Introduction to Ordinals

1. Orderings

We give here a quick presentation of the definitions and basic facts about the ordinals. We start with an "informal" presentation, and then shift to the formal (Von Neumann) definition.

Our presentation will also be slightly informal in that it takes place outside of formal ZFC set theory, but this will suffice.

Definition 1.1. A (strict) linear order $<$ is a binary relation (i.e., $<$ is a set of ordered pairs) on a set X (i.e., $< \subseteq X \times X$) satisfying:

(1) (irreflexive) $\forall x \in X\ \neg(x < x)$.
(2) (connected) $\forall x, y \in X\ (x < y \vee x = y \vee y < x)$
(3) (transitivity) $\forall x, y, z \in X\ ((x < y \wedge y < z) \rightarrow x < z)$

Note that in a linear order exactly one of $x < y$, $x = y$, or $y < x$ holds for any $x, y \in X$. We write $x > y$ to denote $y < x$ as usual. We also write $x \leq y$ to abbreviate $(x < y \vee x = y)$, and likewise for $x \geq y$. We call X the domain of the linear order. Note that X is determined by the linear order $<$, but we nevertheless often write $(X, <)$ or $(X, <_X)$ to emphasize X is the domain of $<$. We also sometimes just write $<_X$ to denote a linear order on X.

Examples of linear orderings include $(\mathbb{N}, \prec)$, $(\mathbb{R}, \prec)$, $(\mathbb{Q}, \prec)$, $(\mathbb{Z}, \prec)$, where $\prec$ in all cases is the ordering induced by the usual ordering on $\mathbb{R}$.

The notion of linear ordering can also be introduced through its non-strict version, which we next give.

Definition 1.2. A (non-strict) linear ordering $\preceq$ on a set X is a binary relation on X satisfying:

(1) (reflexive) $\forall x\ x \preceq x$
(2) (connected) $\forall x, y \in X\ (x \preceq y \vee y \preceq x)$
(3) (transitive) $\forall x, y, z,\ (x \preceq y \wedge y \preceq z \rightarrow x \preceq z)$
(4) $\forall x, y \in X\ ((x \preceq y \wedge y \preceq x) \rightarrow x = y)$

The strict and non-strict versions of the definition of linear order are essentially equivalent by the following exercise.

Exercise 1. Show that if $<$ is a strict linear order then the relation $\preceq$ defined by $x \preceq y$ iff $(x < y \vee x = y)$ is a non-strict linear order. Show that if $\preceq$ is a non-strict linear order then $<$ defined by $x < y$ iff $x \preceq y \wedge \neg(y \preceq x)$ is a strict linear order.

In view of Exercise 1 we can take either Definition 1.1 or Definition 1.2 as the definition of linear order, and use the notations $<$, $\leq$ interchangeably.

The notion of an ordinal is based off of the definition of a wellorder, which is a strengthening of the definition of a linear order. We pause first to give some generalizations of the notion of linear order, which we don't need for the definition of ordinals but are important concepts nonetheless.

Definition 1.3. A *prelinear order* $\preceq$ is a binary relation on a set X satisfying 1–3 in Definition 1.2.

A *partial order* $\preceq$ is a binary relation on a set X satisfying 1 (reflexive) and 3 (transitive) of Definition 1.2.

Thus, a prelinear order is a connected partial order (i.e., also satisfies $\forall x, y\ (x \preceq y \vee y \preceq x)$.

Remark 1.4. We note that sometimes people require a partial order to also satisfy 4 of Definition 1.2, and call our definition a *prepartial order* or *quasi-order*. On the other hand, people using our definition of partial order sometimes refer to a partial order which also satisfies 4 of Definition 1.2 as being a *strict partial order*. In all cases, the main point is that the notion of linear order or prelinear order has the connectedness axiom, while partial order does not.

The notions of linear order and prelinear order have a simple relation according to the following exercise.

Exercise 2. Suppose $\preceq$ is a prelinear order. Define $x \cong y$ iff $(x \preceq y \wedge y \preceq x)$. Show that $\cong$ is an equivalence relation X. Define $<$ on the set of $\cong$-equivalence classes by: $[x] < [y]$ iff $((x \preceq y) \wedge \neg(y \preceq x))$. Show that $<$ is a linear order. Conversely, given an equivalence relation $\cong$ on a set X, and given a linear order $<$ of the set of equivalence classes, if we define $x \preceq y$ iff $(([x] < [y]) \vee ([x] = [y]))$, then $\prec$ is a prelinear order on X. Thus, a prelinear order is just a linear order of equivalence classes.

