Propositional Logic
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We first consider the *syntax* of propositional logic. A *language* $\mathcal{L}$ of propositional logic consists of a set $\mathrm{Var} = \{A_i : i \in \mathcal{I}\}$ of sentence variables, along with the connective symbols $\neg$ and $\to$ and left and right parentheses (,). Here $\mathcal{I}$ is an arbitrary index set (not necessarily countable).

We next define the set WFF of well-formed formulas, or wffs. We also just call these the "formulas" of propositional logic.

Definition 1. We define the set WFF_n for $n \in \mathbb{N}$, by recursion on n as follows. $\mathrm{WFF}_0 = \{A_i : i \in \mathcal{I}\}$ is the set of sentence variables. For $n > 0$ we define

$$\mathrm{WFF}_n = \bigcup_{m<n} \mathrm{WFF}_m \cup \{(\neg\alpha) : \alpha \in \bigcup_{m<n} \mathrm{WFF}_m\} \cup \{(\alpha \to \beta) : \alpha, \beta \in \bigcup_{m<n} \mathrm{WFF}_m\}$$

We set $\mathrm{WFF} = \bigcup_n \mathrm{WFF}_n$.

We have officially only allowed the logical connectives $\neg$ and $\to$ in our language. It is convenient to use other connectives as well, so we make the following notational conventions.

Convention 2. For any two wffs α, β, $(\alpha \vee \beta)$ abbreviates the wff $((\neg\alpha) \to \beta)$. Also, $(\alpha \wedge \beta)$ abbreviates the wff $(\neg(\alpha \to (\neg\beta)))$. Finally, $(\alpha \leftrightarrow \beta)$ abbreviates $((\alpha \to \beta) \wedge (\beta \to \alpha))$.

We next state and prove a unique readabilty result. We require the following lemma.

Lemma 3. *For any string α of logical symbols, let $\ell(\alpha)$ be the number of left parentheses in α, and $r(\alpha)$ the number of right parentheses in α.*

(1) *For any wff φ, $\ell(\varphi) = r(\varphi)$.*
(2) *For any wff φ and any non-empty proper initial segment φ' of φ we have $\ell(\varphi') > r(\varphi')$.*

Proof. We prove both parts of the lemma simultaneously by induction on φ.

Ground case: $\varphi = A_i$ is a sentence variable. (1) follows since $\ell(A_i) = 0 = r(A_i)$. Part (2) of the lemma is trivial as there are no non-empty proper initial segments of φ in this case.

Inductive step: This breaks into two cases as in the inductive definition of WFF_n.

<u>case 1</u>: $\varphi = (\neg\alpha)$ for some wff α. We may assume inductively that the lemma holds for the wff α. So, $\ell(\alpha) = r(\alpha)$. Then $\ell(\varphi) = 1 + \ell(\alpha) = 1 + r(\alpha) = r(\varphi)$. This shows part (1) for φ.

To show part (2) for φ, consider a proper initial segment φ' of φ. If $\varphi' = ($, then $\ell(\varphi') = 1$ and $r(\varphi') = 0$ so (2) holds. If $\varphi' = (\neg$, we also have $\ell(\varphi') = 1$, $r(\varphi') = 0$. If $\varphi' = (\neg\alpha'$, where α' is a proper initial segment of α, then by induction $\ell(\alpha') > r(\alpha')$. Then $\ell(\varphi') = 1 + \ell(\alpha') > 1 + r(\alpha') > r(\alpha') = r(\varphi')$. If $\varphi' = (\neg\alpha$, then $\ell(\varphi') = 1 + \ell(\alpha) = 1 + r(\alpha) > r(\alpha) = r(\varphi')$.

<u>case 2</u>: $\varphi = (\alpha \to \beta)$ for some wffs α, β. We may assume inductively that the lemma holds for α and β. For part (1), we have $\ell(\varphi) = 1 + \ell(\alpha) + \ell(\beta) = 1 + r(\alpha) + r(\beta) = r(\varphi)$.

