, each of Baire class less than α. When considering Polish spaces other than $\mathbb{R}_{\mathrm{std}}$, however, we must modify the definition of the Baire hierarchy at the bottom level slightly to get our main result. So, we take the following definition as our official definition of the Baire hierarchy for any Polish space (we see below that for $\mathbb{R}_{\mathrm{std}}$ this definition is equivalent to the one just stated).

Definition 1.64. Let X, Y be Polish spaces. We say $f\colon X \to Y$ is of Baire class 1 if it is $\mathbf{\Sigma}^0_2$-measurable. For $\alpha > 1$, we say f is of Baire class α if it is a pointwise limit of functions each of which has Baire class less than α.

Exercise 30. Show that if $f, g\colon X \to \mathbb{R}$ are Baire class α, then $f \pm g$ is also Baire class α. Show this also for f, g being $\mathbf{\Sigma}^0_\alpha$ measurable.

The nest theorem is the general statement of the Lebesgue–Hausdorff theorem.

Theorem 1.65. *Let X, Y be Polish spaces, and $\alpha \geqslant 1$ a countable ordinal. Then $f\colon X \to Y$ is Baire class α iff f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable.*

Proof. The theorem is true by definition for $\alpha = 1$, so assume $\alpha > 1$. Suppose first that $f\colon X \to Y$ is of Baire class α. Let $f = \lim_n f_n$ where each f_n is of Baire class $\beta_n < \alpha$. By induction, each f_n is $\mathbf{\Sigma}^0_{\beta_n+1}$-measurable, and hence $\mathbf{\Sigma}^0_\alpha$ measurable. Let $U \subseteq Y$ be open. Write $U = \bigcup V_n$ where each V_n is an open set with $\overline{V_n} \subseteq U$. Then,

$$x \in f^{-1}(U) \leftrightarrow \exists k\ \exists n\ \forall m \geqslant n\ (f_m(x) \in \overline{V_k}).$$

Since $\{x\colon f_m(x) \in \overline{V_k}\}$ is $\mathbf{\Pi}^0_{\beta_m}$, this shows that $f^{-1}(U) \in \mathbf{\Sigma}^0_{\alpha+1}$.

Suppose next that f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable, and we show f is of Baire class α. Consider first the following special case. Suppose $C = \{y_0, y_1, \dots\} \subseteq Y$ is countable and $A_0, A_1, \dots$ are pairwise disjoint $\mathbf{\Delta}^0_{\alpha+1}$ sets which partition X, and let f be the function which takes value y_n on A_n. For each n, by lemma 1.63 let A^m_n be a sequence of sets such that $A_n = \lim_m A^m_n$, and each $A^m_n \in \mathbf{\Delta}^0_\alpha$ if α is a successor and in $\bigcup_{\beta<\alpha} \mathbf{\Delta}^0_\beta$ if α is limit. Define for each m the function f_m by $f_m(x) = y_n$ where n is the least integer $\leqslant m$ such that $x \in A^m_n$, if one exists, and $f_m(x) = y_0$ otherwise. It is easy to check that f_m is $\mathbf{\Delta}^0_\alpha$-measurable for α successor, and $\mathbf{\Delta}^0_\beta$ measurable for some $\beta < \alpha$ if α is a limit. In either case, f_m has Baire class $< \alpha$. Also, $f = \lim_m f_m$, since if n is such that $x \in A_n$, then for large enough m, n will also be least such that $x \in A^m_n$. Thus, f is Baire class α.

We return now to the general case. For each $\epsilon_n = \frac{1}{2^n}$, let $C_n = \{y^0_n, y^1_n, \dots\}$ be an ϵ_n net in Y (i.e., every point in Y is within ϵ_n of a point in C_n). Let

$$A^i_n = \{x \in X\colon \rho(f(x), y^i_n) < \epsilon_n\}.$$

So, each A^i_n is $\mathbf{\Sigma}^0_{\alpha+1}$, and for each n, the $\{A^i_n\}_{i\in\omega}$ partition X. By ω-reduction, let $B^i_n \subseteq A^i_n$ be $\mathbf{\Sigma}^0_{\alpha+1}$ and such that $\{B^i_n\}_{i\in\omega}$ are pairwise disjoint and partition X. It follows that each B^i_n is $\mathbf{\Delta}^0_{\alpha+1}$ (its complement is a countable union of the B^j_n for $j \neq i$). For each n, let g_n be the function which is equal to y^i_n on B^i_n. Let g^m_n be the sequence of functions constructed as in the special case considered above. So, each g^m_n is of Baire class $< \alpha$ and $\lim_m g^m_n = g_n$. Define the function f_n as follows. Consider the sequence of points

$$g^n_0(x), g^n_1(x), \dots, g^n_n(x).$$

Let $a^n_0(x) = g^n_0(x)$, and define inductively, $a^n_{i+1}(x) = g^n_{i+1}(x)$ if $\rho(g^n_{i+1}(x), a^n_i(x)) < 2\epsilon_n$, and $a^n_{i+1}(x) = a^n_i(x)$ otherwise. Then set $f_n(x) = a^n_n(x)$. Since each g^n_i is $\mathbf{\Delta}^0_\alpha$-measurable (if α is a limit then it is $\mathbf{\Delta}^0_\beta$-measurable for some $\beta < \alpha$), and each a^n_i is a Boolean combination of the g^n_j for $j \leqslant i$ (recall here that the range of each g^n_i is finite, and the preimage of every point is $\mathbf{\Delta}^0_\alpha$), it follows that f_n is also $\mathbf{\Delta}^0_\alpha$-measurable ($\mathbf{\Delta}^0_\beta$ for $\beta < \alpha$ if α is a limit).

Finally, we show that $\lim_n f_n = f$. Let $x \in X$, and let $i_0, i_1, \dots$ be such that $x \in B_0^{i_0}$, $B_1^{i_1}, \dots$. For every k and large enough n we have that $g_k^n(x) = y_k^{i_k}$. So for large enough n we will have that for all $j < k$ that $\rho(g_j^n(x), g_{j+1}^n(x)) = \rho(y_j^{i_j}, y_{j+1}^{i_{j+1}}) < 2\epsilon_j$ since $\rho(y_j^{i_j}, f(x)) < \epsilon_j$ as $x \in B_j^{i_j}$. Thus, for all k we have that for all large enough n that $a_k^n(x) = g_k^n(x) = y_k^{i_k}$, and thus $\rho(f_n(x), f(x)) \leqslant \rho(f(x), a_k^n(x)) + \rho(a_k^n(x), f_n(x)) < \epsilon_k + \sum_{j=k+1}^n \epsilon_j \leqslant 2\epsilon_k$. Thus, $\lim_n f_n = f$. $\square$

In definition 1.64 we defined a function to have Baire class 1 if it is $\mathbf{\Sigma}_2^0$ measurable. This, of course, made the $\alpha = 1$ case of theorem 1.65 trivial, but we are still left with the question of whether every $\mathbf{\Sigma}_2^0$ measurable function is the pointwise limit of a sequence of continuous functions. The argument as in the easy direction of theorem 1.65 shows that every pointwise limit of continuous functions is $\mathbf{\Sigma}_2^0$ measurable. The other direction is not true in general. For example, if $X = \mathbb{R}$ and $Y = \{0, 1\}$, and $f\colon X \to Y$ is the characteristic function of a non-trivial $\mathbf{\Delta}_2^0$ set, then f is $\mathbf{\Sigma}_2^0$ measurable, but f is not the limit of a sequence of continuous functions (every continuous function from X to Y is constant). However, in the case of classical interest, that is when $X = Y = \mathbb{R}$, this direction is also true. In fact we have the following.

Theorem 1.66. *If either X is 0-dimensional or $Y = \mathbb{R}$, then every $f\colon X \to Y$ which is $\mathbf{\Sigma}_2^0$ measurable is the pointwise limit of a sequence of continuous functions.*

Proof. If X is 0-dimensional, the proof of theorem 1.65 still works, since in this case every $\mathbf{\Delta}_2^0$ set is the limit of a sequence of $\mathbf{\Delta}_1^0$ sets. So, suppose $Y = \mathbb{R}$. Suppose that we know the result for functions from $X \to (0, 1)$. Let $f\colon X \to \mathbb{R}$ be $\mathbf{\Sigma}_2^0$ measurable. Let $\pi\colon \mathbb{R} \to (0, 1)$ be a homeomorphism. Let $f' = \pi \circ f\colon x \to (0, 1)$. Let $g_n\colon X \to \mathbb{R}$ be continuous with $f' = \lim_n g_n$. Let

$$g'_n(x) = \begin{cases} 1 - \frac{1}{n} & \text{if } g_n(x) > 1 - \frac{1}{n} \\ g_n(x) & \text{if } \frac{1}{n} \leqslant g_n(x) \leqslant 1 - \frac{1}{n} \\ \frac{1}{n} & \text{if } g_n(x) < \frac{1}{n} \end{cases}$$

then $g'_n\colon x \to (0, 1)$ is also continuous and $f' = \lim_n g'_n$. Let $f_n = \pi^{-1} \circ g'_n$. Then $f_n\colon X \to \mathbb{R}$ is continuous and $f = \lim_n f_n$.

So, we may assume $f\colon X \to (0, 1)$. As in the proof of theorem 1.65, let for each $\epsilon_n = \frac{1}{2^n}$ $C_n = \{y_n^0, y_n^1, \dots, y_n^k\}$ be a (finite) ϵ_n net in $(0, 1)$. Define the A_n^i and B_n^i as before. Also as before, let $g_n = y_n^i$ on B_n^i. Let $h_0 = g_0$, and $h_i = g_i - g_{i-1}$ for $i > 0$. Clearly $|h_i| \leqslant \frac{1}{2^{i-2}}$. Also, $f = \sum_{i=0}^\infty h_i$. Note that each h_i takes on only finitely many values, decided by a $\mathbf{\Delta}_2^0$ partition of X. We show in a moment that each such function h_i is a limit of continuous functions. Given this, note that each h_i is then actually a limit of continuous functions, each of which is at most $\frac{1}{2^{i-1}}$ in absolute value (by truncating the continuous functions to lie in this range). Say $h_i = \lim_j g_i^j$ where g_i^j is continuous and $|g_i^j| \leqslant \frac{1}{2^{i-2}}$. Define then $f_n = \sum g_i^n$. It is straightforward to check that f_n is well-defined and $f = \lim_n f_n$.

Finally, suppose $B_1, \dots, B_k$ are a partition of X into $\mathbf{\Delta}_2^0$ sets, and h takes the constant value y^i on B_i. For each i write $B_i = \bigcup_k F_k$, $X - B_i = \bigcup_k E_k$ as increasing unions of closed sets. For each k, by Urysohn's lemma let $f_k^i\colon X \to (0, 1)$ be such that f_k^i is 1 on F_k and 0 on E_k. Let $f_k = \sum f_k^i$. Each f_k is continuous and $\lim_k f_k = h$. $\square$

2. Analytic, Co-analytic, And Projective Sets

Recall X, Y denote Polish spaces.

Definition 2.1. A set $A \subseteq X$ is $\boldsymbol{\Sigma}^1_1$, or *analytic*, if there is a closed set $F \subseteq X \times \omega^\omega$ such that for all x, $A(x) \leftrightarrow \exists y \in \omega^\omega\ F(x, y)$. A set A is $\boldsymbol{\Pi}^1_1$ or *coanalytic* if $X - A$ is $\boldsymbol{\Sigma}^1_1$. We say A is $\boldsymbol{\Delta}^1_1$ if it is both $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$.

That is, the $\boldsymbol{\Sigma}^1_1$ sets are the projections of closed sets in $X \times \omega^\omega$. If we require the set F to be closed as we did, then it is important that we use ω^ω in the second coordinate rather than say 2^ω or $\mathbb{R}$. That is because since 2^ω is compact, any closed set $F \subseteq X \times 2^\omega$ will project to a closed set in X. Similarly, since $\mathbb{R}$ is σ-compact, any closed set in $X \times \mathbb{R}$ will project to a $\boldsymbol{\Sigma}^0_2$ set in X.

Even with the right definition 2.1 it is perhaps not immediately clear that every Borel set is analytic. We could have made this trivial by allowing the F in definition 2.1 to be Borel (which we see shortly gives an equivalent definition), but we want to have the representation afforded by definition 2.1.

Exercise 31. Show that every Borel set A in a Polish space X is a continuous image image of a closed set $F \subseteq \omega^\omega$ (note: a stronger version of this is proved below). Deduce that every Borel set in a Polish space is $\boldsymbol{\Sigma}^1_1$. [hint: Prove this by induction on the Borel rank of A. If A is open or closed, use lemma 1.24. If $A = \bigcup_n A_n$ and $A_n = f_n(F_n)$, define $F \subseteq \omega^\omega$ by piecing together disjoint copies of the F_n, and define f on F in the obvious manner. If $A = \bigcap_n A_n$, where $A_n = f_n(F_n)$, let $F \subseteq \omega^\omega$ be the set of x such that $\forall n\ (x)_n \in F_n$ and $\forall i, j\ f_i((x)_i) = f_j((x)_j)$. Show that F is closed. For $x \in F$, let $f(x) = f_0((x)_0)$.]

Using logical notation we may rephrase the definitions of analytic and coanalytic by saying $\boldsymbol{\Sigma}^1_1 = \exists^{\omega^\omega} \boldsymbol{\Pi}^0_1$ and $\boldsymbol{\Pi}^1_1 = \neg \boldsymbol{\Sigma}^1_1$. Of course, more precisely we mean $\boldsymbol{\Sigma}^1_1 \restriction X = \exists^{\omega^\omega} \boldsymbol{\Pi}^0_1 \restriction (X \times \omega^\omega)$. This suggests a natural extension of these pointclasses, which gives the *projective hierarchy.*

Definition 2.2 (Projective Hierarchy)**.** For any Polish space X and $n > 1$ we define $\boldsymbol{\Sigma}^1_n = \exists^{\omega^\omega} \boldsymbol{\Pi}^1_{n-1}$, $\boldsymbol{\Pi}^1_n = \neg \boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Delta}^1_n = \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$.

For example, a set $A \subseteq X$ is $\boldsymbol{\Sigma}^1_3$ if it of the form $A(x) \leftrightarrow \exists y\ \forall z\ \exists w\ F(x, y, z, w)$, where $F \subseteq X \times (\omega^\omega)^3$ is closed, and all the y, z, w quantifiers range over ω^ω. Similarly, A is $\boldsymbol{\Pi}^1_3$ if it is of the form $A(x) \leftrightarrow \forall y\ \exists z\ \forall w\ \neg F(x, y, z, w)$, where again F is closed. Thus, a bound for the level of the set A in the projective hierarchy is obtained by counting quantifier alternations in the logical description of the set.

Lemma 2.3. *For any Polish space X, all of the $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$ are pointclasses and are closed under countable unions and intersections. The pointclass $\boldsymbol{\Sigma}^1_n$ is closed under existential quantification over Polish spaces, and $\boldsymbol{\Pi}^1_n$ is closed under universal quantification over Polish spaces. Furthermore, $\boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n \subseteq \boldsymbol{\Delta}^1_{n+1}$.*

Proof. It follows easily that they are all pointclasses. For example, to see this for $\boldsymbol{\Sigma}^1_1$, suppose $B \subseteq Y$ is $\boldsymbol{\Sigma}^1_1$ and $f \colon X \to Y$ is continuous. Let $A = f^{-1}(X)$. Then $A(x) \leftrightarrow B(f(x)) \leftrightarrow \exists z \in \omega^\omega\ F(f(x), z)$ where $F \subseteq Y \times \omega^\omega$ is closed. Since $\boldsymbol{\Pi}^0_1$ is a pointclass, $\{(x, z) \colon F(f(x), z)\}$ is closed, and so $A \in \boldsymbol{\Sigma}^1_1$.

To show $\boldsymbol{\Sigma}^1_1$ is closed under countable unions, suppose $A_n \subseteq X$ are $\boldsymbol{\Sigma}^1_1$, and let $F_n \subseteq X \times \omega^\omega$ be closed such that $A_n(x) \leftrightarrow \exists y \in \omega^\omega\ F_n(x, y)$. Define $F \subseteq X \times \omega^\omega$ by

$F(x,y) \leftrightarrow F_{y(0)}(x,y')$, where $y'(n) = y(n+1)$. F is easily closed. Let $A = \bigcup_n A_n$. Then $A(x) \leftrightarrow \exists y \in \omega^\omega F(x,y)$, so $A \in \mathbf{\Sigma}^1_1$.

To show $\mathbf{\Sigma}^1_1$ is closed under countable intersections, let A_n, F_n be as above, and let $A = \bigcap_n A_n$. Then,

$$\begin{aligned} A(x) &\leftrightarrow \forall n\ \exists y\ F_n(x,y) \\ &\leftrightarrow \exists y\ \forall n\ F_n(x,(y)_n) \end{aligned}$$

Since $\{(x,y) : \forall n\ F_n(x,(y)_n)\}$ is closed, this shows that $A \in \mathbf{\Sigma}^1_1$.

The proof that $\mathbf{\Sigma}^1_n$, for $n > 1$, is closed under countable unions and intersections is almost identical. It follows that all the $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$ classes are closed under countable unions and intersections.

To see that $\mathbf{\Sigma}^1_n$ is closed under $\exists^Y$ for any Polish space Y, let $B \subseteq X \times Y$ be $\mathbf{\Sigma}^1_n$, and let $A(x) \leftrightarrow \exists y \in Y\ B(x,y)$. Let $\pi \colon \omega^\omega \to Y$ be continuous and onto. Define $B' \subseteq X \times \omega^\omega$ by $B'(x,z) \leftrightarrow B(x,\pi(z))$. So, $B' \in \mathbf{\Sigma}^1_n \restriction X \times \omega^\omega$. Write $B'(x,z) \leftrightarrow \exists w \in \omega^\omega C(x,z,w)$, where $C \in \mathbf{\Pi}^1_{n-1}$ (we assume $n > 1$, the case $n = 1$ being similar). Then

$$\begin{aligned} A(x) &\leftrightarrow \exists y \in Y\ B(x,y) \\ &\leftrightarrow \exists z \in \omega^\omega\ B(x,\pi(z)) \\ &\leftrightarrow \exists z \in \omega^\omega\ \exists w \in \omega^\omega\ C(x,\pi(z),w) \\ &\leftrightarrow \exists u \in \omega^\omega\ C(x,\pi((u)_0),(u)_1) \end{aligned}$$

Now, $\{(x,u) : C(x,\pi((u)_0),(u)_1)\}$ is $\mathbf{\Pi}^1_{n-1}$ and this shows $A \in \mathbf{\Sigma}^1_n$. The closure of $\mathbf{\Pi}^1_n$ under $\forall^Y$ follows from this.

The fact that $\mathbf{\Sigma}^1_n \subseteq \mathbf{\Pi}^1_{n+1}$ is essentially trivial: if $A \in \mathbf{\Sigma}^1_n$ then $A(x) \leftrightarrow \forall y\ B(x,y)$ where $B(x,y) \leftrightarrow A(x)$. To see that $\mathbf{\Sigma}^1_n \subseteq \mathbf{\Sigma}^1_{n+1}$ it suffices by induction to show that $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^1_1$, that is, $\mathbf{\Sigma}^1_1$ contains all the open sets. However $\mathbf{\Sigma}^1_1$ contains all the Borel sets since $\mathbf{\Sigma}^1_1$ contains all the closed sets and is closed under countable unions and intersections. This shows $\mathbf{\Sigma}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$ and thus also $\mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$. □

Exercise 32. Suppose $Y \subseteq X$ are Polish spaces (Y with subspace topology from X). Show that a set $A \subseteq Y$ is $\mathbf{\Sigma}^1_n \restriction Y$ iff A is $\mathbf{\Pi}^1_n \restriction X$, and likewise for $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$. [hint: Use the fact that Y is a G_δ in X, and thus every set $A \subseteq Y \times \omega^\omega$ is Borel in $Y \times \omega^\omega$ iff it is Borel in $X \times \omega^\omega$. Use the closure properties of the $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$.]

Exercise 33. Show that if $f \colon X \to Y$ and $g \colon Y \to Z$ are both $\mathbf{\Sigma}^1_n$-measurable functions, then $g \circ f$ is also $\mathbf{\Sigma}^1_n$-measurable.

The next lemma records a few simple reformulations of the notion of analytic.

Lemma 2.4. *For any Polish space X the following are equivalent.*

(1) *$A \subseteq X$ is analytic.*
(2) *$A \subseteq X$ is the continuous image of a closed set $F \subseteq \omega^\omega$.*
(3) *$A \subseteq X$ is the image of a Borel set $B \subseteq Y$ in a Polish space Y by a Borel function $f \colon Y \to X$.*

Proof. Clearly (1) ⇒ (2) ⇒ (3). So, suppose $B \subseteq Y$ is Borel, $f \colon Y \to X$ is a Borel function, and $A = f[B]$. Then $A(x) \leftrightarrow \exists y \in Y\ (f(y) = x)$. Now $\{(x,y) : f(x) = y\}$ is a Borel set in $X \times Y$ since it is the preimage of the closed equality relation by the Borel function $(x,y) \mapsto (f(x),y)$. By the closure of $\mathbf{\Sigma}^1_1$ under $\exists^Y$ it follows that $A \in \mathbf{\Sigma}^1_1$. □

Lemma 2.5. *For every Polish space X and every n, the pointclasses $\mathbf{\Sigma}^1_n \restriction X$, $\mathbf{\Pi}^1_n \restriction X$ have universal sets $U \subseteq 2^\omega \times X$. For every uncountable Polish space X and every n, there is a set in $\mathbf{\Sigma}^1_n \restriction X - \mathbf{\Pi}^1_n \restriction X$ and vice-versa. In particular, all of the inclusions for the projective hierarchy are proper.*

Proof. Recall there is a closed $F \subseteq 2^\omega \times X$ which is universal for $\mathbf{\Pi}^0_1 \restriction X$, for any Polish space X. Let $F \subseteq 2^\omega \times (X \times \omega^\omega)$ be universal for $\mathbf{\Pi}^0_1 \restriction X \times \omega^\omega$. Define $U(x, y) \leftrightarrow \exists z\ F(x, y, z)$. So, $U \in \mathbf{\Sigma}^1_1 \restriction (2^\omega \times X)$. Suppose that $A \subseteq X$ is $\mathbf{\Sigma}^1_1$, say $A(y) \leftrightarrow \exists z\ E(y, z)$ where $E \subseteq X \times \omega^\omega$ is closed. Let $x \in 2^\omega$ be such that $E = F_x$. Then $U_x(y) \leftrightarrow U(x, y) \leftrightarrow \exists z\ F(x, y, z) \leftrightarrow \exists z\ E(y, z) \leftrightarrow A(y)$. Thus, U is universal for $\mathbf{\Sigma}^1_1 \restriction X$.

The inductive step is entirely similar. Suppose $V \subseteq 2^\omega \times (X \times \omega^\omega)$ is universal for $\mathbf{\Pi}^1_{n-1} \restriction (X \times \omega^\omega)$. Define $U(x, y) \leftrightarrow \exists z\ V(x, y, z)$. Suppose that $A \subseteq X$ is $\mathbf{\Sigma}^1_n$. Write $A(y) \leftrightarrow \exists z\ E(y, z)$ where $E \subseteq X \times \omega^\omega$ is $\mathbf{\Pi}^1_{n-1}$. Let $x \in 2^\omega$ be such that $E = V_x$. Then $U_x(y) \leftrightarrow U(x, y) \leftrightarrow \exists z\ V(x, y, z) \leftrightarrow \exists z\ E(y, z) \leftrightarrow A(y)$. So, U is universal for $\mathbf{\Sigma}^1_n \restriction X$. It follows immediately that $\mathbf{\Pi}^1_n \restriction X$ also has a universal set.

If X is an uncountable Polish space, then X contains a homeomorphic copy of 2^ω. So, we may view $2^\omega \subseteq X$, and 2^ω is closed in X (since it is compact). Any closed set in $2^\omega \times X$ is still closed as a subset of $X \times X$. It follows that the universal sets $U \subseteq 2^\omega \times X$ constructed above for the $\mathbf{\Sigma}^1_n \restriction X$, $\mathbf{\Pi}^1_n \restriction X$ are still $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ sets in the space $X \times X$, and are clearly still universal. From theorem 1.39 it follows that any such universal set $U \subseteq X \times X$ for $\mathbf{\Sigma}^1_n \restriction X$ cannot lie in $\mathbf{\Pi}^1_n \restriction X$. $\square$

We next discuss a useful representation for $\mathbf{\Sigma}^1_1$ sets, in terms of the *Suslin operation $\mathcal{A}$*. For sets $A \subseteq \omega^\omega$, this will have an especially important and useful form, and will motivate a key definition, that of a Suslin representation.

We first define the Suslin operation for a general Polish space.

