Introduction to Ordinals

1. Ordinals

We give here a quick presentation of the definitions and basic facts about the ordinals. Our presentation will be informal, that is, outside of formal ZFC set theory, but will suffice.

Definition 1.1. A linear ordering $(X, \prec_X)$ is a set X and a binary relation $\prec_X$ on X satisfying (we write $\prec$ for $\prec_X$ when X is understood):

(1) $\forall x\ x \not\prec x$ (irreflexive)
(2) $\forall x \forall y (x \prec y \vee x = y \vee y \prec x)$ (connected)
(3) $\forall x \forall y \forall z (x \prec y \wedge y \prec z \rightarrow x \prec z)$ (transitive)

Note that exactly one of the cases holds in axiom 2 above.

Examples of linear orderings include $(\mathbb{N}, \prec)$, $(\mathbb{R}, \prec)$, $(\mathbb{Q}, \prec)$, $(\mathbb{Z}, \prec)$, where $\prec$ in all cases is the ordering induceed by the usual ordering on $\mathbb{R}$.

Definition 1.2. A well-ordering $(X, \prec_X)$ is a linear ordering such that for every $S \subseteq X$, $S \neq \emptyset$, S has a $\prec$ least element. That is, $\exists x \in S\ \forall y \in S\ \neg(y \prec x)$.

With their usual orderings, $\mathbb{N}$ is well-ordered, but $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not.

More generally, if R is a binary relation on a set X, we say R is well-founded if for every $S \subseteq X$, $S \neq \emptyset$, $\exists x \in S\ \forall y \in S\ \neg(yRx)$. Thus, a well-oredering is just a linear ordering which is well-founded.

Fact 1.3. (DC) A relation R on a set X is well-founded iff there does not exist an infinite decreasing chain, i.e., elements $x_0, x_1, \dots$ of X with $x_{n+1} R x_n$ for all n.

Proof. If R is well-founded, and $x_0, x_1, \dots$ is a sequence form X, let x_n be an R-minimal element of $S \doteq \{x_0, x_1, \dots\}$. Then $\neg x_{n+1} R x_n$. (This direction did not use AC).

Suppose now (X, R) has no infinite decreasing chains, and let $S \subseteq X$, $X \neq \emptyset$. For each finite decreasing chain $x_n R x_{n-1} R \dots R x_2 R x_1 R x_0$ of elements from S, "pick" an $x_{n+1} \in S$ such that $x_{n+1} R x_n$. This exixts by the assumption on S. This defines a decresing chain $x_0, x_1, \dots$.

More formally, for each finite decreasing chain $\vec{x} = x_n R x_{n-1} R \dots R x_2 R x_1 R x_0$ of elements from S, let $A_{\vec{x}} = \{y \in S : yRx_n\}$. By assumption, $A_{\vec{x}} \neq \emptyset$. By DC, there is an infinite sequence $x_0, x_1, \dots$ such that $\forall n\ x_{n+1} \in A_{x_0, \dots, x_n}$, i.e., $x_{n+1} R x_n$. □

Though we are mainly concerned with well-orderings, we can make the following definitions for linear orderings.

Definition 1.4. If $(X, \prec_X)$ is a linear ordering and $x \in X$, let $I_x^{\prec_X}$ (or just I_x if $\prec_X$ is understood) denote the initial segment determined by x. That is, $I_x = \{y \in X : y \prec x\}$. An initial segment of X means a set $I \subseteq X$ such that $\forall x, x' \in X\ (x \in I \wedge x' \prec x \rightarrow x' \in I)$. An initial segment I of X is said to be proper if $I \neq X$.

Definition 1.5. Let $(X, \prec_X)$, $(Y, \prec_Y)$ be linear orderings. We say a map $\pi : A \rightarrow Y$, where $A \subseteq X$, is order-preserving if for all $x_1, x_2 \in A$, $x_1 \prec_X x_2 \leftrightarrow \pi(x_1) \prec_Y \pi(x_2)$. We say π is an order-isomorphism from X to Y if π is an order-preserving bijection.

We frequently just say $\prec_X$, $\prec_Y$ are isomorphic, written $\prec_X \cong \prec_Y$, to abbreviate "order-isomorphic".

Exercise. Show that if $(X, \prec)$ is a well-ordering and I is a proper initial segment of X, then $\exists x \in X \; I = I_x$.