Given a partial order $\preceq$, we define $x \prec y$ iff $x \preceq y$ and $\neg(y \preceq x)$. Then $\prec$ is irreflexive (i.e., $\forall x\ \neg(x \prec x)$) and transitive. Conversely, if $\prec$ is irreflexive and transitive, and $\preceq$ is defined by $x \preceq y$ iff $x \prec y$ or $x = y$, then $\preceq$ is a partial-order. Also, if $\preceq$ is a strict partial-order and $\prec$ is obtained from $\preceq$ and $\preceq'$ is obtained from $\prec$ as above, then $\preceq = \preceq'$. Thus, it makes no difference whether we consider $\preceq$ or $\prec$.

We now turn the the definition of a wellordering, which is capturing the essence of being an ordinal.

Definition 1.5. A wellordering $(X, \prec_X)$ is a linear ordering such that for every $S \subseteq X$ with $S \neq \emptyset$ we have that S has a $\prec$ least element. That is, $\exists x \in S\ \forall y \in S\ \neg(y \prec x)$.

The axiom of choice is particularly relevant when discussing wellorderings as it is equivalent to the following, which we take as our officiial definition.

Definition 1.6. (AC) The axiom of choice AC is the statement that every set X can be wellordered. That is, there is a wellordering $<$ with domain X.

Remark 1.7. In can be shown (working in ZF set theory) that AC is equivalent to the following "choice principle": For every non-empty relation R (i.e., R is a set of ordered pairs), there is a function $F \subseteq R$ with $\mathrm{dom}(F) = \mathrm{dom}(R) = \{x \colon \exists y\ \langle x, y\rangle \in R\}$. Thus, for all $x \in \mathrm{dom}(R)$, F "chooses" the element $F(x)$ from the section $R_x = \{y \colon \langle x, y\rangle \in R\}$. This can be reworded as: for every non-empty set $\mathcal{I}$, and for every function f with $\mathrm{dom}(f) = \mathcal{I}$ such that $f(x) \neq \emptyset$ for all $x \in \mathcal{I}$, there is a function (a "choice function") g with $\mathrm{dom}(g) = \mathrm{dom}(f) = \mathcal{I}$ such that $g(x) \in f(x)$ for all $x \in \mathcal{I}$ (take R to be the set of all $\langle x, y\rangle$ such that $x \in \mathcal{I}$ and $y \in f(x)$).

With their usual orderings, $\mathbb{N}$ is wellordered, but $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not.

More generally, if R is a binary relation on a set X, we say R is *wellfounded* if for every $S \subseteq X$ with $S \neq \emptyset$, $\exists x \in S\ \forall y \in S\ \neg(yRx)$. Thus, a wellordering is just a linear ordering which is wellfounded.

When talking about wellfounded relations, a weak form of the axiom of choice is relevant, which is called the axiom of *dependent choice*.

Definition 1.8. (DC) The axiom of dependent choice DC is the statement that if X is a set and R is a non-empty binary relation on $X^{<\omega}$ such that $\forall n\, \forall x_1, \dots, x_n \in X\ (R(\langle x_1, \dots, x_n\rangle \to \exists y \in X\ R(\langle x_1, \dots, x_n, y\rangle)))$, then $\exists \vec{x} \in X^\omega$ such that $\forall n\ R(\vec{x} \restriction n)$.

Fact 1.9. (DC) A relation R on a set X is wellfounded iff there does not exist an infinite decreasing chain, i.e., elements $x_0, x_1, \dots$ of X with $x_{n+1} R x_n$ for all n.

Proof. If R is wellfounded, and $x_0, x_1, \dots$ is a sequence form X, let x_n be an R-minimal element of $S \doteq \{x_0, x_1, \dots\}$. Then $\neg x_{n+1} R x_n$. So, there are no infinite R-decreasing chains. (This direction did not use any form of AC).

Suppose now (X, R) has no infinite decreasing chains, and let $S \subseteq X$, $S \neq \emptyset$. Suppose S has no R-minimal element. For each finite decreasing chain

$$x_n R x_{n-1} R \dots R x_2 R x_1 R x_0$$

of elements from S, "pick" using DC an $x_{n+1} \in S$ such that $x_{n+1} R x_n$. This exists by the assumption on S. This defines an infinite decreasing chain $x_0, x_1, \dots$.