Definition 2.6. Let X be Polish and $\{B_s\}_{s \in \omega^{<\omega}}$ be a family of subsets of X indexed by $\omega^{<\omega}$. Then $\mathcal{A}_s(\{B_s\})$ is defined by

$$x \in \mathcal{A}(\{B_s\}) \leftrightarrow \exists y \in \omega^\omega\ \forall n\ (x \in B_{y \restriction n}).$$

When there is no danger of confusion we will just write $\mathcal{A}(\{B_s\})$.

It is important in this definition to use $\omega^{<\omega}$ as the index set, as opposed to say $2^{<\omega}$ as the next exercise shows.

Exercise 34. Suppose $\{B_s\}_{s \in 2^{<\omega}}$ is a family of Borel subsets of the Polish space X. Define the analog of the Suslin operation in the obvious manner, that is, $x \in \mathcal{A}'(\{B_s\}) \leftrightarrow \exists y \in 2^\omega\ \forall n\ (x \in B_{y \restriction n})$. Show that $\mathcal{A}'(\{B_s\})$ is still a Borel set. [hint: Using the compactness of 2^ω (i.e., König's lemma, the fact that every infinite finitely splitting tree has an infinite branch), show that $x \in \mathcal{A}'(\{B_s\})$ iff $\forall n\ \exists s \in 2^n\ (x \in B_s)$. Show this gives a Borel computation of $\mathcal{A}'(\{B_s\})$.]

Note that there is no loss of generality is assuming that if t extends s then $F_t \subseteq F_s$, since we can replace the closed sets F_s with $F'_s = \bigcap_{i < \mathrm{lh}(s)} F_{s \restriction i}$. Clearly $\mathcal{A}(\{F_s\}) = \mathcal{A}(\{F'_s\})$.

Theorem 2.7. *For any Polish space X, $\mathbf{\Sigma}^1_1 \restriction X = \mathcal{A}(\mathbf{\Pi}^0_1 \restriction X)$. That is, $A \subseteq X$ is $\mathbf{\Sigma}^1_1$ iff it is of the form $A = \mathcal{A}(\{F_s\})$, where the $F_s \subseteq X$ are closed.*

Proof. Suppose first that $A = \mathcal{A}(\{F_s\})$ where the F_s are closed. So, $A(x) \leftrightarrow \exists y \in \omega^\omega\ \forall n\ (x \in F_{y \restriction n})$. The relation $R(x, y, n) \leftrightarrow (x \in F_{y \restriction n})$ is closed since for any n, $\{(x, y) \colon x \in F_{y \restriction n}\}$ is closed. This shows $A \in \mathbf{\Sigma}^1_1 \restriction X$.

Suppose next that $A \in \mathbf{\Sigma}^1_1 \restriction X$. Say, $A(x) \leftrightarrow \exists y \in \omega^\omega F(x,y)$ where $F \subseteq X \times \omega^\omega$ is closed. For $s \in \omega^{<\omega}$ let $F_s = \overline{A_s}$ where $A_s(x) \leftrightarrow \exists y\ (y \restriction \mathrm{lh}(s) = s \wedge F(x,y))$. We show that $A = \mathcal{A}(\{F_s\})$ which suffices. If $x \in A$, let $y \in \omega^\omega$ be such that $F(x,y)$. Then for any n, $x \in A_{y\restriction n} \subseteq F_{y \restriction n}$, using y as a witness in the definition of $A_{y\restriction n}$. Suppose next that $x \in \mathcal{A}(\{F_s\})$. Let $y \in \omega^\omega$ be such that for all n, $x \in F_{y\restriction n}$. So, for each n there is an x_n and a y_n such that $\rho(x,x_n) < \frac{1}{n}$ and $y_n \restriction n = y \restriction n$ such that $F(x_n, y_n)$ (ρ is a compatible complete metric for X). But clearly $(x_n, y_n) \to (x,y)$, and since F is closed this gives $(x,y) \in F$. So, $x \in A$. □

Exercise 35. Show that for any pointclass $\mathbf{\Gamma}$, $\mathcal{A}(\mathbf{\Gamma})$ is also a pointclass and is closed under countable unions and intersections. Show also that if $\mathbf{\Gamma}\restriction X$ has a universal set $U \subseteq \omega^\omega \times X$, then so does $\mathcal{A}(\mathbf{\Gamma})$. Deduce in this case that $\mathbf{\Gamma}\restriction X$ is non-selfdual.

Exercise 36. Show that for any pointclass $\mathbf{\Gamma}$, $\mathcal{A}(\mathcal{A}(\mathbf{\Gamma})) = \mathcal{A}(\mathbf{\Gamma})$. [hint: Let $A \in \mathcal{A}(\mathcal{A}(\mathbf{\Gamma}))$. Say, $A = \mathcal{A}_s(\{B_s\})$, where $B_s = \mathcal{A}_t(\{C^s_t\})$, and each $C^s_t \in \mathbf{\Gamma}$. Then $A(x) \leftrightarrow \exists y\ \forall n\ \exists z\ \forall m\ (x \in C^{y\restriction n}_{z \restriction m}) \leftrightarrow \exists y\ \exists z\ \forall n\ \forall m\ (x \in C^{y\restriction n}_{(z)_n \restriction m}) \leftrightarrow \exists w\ \forall k\ (x \in C^{(w)_0 \restriction (k)_0}_{((w)_1)_{(k)_0} \restriction (k)_1})$. Note that $(w)_0 \restriction (k)_0$ and $((w)_1)_{(k)_0} \restriction (k)_1$ depend only on $w \restriction k$. So define $D_s = C^u_v$ where $u = (s)_0 \restriction (\ln(s))_0$ and $v = ((s)_1)_{(\ln(s))_0} \restriction (\ln(s))_1$. Then $A = \mathcal{A}(\{D_s\})$.]

Although $\mathbf{\Sigma}^1_1 = \mathcal{A}(\mathbf{\Pi}^0_1)$, it is not true that $\mathbf{\Sigma}^1_2 = \mathcal{A}(\mathbf{\Pi}^1_1)$. In fact, $\mathcal{A}(\mathbf{\Pi}^1_1) \subsetneq \mathbf{\Delta}^1_2$ (we give this computation later). If we iterate taking the operation $\mathcal{A}$ and complements ω times, we generate the smallest σ-algebra closed under the operation $\mathcal{A}$. This collection is sometimes referred to as the C-sets. It is properly contained within $\mathbf{\Delta}^1_2$.

Given a Suslin scheme $\{A_s\}$, we may improve it slightly as follows.

Definition 2.8. We say a Suslin scheme $\{A_s\}_{s \in \omega^{<\omega}}$ is *good* if it satisfies:

(1) If t extends s then $A'_t \subseteq A'_s$,
(2) $\mathrm{diam}(A'_s) < \frac{1}{2^{\mathrm{lh}(s)}}$.

Lemma 2.9. *Suppose $\mathbf{\Gamma}$ is closed under intersections with closed sets, and let $\{A_s\}_{s\in\omega^{<\omega}}$ be a Suslin scheme with each $A_s \in \mathbf{\Gamma}$. Then there is a good Suslin scheme $\{A'_s\}$ with each $A'_s \in \mathbf{\Gamma}$ with $\mathcal{A}(\{A_s\}) = \mathcal{A}(\{A'_s\})$.*

Proof. Let V_i be a base for X. Say a sequence $u \in \omega^{<\omega}$ is good if

$$\forall i < \mathrm{lh}(u)\ \mathrm{diam}(V_{u(i)}) < \frac{1}{2^i} \text{ and } \forall i < \mathrm{lh}(u) - 1\ \bar{V}_{u(i+1)} \subseteq V_{u(i)}.$$

For $(u,s) \in (\omega^{<\omega})^2$ with $\mathrm{lh}(u) = \mathrm{lh}(s)$, Set $B_{(u,s)} = A_s \cap \bar{V}_{u(\mathrm{lh}(u)-1)}$ if u is good and $B_{(u,s)} = \emptyset$ otherwise.

If $x \in \mathcal{A}(\{B_{u,s}\})$, then there is a z and a y such that $\forall n\ x \in B_{z\restriction n, y \restriction n} \subseteq A_{y\restriction n}$, and so $x \in \mathcal{A}(\{A_s\})$. If $x \in \mathcal{A}(\{A_s\})$, let $y \in \omega^\omega$ be such that $\forall n\ x \in A_{s \restriction n}$. Fix a $z \in \omega^\omega$ such that for all n, $z\restriction n$ is good and $x \in V_{z(n)}$ (which we can easily do since $\{V_i\}$ is a base). Then for all n, $x \in B_{z\restriction n, y \restriction n}$, so $x \in \mathcal{A}(\{B_{u,s}\})$. Finally, by taking a bijection between $\omega \times \omega$ and ω allows us to reorganize the Suslin scheme into one with index set $\omega^{<\omega}$. That is, define $A'_t = B_{u,s}$, where $u(i) = (t(i))_0$ and $s(i) = (t(i))_1$. □

We next re-interpret the notion of analytic and Suslin operation when $X = \omega^\omega$. Here the concepts simplify down to a basic combinatorial notion, that of a Suslin representation. Recall that if T is a tree on a set Z, then $[T] \subseteq Z^\omega$ is the set of infinite branches through T, that is, $[T] = \{f \in Z^\omega : \forall n\ f\restriction n \in Z\}$.

Definition 2.10. Let T be a tree on $Y \times Z$. We let

$$\begin{aligned} p[T] &= \{f \in Y^\omega : \exists g \in Z^\omega\ (f,g) \in [T]\} \\ &= \{f \in Y^\omega : \exists g \in Z^\omega\ \forall n\ (f\restriction n, g\restriction n) \in T\}. \end{aligned}$$

Note that we identify elements of a tree T on $Y \times Z$ with pairs of sequences (s,t) with $s \in Y^{<\omega}$, $t \in Z^{<\omega}$, and $\mathrm{lh}(s) = \mathrm{lh}(t)$.

Theorem 2.11. *A set $A \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_1$ iff there is a tree T on $\omega \times \omega$ such that $A = p[T]$.*

Proof. first suppose T is a tree on $\omega \times \omega$ and $A = p[Y]$. So, $A(x) \leftrightarrow \exists y \in \omega^\omega\ \forall n\ (x\restriction n, y\restriction n) \in T$. For each n, $\{(x,y) : (x\restriction n, y\restriction n) \in T\}$ is a clopen set in $\omega^\omega \times \omega^\omega$. So, $\{(x,y) : \forall n\ (x\restriction n, y\restriction n) \in T\}$ is a closed set in $\omega^\omega \times \omega^\omega$. This shows $A \in \mathbf{\Sigma}^1_1$.

Suppose next that $A \in \mathbf{\Sigma}^1_1$, $A \subseteq \omega^\omega$. Let $F \subseteq \omega^\omega \times \omega^\omega$ be closed such that $A(x) \leftrightarrow \exists y\ F(x,y)$. Let T be the tree on $\omega \times \omega$ corresponding to F, that is, $(s,t) \in T$ iff $\exists x, y\ (x\restriction \mathrm{lh}(s) = s \wedge y\restriction \mathrm{lh}(y) = y \wedge (x,y) \in F\}$. Since F is closed, $F = [T]$. Thus $A(x) \leftrightarrow \exists y\ (x,y) \in [T]$ and we are done. $\square$

This representation suggests a natural generalization. We emphasize that the next definition is made in ZF.

Definition 2.12. Let $\kappa \in \mathrm{On}$. We say $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ such that $A = p[T]$.

Exercise 37. Show that if $\alpha \in \mathrm{On}$ and $\kappa = |\alpha|$, then a set is α-Suslin iff it is κ-Suslin. We let $S(\kappa)$ denote the collection of κ-Suslin sets.

In view of exercise 37, there is no loss of generality in restricting to cardinals in definition 2.12.

Thus, $S(\omega) = \mathbf{\Sigma}^1_1$. It is not immediately clear for which cardinals κ we pick up new sets. We make this into the following definition.

Definition 2.13. We say κ is a *Suslin cardinal* if $S(\kappa) - \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset$.

Exercise 38. Show that for any cardinal κ that $S(\kappa)$ is a pointclass closed under countable unions, countable intersections, and $\exists^{\omega^\omega}$.

Assuming AC, every set $A \subseteq \omega^\omega$ is $\mathfrak{c} = 2^\omega$-Suslin, but these trivial Suslin representations are of no interest [define the tree T on $\omega \times \mathfrak{c}$ by $(s, \vec{\alpha}) \in T$ iff $x_\alpha\restriction(\mathrm{lh}(s) = s$, where $\{x_\alpha\}_{\alpha<\mathfrak{c}}$ is a wellordering of A.] We will discuss Suslin sets and Suslin cardinals in more detail later, but now we return to the theory of $\mathbf{\Sigma}^1_1$ sets.

We next show Suslin's theorem that $\mathbf{\Delta}^1_1$ =Borel. We prove this in the form of a more general separation theorem.

Theorem 2.14 (Suslin)**.** *Let A, B be disjoint $\mathbf{\Sigma}^1_1$ sets in a Polish space X. Then there is a Borel set C with $A \subseteq C$ and $C \cap B = \emptyset$.*

Proof. Let $A = \mathcal{A}(\{F_s\})$, $B = \mathcal{A}(\{H_t\})$, where F_s, H_t are closed, and $A \cap B = \emptyset$. We may assume the F_s and H_t are good Suslin schemes. We may also assume $F_\emptyset = H_\emptyset = X$. Define T to be the tree on $\omega \times \omega$ given by:

$$(s,t) \in T \leftrightarrow (\mathrm{lh}(s) = \mathrm{lh}(t) \wedge F_s \cap H_t \neq \emptyset).$$

We first claim that T is wellfounded. For suppose $(x,y) \in [T]$. For each n, let $z_n \in F_{x\restriction n} \cap H_{y\restriction n}$. Since the Suslin schemes are good, $\{z_n\}$ is Cauchy, so converges to some $z \in X$. Since the F_s and H_t are decreasing along any branch (from goodness), and the F_s, H_t are closed, we have $z \in \bigcap_n F_{x\restriction n}$, so $z \in A$. Likewise, $z \in B$, a contradiction since $A \cap B = \varnothing$.

By induction on the rank of (s,t) in the tree T we define a Borel set $C_{(s,t)} \subseteq X$ which separates $A_s \doteq \{z\colon \exists x\,(x\restriction \mathrm{lh}(s) = n \wedge \forall n\ (z \in F_{x\restriction n}))$ from B_t (with a similar definition). If (s,t) has rank 0 (which we take to mean is not in T), then $F_s \cap H_t = \varnothing$. Then $C_s = F_s$ is a closed set which contains A_s and is disjoint from B_s (since $A_s \subseteq F_s$ and $B_s \subseteq H_s$).

For the general inductive step, fix $(s,t) \in T$. By induction, for each $i,j \in \omega$ there is a Borel set $C_{s^\frown i,t^\frown j}$ separating $A_{s^\frown i}$ from $B_{t^\frown j}$. Note that $A_s = \bigcup_i A_{s^\frown i}$ and $B_t = \bigcup_j B_{t^\frown j}$. Define then $C_{(s,t)} = \bigcup_i \bigcap_j C_{s^\frown i,t^\frown j}$. This clearly works, noting that for each i, $\bigcap_j C_{s^\frown i,t^\frown j}$ separates $A_{s^\frown i}$ from $\bigcup_j B_{t^\frown j}$.

Finally, $C = C_{(\varnothing,\varnothing)}$ is the desired Borel set. □

Corollary 2.15. *For every Polish space X, $\boldsymbol{\Delta}^1_1\restriction X$ is the collection of Borel sets in X.*

Proof. If $A \subseteq X$ is $\boldsymbol{\Delta}^1_1$, apply the previous theorem to A and $B \doteq X - A$. The separating set C must equal A. □

Corollary 2.16. *For every Polish space X,* $\mathrm{sep}(\boldsymbol{\Sigma}^1_1)$.

Corollary 2.17. *A function $f\colon X \to Y$ is Borel (that is, Borel measurable) iff it has a Borel graph $G_f \subseteq X \times Y$.*

Proof. First suppose that $f\colon X \to Y$ is Borel measurable. Let $\{V_n\}$ be a base for Y. Then $f(x) = y$ iff $\forall n\ (x \in f^{-1}(V_n) \leftrightarrow y \in V_n)$. Since each $f^{-1}(V_n)$ is Borel, this shows G_f is Borel.

Suppose next that G_f is Borel. Let $V \subseteq Y$ be open. Then $f(x) \in V \leftrightarrow \exists y\ (G_f(x,y) \wedge y \in V) \leftrightarrow \forall y\ (G_f(x,y) \to y \in V)$. This shows that $f^{-1}(V)$ is both $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$, and hence Borel. □

Corollary 2.18. *If $f\colon X \to Y$ is a Borel function and $A \subseteq Y$ is $\boldsymbol{\Sigma}^1_n$ (or $\boldsymbol{\Pi}^1_n$), then $f^{-1}(A)$ is $\boldsymbol{\Sigma}^1_n$ (or $\boldsymbol{\Pi}^1_n$).*

Proof. If $A \in \boldsymbol{\Sigma}^1_n$, then $x \in f^{-1}(A) \leftrightarrow \exists y\ (y = f(x) \wedge y \in A)$, which shows $f^{-1}(A) \in \boldsymbol{\Sigma}^1_n$. If $A \in \boldsymbol{\Pi}^1_n$, then $x \in f^{-1}(A) \leftrightarrow \forall y\ (y = f(x) \to y \in A)$, which shows $f^{-1}(A) \in \boldsymbol{\Pi}^1_n$. □

Thus, the projective hierarchy begins at precisely the point where the Borel ends, at the collection of Borel sets $\mathcal{B} = \boldsymbol{\Delta}^1_1$.

The next result on Borel sets strengthens exercise 31 and is an important fact in its own right. We will see below that the converse of this theorem also holds.

Theorem 2.19. *Every Borel set in a Polish space is the continuous one-to-one image of a closed subset of ω^ω.*

Remark 2.20. It doesn't matter whether we view the conclusion as saying there is a closed $F \subseteq \omega^\omega$ and a continuous $f\colon F \to X$ with $A = f(F)$, or as saying there is a closed $F \subseteq \omega^\omega$ and a continuous $f\colon \omega^\omega \to X$ with $A = f(F)$. This is because every closed set $F \subseteq \omega^\omega$ is a retract of ω^ω (see exercise 20).

Proof. Let $A \subseteq X$ be Borel. If A is open or closed, then it is Polish (with the subspace topology), and hence the result follows from lemma 1.26. From exercise 15 it suffices to show that the family $\mathcal{F}$ of subsets of X which are continuous one-to-one images of closed subsets of ω^ω is closed under countable disjoint unions and countable intersections.

Suppose $A = \bigcup_n A_n$, a disjoint union, where each $A_n \in \mathcal{F}$. Say, $A_n = f_n(F_n)$ where $F_n \subseteq \omega^\omega$ is closed and $f_n \colon F_n \to X$ is continuous and one-to-one. Define $F \subseteq \omega^\omega$ by $F(x) \leftrightarrow x' \in F_{x(0)}$, where $x'(n) = x(n+1)$. Easily F is closed. Define $f \colon F \to X$ by $f(x) = f_{x(0)}(x')$. Since each f_n in one-to-one on F_n, f is also one-to-one $\{x \colon x(0) = n\}$. Since the A_n are pairwise disjoint, it then follows that f is one-to-one on F. Since f is continuous on each of the relatively clopen pieces $\{x \colon x(0) = n\}$ it follows that f is continuous on F. Clearly, $f[F] = \bigcup_n f_n[F_n] = \bigcup_n A_n = A$.

Suppose next that $A = \bigcap_n A_n$, and $A_n = f_n[F_n]$ where again $F_n \subseteq \omega^\omega$ is closed. Define $F \subseteq \omega^\omega$ by

$$F(x) \leftrightarrow [(\forall n\ (x)_n \in F_n) \wedge \forall i, j\ (f_i((x)_i) = f_j((x)_j))].$$

Clearly F is closed. Define f on F by $f(x) = f_0((x)_0)$ ($= f_i((x)_i)$ for all i). If $x \in F$, then since $f(x) = f_i((x)_i)$, we have $f(x) \in A_i$. So, $f[F] \subseteq \bigcap_n A_n$. If $y \in \bigcap_n A_n$, then for each i let $x_i \in A_i$ with $f_i(x_i) = y$. Let $x \in \omega^\omega$ be such that for all i, $(x)_i = x_i$. Then $x \in F$ and $f(x) = y$.

Finally, f is one-to-one on F. For suppose $x, y \in F$ and $x \neq y$. Then for some n, $(x)_n \neq (y)_n$, and both are in F_n. Since f_n is one-to-one on F_n, $f_n((x)_n) \neq f_n((y)_n)$. Thus $f(x) = f_n((x)_n) \neq f_n((y)_n) = f(y)$. □

Theorem 2.21 (Lusin)**.** *If $f \colon X \to Y$ is Borel and one-to-one, and $A \subseteq X$ is Borel, then $f[A]$ is Borel.*

Proof. We first do the special case where f is continuous. We have already shown in theorem 2.19 that every Borel set is the continuous one-to-one image of a closed set in ω^ω. Thus, it suffices to show that a continuous one-to-one image of a closed set in ω^ω is Borel.

Suppose $F \subseteq \omega^\omega$ is closed, and $A = f[F]$ where $f \colon F \to X$ is continuous and one-to-one. Let $F = [T]$ where T is a tree on ω. For $s \in T$, let $F_s = F \cap N_s$, and let $A_s = f[F_s] \subseteq A$. Since f is one-to-one, if $s \perp t$, then $A_s \cap A_t = \varnothing$. Using Suslin's theorem, we define Borel sets B_s with $A_s \subseteq B_s$ and such that if $s \perp t$ then $B_s \cap B_t = \varnothing$. If B_s is defined, using Suslin's theorem for each i let $C_{s^\frown i} \supseteq B_{s^\frown i}$ be Borel and such that if $i \neq j$ then $C_{s^\frown i} \cap C_{s^\frown j} = \varnothing$ (separate $A_{s^\frown i}$ from $\bigcup_{j \neq i} A_{s^\frown j}$, and then disjointize these separating Borel sets). Let $B_{s^\frown i} = C_{s^\frown i} \cap B_s$. Finally, by replacing each B_s with $B_s \cap \overline{A}_s$, we may assume that $B_s \subseteq \overline{A}_s$ for each s.

We next show that $A = \mathcal{A}(\{B_s\})$. First assume $x \in A$, say $x = f(y)$ where $y \in [T]$. then for all n, $y \in F_{y \restriction n}$, and so $x \in A_{y \restriction n} \subseteq B_{y \restriction n}$, so $x \in \mathcal{A}(\{B_s\})$. Next suppose $x \in \mathcal{A}(\{B_s\})$. Fix $y \in \omega^\omega$ such that] for all n, $x \in B_{y \restriction n}$. Since $B_{s \restriction n} \subseteq \overline{A}_{y \restriction n}$, there is for each n a $y_n \in \omega^\omega$ such that $y_n \restriction n = y \restriction y$ and $\rho(f(y(n), x) < \frac{1}{2^n}$. Since $y_n \to y$, $y \in F$. Since f is continuous, $f(y) = \lim_n f(y_n) = x$. Thus, $x \in A$.

Since the B_s are Borel and have the property that whenever $s \perp t$ then $B_s \cap B_t = \varnothing$, it follows that $\mathcal{A}(\{B_s\})$ is Borel. To see this, note that

$$\begin{aligned} x \in \mathcal{A}(\{B_s\}) &\leftrightarrow \exists y\ \forall n\ (x \in B_{y\restriction n}) \\ &\leftrightarrow \forall k\ \exists s \in \omega^k\ (x \in B_s). \end{aligned}$$

The last equivalence follows from the disjointness assumption on the B_s, and shows that $\mathcal{A}(\{B\})$ is Borel.

For the general case where $f\colon X \to Y$ is Borel, enlarge the topology τ on X to a finer Polish topology τ' making $f\colon (X, \tau') \to Y$ continuous, and then apply the above case. To do this, consider a basis $\{V_n\}_{n\in\omega}$ for Y. Get τ' to make all of the $f^{-1}(V_n)$ clopen. This makes f continuous. □

In corollary 2.16 we observed $\mathrm{sep}(\boldsymbol{\Sigma}^1_1)$. This leads us to suspect that $\mathrm{red}(\boldsymbol{\Pi}^1_1)$ and even $\mathrm{pwo}(\boldsymbol{\Pi}^1_1)$ might be true. In fact an even stronger property property called the *scale* property holds for $\boldsymbol{\Pi}^1_1$. We first define the notion of scale.

Definition 2.22. A *semi-scale* $\{\phi_n\}_{n\in\omega}$ on a set $A \subseteq X$ is a sequence of norms $\phi_n\colon A \to \mathrm{On}$ such that if $\{x_m\}_{m\in\omega} \subseteq A$, $x_m \to x$, and for each n the norms $\phi_n(x_m)$ are eventually constant (that is, there is a $\lambda_n \in \mathrm{On}$ such that $\phi_n(x_m) = \lambda_n$ for all large enough m), then $x \in A$.