Exercise. Suppose $(X, \prec_X)$, $(Y, \prec_Y)$ are well-orderings. Let $\prec_X \oplus \prec_Y$ be the ordering with domain $(X \times \{0\}) \cup (Y \times \{1\})$ and ordered by: $(z_1, i_1) \prec (z_2, i_2)$ iff $(i_1 < i_2) \vee (i_1 = i_2 = 0 \wedge z_1 \prec_X z_2) \vee (i_1 = i_2 = 1 \wedge z_1 \prec_Y z_2)$. Show that $\prec_X \oplus \prec_Y$ is also a well-ordering.

Exercise. Suppose $(X, \prec_X)$, $(Y, \prec_Y)$ are well-orderings. Let $\prec_X \otimes \prec_Y$ be the ordering with domain $X \times Y$ and ordered by: $(x_1, y_1) \prec (x_2, y_2)$ iff $(y_1 \prec_Y y_2) \vee (y_1 = y_2 \wedge x_1 \prec_X x_2)$. Show that $\prec_X \otimes \prec_Y$ is also a well-ordering.

We develop some of the basic facts about well-orderings.

Lemma 1.1. *If the well-orderings $(X, \prec_X)$, $(Y, \prec_Y)$ are order-isomorphic, then the order-isomorphism between them is unique.*

Proof. Suppose $f, g \colon x \to Y$ are both order-isomorphisms. We show that $f = g$. If not, let $x_0 \in X$ be the $\prec_X$ least x such that $f(x) \neq g(x)$. Without loss of generality, Suppose $f(x_0) \prec_Y g(x_0)$. Let x_1 be such that $g(x_1) = f(x_0)$. Clearly $x_1 \neq x_0$. If $x_1 \prec_X x_0$, then by minimality of x_0, $g(x_1) = f(x_1) \prec_X f(x_0)$, a contradiction. Thus, $x_0 \prec_X x_1$. However, this contradicts g being order-preserving. □

Lemma 1.2. *If $(X, \prec_X)$ is a well-ordering, then X is not order-isomorphic to any properinitial segment of itself.*

Proof. Suppose $\pi \colon I \to X$ is an order-isomorphism between the proper initial segment I of X and all of X. We cannot have $\pi(x) = x$ for all $x \in I$, as then π would not be onto. Let x_0 be the least element of I such that $\pi(x) \neq x$. We can't have $\pi(x_0) \prec x_0$ since then $\pi(\pi(x_0)) = \pi(x_0)$, and thus π is not one-to-one. So, $x_0 \prec \pi(x_0)$.

Let $x_1 \in I$ be such that $\pi(x_1) = x_0$. Clearly $x_1 \neq x_0$ (since $\pi(x_0) \neq x_0$). If $x_1 \prec x_0$, then $\pi(x_1) = x_1 \prec x_0$, which is impossible. If $x_0 \prec x_1$, then $\pi(x_1) = x_0 \prec \pi(x_0)$, which contradicts π being order-preserving. □

Exercise. Show that if π is an isomorphism from $\prec_X$ to $\prec_Y$ and $x \in X$, then $\pi \restriction I_x^{\prec_X}$ is an isomorphism between $I_x^{\prec_X}$ and $I_y^{\prec_Y}$.

Theorem 1.6. *Let $(X, \prec_X)$, $(y, \prec_Y)$ be well-orderings. Then exactly one of the following holds.*

1. *$(X, \prec_X)$ is isomorphic to a proper initial segment of $(Y, \prec_Y)$.*
2. *$(Y, \prec_Y)$ is isomorphic to a proper initial segment of $(X, \prec_X)$.*
3. *$(X, \prec_X)$ is isomorphic to $(y, \prec_Y)$.*

Proof. For $x \in X, y \in Y$, define $R(x, y)$ iff $I_x^{\prec_X} \cong I_y^{\prec_Y}$.

First note that for all $x \in X$ and $y_1, y_2 \in Y$, if $R(x, y_1)$ and $R(x, y_2)$, then $y_1 = y_2$. If not, say w.l.o.g $y_1 \prec_Y y_2$. But then, $I_x^{\prec_X} \cong I_{y_1}^{\prec_Y} \cong I_{y_2}^{\prec_Y}$. This violates lemma 1.2.

Thus, R is a partial function. Likewise, R is one-to-one, since if $R(x_1, y), R(x_2, y)$ but (w.l.o.g.) $x_1 \prec_X x_2$, then $I_{x_1}^{\prec_X} \cong I_y^{\prec_Y} \cong I_{x_2}^{\prec_X}$.

