We define the notion of a metric.

Definition 0.1. A metric ρ on a set X is a fuction $\rho\colon X^2 \to \mathbb{R}$ satisfying the following:

(1) $\rho(x,y) \geq 0$ for all $x, y \in X$, and $\rho(x,y) = 0$ iff $x = y$.
(2) (symmetry) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$.
(3) (triangle inequality) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$.

Remark 0.2. If we remove the requirement that $\rho(x,y) = 0$ iff $x = y$, then we get the notion of a *pseudometric*.

If ρ is a metric on a set X, then we define the ρ-metric topology τ_ρ on X as follows. $U \subseteq X$ is τ_ρ iff $\forall x \in U\ \exists \epsilon > 0\ (B_\rho(x,\epsilon) \subseteq U)$, where $B_\rho(x,\epsilon) = \{y \in X \colon \rho(x,y) < \epsilon\}$.

Show this is a topology on X. Note, this does not use the triangle inequality, or the other axioms of a metric, only the fact that if $\epsilon_1 < \epsilon_2$, then $B_\rho(x,\epsilon_1) \subseteq B_\rho(x,\epsilon_2)$, which is immediate from the definition of of $B_\rho(x,\epsilon)$.

On the other hand, using the triangle inequality we have the following basic fact.

Fact 0.3. For any metric ρ on a set X, the ball $B_\rho(x,\epsilon)$ is open.

Proof. Let $y \in B_\rho(x,\epsilon)$. So, $\rho(x,y) < \epsilon$. Let $\delta = \epsilon - \rho(x,y) > 0$. If $z \in B_\rho(y,\delta)$, then $\rho(x,z) \leq \rho(x,y) + \rho(y,z) < \rho(x,y) + \delta = \epsilon$. This shows $B_\rho(x,\epsilon)$ is open in τ_ρ. □

Example 1. Some examples:

(1) $X = \mathbb{R}$, $\rho(x,y) = |x-y|$, the standard metric on $\mathbb{R}$. This gives the standard topology τ_{std} on $\mathbb{R}$. We call the space $\mathbb{R}_{\text{std}}$.
(2) $X = \mathbb{R}^n$. The standard Euclidean metric is

$$\rho(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}.$$

(3) On $X = \mathbb{R}^n$ we also have the following metrics.

$$\rho_2(\vec{x},\vec{y}) = |x_1-y_1| + \cdots + |x_n-y_n|$$
$$\rho_3(\vec{x},\vec{y}) = \max\{|x_1-y_1|, \dots, |x_n-y_n|\}$$

We show below that these metrics are equivalent, that is, they give the same topology on $\mathbb{R}^n$.

(4) X any set. The *discrete metric* on X is the metric given by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

Show this is a metric, and it gives the discrete topology on X.

Example 2. If (X,ρ) is a metric space and $Y \subseteq X$, then $\rho \restriction (Y \times Y)$ is also a metric.

Proof. It is trivial that the axioms go down from X to Y as they are all universal. □

We say two metrics ρ, d on a set X are *equivalent* if they give the same topology on X.

Fact 0.4. If ρ, d are metrics on X, then $\tau_\rho \subseteq \tau_d$ iff

$$\forall x \in X\ \forall \epsilon > 0\ \exists \delta > 0\ (B_d(x,\delta) \subseteq B_\rho(x,\epsilon)). \tag{1}$$

Proof. Suppose first that (1) holds. Let $U \in \tau_\rho$. Let $x \in U$. Since $U \in \tau_\rho$, there is an $\epsilon > 0$ such that $B_\rho(x,\epsilon) \subseteq U$. From (1), there is a $\delta > 0$ such that $B_d(x,\delta) \subseteq B_\rho(x,\epsilon) \subseteq U$, which shows $U \in \tau_d$.

Conversely, suppose $\tau_\rho \subseteq \tau_d$. We show (1). Let $x \in X$ and fix $\epsilon > 0$. Since $B_\rho(x,\epsilon) \in \tau_\rho \subseteq \tau_d$, we have from the definition of τ_d that there is a $\delta > 0$ such that $B_d(x,\delta) \subseteq U$. This shows (1).

□

Corollary 0.5. *Given two metrics ρ, d on a set X, we have that $\tau_\rho = \tau_d$ iff $\forall x \in X\ \forall \epsilon > 0\ \exists \delta > 0\ (B_d(x,\delta) \subseteq B_\rho(x,\epsilon))$ and $\forall x \in X\ \forall \epsilon > 0\ \exists \delta > 0\ (B_\rho(x,\delta) \subseteq B_d(x,\epsilon))$.*

We have the particular special case of Fact 0.4.