A *scale* $\{\phi_n\}_{n\in\omega}$ is a semi-scale with the additional *semi-continuity property*: for all n, $\phi_n(x) \leqslant \lambda_n = \lim_{m\to\infty} \phi_n(x_m)$.

For sets in the Baire space, we first show that semi-scales are equivalent to Suslin representations.

If T is an illfounded tree on a wellordered set $(X, <)$, then the *left-most branch* of T, $\ell(T) = (\ell(0), \ell(1), \dots)$ is defined as follows: $\ell(0)$ is the $<$ least element of X such that $\exists f \in X^\omega\ (x\restriction 1 = (\ell(0)) \wedge f \in [T])$. In general we inductively define $\ell(n)$ to be the $<$ least element of X such that $\exists f \in X^\omega\ (f\restriction n\ = \ell\restriction n \wedge f \in [T])$. This is well-defined, and the resulting branch $\ell \in [T]$ is leftmost in the sense that if $f \in [T]$ and $\ell \neq f$, then for the least n such that $\ell(n) \neq f(n)$ we have $\ell(n) < f(n)$.

Lemma 2.23. *$A \subseteq \omega^\omega$ is κ-Suslin iff A admits a semi-scale with norms into κ.*

Proof. Suppose first that A is κ-Suslin, say $A = p[T]$ where T is a tree on $\omega \times \kappa$. For each $x \in p[T]$, let $\ell^x = \langle \ell^x(0), \ell^x(1), \dots\rangle$ be the left-most branch of $T_x = \{\vec{\alpha}\colon (x\restriction \mathrm{lh}(\vec{\alpha}), \vec{\alpha}) \in T\}$. Let $\phi_n(x) = \ell^x(n)$. To see this works, suppose $x_m \in A$, $x_m \to x$, and for each n, $\lim_m \phi_n(x_m) = \lambda_n$. Then for each n and and all large enough m we have that $x_m\restriction n = x\restriction n$ and $\ell^{x_m}\restriction n = (\lambda_0, \dots, \lambda_{n-1})$. In particular, since $(x_m, \ell^{x_m}) \in [T]$, we have $(x\restriction n, (\lambda_0, \dots, \lambda_{n-1})) \in T$. Thus, $(x, \vec{\lambda}) \in [T]$, so $x \in A$.

For the other direction, suppose $\{\phi_n\}_{n\in\omega}$ is a semi-scale on A. Define the *tree of the semiscale* T_ϕ by:

$$(s, \vec{\alpha}) \in T_\phi \leftrightarrow \exists x\ (x\restriction \mathrm{lh}(s) = s \wedge \phi_0(x) = \alpha_0 \wedge \cdots \wedge \phi_{\mathrm{lh}(s)-1}(x) = \alpha_{\mathrm{lh}(s)-1}).$$

If $x \in A$, then $x \in p[T_\phi]$, since $(x, (\phi_0(x), \phi_1(x), \dots)) \in [T_\phi]$. On the other hand, suppose $x \in p[T]$, say $(x, f) \in [T]$. for each n, $(x\restriction n, f\restriction n) \in T$, and so there is an $x_n \in A$ with $x_m\restriction n = x\restriction n$ and $\phi_0(x_m) = f(0), \dots, \phi_{n-1}(x_m) = f(n-1)$. Then $x_m \to x$ and for each n and all $j > n$, $\phi_n(x_j) = f(n)$, so all the norms are eventually constant. Since $\{\phi_n\}$ is a semi-scale, we have $x \in A$. □

If $A = p[T]$ and $\{\phi_n\}$ is the semi-scale derived from T, we can patch-up $\{\phi_n\}$ to be a scale in the following manner. Say each ϕ_n is a norm into $\kappa \in \mathrm{On}$. Let $\langle \alpha_0, \dots, \alpha_{i-1} \rangle$ denote the rank of $(\alpha_0, \dots, \alpha_{i-1})$ in the lexicographic ordering on κ^i (so this lexicographic ordering has length the ordinal κ^i). Define now the norms ψ_n on A by:

$$\psi_n(x) = \langle \phi_0(x), \phi_1(x), \dots, \phi_n(x) \rangle.$$

Claim. $\{\psi_n\}_{n \in \omega}$ is a scale on A.

Proof. Suppose $\{x_m\} \subseteq A$, $x_m \to x$, and and for each n, $\lim_m \psi_n(x_m) = \lambda_n$. Say $\lambda_n = \langle \alpha_0^n, \alpha_1^n, \dots, \alpha_n^n \rangle$. So, for large enough m we have $\phi_0(x_m) = \alpha_0^n, \dots, \phi_n(x_m) = \alpha_n^n$. In particular, for all $n, n' \geqslant i$, $\alpha_i^n = \alpha_i^{n'} = \alpha_i \doteq \lim_m \phi_i(x_m)$. So, for large enough m, $\phi_i(x_m) = \alpha_i$. Since $\{\phi_n\}$ is a semiscale on A, $x \in A$. Moreover, $(x, \vec{\alpha}) \in [T]$. For any n we then have $\psi_n(x) = \langle \ell^x(0), \dots, \ell^x(n) \rangle \leqslant \langle \alpha_0, \dots, \alpha_n \rangle = \lambda_n$ since ℓ^x is the left-most branch of T_x. □

We call $\{\psi_n\}$ the scale derived from T. Note that if T is a tree on $\omega \times \kappa$, then the derived semi-scale has norms into κ, but the derived scale has norms on the slightly bigger ordinal κ^ω (ordinal exponentiation).

Exercise 39. Let $\{\phi_n\}_{n \in \omega}$ be a semi-scale on $A \subseteq \omega^\omega$. Show that we may directly define the scale $\{\psi_n\}_{n \in \omega}$ on A by:

$$\begin{aligned}\psi_n(x) = \min\{\langle \alpha_0, \dots, \alpha_n \rangle \colon & \exists \vec{\beta}\ \forall m\ \exists y \in A\ (y \restriction m = x \restriction m \wedge \\ & \phi_0(y) = \alpha_0 \wedge \dots \wedge \phi_n(y) = \alpha_n \wedge \forall k > n\ \phi_k(y) = \beta_{k-n-1})\}\end{aligned}$$

[hint: you can trace through the definition of ψ given before from the tree T_ϕ, or you can proceed directly as follows. Let $x_m \in A$ and assume $x_m \to x$ and for each n, the $\psi_n(x_m)$ are eventually equal to $\langle \alpha_0^n, \dots, \alpha_n^n \rangle$. Argue that for $i, j \geqslant n$ that $\alpha_n^i = \alpha_n^j$. Let α_n be the common value of the α_n^i for $i \geqslant n$. Then for all n and all m there is a y with $y \restriction m = x \restriction n$ and $\phi_0(y) = \alpha_0, \dots, \phi_n(y) = \alpha_n$. Since ϕ is a semi-scale, $x \in A$. From the definition of ψ, it follows easily that $\phi_n(x) \leqslant \langle \alpha_0, \dots, \alpha_n \rangle$.]

Exercise 40. Show that if we start with a scale $\phi = \{\phi_n\}_{n \in \omega}$ on a set A, form the tree T_ϕ of the scale, and then let $\psi = \{\psi_n\}_{n \in \omega}$ be the semi-scale of the tree T_ϕ, we have $\psi = \phi$.

Exercise 41. Show that if we start with a tree T on $\omega \times \kappa$, let $\phi = \{\phi_n\}_{n \in \omega}$ be the semi-scale of the tree T (on the set $A = p[T]$), and then let T' be the tree of the semi-scale ϕ, then $T' \subseteq T$. Give an example where $T' \neq T$.

The canonical scale on $A = p[T]$ has, with a slight modification of the definition, a few extra properties which we abstract in the following definition.

Definition 2.24. We say the scale $\{\phi_n\}_{n \in \omega}$ on the set A is *good* if whenever $x_m \in A$ and for each n the norms $\phi_n(x_m)$ are eventually constant, then the x_m converge to some x (and so $x \in A$).

We say $\{\phi_n\}_{n \in \omega}$ is *very good* if it is good and for all $x, y \in A$, and $i \leqslant j$, if $\phi_j(x) \leqslant \phi_j(y)$, then $\phi_i(x) \leqslant \phi_i(y)$.

We say $\{\phi_n\}_{n \in \omega}$ is *excellent* if it is very good and whenever $x, y \in A$ and $\phi_n(x) = \phi_n(y)$ then $x \restriction n = y \restriction n$.

Lemma 2.25. *Suppose $A = p[T]$ where T is a tree on $\omega \times \kappa$. For $x \in A$, define*

$$\psi_n(x) = \langle \ell^x(0), x(0), \ell^x(1), x(1), \dots, \ell^x(n), x(n) \rangle,$$

where $\ell^x = (\ell^x(0), \ell^x(1), \dots)$ *is the left-most branch of* T_x. *Then* ψ *is an excellent scale on* A *with norms into* κ^ω *(ordinal exponentiation).*

Proof. Clearly ψ satisfies the excellence condition. It follows that if $\{x_m\} \subseteq A$ are such that all the norms $\phi_n(x_m)$ are eventually constant, then the x_m converge to some x. As before, we must have $x \in A$. The previous proof of the scale property still holds here since $x \restriction n$ will also equal the limiting values of the $x_m \restriction n$. The very good condition follows from the definition of ψ_n. □

We can also go directly from a scale to an excellent scale.

Lemma 2.26. *Let* $A \subseteq \omega^\omega$ *and* $\{\phi_n\}$ *be a scale on* A*, with say norms into* κ*. Define* ψ_n *on* A *by* $\psi_n(x) = \langle \phi_0(x), x(0), \phi_1(x), x(1), \dots, \phi_n(x), x(n)\rangle$ *(the rank in the lexicographic ordering on* $2n+2$ *tuples from* κ*). Then* $\{\psi_n\}$ *is an excellent scale on* A*.*

Proof. The excellence and very good conditions easily hold. If $\{x_m\} \subseteq A$ and for each n the norms $\psi_n(x_m)$ are eventually constant, then also the $\phi_n(x_m)$ are eventually constant, and $x_m \to x$ for some x. Since $\{\phi_n\}$ is a scale, it follows that $x \in A$ and $\phi_n(x) \leqslant \lim_m \phi_n(x_m)$. It follows that $\psi_n(x) \leqslant \lim_m \psi_n(x_m)$. □

Although the canonical scale (or excellent scale) derived from a Suslin representation $A = p[T]$, T a tree on $\omega \times \kappa$, has norms into a slightly bigger ordinal than κ (though of the same cardinality as κ), we nevertheless have the following.

Theorem 2.27. *For any* $A \subseteq \omega^\omega$ *and cardinal* κ*, the following are equivalent.*

(1) *A is κ-Suslin.*
(2) *A admits a semi-scale with norms into κ.*
(3) *A admits an excellent scale with norms into κ.*

Proof. It is clear that (3) ⇒ (2), and we have already shown that (2) ⇒ (1). So, assume $A = p[T]$ with T a tree on $\omega \times \kappa$, and we show that A admits an excellent scale with norms into κ.

We consider two cases. First assume that $\mathrm{cof}(\kappa) > \omega$. Let T' be the tree on $\omega \times \kappa$ defined by

$$\begin{aligned}(s, (\alpha_0, \dots, \alpha_{n-1})) \in T' \leftrightarrow [&\alpha_0 = \max\{\alpha_0, \dots, \alpha_{n-1}\} \wedge \\ &(s \restriction n-1, (\alpha_1, \dots, \alpha_{n-1})) \in T].\end{aligned}$$

Since $\mathrm{cof}(\kappa) > \omega$ we easily have that $p[T] = p[T']$. Let

$$\psi_n(x) = \langle \ell^x(0), x(0), \ell^x(1), x(1), \dots, \ell^x(n), x(n)\rangle,$$

where now this refers to the rank of the tuple in lexicographic ordering on the tuples of length $2n+2$ from κ having the first element maximal. The rank of any such tuple is less than κ, so $\psi_n(x) < \kappa$. As before, $\{\psi_n\}$ is an excellent scale on $A = p[T'] = p[T]$.

Suppose next that $\mathrm{cof}(\kappa) = \omega$. Let $\{\kappa_n\}_{n \in \omega}$ be an increasing sequence of cardinals with $\kappa = \sup_n \kappa_n$. Let T' be the tree on $\omega \times \kappa$ consisting of all (s, t) with $\mathrm{lh}(s) = \mathrm{lh}(t)$ and t an initial segment of a sequence of the form

$$\langle n_0, \underbrace{0, \dots, 0}_{n_0+1}, \alpha_0, n_1, \underbrace{0, \dots, 0}_{n_1+1}, \alpha_1, \dots\rangle$$

where $(s, (\alpha_0, \alpha_1, \dots, \alpha_{\mathrm{lh}(s)-1})) \in T$ and each $t(i) < \kappa_i$. Clearly, $p[T] = p[T]'$. Let $\{\psi_n\}$ be the excellent scale derived from T', where for ψ_n we use lexicographic ordering on $n+1$ tuples $(\beta_0, \dots, \beta_{n+1})$ from κ satisfying $\beta_j < \kappa_j$ for all j. So, ψ_n maps into $(\kappa_j)^+ < \kappa$. $\square$

One use of scales is that they provide a way to uniformize relations. To see this, suppose $R \subseteq \omega^\omega \times \omega^\omega$ is κ-Suslin, say $R = p[T]$ where T is a tree on $\omega \times \omega \times \kappa$. Note that $x \in \operatorname{dom}(R)$ iff $\exists y\ T_{x,y}$ is illfounded iff T_x is illfounded. For $x \in \operatorname{dom}(R)$, let $\ell^x = (\ell_0^x, \ell_1^x)$ be the leftmost branch of T_x. The map $x \mapsto y = \ell_0^x$ is a uniformization of R. We can also give the previous uniformization argument directly in terms of scales. We state this in the following lemma.

Lemma 2.28. *Let $R \subseteq \omega^\omega \times \omega^\omega$ and let $\{\phi_n\}_{n \in \omega}$ be an excellent scale on R. For each $x \in \operatorname{dom}(R)$ and $n \in \omega$, let*

$$\begin{aligned} A_n^x &= \{y \colon (x,y) \in R \wedge \phi_n(x,y) \text{ is minimal}\} \\ &= \{y \colon \forall z\ (x,y) \leqslant_n^* (x,z)\}, \end{aligned}$$

where $\leqslant_n^*$ is the norm relation corresponding to ϕ_n. Then for each $x \in \operatorname{dom}(R)$, $A_0^x \supseteq A_1^x \supseteq \cdots$, and there is unique point $f(x)$ in $\bigcap_n A_n^x$. Also, $R(x, f(x))$ for all $x \in \operatorname{dom}(R)$. That is, f is a uniformization for R.

Proof. Fix $x \in \operatorname{dom}(R)$, and let A_n^x be as above. If $y \in A_n^x$, then for any z such that $R(x,z)$ we have $\phi_n(y,y) \leqslant \phi_n(x,z)$. Since ϕ is very good, we have $\phi_i(x,y) \leqslant \phi_i(x,z)$ for all $i \leqslant n$ as well. Thus, $y \in A_i^x$ for all $i \leqslant n$. Let λ_n be the minimal value of $\phi_n(x,y)$, among all $y \in R_x$. Let $y_n \in A_n^x$ for each n. So, $\phi_n(x, y_n) = \lambda_n$. By very goodness, we also have $\phi_i(y_n) = \lambda_i$ for all $i \leqslant n$. So, for each i, the norms $\phi_i(x, y_n)$ are eventually equal to λ_i. Since ϕ is a good scale, $R(x,y)$. From the scale property, $\phi_n(x,y) \leqslant \lambda_n$, and hence $\phi_n(x,y) = \lambda_n$ for all n. If $z \in \bigcap_n A_n^x$, then for all n, $\phi_n(x,z) = \phi_n(x,y) = \lambda_n$. By the excellence condition, $z = y$. Thus, $\bigcap_n A_n^x$ is a singleton set $\{f(x)\}$, and thus the map $x \mapsto f(x)$ is a uniformization of R. $\square$

We next identify some useful coding sets.

Definition 2.29. LO is the set of $x \in 2^\omega$ such that $\{(n,m) \colon x(\langle n,m \rangle = 1\}$ is a linear order of $\operatorname{field}(x) \doteq \{n \colon \exists m\ x(\langle n,m \rangle) = 1 \vee x(\langle m,n \rangle) = 1\}$.

WO is the set of $x \in 2^\omega$ such that $\{(n,m) \colon x(\langle n,m \rangle = 1\}$ is a wellorder of $\operatorname{field}(x)$.

WF is the set of $x \in 2^\omega$ such that $\{(n,m) \colon x(\langle n,m \rangle = 1\}$ is wellfounded.

Thus, WO $\subseteq$ LO, and WO $\subseteq$ WF. We let $<_x$ denote the binary relation $\{(n,m) \colon x(\langle n,m \rangle = 1\}$. So, $x \in$ LO iff $<_x$ is a linear order, and $x \in$ WO iff $<_x$ is a wellorder.

Lemma 2.30. *$LO \in \boldsymbol{\Pi}_1^0$, WO, $WF \in \boldsymbol{\Pi}_1^1$.*

Proof. $x \in$ LO iff the following conditions defining a linear ordering are satisfied:

(1) $\forall n\ (x(\langle n,n \rangle) = 0)$
(2) $\forall n,m \in \operatorname{field}(x)\ (x(\langle n,m \rangle) = 1 \vee n = m \vee x(\langle m,n \rangle) = 1)$
(3) $\forall n,m \in \operatorname{field}(x)\ (x(\langle n,m \rangle) = 1 \rightarrow \neg x(\langle m,n \rangle) = 1)$
(4) $\forall n,m,k \in \operatorname{field}(x)\ (x(\langle n,m \rangle) = 1 \wedge x(\langle m,k \rangle) = 1 \rightarrow x(\langle n,k \rangle) = 1)$

Each of these is a $\boldsymbol{\Pi}_1^0$ condition, so LO $\in \boldsymbol{\Pi}_1^0$. Since $x \in \text{WO} \leftrightarrow x \in \text{LO} \wedge x \in \text{WF}$, it suffices to show that WF $\in \boldsymbol{\Pi}_1^1$. But, $x \in \text{WF} \leftrightarrow \neg \exists y\ \forall n\ (x(\langle y(n+1), y(n) \rangle) = 1)$, which shows WF $\in \boldsymbol{\Pi}_1^1$. $\square$

We show below that WO and WF are $\boldsymbol{\Pi}_1^1$-complete sets, which we introduce in the following definition.

Definition 2.31. If A, $B \subseteq \omega^\omega$ we say A is *Wadge reducible* to B if there is a continuous function $f \colon \omega^\omega \to \omega^\omega$ such that $\forall x\ (x \in A \leftrightarrow f(x) \in B)$. If $\boldsymbol{\Gamma}$ is a pointclass, we say B is $\boldsymbol{\Gamma}$-*hard* if every $A \in \boldsymbol{\Gamma}$ is Wadge reducible to B. We say B is $\boldsymbol{\Gamma}$-*complete* if $B \in \boldsymbol{\Gamma}$ and B is $\boldsymbol{\Gamma}$-hard.

The following general definition shows how to turn tree orderings into linear orderings, and is useful in a variety of settings.

Definition 2.32. Let T be tree on a wellordered set $(X, <_X)$. The *Kleene-Brouwer* ordering $<_{\mathrm{KB}}$ on T is defined by:

$$s <_{\mathrm{KB}} t \leftrightarrow [s \text{ extends } t \vee \exists i < \min\{\mathrm{lh}(s), \mathrm{lh}(t)\}\ (s(i) \neq t(i) \wedge s(i) <_X t(i) \\ \wedge \forall j < i\ (s(j) = t(j)))].$$

Thus, the Kleene-Brouwer ordering is like lexicographic ordering except that extensions of sequences are smaller in the Kleene-Brouwer order.

Theorem 2.33. *WF, WO are* $\boldsymbol{\Pi}^1_1$*-complete.*

Proof. We first show WF is $\boldsymbol{\Pi}^1_1$-complete. Let $A \subseteq \omega^\omega$ be $\boldsymbol{\Pi}^1_1$. Let T be a tree on $\omega \times \omega$ with $A(x) \leftrightarrow T_x$ is wellfounded. For $x \in \omega^\omega$ we define a linear-order $<_x$ as follows. Let $\pi \colon \omega \to \omega^{<\omega}$ be a bijection. Define

$$n <_x m \leftrightarrow (\pi(n), \pi(m) \in T_x \wedge \pi(n) \text{ extends } \pi(m)).$$

since T_x is wellfounded, so is $<_x$. Let $f(x) \in 2^\omega$ be the real coding $<_x$, that is, $f(x)(i) = 1$ iff $(i)_0 <_x (i)_1$. So, $x \in A$ iff $f(x) \in$ WF. Note finally that $x \mapsto f(x)$ is continuous. This is because $f(x)(i)$ is determined by T_x restricted to sequences of length at most $k \doteq \max\{\mathrm{lh}(\pi((i)_0), \pi((i))_1)\}$. This, in turn, is determined by $x \restriction k$. Thus, f is a continuous reduction of A to WF, which shows WF is $\boldsymbol{\Pi}^1_1$-complete.

Consider next WO. Define $<_x$ now using the Kleene-Brouwer order on T_x:

$$n <_x m \leftrightarrow (\pi(n), \pi(m) \in T_x \wedge \pi(n) <_{\mathrm{KB}} \pi(m)).$$

Again let $f(x)$ code $<_x$. As before, f is continuous. Note that $f(x) \in$ LO for all x. Also, $x \in A$ iff T_x is wellfounded iff $<_x$ is a wellordering iff $f(x) \in$ WO. This shows WO is $\boldsymbol{\Pi}^1_1$-complete. $\square$

We can give a version of this for general Polish spaces as the next exercise shows.

Exercise 42. Let X be Polish and $A \subseteq X$ be $\boldsymbol{\Pi}^1_1$. Show that there is a Borel function $f \colon X \to 2^\omega$ such that $\forall x \in X\ (x \in A \leftrightarrow f(x) \in \mathrm{WF})$, and likewise for WO. Show that in fact we may take f to be $\boldsymbol{\Sigma}^0_2$-measurable. [hint: Say $X - A = \mathcal{A}(\{F_s\})$, where $\{F_s\}_{s \in \omega^{<\omega}}$ is a closed good Suslin scheme. Let $\pi \colon \omega \to \omega^{<\omega}$ again be a bijection. Define $<_x$ by: $n <_x m$ iff $x \in F_{\pi(n)} \wedge x \in F_{\pi(m)} \wedge \pi(n)$ extends $\pi(m)$.]

We use these ideas to show the prewellordering property for $\boldsymbol{\Pi}^1_1$.

Theorem 2.34. *For any Polish space* X*,* $\mathrm{pwo}(\boldsymbol{\Pi}^1_1)$*.*

Proof. Let $A \subseteq X$ be $\boldsymbol{\Pi}^1_1$. Let $f \colon X \to$ LO be Borel such that $\forall x\ (x \in A \leftrightarrow f(x) \in$ WO). Define the norm ϕ on A by $\phi(x) = |f(x)| =$ the rank of the wellordering $f(x)$. So, $\phi \colon A \to \omega_1$. We show that ϕ is a $\boldsymbol{\Pi}^1_1$-norm. We have

$$x <^* y \leftrightarrow [f(x) \in \mathrm{WO} \wedge \neg \exists z\ (z \text{ codes an order-preserving map from } <_{f(y)} \text{ to } <_{f(x)}) \\ \leftrightarrow [f(x) \in \mathrm{WO} \wedge \neg \exists z\ (\forall n, m\ (x <_{f(y)} n \to z(n) <_{f(x)} z(m)))$$

Also,

$$\begin{aligned} x \leqslant^* y \leftrightarrow & [f(x) \in \mathrm{WO} \wedge \neg\exists z \ (z \text{ codes an order-preserving map from } <_{f(y)} \text{ to} \\ & \quad \text{a proper initial segment of } <_{f(y)}) \\ \leftrightarrow & [f(x) \in \mathrm{WO} \wedge \neg\exists a \in \mathrm{dom}(<_{f(x)}) \ \exists z \ (\forall n, m \ (n <_{f(y)} m \rightarrow \\ & \quad z(n) <_{f(x)} z(m) <_{f(x)} a)) \end{aligned}$$

The above computations show that $<^*$ and $\leqslant^*$ are both $\mathbf{\Pi}^1_1$, and thus ϕ is a $\mathbf{\Pi}^1_1$-norm. □

From lemma 1.57 we have:

Corollary 2.35. $\mathbf{\Pi}^1_1$ *has the number uniformixation property (and hence the ω-reduction and reduction properties).*

We have the following boundedness principle for norms.

Lemma 2.36. *Let $\mathbf{\Gamma}$ be a nonselfdual pointclass closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and assume* $\mathrm{pwo}(\mathbf{\Gamma})$. *Let ϕ be a $\mathbf{\Gamma}$-norm on a $\mathbf{\Gamma}$-complete set A. If $B \subseteq A$ is in $\mathbf{\Delta}$, then* $\sup\{\phi(x) \colon x \in B\} < \sup\{\phi(x) \colon x \in A\}$.