We next claim that $\mathrm{dom}(R)$ is an initial segment of $\prec_X$. Suppose $x_2 \in \mathrm{dom}(R)$ and $x_1 \prec_X x_2$. Say $R(x_2, y_2)$, that is, $I_{x_2}^{\prec_X} \cong I_{y_2}^{\prec_Y}$. Let π be an isomorphism from $I_{x_2}^{\prec_X}$ to $I_{y_2}^{\prec_Y}$. By the exercise above, $I_{x_1}^{\prec_X} \cong I_{y_1}^{\prec_Y}$, where $y_1 = \pi(x_1)$. Thus,

$R(x_1, y_1)$, and so $x_1 \in \mathrm{dom}(R)$. We have also shown that R is order-preserving from $\prec_X$ to $\prec_Y$.

An exactly similar argument shows likewise that $\mathrm{ran}(R)$ is an initial segment of $\prec_Y$.

We have shown so far that R is an isomorphism from an initial segment of $\prec_X$, say I, to an initial segment of $\prec_Y$, say J.

We now consider cases.

If $I = X$ but $J \neq Y$, Then case (1) of the theorem holds. If $I \neq X$ but $J = Y$, then case (2) of the theorem holds. If $I = X$ and $J = Y$, then clearly Then case III of the theorem holds. Suppose finally that $I \neq X$ and $J \neq Y$. We show that this case does not occur. by an exercise, let $I = I_x^{\prec_x}$ and $J = I_y^{\prec_Y}$. Since R is an isomorphism between I and J, by definition we have $R(x, y)$. Thus $x \in \mathrm{dom}(R)$, so $x \in I_x$, a contradiction.

We have now shown that one of the three cases of the theorem holds. Uniqueness of the case follows immediately from lemma 1.1. □

We now state our (slightly informal) definition of ordinal.

Definition 1.7. An ordinal α is an equivalence class of a wellordering $(X, \prec_X)$ under order-isomorphism. Thus, $\alpha = [(X, \prec_X)]$.

Remark 1.8. The informality in the above definition lies in some set theoretic subtleties. Namely, the "equivalence classes" as defined above are actually too large to be sets, they are proper classes. Thus, from a formal set theoretic point of view, the definition doesn't make sense (this is actually a minor problem that plagues many common definitions in mathematics). However, the problem is easy to correct if one does the formal development of set theory, and will not bother us.

We frequently use lower case Greek letters like α, β, γ for ordinals. ON denotes the (proper class) of all ordinals. Suppose α, β are ordinals. Say $\alpha = [(X, \prec_X)]$ and $\beta = [(X, \prec_X)]$. We say $\alpha < \beta$ iff $(X, \prec_X) \cong (Y, \prec_Y)$. This is clearly well-defined.

Theorem 1.6 may then be restated as saying for any two ordinals α, β, exactly one of the following holds: $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$. Note that if $\alpha = [(X, \prec_X)]$ and $\beta < \alpha$, then we may represent β as $\beta = [(I_x, \prec_X)]$ for some proper initial segment I_x of $\prec_X$.

The following theorem gives a fundamental property of ordinals. It says, in effect, that the collection of all ordinals behaves like an ordinal.

Exercise. Let $\alpha, \beta \in ON$. Suppose that $\forall \gamma < \alpha\ \exists \delta < \beta\ \gamma \leq \delta$. Show that $\alpha \leq \beta$ (hint: use theorem 1.6 and lemma 1.2).

Exercise. Let $\alpha, \beta \in ON$. Suppose there is an order-preserving map π from α to β. Show that $\alpha \leq \beta$.

Theorem 1.9. *Let $S \subseteq ON$ be a non-empty set of ordinals. Then $\exists \alpha \in S$ which is minimal, i.e., $\forall \beta \in S\ (\beta = \alpha \vee \beta > \alpha)$.*

Proof. Let $S \subseteq ON$ be non-empty, and let $\alpha \in S$. Say $\alpha = [(X, \prec_X)]$. If α is the least element in S we are done, so suppose not. For each $\beta < \alpha$ in S, let $x_\beta \in X$ be such that $I_{x_\beta}^{\prec_X} \cong \beta$. Let $A = \{x_\beta \colon \beta < \alpha \wedge \beta \in S\}$. Let x_{β_0} be the $\prec_X$ least element of A (so $\beta_0 < \alpha$ and $\beta_0 \in S$). Then β_0 is the least ordinal in S. For if $\gamma \in S$, then either $\gamma \geq \alpha$ in which case the result is clear, or $\gamma < \alpha$ in which case $\gamma \cong I_{x_\gamma}$ for $x_\gamma \in A$. But then $x_\beta = x_\gamma$ or $x_\beta \prec_x x_\gamma$. Thus either $\beta = \gamma$ or $\beta < \gamma$. □

Definition 1.10. An ordinal α is a successor ordinal if $\{\beta \colon \beta < \alpha\}$ has a largest element. Otherwise α is called a limit ordinal.