Fact 0.6. If ρ, d are metrics on a set X and there is a constant $C \in \mathbb{R}$ such that $\rho(x,y) \leq Cd(x,y)$ for all $x,y \in X$, then $\tau_\rho \subseteq \tau_d$.

Proof. Let $x \in X$, and fix $\epsilon > 0$. We need to find a $\delta > 0$ such that $B_d(x,\delta) \subseteq B_\rho(x,\epsilon)$. Let $\delta = \frac{\epsilon}{C}$ (note that wlog $C > 0$). Then if $y \in B_d(x,\delta)$ we have $d(x,y) < \delta = \frac{\epsilon}{C}$, so $\rho(x,y) < C\frac{\epsilon}{C} = \epsilon$, so $y \in B_\rho(x,\epsilon)$. □

Corollary 0.7. *If ρ, d are metrics on X and there are constants C, D such that $\rho \leq Cd$ and $d \leq D\rho$, then $\tau_\rho = \tau_d$, that is, ρ and d are equivalent metrics.*

Using the above we have the following fact.

Fact 0.8. Let ρ be a metric on X. Let $f\colon X \to \mathbb{R}$. Define d by $d(x,y) = \rho(x,y) + |f(x) - f(y)|$. Then d is a metric on X and $\tau_\rho \subseteq \tau_d$.

Proof. It is easy to show that d is a metric on X. Since we clearly have that $\rho \leq d$, it follows that $\tau_\rho \subseteq \tau_d$. □

Example 3. Let $\rho(x,y) = |x-y|$ be the standard metric on $\mathbb{R}$, and let $d(x,y) = \rho(x,y) + |x^2 - y^2|$. From Fact 0.8 we have that $\tau_\rho \subseteq \tau_d$. Using the fact that the function $f(x) = x^2$ is continuous, it is easy to see that $\tau_d \subseteq \tau_\rho$. So, ρ and d are equivalent metric on R. However, it is not the case that there is a constant C such that $d \leq C\rho$ (easy to see).

Fact 0.9. The metrics ρ, ρ_2, and ρ_3 on $\mathbb{R}^n$ of Example 1 are all equivalent.

Proof. The three metrics are all bounded in terms of each other. We have $\rho_3 \leq \rho \leq \rho_2 \leq n\rho_3$ (for the second inequality, square both sides).

□

Example 4. Let $X = \mathbb{R}^\omega$ be the set of all sequences $x = (x_0, x_1, \dots,)$ of real numbers. For $1 \leq p \leq \infty$, we define the map $x \in X \mapsto \|x\|_p \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ as follows. If $p < \infty$ then we set $\|x\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}}$ (if the sum diverges then we set $\|x\| = \infty$). For $p = \infty$ we define $\|x\|_\infty = \sup\{|x_i|\}$, where we take the "sup" to be ∞ if the sequence of $|x_i|$ is not bounded. For $1 \leq p \leq \infty$ we let $\ell_p \subseteq X$ be the set of sequences x such that $\|x\|_p < \infty$. It can be shown that this is *norm* on ℓ_p, that is we have:

(1) $\|cx\|_p = |c|\|x\|_p$ for all $c \in \mathbb{R}$.

(2) $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ for all $x, y \in \ell_p$.

From these it follows that we have a metric d_p on ℓ_p defined by $d_p(x,y) = \|x-y\|_p$ (note that addition and scalar multiplication are defined on ℓ_p, and that it is a vector space over $\mathbb{R}$). For $p = 1, 2, \infty$, these metrics generalize the metrics of Example 1 from $\mathbb{R}^n$ to infinite dimensional versions of these spaces. For finite dimensional spaces, the $\mathbb{R}^n$, these metrics are equivalent as shown in Fact 0.9. On $\mathbb{R}^\omega$, however, they are not equivalent. If $1 \leq p \leq q \leq \infty$ we do have $\|x\|_q \leq \|x\|_p$, and so $\ell_p \subseteq \ell_q$. Thus, as topological (metric) spaces we have that $\tau_{d_q} \subseteq \tau_{d_p}$ (on the space ℓ_p) To see this norm inequality note that we may assume wlog that $p < \infty$ and $\|x\|_p < \infty$, in which case the terms $\|x_i\|$ must tend to 0. So, $B = \max\{|x_i|\}$ is well-defined. Since both norms scale under scalar multiplication, we may assume that $B = 1$. We then have:

$$\|x\|_q = \left(\sum_i |x_i|^q\right)^{\frac{1}{q}} \leq \left(\sum_i |x_i|^p\right)^{\frac{1}{q}} = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}\frac{p}{q}} = (\|x\|_p)^{\frac{p}{q}} \leq \|x\|_p.$$