Proof. Suppose $B \subseteq A$ is in $\mathbf{\Delta}$ and $\sup\{\phi(x) \colon x \in B\} = \sup\{\phi(x) \colon x \in A\}$. Then we would have $x \in A \leftrightarrow \exists y \in B \ (x \leqslant_{\check{\mathbf{\Gamma}}} y)$. This computes A to be in $\check{\mathbf{\Gamma}}$, a contradiction. □

Corollary 2.37. *If $B \subseteq WO$ is Borel, then* $\sup\{|x| \colon x \in B\} < \omega_1$.

We pause to note that the prewellordering property propagates in ZF by existential quantification.

Lemma 2.38. *Let $\mathbf{\Gamma}$ be a pointclass closed under $\forall^{\omega^\omega}$, and assume* $\mathrm{pwo}(\mathbf{\Gamma})$. *Then* $\mathrm{pwo}(\exists^{\omega^\omega}\mathbf{\Gamma})$.

Proof. Let $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$, say $A(x) \leftrightarrow \exists y \ B(x, y)$ where $B \subseteq X \times \omega^\omega$ is $\mathbf{\Gamma}$. Let ϕ be a $\mathbf{\Gamma}$-norm on B. For $x \in A$ define $\psi(x) = \inf\{\phi(x, y) \colon (x, y) \in B\}$. We have:

$$x <^*_\psi y \leftrightarrow \exists z \ \forall w \ ((x, z) <^*_\phi (y, w)),$$

which shows $\leqslant^*_\psi \in \mathbf{\Gamma}$. We also have

$$x \leqslant^*_\psi y \leftrightarrow \exists z \ \forall w \ ((x, z) \leqslant^*_\phi (y, w)),$$

so $\leqslant^*_\psi$ is also in $\mathbf{\Gamma}$. □

Corollary 2.39. $\mathrm{pwo}(\mathbf{\Sigma}^1_2)$.

So, $\mathbf{\Sigma}^1_2$ has the ω-reduction and reduction properties, and $\mathbf{\Pi}^1_2$ has the separation property.

We next introduce the notion of a $\mathbf{\Gamma}$-scale which adds a definability hypothesis to the Suslin representation. Viewing the Suslin representaion via the norms (i.e., considering the corresponding scale) suggests how to add the definability hypothesis.

Definition 2.40. Let $\mathbf{\Gamma}$ be a pointclass and $A \subseteq X$. A sequence of norms $\{\phi_n\}_{n\in\omega}$ on A is said to be a $\mathbf{\Gamma}$-semiscale (or scale, very good scale, etc.) if it is a semiscale (or scale, etc.) and each of the norms ϕ_n is a $\mathbf{\Gamma}$-norm.

Definition 2.41. We say a pointclass $\boldsymbol{\Gamma}$ has the *scale property*, scale($\boldsymbol{\Gamma}$), if every $A \in \boldsymbol{\Gamma}$ admits a $\boldsymbol{\Gamma}$-scale.

The notions of $\boldsymbol{\Gamma}$-scale and the scale property were introduced by Moschovakis and have their motivation in the methods used in the Novikov-Kondo solution to the uniformization problem.

The next lemma says that passing from a scale to a very good or excellent scale does not usually increase the complexity of the scale.

Lemma 2.42. *Let $\boldsymbol{\Gamma}$ be a pointclass closed under $\wedge$, $\vee$. If $A \subseteq \omega^\omega$ and A admits a $\boldsymbol{\Gamma}$-scale, then A admits an excellent $\boldsymbol{\Gamma}$-scale.*

Proof. Let $\{\phi_n\}$ be a $\boldsymbol{\Gamma}$-scale on A. As before, define

$$\psi_n(x) = \langle \phi_0(x), x(0), \phi_1(x), x(1), \ldots, \phi_n(x), x(n) \rangle.$$

We showed in lemma 2.26 that $\{\psi_n\}$ is an excellent scale on A. The definability conditions are easily checked, for example:

$$\begin{aligned} x <^*_{\psi,m} y \leftrightarrow\ & x <^*_{\phi,0} y \\ & \vee (x \leqslant^*_{\phi,0} y \wedge y \leqslant^*_{\phi,0} x \wedge x(0) < y(0)) \\ & \vee (x \leqslant^*_{\phi,0} y \wedge y \leqslant^*_{\phi,0} x \wedge x(0) = y(0) \wedge x <^*_{\phi,1} y) \ldots \\ & \vee (x \leqslant^*_{\phi,0} y \wedge y \leqslant^*_{\phi,0} x \wedge x(0) = y(0) \wedge \cdots \wedge x(n) < y(n)) \end{aligned}$$

□

Exercise 43. Show that if $\boldsymbol{\Gamma}$ is closed under $\wedge$, $\vee$, and contains $\boldsymbol{\Sigma}^0_2$, then if every $A \subseteq X$ admits a $\boldsymbol{\Gamma}$-scale, then every $A \subseteq X$ admits a $\boldsymbol{\Gamma}$-excellent scale. Here the excellemce condition in interpreted as meaning that if $\psi_n(x) = \psi_n(y)$, then $\rho(x, y) < \frac{1}{2^n}$. [hint: For $x \in A$ let $\psi_n(x) = \langle \phi_0(x), \chi_0(x), \phi_1(x), \chi_1(x), \ldots, \phi_n(x), \chi_n(x) \rangle$, where the χ_n are define inductively as follows. Let $\{V_i\}$ be a base for X. Let $\chi_0(x)$ be the least $i \in \omega$ such that $x \in V_i$ and $\operatorname{diam}(V_i) < \frac{1}{2^0}$. Let $\chi_n(x)$ be the least $i \in \omega$ such that $x \in \overline{V_i} \subseteq V_{\chi_{n-1}(x)}$ and $\operatorname{diam}(V_i) < \frac{1}{2^n}$. Show that each ψ_n is a $\boldsymbol{\Gamma}$-norm and $\{\psi_n\}$ is an excellent scale on A.]

We next show the scale property for $\boldsymbol{\Pi}^1_1$, which is the essence of the solution to the uniformization problem.

Theorem 2.43. scale($\boldsymbol{\Pi}^1_1$).

Proof. We prove it for the case $X = \omega^\omega$, leaving the general case as an exercise. Let $A \subseteq \omega^\omega$ be $\boldsymbol{\Pi}^1_1$, and let $f : \omega^\omega \to \mathrm{LO}$ be the continuous function of theorem 2.33 so that $\forall x\ (x \in A \leftrightarrow f(x) \in \mathrm{WO})$. For $x \in A$, let $\phi_0(x) = |f(x)|$ (the rank of the wellordering $f(x)$). For $n > 0$ let $\phi_n(x) = \langle \phi_0(x), |n-1|_{<_{f(x)}} \rangle$, where $|n-1|_{<_{f(x)}}$ is the rank of $n-1$ in the wellordering $<_{f(x)}$ if $n-1 \in \operatorname{dom}(<_{f(x)})$, and is 0 otherwise.

We fist show that each ϕ_n is a $\boldsymbol{\Pi}^1_1$-norm on A. We have:

$$\begin{aligned} x <^*_n y \leftrightarrow (x <^*_0 y) \vee (x \leqslant^*_0 \wedge y \leqslant^*_0 x \wedge \neg \exists z\ (& z \text{ codes an order-preserving map} \\ & \text{from } I_n^{f(y)} \text{ to } I_n^{f(x)})), \end{aligned}$$

where $I_n^{f(x)} = \{m : m <_{f(x)} n\}$ is the initial segment of the order $<_{f(x)}$ determined by m. In theorem 2.34 we showed that ϕ_0 is a $\boldsymbol{\Pi}^1_1$-norm, so $<^*_0$, $\leqslant^*_0$ are $\boldsymbol{\Pi}^1_1$ relations. It follws that $\leqslant^*_n$ is also $\boldsymbol{\Pi}^1_1$. A similar computation shows that $\leqslant^*_n$ is $\boldsymbol{\Pi}^1_1$.

Next we show that $\{\phi_n\}$ is a scale on A. Suppose $\{x_m\} \subseteq A$, $x_n \to x$, and for all n, $\phi_n(x_m)$ is eventually equal to $\lambda_n = \langle \alpha_n, \beta_n \rangle$. We must show that $<_{f(x)}$ is wellfounded. To see this, note that if $k <_{f(x)} l$, then (since f is continuous) for all large enough m we have $k <_{f(x_m)} l$, and hence $|k|_{<_{f(x_m)}} < |l|_{<_{f(x_m)}}$. It follows that $\beta_{k+1} < \beta_{l+1}$. Thus, $n \to \beta_{n+1}$ is an order-preserving map from $<_{f(x)}$ to On, and so $<_{f(x)}$ is wellfounded.

Finally, we show the lower semi-continuity property. Note that $\alpha = \alpha_n$ doesn't really depend on n, it is the limiting value of $\phi_0(x_m)$. Also, $\beta_n < \alpha$ for each n. Since $n \to \beta_{n+1}$ is order-preserving on $<_{f(x)}$, this shows that $\phi_0(x) = f(x) \leqslant \alpha$. Similarly, for each n the map $k \mapsto \beta_{k+1}$ is order-preserving from $I_{n-1}^{f(x)}$ to β_n. Thus, $|n-1|_{<_{f(x)}} \leqslant \beta_n$. So, $\phi_n(x) = \langle \phi_0(x), |n-1|_{<_{f(x)}} \rangle \leqslant \langle \alpha, \beta_n \rangle = \lambda_n$. □

Corollary 2.44. *Every $\boldsymbol{\Pi}^1_1$ set $A \subseteq \omega^\omega$ admits an excellent $\boldsymbol{\Pi}^1_1$-scale.*

Corollary 2.45. *Every $\boldsymbol{\Pi}^1_1$ relation $R \subseteq X \times Y$ has a $\boldsymbol{\Pi}^1_1$ uniformization.*

Proof. First suppose $X = Y = \omega^\omega$. From corollary 2.44, let $\{\phi_n\}$ be an excellent scale on R. Let $R' \subseteq R$ be the uniformization from the scale ϕ as in lemma 2.28. Thus,

$$R'(x,y) \leftrightarrow \forall n\ \forall z\ ((x,y) \leqslant^*_m (x,z)).$$

Since $\leqslant^*_n$ is $\boldsymbol{\Pi}^1_1$, $R' \in \boldsymbol{\Pi}^1_1$.

Consider now the general case $R \subseteq X \times Y$. Let $\pi_1 \colon \omega^\omega \to X$ and $\pi_2 \colon \omega^\omega \to Y$ be Borel bijections. Let $S \subseteq \omega^\omega \times \omega^\omega$ be defined by $S(u,v) \leftrightarrow R(\pi_1(x), \pi_1(y))$. Note that S is $\boldsymbol{\Pi}^1_1$ as it is the inverse image of a $\boldsymbol{\Pi}^1_1$ set by a Borel function (c.f. corollary 2.18). Let S' be a $\boldsymbol{\Pi}^1_1$ uniformization of S. Define R' by $R'(x,y) \leftrightarrow S'(\pi_1^{-1}(x), \pi_2^{-1}(y))$. Then R' is $\boldsymbol{\Pi}^1_1$ as π_1^{-1}, π_2^{-1} are also Borel functions. Clearly R' is a uniformization of R. □

It makes sense to ask whether a $\boldsymbol{\Pi}^1_1$ set in an arbitrary Polish space X admits a $\boldsymbol{\Pi}^1_1$-scale (though the analagous question about Suslin representations doesn't make sense). The next exercise shows that this is the case.

Exercise 44. Show that if $A \subseteq X$ is $\boldsymbol{\Pi}^1_1$, then A admits a $\boldsymbol{\Pi}^1_1$-scale. [hint: let $\{F_s\}_{s \in \omega^{<\omega}}$ be a closed good Suslin scheme such that $X - A = \mathcal{A}(\{F_s\})$, that is, $A(x)$ iff $T(x)$ is wellfounded, where $T(x) \doteq \{s \in \omega^{<\omega} \colon x \in F_s\}$. For $x \in A$, let $\phi_0(x) = |T(x)| =$ the rank of $T(x)$. For $n > 0$, let $\phi_n(x) = |\pi(n-1)|_{T(x)}$, where $\pi \colon \omega \to \omega^{<\omega}$ is a bijection. Let $\{V_i\}$ be a base for X. Let $\chi_n(x) = -1$ if $x \in F_{\pi(n)}$ (we allow -1 as a value of the norms for convenience), and otherwise $\chi_n(x) =$ the least i such that $x \in \overline{V}_i \subseteq X - F_{\pi(n)}$. For $x \in A$, let $\psi_n(x) = \langle \phi_0(x), \chi_0(x), \dots, \phi_n(x), \chi_n(x) \rangle$. Show that each ψ_n is a $\boldsymbol{\Pi}^1_1$-norm on A. To see that it is a scale on A, suppose $\{x_m\} \subseteq A$, $x_m \to x$, and for each n the $\psi_n(x_m)$ are eventually equal to λ_n. So, the $\phi_n(x_m)$ are eventually constant as well. Using the χ_i, show that the trees $T(x_i)$ converge to the tree $T(x)$ in the sense that for every $s \in \omega^{<\omega}$, $s \in T(x)$ iff $s \in T(x_m)$ for large enough m. Let $\langle \alpha, \beta_n \rangle$ be the limiting value of the $\phi_n(x_m)$. As in theorem 2.43, show that the map $s \mapsto \beta_{\pi^{-1}(s)}$ is order-preserving and follow the proof of theorem 2.43].

Exercise 45. Show that if $A \subseteq X$ is $\boldsymbol{\Pi}^1_1$, then A admits an excellent $\boldsymbol{\Pi}^1_1$-scale. Here excellent means very good and with the property that if $x, y \in A$ and $\phi_n(x) = \phi_n(y)$, then $\rho(x,y) < \frac{1}{2^n}$, where ρ is a compatible complete metric for X. [hint: let $\psi_n(x) = \langle \phi_0(x), \chi_0(x), \sigma_0(x), \dots, \phi_n(x), \chi_n(x), \sigma_n(x) \rangle$ where the $\phi_n(x)$, $\chi_n(x)$ are above and $\sigma_n(x)$ is defined recursively as follows. Let $\sigma_0(x) = 0$. Let $\sigma_{n+1}(x)$ be the least $i \in \omega$ such that $x \in \overline{V}_i \subseteq V_{\sigma_n(x)}$ and $\operatorname{diam}(V_i) < \frac{1}{2^{n+1}}$].

Exercise 46. Show directly using exercise 45 that every $\mathbf{\Pi}^1_1$ relation $R \subseteq X \times Y$ has a $\mathbf{\Pi}^1_1$ uniformization.

The next fact shows that the uniformization property passes up through existential quantification. Let $\operatorname{unif}(\mathbf{\Gamma})$ be the statement that every $R \subseteq X \times Y$ in $\mathbf{\Gamma}$ has a uniformization $R' \subseteq R$ in $\mathbf{\Gamma}$.

Lemma 2.46. *Let* $\mathbf{\Gamma}$ *be a pointclass and assume* $\operatorname{unif}(\mathbf{\Gamma})$. *Then* $\operatorname{unif}(\exists^{\omega^\omega}\mathbf{\Gamma})$.

Proof. Let $R \subseteq X \times Y$ be in $\exists^{\omega^\omega}\mathbf{\Gamma}$, say $R(x, y) \leftrightarrow \exists z \in \omega^\omega\ S(x, y, z)$ with $S \in \mathbf{\Gamma}$. View S as a subset of $X \times (Y \times \omega^\omega)$ (i.e., $(X \times Y) \times \omega^\omega$ is homeomorphic to $X \times (Y \times \omega^\omega)$). Let $S' \subseteq X \times (Y \times \omega^\omega)$ be in $\mathbf{\Gamma}$ and uniformize S. Then define $R'(x, y) \leftrightarrow \exists z\ S'(x, y, z)$. Clearly R' is a uniformization of R. $\square$

Corollary 2.47. $\operatorname{unif}(\mathbf{\Sigma}^1_2)$.

Likewise, it is easy to show that the scale property propagates upward through existential quantification, according to the next lemma. Note that the scale property for $\mathbf{\Sigma}^1_2$, for example, does not directly imply uniformization for $\mathbf{\Sigma}^1_2$ (the proof of uniformization from scales required $\mathbf{\Gamma}$ to be closed under $\forall^{\omega^\omega}$; see the proof of corollary 2.45). However, establishing $\operatorname{scale}(\mathbf{\Sigma}^1_2)$ is important for propagating the scale property further (assuming stronger axioms).

Lemma 2.48. *Let* $\mathbf{\Gamma}$ *be a pointclass closed under* $\forall^{\omega^\omega}$, $\wedge$, $\vee$. *Assume* $\operatorname{scale}(\mathbf{\Gamma})$. *Then* $\operatorname{scale}(\exists^{\omega^\omega}\mathbf{\Gamma})$.

Proof. Let $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$, say $A(x) \leftrightarrow \exists y\ B(x, y)$ where $B \in \mathbf{\Gamma}$. Let $\{\phi_n\}$ be a $\mathbf{\Gamma}$-scale on B, and without loss of generality we may assume that $\{\phi_n\}$ is very good. Define ψ_n on A by:

$$\psi_n(x) = \inf\{\phi_n(x, y) \colon (x, y) \in B\}.$$

Note that $x \leqslant^*_{\psi_n} y \leftrightarrow \exists z\ \forall w\ (x, z) \leqslant^*_{\phi_n} (y, w)$, so $\leqslant^*_{\psi_n}$ is in $\exists^{\omega^\omega}\mathbf{\Gamma}$. Suppose $\{x_m\} \subseteq A$ and for each n, $\psi_n(x_m)$ is eventually equal to λ_n. For each m, let y_m be such that $\phi_m(x_m, y_m) = \psi_m(x_m)$. By very goodness, $(x_{,m}, y_m)$ has minimal value for all the norms ϕ_i for $i \leqslant m$ (amongst all $y \in A_x$). So, for all n we have that for all large enough m that $\phi_n(x_m, y_m) = \lambda_n$. Thus, (x_m, y_m) converges to some $(x, y) \in B$, and so $x_m \to x \in A$. Also, $\phi_n(x, y) \leqslant \lambda_n$ as $\{\phi_n\}$ is a scale. Thus, $\psi_n(x) \leqslant \phi_n(x, y) \leqslant \lambda_n$ and so $\{\psi_n\}$ is a scale. $\square$

The scale on a $\mathbf{\Pi}^1_1$ set in ω^ω constructed in theorem 2.43 implicitly also builds a Suslin representation for $\mathbf{\Pi}^1_1$ sets. We construct directly now a related, but somewhat different, Suslin representation called the *Shoenfield tree* which also has other useful properties, specifically it is a homogeneous tree (we define this concept later).

Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_1$, and let T be a tree on $\omega \times \omega$ such that $\omega^\omega - A = p[T]$, that is, $A(x)$ iff T_x is wellfounded. Let $\pi \colon \omega \to \omega^{<\omega}$ be a reasonable bijection, say with $\pi^{-1}(s) \geqslant \operatorname{lh}(s)$ for all s. Note that $\pi(0) = \varnothing$ (which we consider to be the root node of any tree). Define the tree S on $\omega \times \omega_1$ by:

$$\begin{aligned}(s, (\alpha_0, \dots, \alpha_n)) \in S \text{ iff } \forall 0 \leqslant i, j \leqslant n\ (\alpha_i < \alpha_j) \leftrightarrow\ & \\ & [(\pi(i), \pi(j) \in T_s \wedge \pi(i) <_{\mathrm{KB}} \pi(j)) \\ & \vee (\pi(i) \notin T_s \wedge \pi(j) \in T_s) \\ & \vee (\pi(i), \pi(j) \notin T_s \wedge i < j)]\end{aligned}$$

We refer to the tree S defined above as the Shoenfield tree for A. Note that S only depends on the tree T projecting to $\omega^\omega - A$. We claim that $A = p[S]$. If $x \in A$, then T_x is wellfounded. Let $\prec_x$ be the ordering on ω defined by the last three disjuncts in the above definition of S. $\prec_x$ is easily a wellorder using the fact that $<_{\mathrm{KB}} \restriction T_x$ is a wellorder in this case. Let $\phi(n)$ be the rank of n in $\prec_x$. Then $(x, (\phi(0), \phi(1), \dots))$ is a branch through S, so $x \in p[S]$. If $x \notin A$, then T_x is illfounded, say $(x, y) \in [T]$. Then S_x must be wellfounded, for suppose $(x, \vec{\alpha}) \in [S]$. Let $k_i = \pi^{-1}(y \restriction i)$. Then $\alpha_{k_0} > \alpha_{k_1} > \dots$, a contradiction.

Thus, for all x we have S_x is illfounded iff T_x is wellfounded. In particular, $x \in A$ iff T_x is wellfounded iff S_x is illfounded, that is, $A = p[S]$.

We state two important properties of the Shoenfield tree. The first is that the Shoenfield tree construction is absolute to transitive models of enough of ZF containing ω_1. That is, if $V_1 \subseteq V_2$ are transitive models of enough of ZF and $(\omega_1)^{V_1} = (\omega_1)^{V_2}$, and $T \in V_1$ is a tree on $\omega \times \omega$, then $(S)^{V_1} = (S)^{V_2}$, where $(S)^{V_1}$ denotes the Shoenfield tree computed in V_1 and likewise for V_2. This is immediate from the fact that both models compute the same set of countable ordinals, and the fact that the conditions for $(s, \vec{\alpha})$ to be in S are clearly absolute.

Secondly, note that S is *homogeneous*, which here means that if $(s, \vec{\alpha}) \in S$ and $\vec{\beta}$ is order-isomorphic to $\vec{\alpha}$, then $(s, \vec{\beta}) \in S$. That is, being in the tree S only depends on the relative ordering of the ordinals in $\vec{\alpha}$ (we give the general definition of a homogeneous tree later).

Note also that if $(s, (\alpha_0, \dots, \alpha_n)) \in S$ then $\alpha_0 > \alpha_1, \dots, \alpha_n$ (since $\pi(0) = \varnothing$ is the maximal element in the ordering $\prec_x$).

The Shoenfield tree construction also give a Suslin representation for $\boldsymbol{\Sigma}^1_2$ sets. For suppose $A \subseteq \omega^\omega$ is $\boldsymbol{\Sigma}^1_2$, say $A(x) \leftrightarrow \exists y\ B(x, y)$ where $B \in \boldsymbol{\Pi}^1_1$. Let T be a tree on $\omega \times \omega \times \omega$ such that $B(x, y) \leftrightarrow T_{x,y}$ is wellfounded. Let S be the Shoenfield tree on $\omega \times \omega \times \omega_1$ constructed from T in the obvious manner. So, $B(x, y) \leftrightarrow S_{x,y}$ is illfounded. But then we also have $A(x) \leftrightarrow \exists y\ S_{x,y}$ is illfounded $\leftrightarrow S_x$ is illfounded (here S_x is a tree on $\omega \times \omega_1$). If we identify $\omega \times \omega_1$ with ω_1, then S may be viewed as a tree on $\omega \times \omega_1$ (identifying the second and third coordinates with a single coordinate). So, $A = p[S]$ which gives a Suslin representation for A. Again, the operation $T \mapsto S$ is absolute between transitive models of enough of ZF having the same ω_1. Now however, S is no longer homogeneous but rather weakly homogeneous, which in this case we can take to mean that S is isomorphic to a homogeneous tree on $\omega \times \omega \times \omega_1$ by an identification of the second and third coordinates (we will give a more official and general definition later).

We summarize this discussion in the following theorem.

Theorem 2.49. *If $A \subseteq \omega^\omega$ is $\boldsymbol{\Pi}^1_1$, then the Shoenfield tree S on $\omega \times \omega_1$ is such that $A = p[S]$. Furthermore S is homogeneous and the map $T \mapsto S$ is absolute between transitive models of (enough of) ZF having the same ω_1 (where T is a tree on $\omega \times \omega$*

such that $\omega^\omega - A = p[T]$). If A is $\mathbf{\Sigma}^1_2$, then these facts also hold except S is only weakly homogeneous.

Suppose again that $A \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_1$, T is a tree on $\omega \times \omega$ with $\omega^\omega - A = p[T]$, and S is the Shoenfield tree on $\omega \times \omega_1$ constructed from T. We claim the S has true left-most branches.

Definition 2.50. Let V be a tree on a wellordered set $(X, \prec)$. We say $f \in [V]$ is a *true left-most branch* if $\forall g \in [V]\ \forall n\ (f(n) \preceq g(n))$.

To see this, suppose S_x is illfounded, that is $x \in A$. So, T_x is wellfounded. Again let $\phi(n)$ be the rank of n in the wellordering $\prec_x$ of ω defined above. So, $(x, (\phi(0), \phi(1), \dots)) \in [S_x]$. Suppose $(x, g) \in [S_x]$. Then $n \mapsto g(n)$ is order-preserving from $\prec_x$ to On. It follows that for every $m \in \omega$, $g(m)$ is greater than or equal to the rank of m in $\prec_x$, which is $\phi(m)$.