More formally, Let A be the relation $A(\vec{x}, y)$ iff $\vec{x} = \langle x_1, \dots, x_n\rangle$ where $x_{i+1} R x_i$ for all $1 \leq i \leq n-1$, $x_i \in S$ for all i, $y \in S$, and $y R x_n$. Applying DC to the relation A gives an infinite R-decreasing chain of elements from S, which contradicts R having no increasing chains. ☐

Remark 1.10. The axiom of *countable choice* is the statement that if f is a function with domain $\mathbb{N}$ with $f(n) \neq \emptyset$ for all $n \in \mathbb{N}$, then there is a function g with $\mathrm{dom}(g) = \mathrm{dom}(f) = \mathbb{N}$ with $g(n) \in f(n)$ for all $n \in \mathbb{N}$. DC clearly implies countable choice.

Though we are mainly concerned with wellorderings, we can make the following definitions for linear orderings.

Definition 1.11. If $(X, \prec_X)$ is a linear ordering and $x \in X$, let $I_x^{\prec_X}$ (or just I_x if $\prec_X$ is understood) denote the initial segment determined by x. That is, $I_x = \{y \in X \colon y \prec x\}$. An initial segment of X means a set $I \subseteq X$ such that $\forall x, x' \in X\ (x \in I \wedge x' \prec x \to x' \in I)$. An initial segment I of X is said to be proper if $I \neq X$.

Definition 1.12. Let $(X, \prec_X)$, $(Y, \prec_Y)$ be linear orderings. We say a map $\pi \colon A \to Y$, where $A \subseteq X$, is order-preserving if for all $x_1, x_2 \in A$, $x_1 \prec_X x_2 \leftrightarrow \pi(x_1) \prec_Y \pi(x_2)$. We say π is an order-isomorphism from X to Y if π is an order-preserving bijection.

We frequently just say $\prec_X$, $\prec_Y$ are isomorphic, written $\prec_X \cong \prec_Y$, to abbreviate "order-isomorphic".

Exercise 3. Show that if $(X, \prec)$ is a well-ordering and I is a proper initial segment of X, then $\exists x \in X\ I = I_x$.

Exercise 4. Suppose $(X, \prec_X)$, $(Y, \prec_Y)$ are well-orderings. Let $\prec_X \oplus \prec_Y$ be the ordering with domain $(X \times \{0\}) \cup (Y \times \{1\})$ and ordered by: $(z_1, i_1) \prec (z_2, i_2)$ iff $(i_1 < i_2) \vee (i_1 = i_2 = 0 \wedge z_1 \prec_X z_2) \vee (i_1 = i_2 = 1 \wedge z_1 \prec_Y z_2)$. Show that $\prec_X \oplus \prec_Y$ is also a well-ordering.

Exercise 5. Suppose $(X, \prec_X)$, $(Y, \prec_Y)$ are well-orderings. Let $\prec_X \otimes \prec_Y$ be the ordering with domain $X \times Y$ and ordered by: $(x_1, y_1) \prec (x_2, y_2)$ iff $(y_1 \prec_Y y_2) \vee (y_1 = y_2 \wedge x_1 \prec_X x_2)$. Show that $\prec_X \otimes \prec_Y$ is also a well-ordering.

We develop some of the basic facts about well-orderings.

Lemma 1.13. *If the well-orderings $(X, \prec_X)$, $(Y, \prec_Y)$ are order-isomorphic, then the order-isomorphism between them is unique.*

Proof. Suppose $f, g \colon X \to Y$ are both order-isomorphisms. We show that $f = g$. If not, let $x_0 \in X$ be the $\prec_X$ least x such that $f(x) \neq g(x)$. Without loss of generality, Suppose $f(x_0) \prec_Y g(x_0)$. Let x_1 be such that $g(x_1) = f(x_0)$. Clearly $x_1 \neq x_0$. If $x_1 \prec_X x_0$, then by minimality of x_0, $g(x_1) = f(x_1) \prec_X f(x_0)$, a contradiction. Thus, $x_0 \prec_X x_1$. However, we then have $g(x_1) = f(x_0) \prec g(x_0)$, which contradicts g being order-preserving. □

Lemma 1.14. *If $(X, \prec_X)$ is a well-ordering, then X is not order-isomorphic to any proper initial segment of itself.*

Proof. Suppose $\pi \colon I \to X$ is an order-isomorphism between the proper initial segment I of X and all of X. We cannot have $\pi(x) = x$ for all $x \in I$, as then π would not be onto. Let x_0 be the least element of I such that $\pi(x) \neq x$. We can't have $\pi(x_0) \prec x_0$ since then $\pi(\pi(x_0)) = \pi(x_0)$, and thus π is not one-to-one. So, $x_0 \prec \pi(x_0)$.