For α a successor ordinal, we call the largest $\beta < \alpha$ the predecessor of α.

If $\alpha = [(X, \prec_X)] \in ON$, it is not hard to see from theorem 1.6 that there is a least ordinal larger than α. It fact, it is not hard to see directly what this ordinal is. Namely, let $y \notin X$, and let $X' = X \cup \{y\}$. Extend $\prec_X$ to X' be defining $x \prec_{X'} y$ for all $x \in X$. Then any proper initial $I_z^{\prec_{X'}}$ of X' is either a proper initial segment of X, or else $z = y$ and is all of X. In other words, if $\beta = [(X', \prec_{X'})]$ and $\gamma < \beta$, then $\gamma \leq \alpha$. We usually write $\alpha + 1$ for the successor of α just constructed (an explanation for this notation appears below). Thus, the successor ordinals are precisely those of the form $\alpha + 1$ for some $\alpha \in ON$.

The first few ordinals are denoted $0, 1, 2, 3, \dots$ and are calles the finite ordinals. That is, they are $0 = [(\emptyset, \emptyset)]$, $0 + 1$, $2 = 1 + 1$, etc. The first infinite ordinal is denoted ω (see more below), and the next ordinals after that are $\omega + 1, \omega + 2$, etc.

Although it will not play a role for us, one can extend addition and multiplication on the integers ω to all of the ordinals.

Definition 1.11. Let $\alpha = [(X, \prec_X)]$, $\beta = [(Y, \prec_Y)]$ be ordinals. Then $\alpha + \beta$ is defined to be the ordinal represented by the well-ordering $\prec_X \oplus \prec_Y$ (defined earlier).

Also, $\alpha \cdot \beta$ is defined to be the ordinal represented by the well-ordering $\prec_X \otimes \prec_Y$.

Thus, $\alpha + \beta$ consists of a copy of α followed by a copy of β. $\alpha \cdot \beta$ consists of β copies of α. Thus $\omega + \omega$ and $\omega \cdot 2$ both consist of two copies of ω and are thus isomorphic. That is, as ordinals, $\omega + \omega = \omega \cdot 2$. More generally, the following fact is easy to verify (and left to the reder).

Fact 1.12. For all ordinals α, β, γ we have: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$, $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$, $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$.

In general, however, neither $+$ nor $\cdot$ is commutative, and multiplication is not left distributive over addition. For example, $2 + \omega = \omega$, but $\omega + 2 > \omega$. Likewise, $2 \cdot \omega = \omega$, but $\omega \cdot 2 = \omega + \omega > \omega$. Also, $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega \neq \omega + \omega$.

Note though that the notation $\alpha + 1$ for the successor of α is consistent with with the definition of ordinal addition.

2. Countable Ordinals

An ordinal $\alpha = [(X, \prec_X)]$ is said to be countable if X is countable (this is clearly well-defined). In this case, we may w.l.o.g take $X = \omega$. The following is a simple but basic fact about the countable ordinals. The proof, though, does use AC (the result can fail without AC).

Theorem 2.1. *Let $\alpha_0, \alpha_1, \dots$ be a countable set of countable ordinals α_i. Then there is a countable ordinal β such that $\beta > \alpha_i$ for all i.*

Proof. Say (by AC) $\alpha_i = [(\omega, \prec_i)]$. Let $\prec$ be the ordering of $\omega \times \omega$ defined by: $(n_1, m_1) \prec (n_2, m_2)$ iff $(m_1 < m_2) \vee (m_1 = m_2 \wedge n_1 \prec_{m_1} n_2)$. Then $\prec$ is easily a well-ordering of the countable set $\omega \times \omega$, and thus defines a countable ordinal $\beta \doteq [(\omega \times \omega, \prec)]$. Clearly there is an order preserving map from $\prec_i$ into $\prec$ for any i. By the exercise, $\beta \geq \alpha_i$. $\square$

If S is a set of ordinals, let $\sup(S)$ denote the least ordinal α such that $\alpha \geq \beta$ for all $\beta \in S$ (this always exists by theorem 1.6, and the argument of the previous theorem; using some of the axioms of set-theory). The above theorem then says that if S is a countable set of countable ordinals, then $\sup(S)$ is also countable.