We thus have:

Lemma 2.51. *For $A \subseteq \omega^\omega$ in $\mathbf{\Pi}^1_1$ and S the Shoenfield tree for A, S has true left-most branches.*

For $x \in A$, if we define $\phi_n(x) =$ the rank of n in $\prec_x$ as above, then $\{\phi_n\}$ is a scale on A. In fact, this scale is essentially the same one given in the proof of theorem 2.43. The only difference is that we had to take $\psi_n(x) = \langle \phi_0(x), \phi_n(x) \rangle$ to get the scale to be a $\mathbf{\Pi}^1_1$-scale. Thus, the $\mathbf{\Pi}^1_1$-scale on A we constructed is more or less the values of the true left-most branch of S_x.

Exercise 47. Let $A \subseteq \omega^\omega$ be $\mathbf{\Pi}^1_1$. Let $\omega^\omega - A = p[T]$, T a tree on $\omega \times \omega$, and let S be the corresponding Shoenfield tree on $\omega \times \omega_1$. Let $\{\phi_n\}$ be the scale on A given by the true left-most branch (i.e., $\phi_n(x)$ is the rank of n in the wellordering $\preceq_x$ defined above). Let S' be the tree of the scale $\{\phi_n\}$. Show that $S' \subseteq S$.

Although the tree of a scale always has true left-most branches, as does the Shoenfield tree, it is not true that the Shoenfield tree S corresponding to an arbitrary T will be the tree of a scale.

Exercise 48. Show that if a tree T is the tree of a scale, then the scale is uniquely determined by the tree. [hint: Show that if T is the tree of a scale $\{\phi_n\}$ then for $x \in p[T]$, $\phi_n(x)$ is the n^{th} coordinate of the left-most branch of T_x.]

Exercise 49. Consider WF $\subseteq 2^\omega$. Let T be the canonical tree on $2 \times \omega$ with $2^\omega - A = p[T]$ given by $(s, t) \in T$ iff $\forall i < \mathrm{lh}(t)\ \neg(s(\langle t(i), t(i-1) \rangle) = 0)$, that is, if it is consistent with the amount of the ordering $<_s$ determined by s that $t(i) <_s t(i-1)$. Let S be the Shoenfield tree on $2 \times \omega_1$ corresponding to T. Actually, use the slight variation of S which is defined as before except if $(s, \vec{\alpha}) \in S$ and $\pi(i) \notin T_s$, then we require $\alpha_i = -1$ (we will still consider this tree to be homogeneous). Show that S is the tree of a scale. [hint: Fix $(s, (\alpha_0, \dots, \alpha_n)) \in S$. We must show that there is an $x \in$ WF extending s such that $\phi_i(x) = \alpha_i$ for $i \leqslant n$, where $\{\phi_m\}$ is the scale on WF corresponding to the true left-most branch of S_x. That is, $\phi_m(x)$ is the rank of $\pi(m)$ in the ordering $<_{\mathrm{KB}}$ of T_x (we assume $\pi(m) \in T_x$ here). So, show that there is an $x \in$ WF extending s such that for all $i \leqslant n$ (with $\pi(i) \in T_s$) we have that $|i|_{<_x} = \alpha_i$.]

The Shoenfield tree immediately gives an important absoluteness pripciple. First recall the following simple fact. By a "$\mathbf{\Sigma}^1_1$ statement about x" we mean a statement of the form $\phi(x) = \exists y\, \forall n\, R(x\restriction n, y\restriction n)$, where R is a recursive relation on $\omega^{<\omega} \times \omega^{<\omega}$. Thus, $\phi(x) \leftrightarrow \exists y\ (x, y) \in [T]$ where $T(s,t) \leftrightarrow \forall i < \mathrm{lh}(s)\ R(s\restriction i, t\restriction i)$. So, T is in all transitive models of enough of ZF.

Lemma 2.52. *If $M \subseteq N$ are transitive models of enough of ZF, then $\boldsymbol{\Sigma}^1_1$ statements ars absolute between M and N.*

Proof. Let T be a tree on $\omega \times \omega$ with $T \in M$ such that $\phi(x) \leftrightarrow \exists y\ (x, y) \in [T]$. Let $x \in M$. Then $\phi(x)^M \leftrightarrow (T_x \text{ is illfounsded})^M$, and $\phi(x)^N \leftrightarrow (T_x \text{ is illfounsded})^N$. Since $T_x \in M \subseteq N$, by absoluteness of wellfoundedness we have $(T_x \text{ is illfounsded})^M$ iff $(T_x \text{ is illfounsded})^M$. □

Using the Shoenfield tree, we can extend this result as follows. For any tree T on $\omega \times \omega$, and any ordinal α, let $S(T, \alpha)$ be the Shoehfield tree constructed from T, but using the ordinal α instead of ω_1. The same proof as before shows (in ZF) that if $\alpha \geqslant \omega_1$, then T_x is wellfounded iff S_x is illfounded, for all $x \in \omega^\omega$.

Theorem 2.53. *If $M \subseteq N$ are transitive models of enough of ZF and $\omega_1^N \in M$, then $\boldsymbol{\Sigma}^1_2$ statements are absolute between M and N.*

Proof. Let $\phi(x)$ be a $\boldsymbol{\Sigma}^1_2$ statement, that is $\phi(x) = \exists y\ T(x, y)$ is wellfounded, where T is a recursive tree on $\omega \times \omega \times \omega$. So, $T \in M$. Let $S = S(T, \omega_1^N)$, and note that this is the same constucted in M or N, that is, $S^M(T, \omega_1^N) = S^N(T, \omega_1^N) = S$. Let $x \in M$ and assume $\phi(x)^N$, so $(\exists y\ T_{x,y} \text{ is wellfounded}))^N$, and thus $(S_x \text{ is illfounded})^N$. By absoluteness, $(S_x \text{ is illfounded})^M$. So, $\exists y \in M\ (S_{x,y} \text{ is illfounded})^M$, and so $\exists y \in M\ (T_{x,y} \text{ is wellfounded})^M$ since $\omega_1^N \geqslant \omega_1^M$. Thus, $\phi(x)^M$. The other direction is immediate by $\boldsymbol{\Sigma}^1_1$ absoluteness. □

Corollary 2.54. *If V is a transitive model of enough of ZF and $V[G]$ is a generic extension of V, then $\boldsymbol{\Sigma}^1_2$ statements are absolute between V and $V[G]$.*

Exercise 50. Show that there are countab;e transitive models M of ZF_N such that $\boldsymbol{\Sigma}^1_2$ absoluteness fails between M and V. [hint: Suppose $\boldsymbol{\Sigma}^1_2$ absoluteness held between V and every countable transitive model M of ZF_N. Let A be a $\boldsymbol{\Pi}^1_2$ set, say $A(x) \leftrightarrow \phi(x)$ where ϕ is a $\boldsymbol{\Pi}^1_2$ statement. Then we would have $A(x)$ iff $\exists E\ \exists n\ (M = (\omega, E)$ is a countable wellfounded model of ZF_N, $\pi(n) = x$, and $\phi(n)^M)$, which computes A to be $\boldsymbol{\Sigma}^1_2$.]

3. Suslin Representations and the Perfect Set Property

We show how Suslin representations give us structural information about the set. In this section we consider the perfect set property. Recall that in theorem 1.42 we showed that every Borel set in a Polish space is either countable or else contains a perfect set.

Theorem 3.1 (ZF). *Let $A \subseteq \omega^\omega$ be κ-Suslin. Then either $|A| \leqslant \kappa$ (i.e., there is a map of κ onto A), or A contains a perfect set.*

Proof. Let $A = p[T]$, where T is a tree on $\omega \times \kappa$. If $(s, \vec{\alpha}) \in T$, we say T left-splits below $(s, \vec{\alpha})$ if there are $(t, \vec{\beta})$, $(u, \vec{\gamma})$ in T extending $(s, \vec{\alpha})$ with $t \perp u$. In general, for S a tree on $\omega \times \kappa$, let S' be the set of all $(s, \vec{\alpha}) \in S$ such that S left-splits below $(s, \vec{\alpha})$. Let $T_0 = T$, and define the derivatives inductively by $T_{\alpha+1} = (T_\alpha)'$, and for limit α, $T_\alpha = \bigcap_{\beta<\alpha} T_\beta$. Since T has size at most κ, there is a least ordinal $\theta < \kappa^+$ such that $T_\theta = T_{\theta+1}$.

First assume $T_\theta \neq \varnothing$. Then it is easy to build a perfect set in $[T]$. Namely, let $(s_\varnothing, \vec{\alpha}_\varnothing)$ be any node of T_θ. In general, suppose $(s_w, \vec{\alpha}_w) \in T_\theta$ has been defined for $w \in 2^{<\omega}$. Let $(s_{w^\frown 0}, \vec{\alpha}_{w^\frown 0})$, $(s_{w^\frown 1}, \vec{\alpha}_{w^\frown 1})$ be extensions of $(s_w, \vec{\alpha}_w)$ in T_θ with $s_{w^\frown 0} \perp s_{w^\frown 1}$. Define $\pi\colon 2^\omega \to [T]$ by $\pi(x) =$ the union of the $s_{x\restriction n}$.

Suppose next that $T_\theta = \varnothing$. If $x \in [T]$, let ℓ^x be the left-most branch of T_x. Let $\beta < \theta$ be the unique ordinal such that $(x, \ell^x) \in [T_\beta] - [T_{\beta+1}]$. Let $(s, \vec{\alpha})$ be the least initial segment of (x, ℓ^x) such that $(s, \vec{\alpha}) \notin T_{\beta+1}$. We show that the map $x \mapsto (\beta, s, \alpha)$ is one-to-one, which thus maps A into an ordinal of size κ. To see this, suppose x, $y \in A$ and $(x, \ell^x), (y, \ell^y) \in [T_\beta] - [T_{\beta+1}]$ for some β (otherwise we are done). Likewise, we may assume that $(s, \vec{\alpha})$ is the longest initial segment of (x, ℓ^x) in T_β and also for (y, ℓ^y) (or else we are done). Thus, T_β does not left-split below the node $(s, \vec{\alpha})$. This says that $x\restriction n$, $y\restriction n$ are compatible for all n, that is $x = y$. $\square$

Corollary 3.2 (ZF). *Every $\boldsymbol{\Sigma}^1_1$ set in a Polish space is either countable or else contains a perfect set.*

Proof. For $X = \omega^\omega$ the result follows immediately from theorem 3.1. For the general case, let $\pi\colon X \to \omega^\omega$ be a Borel bijection. If $A \subseteq X$ is $\boldsymbol{\Pi}^1_1$, then $A' = \pi[A] \subseteq \omega^\omega$ is also $\boldsymbol{\Pi}^1_1$ (since π^{-1} is Borel). If A' is countable, then so is A. So, suppose A' contains a perfect set P. Then A contains $\pi^{-1}[P]$ which is an uncountable Borel set in X. By theorem 1.42, $\pi^{-1}[P]$ contains a perfect set. $\square$

Similarly we have:

Corollary 3.3 (ZF). *Every $\boldsymbol{\Sigma}^1_2$ set in a Polish space either has size $\leqslant \omega_1$ or else contains a perfect set.*

We also have the following refinement of the perfect set theorem, sometimes called the effective perfect set theorem.

Theorem 3.4 (ZF). *Let $A = p[T]$, where T is a tree on $\omega \times \kappa$. Then either A contains a perfect set or else $A \subseteq L[T]$ and there is in $L[T]$ a one-to-one map from A into κ.*

Proof. Work inside $L[T]$, and define the derivatives T_α as before. Note that the map $S \to S'$ (the left-splitting derivative) is absolute for transitive models of ZF. Thus, the sequence T_α as computed in $L[T]$ is the same as computed in V. Likewise, the least θ such that $T_\theta = T_{\theta+1}$ is the same computed in $L[T]$ or in V. If $T_\theta \neq \varnothing$, then A contains a perfect set as before. Suppose then that $T_\theta = \varnothing$. Let $x \in A$, and let β, $(s, \vec{\alpha})$ be as in theorem 3.1. Then x may be defined in $L[T]$ as the union of all the t such that for some $\vec{\gamma}$, $(t, \vec{\gamma})$ extends $(s, \vec{\alpha})$ and $(t, \vec{\gamma}) \in T_\beta$. This is because on the one hand all such $(t, \vec{\gamma})$ are compatible in their first coordinates, and on the other hand $T(s, \vec{\alpha}) \doteq \{(z, \vec{\gamma}) \in [T_\beta] \colon (z, \vec{\gamma}) \text{ extends } (s, \vec{\alpha})\}$ is non-empty in $L[T]$ by absoluteness. So, $x \in L[T]$. The proof of theorem 3.1 shows that the map $x \mapsto (\beta, s, \vec{\alpha})$ (which is now defined in $L[T]$) is one-to-one. $\square$

The effective perfect set theorem gives more information, for example, we have the following corollary.

Corollary 3.5. *If $\forall x\ ((\omega_1)^{L[x]} < \omega_1)$, then every $\boldsymbol{\Sigma}^1_2$ is either countable or else contains a perfect set.*

Proof. We may assume $A \subseteq \omega^\omega$ is $\mathbf{\Sigma}^1_2$. Say $A(x) \leftrightarrow \exists y\ B(x,y)$, where $B \in \mathbf{\Pi}^1_1$. Let T be a tree on $\omega \times \omega \times \omega$ with $B = p[T]$. T may be coded as a real, so are assuming that $(\omega_1)^{L[T]} < \omega_1$. Let S be the Shoenfield tree corresponding to T, so $S \in L[T]$ (note: in defining S we use $(\omega_1)^V$). By theorem 3.4 it then follows that either A contains a perfect set or else $A \subseteq L[T]$. Since $L[T] \models \mathsf{CH}$, $|A| \leqslant (\omega_1)^{L[T]} < \omega_1$, that is, A is countable. □

On the other hand, the perfect set property for even $\mathbf{\Pi}^1_1$ is not decidable within ZFC. This follows from the previous corollary and the following theorem of Gödel.

Theorem 3.6. *Assume $V = L$. Then there is an uncountable $\mathbf{\Pi}^1_1$ set which does not contain a perfect set.*

Proof. Let $\prec$ be the canonical $\mathbf{\Delta}^1_2$-good wellorder of the reals assuming $V = L$. For each $\alpha < \omega_1$, let x_α be the $\prec$-least element of WO with $|x| = \alpha$. Let $A = \{x_\alpha \colon \alpha < \omega_1\}$. Clearly A is uncountable. Also, $A \in \mathbf{\Sigma}^1_2$ since

$$\begin{aligned} x \in A \leftrightarrow\ & x \in \mathrm{WO} \wedge \exists M\ [M \text{ is a countable transitive wellfounded model} \\ & \quad \text{of } \mathsf{ZFC} + V = L\ \wedge x \in M \wedge (\forall y \prec x\ \neg(y \in \mathrm{WO} \wedge |y| = |x|))^M] \\ \leftrightarrow\ & x \in \mathrm{WO} \wedge \exists E \subseteq \omega \times \omega\ \exists n \in \omega\ [(\omega, E) \text{ is a wellfounded model of} \\ & \quad \mathsf{ZFC} + V = L \wedge \pi(n) = x \wedge (\omega, E) \models \forall z \prec n\ (\neg(z \in \mathrm{WO} \wedge |z| = |n|)], \end{aligned}$$

where π is the transitive collapse map of (ω, E). Let $A(x) \leftrightarrow \exists y\ B(x,y)$ where $B \in \mathbf{\Pi}^1_1$. By uniformization for $\mathbf{\Pi}^1_1$, let $C \subseteq B$ be a $\mathbf{\Pi}^1_1$ uniformization of B. Clearly C is uncountable. Suppose $P \subseteq C$ were perfect. Since P contains a copy of 2^ω, we may assume by shrinking P that P is compact. Let $R = \pi[P]$, where π is the projection map onto the first coordinate. Since $\pi \restriction C$ is one-to-ome, $R \subseteq \mathrm{WO}$ is uncountable and compact, hence Borel. However, a Borel subset of WO is bounded in the ordinals coded by corollary 2.37, contradicting $|R| = \omega_1$. □

We can extract a bit more from the proof of theorem 3.6.

Theorem 3.7. *Assume there is an $x \in \omega^\omega$ such that $\omega_1^{L[x]} = \omega_1$. Then there is an uncountable $\mathbf{\Pi}^1_1$ set with no perfect subset.*

Proof. Suppose $\omega_1^{L[z]} = \omega_1$. We follow the proof of theorem 3.6. Let $\prec_z$ be the canonical wellordering of $L[z]$. Define

$$\begin{aligned} x \in A \leftrightarrow\ & (x \in \mathrm{WO} \cap L[z]) \wedge \exists M\ [M \text{ is a countable transitive wellfounded model} \\ & \quad \text{of } \mathsf{ZFC} + V = L[z]\ \wedge x \in M \wedge (\forall y \prec_z x\ \neg(y \in \mathrm{WO} \wedge |y| = |x|))^M] \\ \leftrightarrow\ & x \in \mathrm{WO} \wedge \exists E \subseteq \omega \times \omega\ \exists m \in \omega\ \exists n \in \omega\ [(\omega, E) \text{ is a wellfounded model of} \\ & \quad \mathsf{ZFC} + V = L[m] \wedge \pi(m) = z \wedge \pi(n) = x \wedge \\ & \quad (\omega, E) \models \forall w \prec_m n\ (\neg(w \in \mathrm{WO} \wedge |w| = |n|)], \end{aligned}$$

where again π is the transitive collapse map. So, $A \in \mathbf{\Sigma}^1_2$ as before. Note that since $\omega_1^{L[z]} = \omega_1$ it follows that A is uncountable. We obtain the $\mathbf{\Pi}^1_1$ set B from A as before, and the same proof shows that B is uncountable and does not contain a perfect set. □

We show next that $\mathsf{ZFC}+$(the perfect set property for $\mathbf{\Pi}^1_1$) is equiconsistent with $\mathsf{ZFC} + \exists$ an inaccessible cardinal.

Lemma 3.8. *Assume the perfect set property for* $\mathbf{\Pi}^1_1$*. Then* ω_1^V *is an inaccessible cardinal in every* $L[x]$*.*

Proof. Clearly ω_1 is regular in $L[x]$. Suppose ω_1 were a successor cardianl in $L[x]$, say $\omega_1 = (\kappa^+)^{L[x]}$ where κ is a cardinal of $L[x]$. since $\kappa < \omega_1^V$, there is a real y such that κ is countable in $L[x, y]$. But then $\omega_1 = (\omega_1)^{L[x,y]}$, contradicting our assumption. Thus, ω_1^V is inaccessible in $L[x]$. □

Corollary 3.9. *CON*(ZFC + *perfect set property*($\mathbf{\Pi}^1_1$)) $\Rightarrow$ *CON*(ZFC + $\exists$ *an inaccessible cardinal*).

Thus, the perfect set property for $\mathbf{\Pi}^1_1$ cannot be proved in ZFC. On the other hand, we have the following. Let $\mathbb{P} = \text{coll}(\omega, < \kappa)$ be the forcing for collapsing all ordinals less than κ to be countable. Recall $\mathbb{P}$ consists of all finitely supported functions f with domain a subset of $\omega \times \kappa$, and such that $f(n, \beta) < \beta$ for all $n \in \omega$ and $\beta < \kappa$. If G is generic for $\mathbb{P}$, we regard G as a sequence $G = \langle g_\alpha \rangle_{\alpha<\kappa}$, where $g_\alpha \colon \omega \to \alpha$. For $\gamma < \kappa$, we let G_γ be those $f \in G$ with domain contained in $\omega \times \gamma$. We regard G_γ as the initial segment $\langle g_\alpha \rangle_{\alpha<\gamma}$ of G. G_γ is generic for $\mathbb{P}_\gamma \doteq \text{coll}(\omega, < \gamma)$.

Lemma 3.10. *Assume* κ *is an inaccessible cardinal. Let* G *be* $\mathbb{P} = \text{coll}(\omega, < \kappa)$ *generic over* V*. Then in* $V[G]$ *we have* $\forall x\ (\omega_1^{L[x]} < \omega_1)$*, and thus we have the perfect set property for* $\mathbf{\Sigma}^1_2$ *in* $V[G]$*.*

Proof. Let $G = \langle g_\alpha \rangle_{\alpha<\kappa}$ be generic over V for $\mathbb{P}$. Since κ is regular, $\mathbb{P}$ is κ-c.c., and so forcing with $\mathbb{P}$ does not collapse κ (but does collapse every $\alpha < \kappa$ to be countable). So, $(\omega_1)^{V[G]} = \kappa$.

Let x be a real in $V[G]$. Since κ is regular and $\mathbb{P}$ is κ-c.c., the usual nice-name argument shows that $x \in V[G_\gamma]$ for some $\gamma < \kappa$. It is enough to show that $(\omega_1)^{V[G_\gamma]} < \omega_1$. Since $|\mathbb{P}_\gamma| = |\gamma|$, forcing with $\mathbb{P}_\gamma$ preserves all cardinals greater than γ. Since κ is a limit cardinal, it follows that $(\omega_1)^{V[G_\gamma]} < \kappa = (\omega_1)^{V[G]}$. □

Thus we can improve corollary 3.9 to an equiconsistency result:

Corollary 3.11. *CON*(ZFC + *perfect set property*($\mathbf{\Pi}^1_1$)) $\Leftrightarrow$ *CON*(ZFC + $\exists$ *an inaccessible cardinal*).

Remark 3.12. In Solovay's model above obtained by forcing with $\mathbb{P} = \text{coll}(\omega, < \kappa)$ with κ inaccessible, we actually have that all ordinal definable sets of reals in $V[G]$ have the perfect set property, and that in the $L(\mathbb{R})$ of this model, every set of reals has the perfect set property. Thus, ZF + (the full perfect set property) still only has the consistency strength of an inaccessible cardinal.

The perfect set property for $\mathbf{\Pi}^1_1$ implies $\forall x\ (\omega_1^{L[x]} < \omega_1)$, and this in turn was enough to get the perfect set property for $\mathbf{\Sigma}^1_2$. However, this is easy to show directly as the following exercise shows.

Exercise 51. Show that if every $\mathbf{\Pi}^1_1$ set has the perfect set property, then every $\mathbf{\Sigma}^1_2$ set has the perfect set property. [hint: If A is an uncountable $\mathbf{\Sigma}^1_2$ set, write $A(x) \leftrightarrow \exists y\ B(x, y)$ where $B \in \mathbf{\Pi}^1_1$. Let B' uniformize B. Use the perfect set property for B'.]

4. Measure and Category

We consider the measure theoretic and topological notions of regularity of sets, namely the notions of measurability and the Baire property. Unlike the perfect set property, the situation now is symmetrical between a pointclass and its dual. We recall first the basic notions.

First consider the case of measure. By a *measure* on a set X we mean a countably additive function $\mu\colon \mathcal{M} \to \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$, where $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ-algebra. By countably additive we mean that if $\{A_n\}_{n\in\omega}$ are pairwise disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$. If X is a topological space and $\mathcal{M}$ is the collection of Borel subsets of X, then we call μ a *Borel measure*. If $\mu(X) = 1$, we call μ a *probability measure*. We say μ is σ-finite if we can write $X = \bigcup_n X_n$ where $\mu(X_n) < \infty$ for each n. There is little difference in the arguments for σ-finite measures and probability measures, so we frequently just consider the case of a probability measure.

Exercise 52. Let μ be a Borel probability measure on a metric space X. Show that μ is regular, that is, show that for any Borel set $B \subseteq X$ and any $\epsilon > 0$, that there is an open set U and a closed set F with $F \subseteq B \subseteq U$ such that $\mu(U) - \mu(F) < \epsilon$. [hint: Prove this by induction on the Borel rank of B. Use the fact that every open set in a metric space is the increasing countable union of closed sets.]

If μ is a Borel probability (or σ-finite) measure on a metric space X, then we extend μ from the Borel sets to a larger σ-algebra, call the *measurable sets*, as follows. First, we define the *null sets* to be those sets Z such that $Z \subseteq A$ for some Borel set A with $\mu(A) = 0$. Clearly the null sets for a σ-ideal (i.e., they are closed under subsets and countable unions). We then define a set $A \subseteq X$ to be measurable if there is a Borel set B such that $A \triangle B = Z$ is null. The collection $\mathcal{M}$ of μ-measurable sets is easily seen to be a σ-algebra containing the Borel sets. Moreover, the measure μ naturally extends from the σ-algebra of Borel sets to the the σ-algebra of measurable sets. Namely, if $A \in \mathcal{M}$ and say $A \triangle B = Z$ is null, set $\mu(A) = \mu(B)$. This is easily well-defined.

Exercise 53. Let μ be a Borel probability measure on a metric space X. Show that the extension of μ to $\mathcal{M}$ is also regular, that is, for every $A \in \mathcal{M}$ and any $\epsilon > 0$, that there is an open set U and a closed set F with $F \subseteq B \subseteq U$ such that $\mu(U) - \mu(F) < \epsilon$.