Let $x_1 \in I$ be such that $\pi(x_1) = x_0$. Clearly $x_1 \neq x_0$ (since $\pi(x_0) \neq x_0$). If $x_1 \prec x_0$, then $\pi(x_1) = x_1 \prec x_0$, which is impossible. If $x_0 \prec x_1$, then $\pi(x_1) = x_0 \prec \pi(x_0)$, which contradicts π being order-preserving. □

Exercise 6. Show that if π is an isomorphism from $\prec_X$ to $\prec_Y$ and $x \in X$, then $\pi \restriction I_x^{\prec_X}$ is an isomorphism between $I_x^{\prec_X}$ and $I_y^{\prec_Y}$.

Exercise 7. Show that there are two countable linear orders, neither of which order-embeds into the other. Thus, there are "incomparable" linear orders. Can you find three countable linear orders, any two of which are incomparable?

Theorem 1.15. *Let $(X, \prec_X)$, $(Y, \prec_Y)$ be well-orderings. Then exactly one of the following holds.*

(1) *$(X, \prec_X)$ is isomorphic to a proper initial segment of $(Y, \prec_Y)$.*
(2) *$(Y, \prec_Y)$ is isomorphic to a proper initial segment of $(X, \prec_X)$.*
(3) *$(X, \prec_X)$ is isomorphic to $(Y, \prec_Y)$.*

Proof. For $x \in X, y \in Y$, define $R(x, y)$ iff $I_x^{\prec_X} \cong I_y^{\prec_Y}$.

First note that for all $x \in X$ and $y_1, y_2 \in Y$, if $R(x, y_1)$ and $R(x, y_2)$, then $y_1 = y_2$. If not, say w.l.o.g $y_1 \prec_Y y_2$. But then, $I_x^{\prec_X} \cong I_{y_1}^{\prec_Y} \cong I_{y_2}^{\prec_Y}$. This violates lemma 1.14.

Thus, R is a partial function. Likewise, R is one-to-one, since if $R(x_1, y), R(x_2, y)$ but (w.l.o.g.) $x_1 \prec_X x_2$, then $I_{x_1}^{\prec_X} \cong I_y^{\prec_Y} \cong I_{x_2}^{\prec_X}$.

We next claim that $\mathrm{dom}(R)$ is an initial segment of $\prec_X$. Suppose $x_2 \in \mathrm{dom}(R)$ and $x_1 \prec_X x_2$. Say $R(x_2, y_2)$, that is, $I_{x_2}^{\prec_X} \cong I_{y_2}^{\prec_Y}$. Let π be an isomorphism from $I_{x_2}^{\prec_X}$ to $I_{y_2}^{\prec_Y}$. By the exercise above, $I_{x_1}^{\prec_X} \cong I_{y_1}^{\prec_Y}$, where $y_1 = \pi(x_1)$. Thus, $R(x_1, y_1)$, and so $x_1 \in \mathrm{dom}(R)$. We have also shown that R is order-preserving from $\prec_X$ to $\prec_Y$.

An exactly similar argument shows likewise that ran(R) is an initial segment of $\prec_Y$.

We have shown so far that R is an isomorphism from an initial segment of $\prec_X$, say I, to an initial segment of $\prec_Y$, say J.

We now consider cases.

If $I = X$ but $J \neq Y$, Then case (1) of the theorem holds. If $I \neq X$ but $J = Y$, then case (2) of the theorem holds. If $I = X$ and $J = Y$, then clearly case (3) of the theorem holds. Suppose finally that $I \neq X$ and $J \neq Y$. We show that this case does not occur. By an exercise, let $I = I_x^{\prec_x}$ and $J = I_y^{\prec_Y}$. Since R is an isomorphism between I and J, by definition we have $R(x, y)$. Thus $x \in \mathrm{dom}(R)$, so $x \in I_x$, a contradiction.

We have now shown that one of the three cases of the theorem holds. Uniqueness of the case follows immediately from lemma 1.13. □

We now state our (slightly informal) definition of ordinal.