Another important fact about the countable ordinals is that they all have "countable cofinality". The concept of cofinality can be defined for a general ordinal, but we content ourselves with the following definition.

Definition 2.2. A limit ordinal α is said to have cofinality ω (written $\mathrm{cof}(\alpha) = \omega$) if there is a map $f \colon \omega \to \alpha$ which is increasing (i.e., order-preserving) and unbounded (i.e., $\forall \beta < \alpha \; \exists n \in \omega \; f(n) \geq \beta$).

We then have:

Theorem 2.3. *Every countable limit ordinal has cofinality ω.*

Proof. Let $\alpha = [(\omega, \prec)]$. Define a sequence of integers $n_0, n_1, \dots$ as follows. Let $n_0 = 0$. Given n_i, let n_{i+1} be the least integer such that I_{n_i} and I_{i+1} are proper initial segments of $I_{n_{i+1}}$. This exists since α is a limit ordinal. Let $\alpha_i = [(I_{n_i}, \prec)]$ be the corresponding ordinal. Thus, $\alpha_0 < \alpha_1 < \dots$ and $\alpha_i < \alpha$ for all i. To finish, we must show that $\sup\{\alpha_i\} = \alpha$. Let $\beta < \alpha$, say $\beta = [(I_m, \prec)]$. Then I_m is a proper initial segment of I_{n_m}, so $\alpha_{n_m} > \beta$. □

3. More On Ordinals

As we mentioned previously, our discussion of ordinals has taken place at a somewhat informal level. To be made literally correct even, the definition needs to be patched up. there is a way of doing this which not only fixes the minor set-theoretic problems, but also results in a much more elegant and usefull definition. The idea of this definition, due to Von Neumann, is to define directly a canonical representative for each equivalence class of well-orderings, so the need to consider equivalence classes disappears. The idea for obtaining a canonical representative is to let the ϵ (set element) relation be the relation $\prec$, that is, only consider orderings of the form (X, ϵ). Also, the ordinals should be canonical representatives in the sense that if $x \in X$, then the initial segment I_x of the ordinal $(X, \prec_X)$ should also be an ordinal, rather than just isomorphic to it. This leads to the following definition.

Definition 3.1. A set X is transitive if $\forall x \in X \; \forall y \in x \; (y \in X)$.

We may now state the official Von Neumann definition of ordinal.

Definition 3.2. An ordinal α is a transitive set which is well-ordered by the ϵ (set element) relation (resticted to $\alpha \times \alpha$).

Note that if $\alpha \in ON$ and $\beta \in \alpha$, then $\beta \in ON$ as well. To see this, first note that (β, ϵ) is well-founded (a subset of a well-ordered set is also well-ordered by the restriction of the well-ordering to that set). We are using here the fact that $\beta \subseteq \alpha$, which follows since α is transitive. We must also show that β is transitive. Suppose $\delta \in \gamma \in \beta$. Since α is transitive, $\delta, \gamma \in \alpha$. Since $\in$ is a transitive relation on α (part of the definition of a linear ordering), we also have $\delta \in \beta$, and we are done.

The fact that representatives are now unique is embodied in the following lemma.

Lemma 3.1. *If $\alpha, \beta \in ON$ and $\alpha \cong \beta$, then $\alpha = \beta$.*

Proof. Let $\pi\colon \alpha \to \beta$ be an isomorphism. It suffices to show that π is the identity map. Suppose not, and let $\alpha_0 \in \alpha$ be least such that $\pi(\alpha_0) \neq \alpha_0$. Thus, $\forall \gamma \in \alpha_0$ $(\pi(\gamma) = \gamma)$. Now, since π is an order-isomporphism from $(\alpha, \in)$ to $(\beta, \in)$, we have $\pi(\alpha_0) = \pi(\{\gamma\colon \gamma \in \alpha_0\}) = \{\pi(\gamma)\colon \gamma \in \alpha_0\}$. But, $\alpha_0 = \{\gamma\colon \gamma \in \alpha_0\}$ (trivially), so $\pi(\alpha_0) = \alpha_0$, a contradiction. $\square$