Exercise 54. Let μ be a Borel probability measure on a metric space X. Show that $A \subseteq X$ is μ-measurable iff for every $\epsilon > 0$ there is an open set U and a closed set F with $F \subseteq B \subseteq U$ such that $\mu(U) - \mu(F) < \epsilon$.

We say a set A in a metric space X is *universally measurable* if it is measurable with respect to every Borel probability measure on X (equivalently, with respect to every σ-finite Borel measure).

We recall the following fact. Recall for a measure μ on a set X, that a point $x \in X$ is said to be an atom if $\mu(\{x\}) > 0$. μ is atomless if there are no atoms. Clearly there can be only countably many atoms. It follows that if μ is a Borel probability measure on the Polish space X then μ can be written as $= mu = \alpha\mu_1 + \beta\mu_2$ where $\alpha + \beta = 1$, μ_1 is a discrete probability measure (i.e., μ_1 concentrates on a countable set), and μ_2 is an atomless Borel probability measure.

Theorem 4.1. *Any two atomless Borel probability mesures on Polish spaces are Borel isomorphic.*

Proof. Let μ be a Borel probability measure on the Polish space X. It suffices to show that μ is Borel isomorphic to Lebesgue measure on $[0,1]$. Clearly X is uncountable as μ is atomless. Since any two Polish spaces are Borel isomorphic, we may assume that $X = [0,1]$. Define $f\colon [0,1] \to [0,1]$ by $f(x) = \mu([0,x])$. Since μ is atomless it follows easily that $f(0) = 0$, $f(1) = 1$, and f is continuous. Also, f in monotonically increasing. We claim that $f(\mu)$ is Lebesgue measure λ on $[0,1]$. To see this, consider $f(\mu)([a,b]) = \mu(f^{-1}([a,b]) = \mu([c,d])$ where $f(c) = a$ and $f(d) = b$. So, $\mu([0,c]) = a$ and $\mu([0,d]) = b$. But then $\mu([c,d]) = \mu((c,d]) = \mu([0,d]) - \mu([0,c]) = f(d) = f(c) = b - a = \lambda([a,b])$. Since $f(\mu)$ and λ agree on all basic open sets, it follows that $f(\lambda)$ and μ agree on all Borel sets in $[0,1]$. The remaining problem is that f need not be one-to-one.

Let $\{I_n\}_{n\in\omega}$ be a maximal family of pairwise disjoint open intervals in $[0,1]$ on which f is constant (such a family is clearly countable). Using countably additivity we easily have that $\mu(I_n) = 0$ for all n (each I_n is a countable union of intervals $[c,d]$ such that f is constant on $[c,d]$. Then $\mu([c,d]) = \mu((c,d]) = \mu([0,d]) - \mu([0,c]) = f(d) - f(c) = 0$). Since μ is atomless, we also have $\mu(\overline{I}_n) = 0$. Let C be the Cantor set in $[0,1]$, so $\lambda(C) = 0$. Let $F = f^{-1}(C)$. Since $\lambda(C) = 0$, $\mu(F) = 0$. Let $E = F \cup \bigcup_n \overline{I}_n$, so $\mu(E) = 0$. Note that if $x, y \in [0,1]$ and $f(x) = f(y)$ then $x, y \in E$. This is because f is constant on (x,y), and so $(x,y) \subseteq I_n$ for some n. Hence, $x, y \in \overline{I}_n$. In particular, f in one-to-one on $[0,1]-E$. Also, f maps $[0,1]-E$ in a continuous one-to-one way to a Borel set B disjoint from F. In fact $[0,1]-B$ is $C \cup D$ where D is countable (the points in D are of the form $f(z)$ where z is an endpoint of an interval $\overline{I}_n$). Since $[0,1]-B$ and E both measure 0 (with respect to λ, μ respectively), it follows that f is a measure preserving bijection between $([0,1-E,\mu)$ and (B,λ). Finally, take a Borel bijection g between E and C. Then the union of f restricted to $[0,1]-E$ and g restricted to E is a measure preserving Borel bijection between $([0,1],\mu)$ and $([0,1],\lambda)$. $\square$

If μ is a Borel probability measure on the metric space X, we let $\mathcal{I}_m = \mathcal{I}_m(\mu)$ denote the σ-ideal of μ-null sets.

We generalize these concepts to an arbitrary ideal on a Polish space.

Definition 4.2. If $\mathcal{I}$ is a countably additive ideal on a Polish space X, we say a set $A \subseteq X$ is $\mathcal{I}$-measurable if there is a Borel set $B \subseteq X$ such that $A \triangle B \in \mathcal{I}$.

The $\mathcal{I}$-measurable sets form a σ-algebra containg the Borel sets in the Polish space X.

Definition 4.3. An ideal $\mathcal{I}$ on a set X is c.c.c. if there does not exist an uncountable collection $\{A_\alpha\}_{\alpha<\omega_1}$ of $\mathcal{I}$-measurable, $\mathcal{I}$-positive sets (i.e., each $A_\alpha \notin \mathcal{I}$) such that $A_\alpha \cap A_\beta \in \mathcal{I}$ whenever $\alpha \neq \beta$.

Lemma 4.4. *For any Borel probability (or σ-finite) measure on a Polish space X, the ideal $\mathcal{I}_m$ of μ-null sets is countably closed and c.c.c.*

Proof. We have already note $\mathcal{I}_m$ is countably closed, as μ is countably additive. To see that $\mathcal{I}_m$ is c.c.c., suppose $\{A_\alpha\}_{\alpha<\omega_1}$ is a sequence of μ-measurable sets such that $\mu(A_\alpha) > 0$ and $\mu(A_\alpha \cap A_\beta) = 0$ for all $\alpha \neq \beta$. We assume μ is a probability measure, the σ-finite case easily following from that. Thinning out the sequence, we may assume that for some $n \in \mathbb{N}^+$ that $\mu(A_\alpha) > \frac{1}{m}$ for all α. It folows that $\mu(A_0 \cup A_n \cup \cdots \cup A_k) > \frac{k}{n}$, which is a contradiction for $k > n$ (since μ is a probability measure). $\square$

Exercise 55. Show how to prove lemma 4.4 in the case where μ is just assumed to be σ-finite.

We consider now the topological notion of category. Recall a set A is a topological space X is aid to be nowhere dense if $\mathrm{int}(\bar{E}) = \varnothing$. This is equivalent to saying that for every open set U, there is an open set $V \subseteq U$ with $V \cap A = \varnothing$. Recall that a set A in a topological space X is said to be *meager* if $A \subseteq \bigcup_n F_n$ where each F_n is closed and nowhere dense. Equivalently, $A = \bigcup_n E_n$ where each E_n is nowhere dense (since the closure of a nowhere dense set is nowhere dense). We say a set A is comeager if $X - A$ is meager.

Clearly a subset of a measger set is meager, and a countable union of meager sets is meager, so the meager sets form a σ-ideal. Let $\mathcal{I}_c$ denote the σ-ideal of meager sets. Recall X is said to be a *Baire space* is every open set is non-meager (that is, every comeager set is dense).

The Baire category theorem says that every complete metric space, and hence every Polisg space, is a Baire space. In particular, in every Polish space the ideal $\mathcal{I}_c$ is proper.

The analog of measurability in the topological context is the Baire property.

Definition 4.5. A set A in a toplogical space X has the Baire property if there is an open set U such that $A \triangle U$ is meager.

In analogy with measure, it perhaps would have seemed more natural in definition 4.5 to require only that there be a Borel set B such that $A \triangle B \in \mathcal{I}_c$. However, the stronger conclusion of definition 4.5 follows from this as the next standard lemma shows.

Lemma 4.6. *Every Borel set in a topological space has the Baire property.*

Proof. If A is open, let $U = A$. If A is closed, let $U = \mathrm{int}(A)$. Note that $A \triangle U = A - U \in \mathcal{I}_c$ since for any closed set F in a topological space, $F - \mathrm{int}(F)$ is (closed) nowhere dense. If $A = \bigcup_n A_n$, by induction let U_n be open such that $A_n \triangle U_n \in \mathcal{I}_c$. Let $U = \bigcup_n U_n$. Then $A \triangle U \subseteq \bigcup_n (A_n \triangle U_n) \in \mathcal{I}_c$ since $\mathcal{I}_c$ is a σ-ideal. Suppose finally that A has the Baire property, and we show $X - A$ does as well. Let U be open such that $A \triangle U = M \in \mathcal{I}_c$. Then $(X - A) \triangle (X - U) = M \in \mathcal{I}_c$ also. Let $V = \mathrm{int}(X - U)$, so $(X - U) \triangle V \in \mathcal{I}_c$. Thus, $(X - A) \triangle V \subseteq (X - A) \triangle (X - U) \cup (X - A) \triangle V \in \mathcal{I}_c$. $\square$

As in the measure case, the meager ideal is countable closed and c.c.c.

Lemma 4.7. *Let X be Polish. Then $\mathcal{I}_c$ is countably closed and c.c.c. That is, there does not exist an uncountable sequence $\{A_\alpha\}_{\alpha<\omega_1}$ of sets with the Baire property and each $A_\alpha \notin \mathcal{I}_c$, and such that $A_\alpha \cap A_\beta \in \mathcal{I}_c$ for all $\alpha \neq \beta$.*

Proof. Let U_α be open such that $A_\alpha \triangle U_\alpha \in \mathcal{I}_c$. Since X is a Baire space and $A_\alpha \notin \mathcal{I}_c$, $U_\alpha \neq \varnothing$. For $\alpha \neq \beta$, we must have $U_\alpha \cap U_\beta = \varnothing$, since $A_\alpha \cap A_\beta \in \mathcal{I}_c$ [note that $(U_\alpha \cap U_\beta) \triangle (A_\alpha \cap A_\beta) \in \mathcal{I}_c$, and so $U_\alpha \cap U_\beta \in \mathcal{I}_c$. Since X is a Baire space, this implies $U_\alpha \cap U_\beta = \varnothing$.] However, in a separable space we cannot have an uncountable sequence of pairwise disjoint open sets. $\square$

The analog of Fubini's theorem in the category context is the Kuratowski-Ulam theorem. The next lemma is the key point. Recall that if $D \subseteq X \times Y$, then D_x denotes the section $D_x = \{y \colon (x, y) \in D\}$.

Lemma 4.8. *Let X, Y be metric spaces with Y second countable. Let $D \subseteq X \times Y$ be dense open (in the product topology). Then $\{x \colon D_x$ is open dense in $Y\}$ is comeager in X.*

Proof. For every $x \in X$, D_x is open in Y. We show that for comeager many $x \in X$ that D_x is dense in Y. Let $\{V_n\}_{n \in \omega}$ be a base for Y. Let U_n be the union of all open sets $U \subseteq X$ such that for some open $V \subseteq V_n$ we have $U \times V \subseteq D$. Then U is dense in X. To see this, let $W \subseteq X$ be open. Then $(W \times V_n) \cap D \neq \varnothing$ since D is dense, and since D is also open we have that $W' \times V \subseteq D$ for some open $W' \subseteq W$ and open $V \subseteq V_n$. Thus, $W' \subseteq U$, showing U is dense.

Since each U_n is dense open, $A \doteq \bigcap_n U_n$ is comeager in X. If $x \in A$, then D_x is dense since for any basic open set V_n, there is an open $V \subseteq V_n$ such that $\{x\} \times V \subseteq D$, that is, $V \subseteq D_x$. □

A useful notation is to write "$\forall^*_X x$" to abbreviate "for comeager many $x \in X$." We also write "$\exists^*_X x$" for "for non-meager many $x \in X$." If the space is clear from the context we just write "$\forall^* x$."

We now prove the Kuratowski-Ulam theorem.

Theorem 4.9. *Let X, Y be Polish spaces. If $C \subseteq X \times Y$ is comeager, then $\{x \in X \colon C_x$ is comeager in $Y\}$ is comeager in X. If $C \subseteq X \times Y$ has the Baire property and $\{x \in X \colon C_x$ is comeager in $Y\}$ is comeager in X, then C is comeager in $X \times Y$.*

Proof. First assume $C \subseteq X \times Y$ is comeager in $X \times Y$. Say $C \supseteq \bigcap_n D_n$ where $D_n \subseteq X \times Y$ is open dense. From lemma 4.8, each $A_n \doteq \{x \in X \colon D_n$ is open dense in $Y\}$ is comeager in X. Thus, $A \doteq \bigcap_n A_n$ is comeager in X. For $x \in A$, each $(D_n)_x$ is open dense in Y, and so C_x is comeager in Y.

Suppose now $C \subseteq X \times Y$ has the Baire property and $\forall^*_X x\ \forall^*_Y y\ (x, y) \in C$. If C is not comeager, then $D \doteq (X \times Y) - C$ is nonmeager. Let W be an open set such that $D \triangle W \in \mathcal{I}$. So, $W \neq \varnothing$. Shrinking W, we may assume $W = U \times V$, and so D is comeager on $U \times V$. Then $E = (D \cap (U \times V)) \cup (X \times Y - U \times V)$ is comeager in $X \times Y$. By the first case, we have $\forall^*_X x\ \forall^*_Y y\ (x, y) \in E$. It follows that $\forall^*_X x\ \forall^*_Y y\ [(x, y) \in C \wedge (x, y) \in E]$. Since X is a Baire space and U is open, fix $x \in U$ such that $\forall^*_Y y\ [(x, y) \in C \wedge (x, y) \in E]$. Since Y is a Baire space and V is open, there is a $y \in Y$ such that $(x, y) \in C \cap E$, a contradiction as C, E are disjoint on $U \times V$. □

To summarize, in both the measure and category contexts, we have a countable additive, c.c.c. ideal $\mathcal{I}$, and the "measurable" sets are those which are equal to a Borel set modulo a set in $\mathcal{I}$ (though in the category case we can improve "Borel" to "open").

Essentially by definition the Borel sets in X are measurable and have the Baire property (when we say measurable we are referring to some Borel probability or σ-finite measure on the Polish space X). We consider the question of which sets are measurable. Most of this discussion is symmetric between measure and category (though not everything is), and the only thing relevant is that the corresponding ideal $\mathcal{I}$ is countably additive and c.c.c. We abstract this in the following definition.

Lemma 4.10. *Let $\mathcal{I}$ be a countably additive, c.c.c. ideal on a Polish space X. Then for every $A \subseteq X$ there is a Borel set B with $A \subseteq B$ and such that for every Borel* *set $C \subseteq B - A$, $C \in \mathcal{I}$. Likewise, there is an $\mathcal{I}$-measurable set N with $A \subseteq N$ and such that for every $\mathcal{I}$-measurable set $C \subseteq N - A$, $C \in \mathcal{I}$.*

Proof. Let $A \subseteq X$. Let $\mathcal{B}$ be a collection of Borel subsets of X which is maximal subject to satisfying:

(1) For every $B \in \mathcal{B}$, $B \cap A = \varnothing$ and $B \notin \mathcal{I}$.
(2) For every $B_1 \neq B_2$ in $\mathcal{B}$, $B_1 \cap B_2 = \varnothing$.

Since $\mathcal{I}$ is c.c.c., $\mathcal{B}$ is countable. Let $C = \cup\mathcal{B}$, so C is Borel, and if we let $B = X - C$, then B is Borel and contains A. By maximality of $\mathcal{B}$, every Borel set contained in $B - A$ must be in $\mathcal{I}$. The proof for $\mathcal{I}$-measurability is the same. □

Theorem 4.11. *Let $\mathcal{I}$ be a countably additive, c.c.c. ideal on the Polish space X. Suppose $A = \mathcal{A}(\{M_s\})$ where each M_s is $\mathcal{I}$-measurable. Then A is $\mathcal{I}$-measurable.*

Proof. For each $s \in \omega^{<\omega}$, recall A_s denotes the result of the Suslin operation starting from M_s, that is, $A_s = \{x \colon \exists y\ (y\restriction \mathrm{lh}(s) = s \wedge \forall n\ x \in M_{y\restriction n}\}$. Thus, $A = A_\varnothing$, and $A_s \subseteq M_s$. For each s, let $N_s \supseteq A_s$ be $\mathcal{I}$-measurable as in lemma 4.10. So, $A_s \subseteq N_s \subseteq M_s$, N_s is $\mathcal{I}$-measurable, and every $\mathcal{I}$-measurable set contained in $N_s - A_s$ is in $\mathcal{I}$.

Let $E_s = N_s - \bigcup_{i\in\omega} N_{s^\frown i}$. Clearly E_s is $\mathcal{I}$-measurable and $E_s \subseteq N_s$. Also, $E_s \cap A_s = \varnothing$ since $A_s = \bigcup_{i\in\omega} A_{s^\frown i}$, and $A_{s^\frown i} \subseteq N_{s^\frown i}$. Thus, $E_s \in \mathcal{I}$. Let $E = \bigcup_s E_s$, so $E \in \mathcal{I}$ as well. We claim that $N_\varnothing - A_\varnothing \subseteq E$ and so is in $\mathcal{I}$, which shows that $A = A_\varnothing$ is $\mathcal{I}$-measurable. Toward a contradiction, suppose $x \in N_\varnothing - A_\varnothing$, and $x \notin E$. Since $x \notin E_\varnothing$, there is an $i_0 \in \omega$ such that $x \in N_{i_0}$. Since $x \notin A_\varnothing$, we must have $x \notin A_{i_0}$. So, $x \in N_{i_0} - A_{i_0}$. Continuing, we define a $y \in \omega^\omega$ such that for all n, $x \in N_{y\restriction n} - A_{y\restriction n}$. Since $N_{y\restriction n} \subseteq M_{y\restriction n}$, this shows that $x \in \mathcal{A}(\{M_s\})$. That is, $x \in A = A_\varnothing$, a contradiction. □

Corollary 4.12. *Every $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ set in a Polish space is universally measurable and has the Baire property.*

We earlier alluded to the following definition.

Definition 4.13. The $\mathcal{C}$ sets in a Polish space X are the smallest σ-algebra closed under the operation $\mathcal{A}$.

Corollary 4.14. *Every $\mathcal{C}$ set in a Polish space X is universally measurable and has the Baire property.*

Exercise 56. Show that every $\mathcal{C}$ set in a Polish space is $\mathbf{\Delta}^1_2$. [hint: Show that if each $B_s \in \mathbf{\Delta}^1_2$ then $A \doteq \mathcal{A}(\{B_s\}) \in \mathbf{\Delta}^1_2$. A direct computation will show that $A \in \mathbf{\Sigma}^1_2$. To see that $X - A$ is $\mathbf{\Sigma}^1_2$, note that $x \in X - A$ iff T_x is wellfounded, where T_x is the tree on ω given by $T_x = \{s \in \omega^{<\omega} \colon x \in B_s\}$. Then, $x \in X - A$ iff $\exists y \in \omega^\omega$ $[(y \in \mathrm{WF}) \wedge \forall n, m \in \omega\ (y(\langle n, m\rangle) = 1 \leftrightarrow x \in B_{\pi(n)} \wedge x \in B_{\pi(m)} \wedge \pi(n) \text{ extends } \pi(m))]$ where $\pi \colon \omega \to \omega^{<\omega}$ is a bijection.]

In general, $\mathbf{\Sigma}^1_1$ sets in a product space $X \times Y$ cannot be uniformized by $\mathbf{\Sigma}^1_1$ sets, even when $Y = \omega$ (otherwise $\mathbf{\Sigma}^1_1$ would have the reduction property, hence $\mathbf{\Sigma}^1_1$ wouls not have the separation property, which it however does). However, the next result shows that $\mathbf{\Sigma}^1_1$ sets in a product can be uniformized by sets in the σ-algebra generated by the $\mathbf{\Sigma}^1_1$, sets, and in fact by fairly simple sets in this collection. In particular, $\mathbf{\Sigma}^1_1$ sets can be uniformized by sets which, when viewed as functions, are measurable also Baire measurable. The next result is due to Jankov and Von Neumann.

Theorem 4.15. *Let $A \subseteq X \times Y$ be $\boldsymbol{\Sigma}^1_1$. Then there is an $A' \subseteq A$ uniformizing A with $A' \in \bigcap_\omega \bigcup_\omega (\boldsymbol{\Sigma}^1_1 \wedge \boldsymbol{\Pi}^1_1)$.*

Proof. Since $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$ sets are preserved under Borel isomorphisms, and isomorphisms respect the set operations, it is enough to consider the case $X = Y = \omega^\omega$.

So, let $A \subseteq \omega^\omega \times \omega^\omega$ be $\boldsymbol{\Sigma}^1_1$. Let T be a tree on $\omega \times \omega \times \omega$ such that $A(x, y) \leftrightarrow T_{x,y}$ is illfounded. Let $B = \mathrm{dom}(A) = \{x \colon \exists y\ A(x, y)\}$. So, $B(x) \leftrightarrow T_x$ is illfounded. For $x \in B$, let $(\ell_0(x), \ell_1(x))$ be the leftmost branch of T_x (we implicitly identify $\omega \times \omega$ with ω). Let $A'(x, y) \leftrightarrow (y = \ell_0(x))$. Clearly A' is a uniformization of A. We compute the complexity of A':

$$\begin{aligned}
A'(x,y) &\leftrightarrow \forall n \in \omega \exists t \in \omega^n\ [(y{\restriction}n, t) = (\ell_0(x){\restriction}n, \ell_1(x){\restriction}n)] \\
&\leftrightarrow \forall n \in \omega\ \exists t \in \omega^n\ [\exists z, w \in \omega^\omega\ ((z, w) \text{ extends } (y{\restriction}n, t) \wedge (x, z, w) \in [T]) \\
&\quad \wedge \forall s', t' \in \omega^n\ ((s', t') < (y{\restriction}n, t) \to \neg \exists z, w \in \omega^\omega\ ((z, w) \text{ extends} \\
&\quad (s', t') \wedge (x, z, w) \in [T]))].
\end{aligned}$$

The expression in square brackets defines an intersection of a $\boldsymbol{\Sigma}^1_1$ and a $\boldsymbol{\Pi}^1_1$ set. □

Suppose $A \subseteq X \times Y$ is $\boldsymbol{\Sigma}^1_1$. Let $B = \mathrm{dom}(A)$, and let $A' \subseteq A$ be the uniformization produced in theorem 4.15. Let $f \colon B \to \omega^\omega$ be the corresponding uniformizing function, that is, $A'(x, f(x))$ for all $x \in B$. Then as a function from X to Y, f is $\boldsymbol{\Gamma}$-measurable, where $\boldsymbol{\Gamma} = \bigcup_\omega (\boldsymbol{\Sigma}^1_1 \wedge \boldsymbol{\Pi}^1_1)$. To see this, note that $f(x){\restriction}n = s$ iff $\exists t \in \omega^n\ [(s, t) = (\ell_0(x){\restriction}n, \ell_1(x){\restriction}n)]$, and then follow the above computation. It follows that f is a measurable function, and also a Baire neasurable function.

5. Lightface Pointclasses

Up to this point we been discussing only the theory of "boldface poinclasses." Recall a (boldface) pointclass $\boldsymbol{\Gamma}$ is a collection of subsets of Polish spaces which is closed under continuous preimages. In the boldface theory, the simplest sets are open and closed sets, that is, the $\boldsymbol{\Sigma}^0_1$ and $\boldsymbol{\Pi}^0_1$ sets, and the simplest functions are the continuous functions. In the lightface theory we refine these notions. The lightface class Σ^0_1, for example, will be the sets obtined by taking not arbitrary unions of basis open sets but just "effective" or simply computable unions (the precise definition will follow). In fact, for each real $x \in \omega^\omega$, we will define the lightface class $\Sigma^0_1(x)$, which will denote those sets which are unions of basic open sets which can be computed from x. Since every union of basic open sets can be coded by some real, it will follow that $\boldsymbol{\Sigma}^0_1 = \bigcup_{x \in \omega^\omega} \Sigma^0_1(x)$. Similarly, the basic notion of continuous is relaced by the effective notion of recursive, or computable.

The lightface theory has its origins in recursion theory, and many of the main notions draw their motivation from that subject. It is possible to develop the main technical results we need without recourse to recursion theory (and in fact we do this this, see theorems **??**, **??**), but nevertheless an understanding of the underlying recursion theory is helpful. We begin with a quick review of some basic concepts in recursion theroy.

A *total* function from ω to ω is a function whose domaun is ω. A *partial* function is one whose domain is a subset of ω. If f is a partial function we say $f(n)$ is defined, or write $f(n)\downarrow$, if $n \in \mathrm{dom}(f)$, and otherwise say $f(n)$ is not defined, or write $f(n)\uparrow$.

A total function $f\colon \omega \to \omega$ is recursive, intuitively, if there is an algorithm which computes f. That is, the algorithm when started with input n terminates after a finite number of steps with the output $f(n)$. Now, an arbitrary algoritm when started with a given input n does not have to terminate at all (e.g., it can go ino an infinite loop). Thus, a general algorithm correspons to a partial recursive function. There are several ways to make these intuitive notions precise. One way is to give a precise definition of what a "machine computation" is, that is, formulate a mathematically rigorous notion of machine computation. Of course, there are many equivalent ways of doing this. For example, one could takr a formal programming language such as C or Fortran, and use this as the basis for the definition. These languages are designed to be flexible enough to be actually practical, and so lead to very cumbersome formal definitions. The attempt to develop the simnplest possible model of computation (or "programmin language") leads to the notion of a *Turing machine.*

Definition 5.1. A Turing machine is a finite set of quadruples $\langle n, s, a, m\rangle$ where $n, m \in \omega$, $a \in \{0, 1\}$, and a is one of the symbols $\{0, 1, L, R\}$.