Definition 1.16. An ordinal α is an equivalence class of a wellordering $(X, \prec_X)$ under order-isomorphism. Thus, $\alpha = [(X, \prec_X)]$.

Remark 1.17. The informality in the above definition lies in some set theoretic subtleties. Namely, the "equivalence classes" as defined above are actually too large to be sets, they are proper classes. Thus, from a formal set theoretic point of view, the definition doesn't make sense (this is actually a minor problem that plagues many common definitions in mathematics). However, the problem is easy to correct if one does the formal development of set theory and moreover, we will give a better definition not affected by this problem shortly.

We frequently use lower case Greek letters like α, β, γ for ordinals. ON denotes the (proper class) of all ordinals. Suppose α, β are ordinals. Say $\alpha = [(X, \prec_X)]$ and $\beta = [(X, \prec_X)]$. We say $\alpha < \beta$ iff $(X, \prec_X) \cong (Y, \prec_Y)$. This is clearly well-defined.

Theorem 1.15 may then be restated as saying for any two ordinals α, β, exactly one of the following holds: $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$. Note that if $\alpha = [(X, \prec_X)]$ and $\beta < \alpha$, then we may represent β as $\beta = [(I_x, \prec_X)]$ for some proper initial segment I_x of $\prec_X$.

Exercise 8. Let $\alpha, \beta \in$ ON. Suppose that $\forall \gamma < \alpha \; \exists \delta < \beta \; \gamma \leq \delta$. Show that $\alpha \leq \beta$ (hint: use theorem 1.15 and lemma 1.14).

Exercise 9. Let $\alpha, \beta \in$ ON. Suppose there is an order-preserving map π from α to β. Show that $\alpha \leq \beta$.

2. Ordinals

We now give the formal definition of ordinal, due to von Neumann. The previous informal definition (aside from the minor set-theoretic problem alluded to) is acceptable but awkward due to the continuing need to take representatives of equivalence classes. The definition we now give avoids this by giving a canonical representative for each class.

Definition 2.1. A set X is transitive if $\forall x \in X \; \forall y \in x \; (y \in X)$.

We may now state the official Von Neumann definition of ordinal.

Definition 2.2. An ordinal α is a transitive set which is well-ordered by the ϵ (set element) relation (resticted to $\alpha \times \alpha$).

We frequently use lower case Greek letters like α, β, γ for ordinals. ON denotes the (proper class) of all ordinals.

Lemma 2.3. *If $\alpha \in ON$ and $\beta \in \alpha$, then $\beta \in ON$ and $\beta = I^\alpha_\beta$.*

Proof. If $\gamma \in \beta$, then $\gamma \in \alpha$ by transitivity, and thus $\beta = I^\alpha_\beta$. This also shows $(\beta, \in)$ is a well-ordering, as it is an initial segment of a well-ordering. To see β is transitive, suppose $\delta \in \gamma \in \beta$. Since α is transitive, $\delta, \gamma \in \alpha$. Since $\in$ is a transitive relation on α (part of the definition of a linear ordering), we also have $\delta \in \beta$, and we are done. □

The fact that representatives are now unique is embodied in the following lemma.

Lemma 2.4. *If $\alpha, \beta \in ON$ and $\alpha \cong \beta$, then $\alpha = \beta$.*

Proof. Let $\pi \colon \alpha \to \beta$ be an isomorphism. It suffices to show that π is the identity map. Suppose not, and let $\alpha_0 \in \alpha$ be least such that $\pi(\alpha_0) \neq \alpha_0$. Thus, $\forall \gamma \in \alpha_0\ (\pi(\gamma) = \gamma)$. Now, since π is an order-isomporphism from $(\alpha, \in)$ to $(\beta, \in)$, we have $\{\pi(\gamma) \colon \gamma \in \alpha_0\} = \{\delta \in \pi(\alpha_0)\}$. Thus, $\alpha_0 = \{\gamma \colon \gamma \in \alpha_0\} = \{\pi(\gamma) \colon \gamma \in \alpha_0\} = \{\delta \in \pi(\alpha_0)\} = \pi(\alpha_0)$, a contradiction. □

Lemma 2.5. *If α, $\beta \in ON$ then exactly one of the following holds: $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.*

Proof. Uniqueness follows from theorem 1.15. If $(\alpha, \in) \cong (\beta, \in)$, then by lemma 2.4 $\alpha = \beta$. Suppose α is isomorphic to a proper initial segment of β. Let $\beta_0 \in \beta$ be such that $(\alpha, \in) \cong (\beta_0, \in)$ (which is the initial segment determined by β_0 in $(\beta, \in)$). Then since β_0 is an ordinal, $\alpha = \beta_0 \in \beta$. The other case is similar. □

The following theorem gives a fundamental property of ordinals. It says, in effect, that the collection of all ordinals behaves like an ordinal.