The first inequality is from $|x_i| \leq 1$ (since $B = 1$), and the last inequality is from $\|x\|_p \geq 1$ and $\frac{p}{q} \leq 1$. In particular, $\ell_1 \subseteq \ell_2 \subseteq \ell_\infty$. The metrics d_1, d_2, d_∞ (or more generally, d_p, d_q for $1 \leq p \leq q \leq \infty$) are not equivalent metrics on ℓ_1, though we do have that $\tau_{d_\infty} \subseteq \tau_{d_2} \subseteq \tau_{d_1}$ (and more generally $\tau_{d_q} \subseteq \tau_{d_p}$ for $1 \leq p \leq q \leq \infty$ on ℓ_p). To see, for example, that the metrics d_1 and d_2 are not equivalent on ℓ_1 let $x^n \in \mathbb{R}^\omega$ be the point defined by $x^n_m = \begin{cases} \frac{1}{m} & \text{if } m \leq n \\ 0 & \text{otherwise} \end{cases}$. Clearly all of the x^n are in ℓ_1 (so are in all of the ℓ_p) as their supports are finite. For any $\epsilon > 0$, there is an N so that if $m, n \geq N$ then $d_2(x^n, x^m) < \epsilon$, since the series $\sum_i \frac{1}{i^2}$ converges. On the other hand, if we fix $m \geq N$, then if we let n get large enough we have that $d_1(x^n, x^m)$ get arbitrarily large (since $\sum_{i>m} \frac{1}{i}$ diverges). Using this, it follows that $\{x^n\}$ is a closed set in τ_{d_1} but not in τ_{d_2}.

Fact 0.10. For any metric ρ on a set X, $B_{\bar{\rho}}(x,\epsilon) = \{y \colon \rho(x,y) \leq \epsilon\}$ is a closed set.

Proof. Let $y \notin B_{\bar{\rho}}(x,\epsilon)$, so $\rho(x,y) > \epsilon$. Let $\delta = \rho(x,y) - \epsilon > 0$. Then $B_\rho(y,\delta) \cap B_{\bar{\rho}}(x,\epsilon) = \emptyset$. To see this, suppose $z \in B_\rho(y,\delta) \cap B_{\bar{\rho}}(x,\epsilon)$. Then $\rho(x,y) \leq \rho(x,z) + \rho(z,y) \leq \epsilon + \rho(z,y) < \epsilon + \delta = \rho(x,y)$, a contradiction. □

Thus, we always have that $\overline{B_\rho(x,\epsilon)} \subseteq B_{\bar{\rho}}(x,\epsilon)$, but in general we do not have equality. For example, let ρ be the discrete metric on X and let $\epsilon = 1$. Then $\overline{B_\rho(x,\epsilon)} = \{x\}$, but $B_{\bar{\rho}}(x,\epsilon) = X$.

Another example: let $Y = \mathbb{R}^2 - S^1 \subseteq \mathbb{R}^2$, and ρ the standard metric on $\mathbb{R}^2$. Then in the metric space (Y,ρ) we have $\overline{B_\rho((0,0),1)} = B_\rho((0,0),1)$, so $B_\rho((0,0),1)$ is closed. So, the open ball $B_\rho((0,0),1)$ is closed.

Theorem 0.11. *For every metric ρ on a set X, there is an equivalent metric ρ' on X which is bounded, in fact $\rho'(x,y) \leq 1$.*

Proof. Let $\rho'(x,y) = \min\{\rho(x,y), 1\}$. To show the triangle inequality: let $x, y, z \in X$. We must show that $\rho'(x,z) \leq \rho'(x,y) + \rho'(y,z)$. If (at least) one of $\rho'(x,y)$, $\rho'(y,z)$ is ≥ 1 then the RHS is ≥ 1, and the LHS is ≤ 1, so the result is immediate. If both $\rho'(x,y)$, $\rho'(y,z)$ are < 1, then $\rho'(x,y) = \rho(x,y)$ and $\rho'(y,z) = \rho(y,z)$. So, $\rho'(x,z) \leq \rho(x,z) \leq \rho(x,y) + \rho(y,z) = \rho'(x,y) + \rho'(y,z)$.