A Turing machine executes as follows. We picture a tape which is infinite in both directions, and at each place on the tape there may be either a 0 or a 1. The Turing machine initially starts at position 0 of the tape. The machine is viewed as having finitely many *states* that it can be in, each state being identified with an integer. Each quadruple $\langle n, s, a, m\rangle$ is interpreted as the following instruction: if the machin is currently in state n and is scanning symbol s at the current position, then take action a and go into state m. If $a = 0$ or 1, then the action is to print a o or a 1 at the current position of the tape. If $a = L$ or R, the action is to move on position to the left or right respectively. To have the machine be given input n, we start the machine (at position 0) on a tape which is 0 everywhere except for a string of $n + 1$ ones in positions 0 to $n - 1$. It is convenient to designate one of the states as a distinguished "halting state," and declare that everything ceases if and when this state is reached. If it is, then the number of ones on the tape at that moment is declaredf to be the output value (one can require that these ones also be in consecutive positions starting at position 0). We can also view Turing machines as computing k-ary functions with an appropriate input convention, say to compute the value on input $(a_1, \dots, a_k)$ we start with the tape containg a string of $a_1 + 1$ ones, then a 0, then $a_2 + 1$ ones, etc.

It is tedious but straightforward to show that the usual programming constructs can be done at the Turing machine level, and that any operation which can done by a C or Fortran command, for example, can be done by a Turing machine. Then next exercise gives some such simple operations.

Exercise 57. Show that functions $f(n) = n + 1$, $f(n, m) = n + m$, $f(n, m) = n \cdot m$ are Turing computable.

Exercise 58. Show that if $f\colon \omega \to \omega$ is computed by the Turing machine T, then there is a Turing machine T' which also computes f and such that T' never moves to a negative position (i.e., to the left of the starting position). [hint: simulate a bi-infinite tape with a subset of the positive positions on the tape, for example using positions of the form $4k + 10$. Show that any action of T can be simulated by an action of T' which stays on the non-negative portion of the tape.]

Another approach is to give an abstract definition of the class of recursive (total) functions as follows.

Definition 5.2. The collection of (total) recursive functions (of any arity) is the smallest collection of functions (each of which is a function from ω^k to ω for some k) satisfying the following:

(1) $f(n) = n + 1$, $f(n,m) = n + m$, $f(n,m) = n \cdot m$ are recursive. Any constant function is recursive. Any projection function $f(a_1, \dots, a_k) = a_i$ is recursive.
(2) Any composition of recursive functions is recursive. More precisely, if $f : \omega^k \to \omega$ is recursive and $g_1, \dots, g_k \colon \omega^l \to \omega$ are recursive, then $h(a_1, \dots, a_l) = f(g_1(a_1, \dots, a_l), \dots, g_k(a_1, \dots, a_l))$ is recursive.
(3) The collection is closed under *primitive recursion*. That is, suppose $g \colon \omega^k \to \omega$ is recursive and $h \colon \omega^{k+2} \to \omega$ is recursive, Define $h \colon \omega^{k+1} \to \omega$ as follows:
$$f(t, a_1, \dots, a_k) = \begin{cases} g(a_1, \dots, a_k) & \text{if } t = 0 \\ h(t, a_1, \dots, a_k, f(t-1, a_1, \dots, a_k)) & \text{if } t > 0 \end{cases}$$
(4) The collection is closed under *total minimalization*. That is, suppose $g \colon \omega^{k+1} \to \omega$ is recursive and assume that for all $(a_1, \dots, a_k)$ there is a $t \in \omega$ such that $g(\vec{a}, t) = 0$. Define $f(a_1, \dots, a_k) = \mu t \; (g(\vec{a}, t) = 0)$, where μt means "the least t."

We also define a relation $R \subseteq \omega^k$ to be recursive iff its characteristic function χ_R is a recursive function.

It is easy to see that this is a welldefined class of functions (in case (4) this is because of the asumption on g). A moments reflection will convince one that any recursive function by definition 5.2 is machine computable. The other direction is also true, but perhaps not quite as obvious. The following exercises develop some of the basic properties of recursive functions and relations.

Exercise 59. Show that the functions
$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and } a \dot{-} b = \begin{cases} a - b & \text{if } a - b \geqslant 0 \\ 0 & \text{if } a - b < 0 \end{cases}$$
are recursive. [hint: both can be defined by simple primitive recursions.]

Exercise 60. Show that the relations $=$ and $<$ are recursive.

Exercise 61. Show that exponentiation is recursive, that is, the function defined by $f(n,m) = n^m$ $(= 0$ if $n = 0)$ is recursive.

Exercise 62. Show that if $R_1, \dots, R_k$ are recursive relations, then so is any Boolean combination of the R_i.

Exercise 63. Show that a function $f \colon \omega^k \to \omega$ is recursive iff its graph $G_f \subseteq \omega^{k+1}$ is a recursive relation.

Exercise 64. Show that if $R \subseteq \omega^k$ is recursive and $f_1, \dots, f_k$ are reursive functions, then $S(\vec{a}) \leftrightarrow R(f_1(\vec{a}), \dots, f_k(\vec{a}))$ is recursive.

Exercise 65. Show that a function defined by cases is recursive. That is, if $R_1, \dots, R_k$ are recursive relations and $f_1, \dots, f_k$ are recursive functions, then the function

$$f(\vec{a}) = \begin{cases} f_1(\vec{a}) & \text{if } R_1(\vec{a}) \\ f_2(\vec{a}) & \text{if } R_2(\vec{a}) \\ & \vdots \\ f_k(\vec{a}) & \text{if } R_k(\vec{a}) \end{cases}$$

is recursive.

Exercise 66. Show that if $\varphi(x_1, \dots, x_k)$ is a quantifier free formula in the language of number theory (i.e., in the first order language with symbols $+, \cdot, E, S, 0, =, <$) then $R(a_1, \dots a_k) \leftrightarrow \mathbb{N} \models \varphi(a_1, \dots, a_k)$ is recursive (here E is interpreted as exponentiation, s the successor function, and the other symbols have their usual aenings). [hint: Prove this by induction on the formula φ. Collect the previous exercises.]

Exercise 67. Show that the recursive relations are closed under bounded quantification. That is, if $R(\vec{a}, b, n)$ is recursive then so is $S(\vec{a}, b) \leftrightarrow \exists n \leqslant b\ R(\vec{a}, b, n)$. Likewise for bounded universal quantification. Similarly, show that if R is a recursive relation and $f(\vec{a}, b) = \mu\, n \leqslant b\ (R(\vec{a}, b, n))$ (if one exists and $= 0$ otherwise), then f is recursive. [hint: Show more generally that if $f(\vec{a}, b, n)$ is a recursive function, then so is $g(\vec{a}, b) = \sum_{n \leqslant b} f(\vec{a}, b, n)$. Use a primitive recursion to do this.]

Say a formula φ in the language of number theory is Δ^0_0 if it is the smallest class of forulas containing the quantifier free formulas and closed under bounded number quantification.

Exercise 68. Show that if φ is a Δ^0_0 formula then the relation defined by φ is primitive recursive. [hint: proceed inductively on φ as in exercise 67, and note that the minimalization operator is never needed.]

Exercise 69. Show that the relations $\mathrm{Prime}(n) \leftrightarrow (n \text{ is prime})$ and $\mathrm{Seq}(n) \leftrightarrow (n \text{ codes a sequence})$ are recursive, in fact their chacteristic functions are primitive recursive. Show that $f(k) = (\text{the } k^{\text{th}} \text{ prime})$ is primitive recursive. Show that the functions $f(n) = \mu p \geqslant n\ (\mathrm{Prime}(p))$, $\mathrm{lh}(n) = $ (the length of the sequence coded by n if n codes a sequence, and 0 otherwise). are primitive recursive. Show that the deocding function $(n)_k = a_k$ if n codes a sequence $n = \langle a_0, \dots, a_l \rangle$ of length $l \geqslant k$, and $= 0$ otherwise, is also primitive recursive. [hint: the Prime and Seq relations are defined by Δ^0_0 formulas. For the next prime function, use the fact that there is a prime between n and $n! + 1$ (actually by Bertrand's theorem between n and $2n$). Use this and a primitive recursion for the k^{th} prime function.]

Theorem 5.3. *The class of recursive functions coincides with the class of (total) Turing computable functions.*

proof (sketch). The proof that every recursive function is Turing computable is tedious but relatively straightforward, using a few Turing machine "programming tricks" such as exercise 58. Suppose then that $f\colon \omega \to \omega$ is Turing computable (the higher arity case is similar). Recall our coding of finite sequences of integers: $\langle a_0, a_1, \dots, a_k \rangle = 2^{a_0+1} \cdot 3^{a_1+1} \cdots p_k^{a_k+1}$, where p_i denotes the i^{th} prime ($p_0 = 2$). At any stage of a computation, the values on the tape, the position of the machine, and the current state of the machine can all be coded by an integer (note that there are only finitely many nonzero values on the tape at any given time). For eaxmple, this information could all be coded by the integer $\langle s, p, b, c, t\rangle$ where s is an integer giving the current state of the machine, p gives the current position, $-b$ is the leftmost 1, c is the rightmost 1, and t is a sequence of length $b+c+1$ which codes the tape values from positions $-b$ to c. We will call such an integer a stage code. An entire compuation of the machine which starts with input k and ends in a halting state with output value l can then be coded by a finite sequence of these stage codes. The relation $\mathrm{Stage}(n) \leftrightarrow n$ is a stage code) is clearly primitive recursive. The relation

$$\begin{aligned} \mathrm{Step}(T, n, m) \leftrightarrow T \text{ codes a Turing machine} \wedge \mathrm{Stage}(n) \wedge \mathrm{Stage}(m) \wedge \\ (m \text{ is obtained from } n \text{ by a valid step of the Turing machine}) \end{aligned}$$

is primitive recursive, using the above exercises (all quantifiers used in computing this relation are bounded). It follows that the relation

$$\begin{aligned} \mathrm{Comp}(T, n, k, l) \leftrightarrow\ & T \text{ codes a Turing machine} \wedge \mathrm{Seq}(n) \wedge \forall i \leqslant \mathrm{lh}(n)\ (\mathrm{Stage}(n)_i) \wedge \\ & ((n)_0 \text{ codes an input stage with input value } k) \wedge \\ & (n_{\mathrm{lh}(n} \text{ codes a halting stage with output value } l) \wedge \\ & \forall i < \mathrm{lh}(n)\ (n_{i+i} \text{ is obtained from } n_i \text{ by a valid step of the} \\ & \text{Turing machine}) \end{aligned}$$

is also primitive recursive.

Finally, the function f_T computed by the Turing machine can be expressed by $f_T(k) = (h(k))_1$ where $h(k) = \mu m\ \mathrm{Comp}(T, (m)_0, k, (m)_1)$. Note the essential use of the minimalization operator in the definition of h (though Comp is primitive recursive). This is the Kleene normal form for a recursive function. In particular, it shows that that every recursive function can be defined with at most one application of the minimalization operator.

Theorem 5.3 gives a little more information. It shows that there is a *universal* Turing machine. For every $e \in \omega$, let $\{e\}$ denote the partial function from $\omega \to \omega$ computed by e, if e codes a Turing machine, and otherwise $\{e\}$ is the 0 function. That is, there is a Turing machine U computing a partial binary function, such that for all $e, n \in \omega$,

$$U(e, n) = \begin{cases} \{e\}(n) & \text{if } e \text{ codes a Turing machine } \wedge \{e\}(n) \downarrow \\ \text{undefined} & \text{otherwise} \end{cases}$$

Namely, $U(e, n) = (h(e, n))_1$, where $h(e, n) = \mu m\ \mathrm{Comp}(e, (m)_0, n, (m)_1)$. □

Exercise 70 (Halting Problem). Show that the halting function

$$H(e, n) = \begin{cases} 1 & \text{if } U(e, n) \downarrow \\ 0 & \text{if } U(e, n) \uparrow \end{cases}$$

with $U(e, n)$ the partial recursive function above, is not recursive. In particular, the partial recursive function U is not total. [hint: Suppose H were recursive. Define

$f\colon \omega \to \omega$ by a diagonal argument:

$$f(n) = \begin{cases} U(n,n)+1 & \text{if } H(n,n)=1 \\ 0 & \text{if } H(n,n)=0 \end{cases}$$

Since H is recursive, this definition shows that f is also recursive. However, by construction f cannot be equal to any recursive function, say $\{e\}$, since $f(e) \neq \{e\}(e)$.]

We next give Kleene's *s-m-n* theorem. We will shortly show, without using recursion theory, that every pointclass that has a universal set has one for which the *s-m-n* property holds, and this will suffice for applications to descriptive set theory. However, the original fact that this holds for recursive functions is still of interest.

Theorem 5.4 (*s-m-n* theorem)**.** *For every integers $m < n$, there is a total recursive function $s\colon \omega^{m+1} \to \omega$ such that for all partial recursive functions $\{e\}$ we have*

$$\{s(e,a_1,\dots,a_m)\}(a_{m+1},\dots,a_n) = \{e\}(a_1,\dots,a_m,a_{m+1},\dots,a_n).$$

Proof. Let $T(b,a_1,\dots,a_m)$ be the Turing machine which on inputs $(a_{m+1},\dots,a_n)$ shifts those to the right on the tape and adds $a_1,\dots,a_m$ before them so as to end with the tape containing the standard input configuration for $(a_1,\dots,a_n)$, and such that the least state of the machine used is b. It is straightforward to check that T is recursive. Let $s(e,a_1,\dots,a_m)$ be the Turing machine which computes $\{e\}\circ T(e,a_1,\dots,a_m)$, which is essentually the union of the machines $T(e,a_1,\dots,a_m)$ and the Turing machine coded by e. Since $(e,a_1,\dots,a_m) \mapsto T(e,a_1,\dots,a_m)$ is recursive, so is $(e,a_1,\dots,a_m) \mapsto s(e,a_1,\dots,a_m)$. $\square$

We next introduce the arithmetical hierarchy for sets of integers (or subsets of ω^k for some k).

Definition 5.5. A subset of ω^k is Δ^0_1 iff it is recursive. A set $A \subseteq \omega^k$ is said to be Σ^0_1 (also called semi-recursive or recursively enumerable) if it of the form $A(\vec{a}) \leftrightarrow \exists b\, R(\vec{a},b)$, where R is recursive. A set A is Π^0_1 (or co-r.e.) if $\omega^k - A$ is Δ^0_1. In general for $n > 1$ we define $\Sigma^0_n = \exists^\omega \Pi^0_{n-1}$, $\Pi^0_n = \check{\Sigma}^0_n$, and $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$.

The next exercise checks that $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$ as well.

Exercise 71. Show that a set $A \subseteq \omega$ (or ω^k) is recursive iff both A and $\omega - A$ are recursively enumerable. [hint: If A is recursive then so is $B = \omega - A$, and so both are recursively enumerable. Suppose that A and B are both r.e. Say $A(n) \leftrightarrow \exists m\, R(n,m)$ and $B(n) \leftrightarrow \exists m\, S(n,m)$ where R, S are recursive. Let $f(n) = \mu m\ (R(n,m) \vee S(n,m))$. Thus, f is (total) recursive and $A(n) \leftrightarrow R(n,f(n))$.]

The next two exercises give alternate characterizations of Σ^0_1 sets.

Exercise 72. Show that A is Σ^0_1 iff A is the range of a total recursive function. [hint: If f is total recursive then $n \in \operatorname{ran}(f) \leftrightarrow \exists m\ (f(m) = n)$. Since the graph of f is recursive, this shows $\operatorname{ran}(f)$ is Σ^0_1. Suppose then that $A \in \Sigma^0_1$, say $A(n) \leftrightarrow \exists m\ R(n,m)$ where R is recursive. Let $f(a) = (a)_0$ if $R((a)_0,(a)_1)$, and otherwise $f(a) = n_0$, where n_0 is the least element of A.]

Exercise 73. Show that A is Σ^0_1 iff $A = \operatorname{dom}(f)$ for some partial recursive function f. [hint: Ia $A \in \Sigma^0_1$, say $A(n) \leftrightarrow \exists m\ R(n,m)$, then let $f(n) = \mu m\ R(n,m)$ (and

$f(n)$ is undefined if $\forall m \; \neg R(n,m)$). Then f is partial recursive and $A = \mathrm{dom}(f)$. Suppose next that $A = \mathrm{dom}(f)$ for some partial recursive f. Then $A(n) \leftrightarrow \exists m$ (m codes a computation starting with input n which terminates). The relation inside the quantifier is recursive.]

Remark 5.6. The theory of recursive functions can be developed entirely from definition 5.2, that is, without appealing to the notion of Turing machine or machine computability. One must be a little careful this way about some points related to partial functions. For example, it is not true that the class of partial recursive functions is the smallest class as defined in definition 5.2 where we allow all functions to be partial (with the natural conventions on when a composition, primitive recursion, etc. are defined); see the following exercise. One could define the partial recursive functions to be those of the form $f(n) = h(\mu m \; (g(n,m) = 0))$, where f, g are primitive recursive functions. One has to redo the proof of the normal form theorem to show that this includes all the total recursive functions.

Exercise 74. Show that there is a partial recursive f on $\omega \times \omega$ such that

$$\forall n \; \exists m \; (f(n,m) \downarrow \wedge f(n,m) = 0)$$

and the (total) function g defined by

$$g(n) = \mu m \; (f(n,m) \downarrow \wedge f(n,m) = 0)$$

is not recursive. [hint: define $f(n,1) = 0$, and $f(n,0) = \{n\}(n)$ (so $f(n,0)$ is defined iff $\{n\}(n) \downarrow$). If g were recursive, then so would be $\{n \colon g(n) = 0\} = \{n \colon \{n\}(n) \downarrow\}$. This set is not recursive, though, by exercise 70.]

The next lemma summarizes the closure properties of these classes. We write $\Sigma^0_n \restriction \omega^k$ for the Σ^0_n subsets of ω^k. We will shortly extend these arithmetical classes to more general Polish spaces.

Lemma 5.7. *$\Sigma^0_n \restriction \omega^k$ is closed under $\wedge$, $\vee$ and $\exists^\omega$. Π^0_k is closed under $\wedge$, $\vee$ and $\forall^\omega$. Δ^0_n is closed under $\wedge$, $\vee$. All of these classes are closed under recursive substitution, that is, preimages under recursive functions. For eample if $A \subseteq \omega^k$ is Σ^0_n, then so is $B(a_1, \dots, a_n) \leftrightarrow A(f_1(\vec{a}), \dots, f_n(\vec{a}))$.*

Proof. Suppose $A \in \Sigma^0_1 \restriction \omega^k$ and $B(\vec{a}) \leftrightarrow A(f_1(\vec{a}), \dots, f_k(\vec{a}))$ where the f_i are recursive. Write $A(\vec{a}) \leftrightarrow \exists m \; R(\vec{a}, m)$ where R is recursive. Then

$$B(\vec{a}) \leftrightarrow \exists m \; R(f_1(\vec{a}), \dots, f_k(\vec{a}), m).$$

The relation insider the quantifier is still recursive, so $B \in \Sigma^0_1$. It follows immediately that all the Σ^0_n, Π^0_n, and Δ^0_n are closed under recursive substitution. To show closure of Σ^0_1 under $\exists^\omega$, let $A(\vec{a}, m)$ be Σ^0_1, say $A(veca, m) \leftrightarrow \exists n \; R(\vec{a}, m, n)$, where R is recusive. Then $B(\vec{a}) \leftrightarrow \exists m \; A(\vec{a}, m) \leftrightarrow \exists m \exists n \; R(\vec{a}, m, n) \leftrightarrow \exists p \; R(\vec{a}, (p)_0, (p)_1)$, which shows $B \in \Sigma^0_1$. The same argument shows Σ^0_n is closed under $\exists^\omega$, and so Π^0_n is closed under $\forall^\omega$. The other cases are similar. $\square$

We construct universal sets for all of the arithmetical classes Σ^0_n, Π^0_n in ω^k. Define $U^k \subseteq \omega^{k+1}$ by $U^k(e, a_1, \dots, a_k) \leftrightarrow \{e\}(a_1, \dots, a_k) \downarrow$. Since the Σ^0_1 sets are the domains of the partial recursive functions, it follows that U is universal for $\Sigma^0_1 \restriction \omega^k$. Applying the operations $\neg$ and $\exists^\omega$ immediately gives universal sets for all of the classes $\Sigma^0_n\restriction\omega^k$, $\Pi^0_n\restriction\omega^k$. Moreover, the *s-m-n* theorem for recursive functions immediately gives a corresponding result for these universal sets. Namely,

$$\begin{aligned} U^n(e,a_1,\dots,a_n) &\leftrightarrow \{e\}(a_1,\dots,a_n)\downarrow \\ &\leftrightarrow \{s(e,a_1,\dots,a_m)\}(a_{m+1},\dots,a_n)\downarrow \\ &\leftrightarrow U^{n-m}(s(e,a_1,\dots,a_m),a_{m+1},\dots,a_n) \end{aligned}$$

where $s\colon \omega^{m+1}\to\omega$ is the total recursive function from theorem 5.4. Since the universal sets for the other classes are obtained from these by the operations, $\neg$, $\exists^\omega$, it also follows immediately that the universal sets for all the Σ^0_n, Π^0_n class have the *s-m-n* property as well. We summarize this in the following lemma.

Lemma 5.8. *For all of the lightface classes $\Gamma=\Sigma^0_n\restriction\omega^k$ or $\Gamma=\Pi^0_n\restriction\omega^k$, there are universal sets $U^k_\Gamma\subseteq\omega^{k+1}$ for $\Gamma\restriction\omega^k$ such that the following holds. For all $m<n$ there are total recurive functions $s_{m,n}\colon\omega^{m+1}\to\omega$ such that*

$$U^n_\Gamma(e,a_1,\dots,a_m,a_{m+1},\dots,a_n)\leftrightarrow U^{n-m}_\Gamma(s_{m,n}(e,a_1,\dots,a_m),a_{m+1},\dots,a_n),$$

for all $a_1,\dots,a_n\in\omega$.

We will give a different proof, avoiding recursion theory, of the existence of universal sets admitting *s-m-n* functions in a general context.

One of the uses of the *s-m-n* functions is to be able to do operations effectively on the codes for sets. For example, suppose Γ is one of the arithmetical classes Σ^0_n or Π^0_n. then there is a total recursive function $t\colon\omega\times\omega\to\omega$ such that for all $a,b\in\omega$, $U_{t(a,b)}=U_a\cap U_b$ (where U refers to the universal lg sets of lemma 5.8). To see this, note that $A(a,b,n)\leftrightarrow U_a(n)\wedge U_b(n)$ is in Γ. Let $e\in\omega$ be such that $U_e((a,b,n)\leftrightarrow A(a,b,n)$. By the *s-m-n* property, $U_e(a,b,n)\leftrightarrow U_{s(e,a,b)}(n)$, so we can take $t(a,b)=s(e,a,b)$.

We next extend these lightface notion to more general Polish spaces.

Definition 5.9. A polish space (X,ρ) with a dense set $D=\{r_0,r_1,\dots\}$ is said to be *recursively presented* if the relations $R(i,j,k,l)\leftrightarrow\rho(r_i,r_j)<\frac{k}{l}$, $S(i,j,k)\leftrightarrow\rho(r_i,r_j)\leqslant\frac{k}{l}$ are recursive (we assume $l>0$).

When we discuss a recursively presented space $(X,\rho),\mathcal{B}$, we will implicitly use the corresponding basis $\mathcal{B}$ for X given by open sets of the form $B_\rho(r_i,q)$, where $q\in\mathbb{Q}$. More specifically, For $n\in\omega$, let $U^X_n\subseteq X$ be the basic open set

$$U^x_n=B_\rho(r_{(n)_0},\frac{(n)_1}{(n)_2}),$$

so the U^X_n form a basis $\mathcal{B}$ for X. We will just write U_n when X is understood. We will always implicitly use the recursive presentation for ω with $r_i=i$. If (X_1,ρ_1,D_1), (X_2,ρ_2,d_2) are recursive presentations for Polish spaces X_1, X_2, then $(X_1\times X_2,\rho_1\times\rho_2,D_1\times D_2)$ is a recursive presentation for $X_1\times X_2$ (with the product topology) where $\rho_1\times\rho_2((x_1,x_2),(y_1,y_2))=\max\{\rho_1(x_1,y_1),\rho_2(x_2,y_2\}$. We will implictly use this presentation when discussing products.