Theorem 2.6. *The collection of ordinals with the $\in$ relation satisfies the axioms for a linear order. That is, $\forall \alpha \in ON\ \alpha \notin \alpha$, $\forall \alpha, \beta \in ON\ (\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$, $\forall \alpha, \beta, \gamma \in ON\ (\alpha \in \beta \wedge \beta \in \gamma \to \alpha \in \gamma)$. Furthermore, it behaves like a well-odering in that if $S \subseteq ON$ is a non-empty set of ordinals, then there is an $\alpha \in S$ which is minimal, i.e., $\forall \beta \in S\ (\beta = \alpha \vee \alpha \in \beta)$.*

Proof. For any $\alpha \in$ ON, $\alpha \notin \alpha$ as otherwise α would be isomorphic (in fact equal) to a proper initial segment of itself. We have already shown connectedness and transitivity is immediate from the definition of ordinal.

Let $S \subseteq$ ON be non-empty, and let $\alpha \in S$. Let $S' = \{\beta \in \alpha \colon \beta \in S\}$. If S' is empty, we are done. Otherwise, let α_0 be a minimal element of S', which exists as $(\alpha, \in)$ is a well-ordering. Then for all $\gamma \in S$, either $\gamma \in \alpha$ in which case $\alpha_0 = \gamma$ or $\alpha_0 \in \gamma$ by minimality, or $\gamma = \alpha$ in which case $\alpha_0 \in \gamma$, or $\alpha \in \gamma$ in which case $\alpha_0 \in \gamma$ by tansitivity. Thus, α_0 is minimal. □

We now show that every well-ordering is isomorphic to a unique ordinal, which justifies our definition.

Theorem 2.7. *Every well-ordering $(X, \prec)$ is order-isomorphic to a unique ordinal.*

Proof. Uniqueness follows from theorem 1.15 and lemma 2.5. Let $Y = \{x \in X \colon \exists \alpha \in \text{ON}(I_x \cong \alpha)\}$. Note that Y is an initial segment of X, since an initial segment of an ordinal is an ordinal. For $x \in Y$, let $f(x)$ be the unique ordinal such that $I_x \cong f(x)$. We claim that $Y = X$. If not, let $Y = I_{x_0}$, where $x_0 \in X$. Let $A = \{f(x) \colon x \in Y\} \subseteq \mathrm{ON}$. First, f is an order-isomorphism between Y and A. For if $x_1 \prec x_2$ then we must have $f(x_1) \in f(x_2)$ as the other possibilities ($f(x_1) = f(x_2)$ or $f(x_2) \in f(x_1)$) lead to an isomorphism of I_{x_2} with an initial segment of I_{x_1}. Next, note that A is transitive. For suppose $\alpha = f(x)$, where $x \in Y$, and $\beta \in \alpha$. Since $I_x \cong \alpha$, there is a $z \prec x$ such that $I_z \cong \beta$. Thus $\beta = f(z) \in A$. Finally, A is well-ordered by $\in$, as $A \subseteq \mathrm{ON}$. Thus, $A \in \mathrm{ON}$. So, $I_{x_0} \cong A \in \mathrm{ON}$, a contradiction. So, $Y = X$, and thus f is an order-isomorphism between X and an ordinal, and we are done. □

We use the notation $\alpha < \beta$ for ordinals to mean $\alpha \in \beta$, which is reasonable in view of theorem 2.6. Thus, every ordinal is the set of smaller ordinals, that is, $\alpha = \{\beta \in \mathrm{ON} \colon \beta < \alpha\}$.

The first few ordinals are described as follows. The least ordinal is $\emptyset$, which we also denote by 0. The next ordinal is $\{0\} = \{\emptyset\}$, which we also denote by 1. The next ordinal is $\{0, 1\} = \{\emptyset, \{\emptyset\}\}$, which we also denote by 2, etc. The least infinite ordinal is $\{0, 1, 2, 3, \dots\}$, which we denote by ω (as a set, it is thus the set of natural numbers). The next ordinal is $\{0, 1, \dots, \omega\}$ which we denote by $\omega + 1$, etc.