To see the metrics are equivalent, first note that $\rho' \leq \rho$, so $\tau_{\rho'} \subseteq \tau_\rho$. For the other direction, let $x \in X$ and fix $\epsilon > 0$. Let $\delta = \min\{\epsilon, 1\}$. If $\rho'(x,y) < \delta$, then since $\delta \leq 1$ we have $\rho'(x,y) = \rho(x,y)$, so $\rho(x,y) < \delta \leq \epsilon$. So, $B_{\rho'}(x,\delta) \subseteq B_\rho(x,\epsilon)$. This shows $\tau_\rho \subseteq \tau_{\rho'}$.

□

It is also true that we can define $\rho'(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$, and also get an equivalent metric.

Exercise 1. Show that if ρ is a metric on the set X, then so is $\rho' = \frac{\rho}{1+\rho}$. [hint: in showing the triangle inequality, there are are two ways to proceed. One is purely algebraic, just multiply both sides of the desired inequality out and see what it becomes. The other way is use the fact that the function $f(x) = \frac{x}{1+x}$ is increasing and convex. Show that $f(a+b) \leq f(a) + f(b)$.]

If ρ is a metric on a set X, we extend ρ to define $\rho(x, A)$ for $x \in X$ and $A \subseteq X$ (with $A \neq \emptyset$) as follows:

$$\rho(x, A) = \inf\{\rho(x, a) \colon a \in A\}.$$

Fact 0.12. If $F \subseteq X$ is closed in the ρ-metric topology, then for all $x \in X$ we have that $x \in F$ iff $\rho(x, F) = 0$. Conversely, if $F \subseteq X$ and for all $x \in X$ we have that $x \in F$ iff $\rho(x, F) = 0$, then F is closed.

Proof. Suppose F is closed. If $x \in F$ we clearly have that $\rho(x, F) = 0$. If $x \notin F$, then for some $\epsilon > 0$ we have $B_\rho(x, \epsilon) \subseteq F^c$. That is, if $y \in F$, then $\rho(x, y) \geq \epsilon$, and so $\rho(x, F) \geq \epsilon$.

Suppose next that $F \subseteq X$ and for all $x \in X$ we have that $x \in F$ iff $\rho(x, F) = 0$. Suppose $x \in \overline{F}$. Then for any $\epsilon > 0$ we have that $B_\rho(x, \epsilon) \cap F \neq \emptyset$. In particular, this says that $\rho(x, F) \leq \epsilon$ for any $\epsilon > 0$. Thus, $\rho(x, F) = 0$. By assumption this means $x \in F$. So, $\overline{F} = F$ and thus F is closed. □

Example 5. Let $X = \{0\} \cup (1, \infty) \subseteq \mathbb{R}$ and let ρ be the restriction of the standard metric on $\mathbb{R}$ to X. Let $F = (1, \infty) \subseteq X$. Then F is a closed set in X. We have that $d(x, F) = 1$, but $d(x, y) > 1$ for all $y \in F$. So, distances to closed sets are not necessarily attained in metric spaces.

Exercise 2. Show that $\rho(y, A) \leq \rho(x, A) + \rho(x, y)$ for any $A \subseteq X$ (with $A \neq \emptyset$), $x, y \in X$. Deduce that for any A and $\epsilon > 0$ the set $B_\rho(A, \epsilon) = \{x \colon \rho(x, A) < \epsilon\}$ is open, as is the set $\{x \in X \colon \rho(x, A) > \epsilon\}$. Also, the set $B_{\bar{\rho}}(A, \epsilon) = \{x \colon \rho(x, A) \leq \epsilon\}$ is closed, as is the set $\{x \in X \colon \rho(x, A) \geq \epsilon\}$.

Exercise 3. Show that in every metric space (X, ρ), every open set U is a countable increasing union of closed sets $U = \bigcup_n F_n$. [hint: define the F_n by considering distances to the closed set U^c.]

We can further extend a metric ρ on the set X to the non-empty subsets of X by $\rho(A, B) = \inf\{\rho(a, b) \colon a \in A, b \in B\}$. This is not, however, a metric in general. In fact, if X has more that two elements it is not a metric: Let x, y, z be distinct elements of X, and let $A = \{x\}$, $B = \{x, y\}$, and $C = \{y, z\}$. Then $\rho(A, B) = 0$, $\rho(B, C) = 0$, but $\rho(A, C) > 0$.