All of the usual Polish spaces such as ω^ω, 2^ω, $\mathbb{R}$, $C[0,1]$ have recursive presentations, and in fact their usual bases work. In particular, in the case of ω^ω we use $r_i=s_i{}^\frown\bar{0}$ ($\bar{0}$ is the constant 0 real), where s_i is the i^{th} sequence in some out (recursive) coding of sequences into integers. the corresponding basis $\mathcal{B}$ is just the usual basis for the Baire space.

We now extend the lightface classes to a recursively presented space, starting with Σ^0_1.

Definition 5.10. Let (X, ρ), D be a recursively presented space with corresponding basis $\mathcal{B} = \{U_n\}$. A set $A \subseteq X$ is Σ^0_1 if $A = \bigcup\{U_n \colon S(n)\}$ where $S \subseteq \omega$ is Σ^0_1.
We let $\Pi^0_n \restriction X = \check{\Sigma}^0_n \restriction X$, $\Sigma^0_n = \exists^\omega \Pi^0_{n-1}$ for $n > 1$, and $\Delta^0_n \restriction X = \Sigma^0_n \restriction X \cap \Pi^0_n \restriction X$.

Thus, $A \subseteq X$ is Σ^0_1 if there is a recursive function $f \colon \omega \to \omega$ which enumerates codes for basic open sets unioning to A. Note that in the case of the Baire space, there is a simple recursive function $t \colon \omega \to \omega$ such that $t(n) \in \mathrm{Seq}$ for all n and $B_\rho(r_{(n)_0}, \frac{(n)_1}{(n)_2}) = N_s$, where $\langle s \rangle = t(n)$. So the definition of A being in Σ^0_1 in this case is equivalent to saying that there is a recursive function which enumerates the codes of sequences giving basic open sets which union to A.

Note that all Σ^0_1 sets are open, that is $\boldsymbol{\Sigma}^0_1$, and so every Σ^0_n or $\boldsymbol{\Pi}^0_n$ set is $\boldsymbol{\Sigma}^0_n$ or $\boldsymbol{\Pi}^0_n$ respectively. We call the Δ^0_1 subsets of X the recursive subsets, using the same terminology as in the $X = \omega$ case. Of course, if X is connected, like $\mathbb{R}_{\mathrm{std}}$, then there are no non-trivial Δ^0_1 subsets. The next exercise, on the other hand, shows that this is a reasonable terminology in the case of $X = \omega^\omega$.

Exercise 75. Show that $A \subseteq 2^\omega$ is Δ^0_1 iff it is computable in the following sense: there is a Turing machine T such that when started on a tape with the real $x \in \omega^\omega$ on it (i.e., the i^{th} position of the tape has value $x(i)$), then the machine will halt with output value 1 if $x \in A$ and will halt with output value 0 if $x \notin A$. Show also that every $A \subseteq 2^\omega$ which is computable in this sense is Δ^0_1. Show that the same is true for subseteq of ω^ω using a reasonable input convention for elements of ω^ω. [hint: If $A \subseteq 2^\omega$ is Δ^0_1, let f, g be total recursive functions such that $A = \bigcup\{N_{f(n)} \colon n \in \omega\}$, $B = 2^\omega - A = \bigcup\{N_{g(n)} \colon n \in \omega\}$ (here we write N_a for the neighborhood N_s where $\langle s \rangle = a$). Let T be the machine which at even $2n$ computes $f(n)$ and then checks to see if $x \in N_{f(n)}$, and at odd steps $2n + 1$ computes $g(n)$ and checks to see if $n \in N_{g(n)}$. If the machine first gets a positive check at an even stage, it outpus 1, and if at an odd stage, outputs a 0.]

We now define the *relativized* lightface pointclasses. for each $x \in \omega^\omega$ we define the classes $\Sigma^0_n(x)$, $\Pi^0_n(x)$, and $\Delta^0_n(x)$.

Definition 5.11. $A \subseteq X$ is $\Sigma^0_n(x)$ (or $\Pi^0_n(x)$, $\Delta^0_n(x)$) if there is a Σ^0_n set $B \subseteq \omega^\omega \times X$ such that $A(y) \leftrightarrow B(x, y)$ for all $y \in X$. More generally, if Γ is any ω-parametrized pointclass, we define $\Gamma(x)$ in a similar manner.

The following lemma connects the lightface and boldface pointclasses.

Lemma 5.12. $\boldsymbol{\Sigma}^0_1 = \bigcup_{x \in \omega^\omega} \Sigma^0_1(x)$, *and likewise for all of the lightface classes* Σ^0_n, Π^0_n, Δ^0_n, Σ^1_n, Π^1_n, Δ^1_n.

Proof. It suffices to prove the result for Σ^0_1, the other cases following by applying quantifiers or negations. Suppose X is recursively presented and $A \subseteq X$ is $\boldsymbol{\Sigma}^0_1$ (i.e., open). Let $A = \cup\{N_n \colon x(n) = 1\}$, for some $x \in 2^\omega$. Define $B(z, y) \leftrightarrow \exists i \ (z(i) = 1 \wedge y \in N_i)$. Easily $B \in \Sigma^0_1 \restriction \omega^\omega \times X$ [consider the recursive function which enumerates all pairs $(\langle s \rangle, k)$ where s is a finite sequence endind in 1 and $k = \mathrm{lh}(s)$. Then $B = \cup\{N_s \times B_k \colon (s, k) \in \mathrm{ran}(f)\}$.] Then $A(y) \leftrightarrow B(x, y)$, so $A \in \Sigma^0_1(x)$. $\square$

For functions between Polish spaces, in the boldface theory the basic notions are continuity and, more generally, f being $\boldsymbol{\Gamma}$-measurable. The corresponding lightface notions are f being recursive and f being Γ-recursive.

Definition 5.13. Let Γ be an ω-parametrized pointclass, and X, Y recursively presented Polish spaces. We say $f\colon X \to Y$ is Γ-recursive if the relation $R(x,n) \leftrightarrow f(x) \in U^Y_n$ is in $\Gamma \restriction (X \times \omega)$. We say f is recursive if it is Σ^0_1-recursive.

Clearly if f is recursive then f is continuous. If f is continuous, then $\{(x,n)\colon f(x) \in U_n\}$ is open, and so $\Sigma^0_1(x)$ for some x. Thus, f is $\Sigma^0_1(x)$-recursive for some x.

Exercise 76. Show that $f\colon \omega^\omega \to \omega^\omega$ is recursive (that is, Σ^0_1-recursive) iff f is Δ^0_1-recursive. [hint: Let $S = \{(x,s)\colon f(x) \in N_s\}$, so $S \in \Sigma^0_1$. Then $(x,s) \notin S$ iff there is a $t \perp s$ such that $f(x) \in N_t$.]

Exercise 77. Show that id $f\colon X \to Y$ and $g\colon Y \to Z$ are recursive, then so is $g \circ f$, and likewise for $\Sigma^0_1(x)$.

The nest lemma extends lemma 5.7 and gives the basic closure propeties of the lightface classes.

Lemma 5.14. *For any recursively presented Polish space X we have the following. $\Sigma^0_n \restriction X$ is closed under $\wedge$, $\vee$, $\exists^\omega$, and substitution by recursive functions (between recursively presented Polish spaces). $\Pi^0_n \restriction X$ is closed under $\wedge$, $\vee$, $\forall^\omega$, and recursive substitutions. $\Delta^0_n \restriction X$ is closed under $\wedge$, $\vee$, and recursive substitutions. Σ^1_n is closed under $\wedge$, $\vee$, $\exists^\omega$, $\forall^\omega$, $\exists^{\omega^\omega}$, and recursive substitutions. Π^1_n is closed under $\wedge$, $\vee$, $\exists^\omega$, $\forall^\omega$, $\forall^{\omega^\omega}$, and recursive substitutions. Δ^1_n is closed under $\wedge$, $\vee$, $\exists^\omega$, $\forall^\omega$, and recursive substitutions. The same holds also for all of the corresponding relatixized pointclasses.*

Proof. Suppose A, $B \subseteq X$ are Σ^0_1. Say

$$A = \bigcup \{B_\rho(r_{(n)_0}, \frac{(n)_1}{(n)_2})\colon S(n)\}, \qquad B = \bigcup \{B_\rho(r_{(n)_0}, \frac{(n)_1}{(n)_2})\colon T(n)\},$$

where $S, T \subseteq \omega$ are Σ^0_1. Then $A \cap B = \bigcup \{B_\rho(r_{(n)_0}, \frac{(n)_1}{(n)_2})\colon R(n)\}$ where

$$\begin{aligned} R(n) \leftrightarrow \exists a\ \exists b\ (S(a) \wedge T(b) \wedge (\rho(r_{(n)_0}, r_{(a)_0}) + \frac{(n)_1}{(n)_2} < \frac{(a)_1}{(a)_2}) \\ \wedge (\rho(r_{(n)_0}, r_{(b)_0}) + \frac{(n)_1}{(n)_2} < \frac{(b)_1}{(b)_2}). \end{aligned}$$

R is a Σ^0_1 subset of ω using here the definition of a recursive presentation. Unions are easier: $A \cup B = \bigcup \{B_\rho(r_{(n)_0}, \frac{(n)_1}{(n)_2})\colon R(n) \vee S(n)\}$, and $R \cup S$ is Σ^0_1. Suppose that $B \subseteq x \times \omega$ is Σ^0_1, and $A(x) \leftrightarrow \exists m\ B(x,m)$. Say $B = \bigcup \{B_\rho(x_{(n)_0}, \frac{(n)_1}{(n)_2}) \times \{m\}\colon S(n,m)\}$ where $S \in \Sigma^0_1 \restriction (\omega \times \omega)$. Then $A = \bigcup \{B_\rho(x_{(ne)_0}, \frac{(n)_1}{(n)_2})\colon T(n)\}$ where $T(n) \leftrightarrow \exists m\ S(n,m)$, so $T \in \Sigma^0_1$. So, $A \in \Sigma^0_1 \restriction X$. To see Σ^0_1 is closed under recursive substitution, suppose $A \subseteq X$ is Σ^0_1 and $f\colon (Z,d) \to X$ is recursive. Let $B(z) \leftrightarrow A(f(z))$. Say, $A = \bigcup \{B_\rho(r_{(n)_0}, \frac{(n)_1}{(n)_2})\colon S(n)\}$, with $S \in \Sigma^0_1 \restriction \omega$. Then $B(z) \leftrightarrow \exists m\ (S(m) \wedge (f(z) \in U^X_m))$. Since the relation $f(z) \in U^X_m$ is Σ^0_1, it follows from the closure properties already established that $B \in \Sigma^0_1$. The closure properties for the other pointclasses folow from these. □

We next extend lemma 5.8 on universal sets and *s*-*m*-*n* functions to more general space.

Theorem 5.15. *Let X be a recursively presented Polish space, and Γ any of the classes Σ^0_n, Π^0_n, Σ^1_n, or Π^1_n. Then there is a universal set $U \subseteq \omega \times X$ for $\Gamma \restriction X$. First, we have the following.*

Proof. It is enough to show the result for $\Gamma = \Sigma^0_1$. Let $V \subseteq \omega \times \omega$ be universal for $\Sigma^0_1 \restriction \omega$. Define $U \subseteq \omega \times X$ by

$$U(n, x) \leftrightarrow \exists m\ (V(n, m) \wedge x \in U^X_m).$$

U is Σ^0_1 from lemma 5.14. For the definition of $\Sigma^0_1 \restriction X$ it is immediate that U is universal for $\Sigma^0_1 \restriction X$. □

From theorem 5.15 we can obtain universal sets for the boldface classes which respect the lightface classes in a certain sense.

Theorem 5.16. *Let X be a recursively presented Polish space, and Γ any of the classes Σ^0_n, Π^0_n, Σ^1_n, or Π^1_n. Let $\boldsymbol{\Gamma}$ denote the corresponding boldface class. Then there is a $U^X \subseteq \omega^\omega \times X$ in Γ which is universal for $\boldsymbol{\Gamma} \restriction X$ and with the following property. If $A \subseteq X$ is in $\Gamma(z)$, for $z \in \omega^\omega$, then there is a real ϵ recursive in z such that $A = (U^x)_\epsilon$.*

Proof. Let $U \subseteq \omega \times \omega^\omega \times X$ be universal for $\Gamma \restriction (\omega^\omega \times X)$ from theorem 5.15. Define $U^X(\epsilon, x) \subseteq \omega^\omega \times X$ by $U^X(\epsilon, x) \leftrightarrow U(\epsilon(0), \epsilon', x)$, where $\epsilon'(n) = \epsilon(n+1)$. $U^X \in \Gamma$ by closure of Γ under recursive substitutions. Suppose $A \subseteq X$ is $\Gamma(z)$ for $z \in \omega^\omega$. So, $A(x) \leftrightarrow B(z, x)$ where $B \subseteq \omega^\omega \times X$ is in $\Gamma \restriction X$. Let $e \in \omega$ be such that $B(z, x) \leftrightarrow U(e, z, x)$. Then if $\epsilon = \langle e, z \rangle$, we have $U^X(\epsilon, x) \leftrightarrow U(e, z, x) \leftrightarrow B(z, x) \leftrightarrow A(x)$. Also, $\epsilon = \langle e, z \rangle$ is recursive in z. □

We now turn to the existence of *s*-*m*-*n* functions. Now must restrict out attention to spaces for which we have recursive coding and decoding functions (at least if we want to keep the *s*-*m*-*n* functions recursive). We first show the existence of the *s*-*m*-*n* functions using the *s*-*m*-*n* theorem for recursive functions. Then we give a more general abstract argument which avoids recursion theory.

Definition 5.17. We say a Polish space X is *reasonable* if it is a product $X = X_1 \times \cdots \times X_n$ of spaces with each $X_i = \omega$, ω^ω, or 2^ω.

Theorem 5.18. *Let Γ any of the classes Σ^0_n, Π^0_n, Σ^1_n, or Π^1_n. For every $X = X_1 \times \cdots \times X_n$ which a product of reasonable spaces X_i, let U^X be the universal sets from theorem 5.16. Then for every $m < n$ there is a recursive function $s \colon \omega^\omega \times X_1 \times \cdots \times X_m \to \omega^\omega$ such that*

$$U^Y(s(\epsilon, x_1, \ldots, x_m), x_{m+1} \ldots, x_n) \leftrightarrow U^X(\epsilon, x_1, \ldots, x_m, \ldots, x_n)$$

for all $\epsilon \in \omega^\omega$, and $x_i \in X_i$, where $Y = X_1 \times \cdots \times X_m$.

Proof. It is enough to consider the case where each X_i is one of the basic spaces ω, ω^ω or 2^ω. We have $U^X(\epsilon, x_1, \ldots, x_n) \leftrightarrow U(\epsilon(0), \epsilon', x_1, \ldots, x_n)$ where U is universal for the Σ^0_1 subsets of $\omega^\omega \times X_1 \times \cdot \times X_n$. Also, $U(\epsilon(0), \epsilon', x_1, \ldots, x_n) \leftrightarrow \exists n\, V(\epsilon(0), n) \wedge W_n(\epsilon', x_1, \ldots, x_n)$ where W_n refers to the recursive presentation for $\omega^\omega \times X_1 \times \cdots \times X_n$. Let U_i, V_i be the recursive presentations for the spaces $\omega^\omega \times X_1 \times \cdots \times X_m$, and $X_{m+1} \times \cdots \times X_n$ respectively. Let $(a, b) \mapsto r(a, b)$ be recursive such that $W_{r(a,b)} = U_a \times V_b$. Let B_n be the recursive presentation for $\omega^\omega \times X_{m+1} \times \cdots \times X_n$, and C_n the recursive presentation for ω^ω. Let $m \mapsto (a(m), b(m))$ be recursive such that $B_m = C_{a(m)} \times V_{b(m)}$. Let $\pi\colon \omega^\omega \to \omega^\omega \times X_1 \times \cdots \times X_m$ be a recursive bijection, and let $g\colon \omega \to \omega$ be recursive such that $\pi(C_j) = U_{g(j)}$. Define

$$S(e, m) \leftrightarrow \exists n\ (V(e, n) \wedge r(g(a(m)), b(m)) = n).$$

Clearly $S \in \Sigma^0_1 \restriction (\omega \times \omega)$, so by the *s*-*m*-*n* theorem on ω, there is a total recursive function $t\colon \omega \to \omega$ such that $V(t(e), m) \leftrightarrow S(e, m)$. Let $s(\epsilon, x_1, \ldots, x_m) = z$ where $z(0) = t(\epsilon(0))$ and $z' = \pi^{-1}(\epsilon', x_1, \ldots, x_m)$. Clearly s is recursive. Then

$$\begin{aligned}
U^Y(s(\epsilon, x_1, \ldots, x_m), x_{m+1}, \ldots, x_n) &\leftrightarrow \exists m\ (V(t(\epsilon(0)), m) \wedge B_m(z', x_{m+1}, \ldots, x_n)) \\
&\leftrightarrow \exists m\ (S(\epsilon(0), m) \wedge B_m(z', x_{m+1}, \ldots, x_n)) \\
&\leftrightarrow \exists m\ \exists n\ (V(\epsilon(0), n) \wedge (r(g(a(m)), b(m)) = n) \wedge B_m(z', x_{m+1}, \ldots, x_n)) \\
&\leftrightarrow \exists n\ (V(\epsilon(0), n) \wedge W_n(\epsilon', x_1, \ldots, x_n)) \\
&\leftrightarrow U^X(\epsilon, x_1, \cdots, x_n)
\end{aligned}$$

□

We now present a more general abstract srgument for the existence of universal sets admitting *s*-*m*-*n* functions. This argument makes no appeal to recursion theory.

Theorem 5.19. *Let $\boldsymbol{\Gamma}$ be a pointclass with $\boldsymbol{\Gamma} \restriction \omega^\omega$ having a universal set. Then there are universal sets $U^X \subseteq \omega^\omega \times X$ for all reasonable spaces X satisfying the following. For any product of reasonable spaces $X = X_1 \times \cdots \times X_m \times \cdots \times X_n$ and $m < n$ with $Y = X_{m+1} \times \cdots \times X_n$, there is a continuous function $s^{X,Y} : \omega^\omega \times X_1 \times \cdots \times X_m \to \omega^\omega$ such that*

$$U^X(y, x_1, \ldots, x_m, \ldots, x_n) \leftrightarrow U^Y(s(y, x_1, \ldots, x_m), x_{m+1}, \ldots, x_n)$$

for all $y \in \omega^\omega$ and $(x_1, \ldots, x_n) \in X$.

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal for $\boldsymbol{\Gamma} \restriction \omega^\omega$. Define $\Gamma \restriction X$ to those subsets of X which are recursively reducible to U. If $X = X_1 \times \cdots \times X_n$ is a reasonable space, define

$$U^X(y, x_1, \ldots, x_n) \leftrightarrow U((y)_0, \langle (y)_1, x_1, \ldots, x_n \rangle)$$

(recall $y \mapsto ((y)_0, (y)_1)$ is our recursive bijection between ω^ω and $\omega^\omega \times \omega^\omega$, and $\langle \cdots \rangle$ denotes our recursive coding functions). Clearly $U^X \in \Gamma$. To see it is universal, suppose $A \subseteq X$ in in $\boldsymbol{\Gamma}$. Define A' by $A'(x) \leftrightarrow A((x)_1, \ldots, (x)_n)$. Clearly $A' \in \boldsymbol{\Gamma}$, so let ϵ be such that $U(\epsilon, x) \leftrightarrow A'(x)$. Let $y = \langle \epsilon, \bar{0} \rangle$. Then $U^X(y, x_1, \ldots, x_n) \leftrightarrow U((y)_0, \langle (y)_1, x_1, \ldots, x_n \rangle) \to U(\epsilon, \langle \bar{0}, x_1, \ldots, x_n \rangle) \leftrightarrow A(x_1, \ldots, x_n)$.

To construct the *s*-*m*-*n* functions, note that

$$U^X(y, x_1, \ldots, x_n) \leftrightarrow U((y)_0, \langle (y)_1, x_1, \ldots, x_n \rangle)$$

and

$$U^Y(s, x_{m+1}, \ldots, x_n) \leftrightarrow U((s)_0, \langle (s)_1, x_{m+1}, \ldots, x_n \rangle).$$

therefore we can take $s^{X,Y}(y, x_1, \ldots, x_m) = \langle \epsilon, \langle y, x_1, \ldots, x_m \rangle \rangle$, where ϵ is such that $U(\epsilon, \langle \langle y, x_1, \ldots, x_m \rangle, x_{m+1}, \ldots, x_n \rangle) \leftrightarrow U((y)_0, \langle (y)_1, x_1, \ldots, x_n \rangle)$. That is, let ϵ be such that $U(\epsilon, z) \leftrightarrow U(z_{0,0,0}, \langle z_{0,0,1}, z_{0,1}, \ldots, z_{0,m}, z_1, \ldots, z_{n-m} \rangle)$, where $z_{0,0,0}$ abbreviates $(((z)_0)_0)_0$, etc.

□

Definition 5.20. We say universal sets $U^X \subseteq \omega^\omega \times X$, for X a reasonable Polish space, are *good* if there are continuous *s-m-n* functions as in the statement of theorem 5.19.

One important consequence of the existence of *s-m-n* functions in the recursion theorem, which says that in defining a $\mathbf{\Gamma}$ set, we may use the eventual code for the set we trying to define in its definition. a precise statement follows.

Theorem 5.21. *Let $\mathbf{\Gamma}$ be a pointclass and $U^X \subseteq \omega^\omega \times X$ good universal sets for $\mathbf{\Gamma}\restriction X$ for X a reasonable Polish space. For any $A \subseteq \omega^\omega \times X$ in $\mathbf{\Gamma}$, there is an $\epsilon \in \omega^\omega$ such that*

$$U^X(\epsilon, x) \leftrightarrow A(\epsilon, x)$$

for all $x \in X$.

Proof. Let $\epsilon_0 \in \omega^\omega$ be such that $U(\epsilon_0, \delta, x) \leftrightarrow A(s(\delta, \delta), x)$, where $U \subseteq \omega^\omega \times \omega^\omega \times X$ and $s\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ is the corresponding *s-m-n* function. Then, $A(s(\delta, \delta), x) \leftrightarrow U(\epsilon_0, \delta, x) \leftrightarrow U(s(\epsilon_0, \delta), x)$. In particular, for $\delta = \epsilon_0$ we have $U(s(\epsilon_0, \epsilon_0), x) \leftrightarrow A(s(\epsilon_0, \epsilon_0), x)$, and so we can take $\epsilon = s(\epsilon_0, \epsilon_0)$. $\square$

Remark 5.22. It is possible to improve theorem 5.19 to obtain recursive *s-m-n* functions, and so get a reasonable abstract notion of an associated lightface class. One way to do this is as follows. View every real y as coding a Lipschitz continuous function τ_y from ω^ω to ω^ω, say by $\tau_y(s) = y(\langle s \rangle)$. Fix the universal set $U \subseteq \omega^\omega \times \omega^\omega$ for $\mathbf{\Gamma}\restriction\omega^\omega$. Let $U' \subseteq \omega^\omega$ be the image of U under our recursive bijection, $U'(z) \leftrightarrow U((z)_0, (z)_1)$. Define for $X = X_1 \times \cdots \times X_n$ a reasonable space,

$$U^X(y, x_1, \dots, x_n) \leftrightarrow \tau_{(y)_0}(\langle x_1, \dots, x_n, (y)_1 \rangle) \in U'.$$

Clearly each $U^X \in \mathbf{\Gamma}$. It is straightforward to check that U^X is universal for $\mathbf{\Gamma}\restriction X$. To see there are recursive *s-m-n* functions, note that for $m < n$ and $Y = X_{m+1} \times \cdots \times X_n$ we have

$$U^Y(s, x_{m+1}, \dots, x_n) \leftrightarrow \tau_{(s)_0}(\langle x_{m+1}, \dots, x_n, (s)_1 \rangle) \in U'.$$

So, we let $s(y, x_1, \dots, x_m) = \langle z, \langle y, x_1, \dots, x_m \rangle \rangle$, where z will be described shortly, and thus

$$U^Y(s, x_{m+1}, \dots, x_n) \leftrightarrow \tau_z(\langle x_{m+1}, \dots, x_n, \langle y, x_1, \dots, x_m \rangle \rangle) \in U'.$$

It remains to show that there is a recursive map $(y, x_1, \dots, x_m) \mapsto z$ such that

$$\tau_z(\langle x_{m+1}, \dots, x_n, \langle y, x_1, \dots, x_m \rangle \rangle) = \tau_{(y)_0}(\langle x_1, \dots, x_n, (y)_1 \rangle.$$

We may effectively from y, $x_1, \dots, x_m$ use this equation to define z. The point is that if we are given k digits of $\langle x_{m+1}, \dots, x_n, \langle y, x_1, \dots, x_m \rangle \rangle$, using our standard coding maps, then we compute at least the first k digits of $\langle x_1, \dots, x_n, (y)_1 \rangle$ (note that only the digits of $x_{m+1}, \dots, x_n$ are problematic since $(y, x_1, \dots, x_m) \mapsto z$ is not required to be Lipschitz).

References

[1] Steve Jackson and R. Daniel Mauldin, ...