Definition 2.8. An ordinal α is a successor ordinal if $\{\beta \colon \beta < \alpha\}$ has a largest element. Otherwise α is called a limit ordinal.

For α a successor ordinal, we call the largest $\beta < \alpha$ the predecessor of α.

If $\alpha \in \mathrm{ON}$, it is not hard to see from theorem 1.15 that there is a least ordinal larger than α. It fact, it is not hard to see directly what this ordinal is. Namely, let $S(\alpha) = \alpha \cup \{\alpha\}$. Easily $S(\alpha)$ is an ordinal, and if $\beta \in S(\alpha)$, then either $\beta \in \alpha$ or $\beta = \alpha$. So, $S(\alpha)$ is the least ordinal greater than α. We usually write $\alpha + 1$ for the successor of α just constructed (the reason for this notation is given below). Thus, the successor ordinals are precisely those of the form $\alpha + 1$ for some $\alpha \in \mathrm{ON}$.

One can extend addition and multiplication on the integers ω to all of the ordinals.

Definition 2.9. Let α, β be ordinals. Then $\alpha + \beta$ is defined to be the ordinal represented by the well-ordering $(\alpha, \in) \oplus (\beta, \in)$ (defined earlier).

Also, $\alpha \cdot \beta$ is defined to be the ordinal represented by the well-ordering $(\alpha, \in) \otimes (\beta, \in)$.

Thus, $\alpha + \beta$ consists of a copy of α followed by a copy of β. $\alpha \cdot \beta$ consists of β copies of α. Thus $\omega + \omega$ and $\omega \cdot 2$ both consist of two copies of ω and are thus isomorphic. That is, as ordinals, $\omega + \omega = \omega \cdot 2$. More generally, the following fact is easy to verify (and left to the reader).

Fact 2.10. For all ordinals α, β, γ we have: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$, $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$, $\gamma \cdot (\alpha + \beta) = \gamma \cdot \alpha + \gamma \cdot \beta$.

In general, however, neither $+$ nor $\cdot$ is commutative, and multiplication is not right distributive over addition. For example, $2 + \omega = \omega$, but $\omega + 2 > \omega$. Likewise, $2 \cdot \omega = \omega$, but $\omega \cdot 2 = \omega + \omega > \omega$. Also, $(1 + 1) \cdot \omega = 2 \cdot \omega = \omega \neq \omega + \omega$.

Note though that the notation $\alpha + 1$ for the successor of α is consistent with with the definition of ordinal addition.

Exercise 10. Exercise. Show that every ordinal α can be written uniquely in the form $\alpha = \beta + n$ where β is a limit ordinal and $n \in \omega$.

Exercise 11. Identify the first ω many ordinals which are additively closed (we say α is additively closed if $\forall \beta < \alpha\ (\beta + \beta < \alpha)$.

Exercise 12. Show that if $\beta \leq \alpha < \beta + \gamma$, then there is a $\gamma' < \gamma$ such that $\alpha = \beta + \gamma'$.

Is S is a set of ordinals, let $\cup S$ denote the union of S, that is, $x \in \cup S \leftrightarrow \exists \alpha \in S\ x \in \alpha$. Then clearly $\cup S$ is a set of ordinals, and so is well-founded. Also, S is easily transitive and hence is an ordinal.

Exercise 13. Show $\cup S$ is the least ordinal greater than or equal to all of the ordinals in S.

We often write $\sup(S)$ in place of $\cup S$.

3. Countable Ordinals

An ordinal α is said to be countable if α is countable as a set. The following is a simple but basic fact about the countable ordinals. The proof, though, does use AC (the result can fail without AC).

Theorem 3.1. *Let $\alpha_0, \alpha_1, \dots$ be a countable set of countable ordinals α_i. Then there is a countable ordinal β such that $\beta > \alpha_i$ for all i.*

Proof. It suffices to show that $\cup\{\alpha_0, \alpha_1, \dots\}$ is countable. This follows from the fact that (using AC) a countable union of countable sets is countable. □

Another important fact about the countable ordinals is that they all have "countable cofinality". The concept of cofinality can be defined for a general ordinal, which we give later.

Definition 3.2. A limit ordinal α is said to have cofinality ω (written $\mathrm{cof}(\alpha) = \omega$) if there is a map $f \colon \omega \to \alpha$ which is increasing (i.e., order-preserving) and unbounded (i.e., $\forall \beta < \alpha\ \exists n \in \omega\ f(n) \geq \beta$).

We then have:

Theorem 3.3. *Every countable limit ordinal has cofinality ω.*

Proof. Let $\pi \colon \omega \to \alpha$ be a bijection. Define $f(n) = \sup\{\pi(0), \dots, \pi(n)\} + n$. Then f is strictly increasing and cofinal in α. □

4. Ordinal Exponentiation

The exponentiation operation on the integers can also be extended to the ordinals. Our definition here is slightly informal, since we take for granted that a definition by transfinite recursion is legitimate (will show that later). An equivalent definition can be given without using recursion, but the recursive definition is perhaps the most natural. One should be careful not to confuse ordinal exponentiation with cardinal exponentiation (which, unfortunately, uses the same notation).

For α, β ordinals, we define the ordinal α^β be recusion on β as follows:

Definition 4.1. $\alpha^0 = 1$. If $\beta = \gamma + 1$, then $\alpha^\beta = \alpha^\gamma \cdot \alpha$. If β is a limit, $\alpha^\beta = \sup_{\gamma < \beta} \alpha^\gamma$.

Lemma 4.2. *For all ordinals α, β, γ: $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$, $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.*

Proof. We show by induction on γ that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$. For $\gamma = 0$ it is immediate. Suppose the equation holds for all $\delta < \gamma$. If $\gamma = \delta + 1$, then

$$\alpha^{\beta+\gamma} = \alpha^{\beta+(\delta+1)} = \alpha^{(\beta+\delta)+1} = \alpha^{\beta+\delta}\cdot\alpha = (\alpha^\beta\cdot\alpha^\delta)\cdot\alpha = \alpha^\beta\cdot(\alpha^\delta\cdot\alpha) = \alpha^\beta\cdot\alpha^{\delta+1} = \alpha^\beta\cdot\alpha^\gamma$$

□

Exercise 14. Show the second part of the lemma.

Exercise 15. Show that an ordinal is additively closed iff it is of the form ω^β for some $\beta \in \mathrm{ON}$.

Exercise 16. Identify the least ordinal closed under ordinal exponentiation.

Theorem 4.3. *(Cantor Normal Form) Every ordinal α can be written uniquely in the form*

$$\alpha = \omega^{\beta_0} \cdot k_0 + \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$$

where $\beta_0 > \beta_1 \cdots < \beta_n$ and $k_0, \ldots, k_n \in \omega$.

Proof. First note that for every $\alpha \in \mathrm{ON}$ there is a largest ordinal β such that $\omega^\beta \leq \alpha$. For let γ be the least ordinal such that $\omega^\gamma > \alpha$. Then γ is a successor, as otherwise $\omega^\gamma = \sup_{\delta<\gamma} \omega^\delta \leq \sup_{\delta<\gamma} \alpha = \alpha$. So, $\gamma = \beta + 1$ for some β, and β is as desired. Let us call β the companion to α.

We prove the existence of the normal form by induction on the size of the companion β_0 to α. By definition $\omega^{\beta_0} \leq \alpha$. If equality holds we are done, so assume $\omega^{\beta_0} < \alpha$. Since $\alpha < \omega^{\beta_0+1} = \omega^{\beta_0} \cdot \omega = \sup_k \omega^{\beta_0} \cdot k$, let k_0 be the largest integer such that $\omega^{\beta_0} \cdot k_0 \leq \alpha$. Then $\alpha = \omega^{\beta_0} \cdot k_0 + \alpha'$, where $\alpha' < \omega^{\beta_0}$. Thus, α' has a smaller comapnion than α, and by induction α' has a normal form. If $\alpha' = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$, then substituting we are done. □

Exercise 17. Show that if $\alpha_0, \ldots, \alpha_n$ are countable ordinals, then any expression built up from them using ordinal sums, products, and exponentiation is also countable.

Exercise 18. Show that $\omega^\alpha + \omega^\beta$ equals ω^β if $\alpha < \beta$.

Exercise 19. We say γ is a *mingling* of α and β if γ can be written as the disjoint union of A and B where A is order-isomorphic to α and B is order-isomorphic to β. Find a mingling γ of two ordinals α, β such that $\gamma > \max\{\alpha \cdot 2, \beta \cdot 2\}$.

Exercise 20. Show that if γ is a mingling of α and β that $\gamma < \max\{\alpha \cdot 3, \beta \cdot 3\}$. (Hint: use Cantor nornal form).