Part (2) is again done by considering the possibilities for a proper initial segment φ' of φ. In all cases we have $\ell(\varphi') > r(\varphi')$. □

Exercise 1. Give the details of the inductive proof of part (2) of the lemma in the case $\varphi = (\alpha \to \beta)$.

We next show we have unique readability in the syntax.

Theorem 4 (Unique Readability for Propositional Logic)**.** *For every $\varphi \in$ WFF, there is a unique way to write φ as either (1) $\varphi = A_i$, (2) $\varphi = (\neg\alpha)$, or (3) $\varphi = (\alpha \Rightarrow \beta)$ where $\alpha, \beta \in$ WFF.*

Proof. Every $\varphi \in$ WFF can be written in one these forms by definition, we must show the uniqueness part. If $\varphi = A_i$, then clearly φ cannot be written as in cases (2), (3), as in those cases the wff begins with the symbol (. Also, the variable symbol A_i is determined uniquely by φ (since $\varphi = A_i$).

Suppose next $\varphi = (\neg\alpha)$ for some $\alpha \in$ WFF. We must show that φ is not of the form (1) or (2), and that if $\varphi = (\neg\alpha')$ then $\alpha = \alpha'$. We have already shown that φ cannot be of case (1), so suppose $\varphi = (\beta \to \gamma)$ for $\beta, \gamma \in$ WFF. Thus,

$$(\neg\alpha) = (\beta \to \gamma).$$

Removing the first symbol gives that

$$\neg\alpha = \beta \to \gamma$$

Thus β begins with the symbol $\neg$. However, an easy induction shows that every wff cannot begin with the symbol $\neg$. Assume next that $\varphi = (\neg\alpha')$. Thus, $(\neg\alpha) = (\neg\alpha')$. Removing the first and least symbol from each string gives $\neg\alpha = \neg\alpha'$, and then removing the first symbil $\neg$gives that $\alpha = \alpha'$.

Suppose now that $\varphi = (\alpha \to \beta)$. We have already shown that φ cannot be of cases (1) or (2). Supppose that

$$(\alpha \to \beta) = (\alpha' \to \beta').$$

Removing the first and last symbols from each string gives

$$\alpha \to \beta = \alpha' \to \beta''$$

Now, α is the smallest initial segment s of the left-hand side for which $\ell(s) = r(s)$ (i.e., is balanced) by Lemma 3. But, the smallest initial segment of the right-hand side which is balanced is α'. Thus, $\alpha = \alpha'$. Removing $\alpha \to$ from the beginning of both sides, and) from the end of both sides gives $\beta = \beta'$.

□

From Theorem 4, when we prove facts about wff by induction, we are always unambiguously in one of the three cases, with the subformulas α (and β in case (3)) uniquely determined.

We next consider the *semantic* notions of a truth value assignment and satisfaction, and then we introduce the *syntactic* notion of a proof or deduction. These two notions will be related by the completeness theorem for propositional logic.

Definition 5. A *truth value assignment* (or tva) is a map $\nu\colon \mathrm{Var} \to \{0, 1\}$.

When considering truth value assignments, we often write F for 0 and T for 1. The intended meaning is "false" for 0 (or F) and "true" for 1 (or T).

We next give the inductive definition of "satisfaction" of a wff by a tva.

Definition 6. Let $\nu\colon \mathrm{Var} \to \{0, 1\}$ be a tva. We extend ν to ν' on WFF as follows.

(1) If $\varphi = A_i$, then $\nu'(A_i) = \nu(A_i)$.

(2) If $\varphi = (\neg\alpha)$, then $\nu'(\varphi) = 1 - \nu'(\alpha)$.
(3) If $\varphi = (\alpha \to \beta)$ then $\nu'(\varphi) = 1$ iff $\nu'(\alpha) = 0$ or $\nu'(\beta) = 1$.

We will henceforth just write ν instead of ν', which should cause no confusion as $\nu' \restriction \mathrm{Var} = \nu \restriction \mathrm{Var}$.

We say a tva ν *satisfies* $\varphi \in \mathrm{WFF}$ if $\nu(\varphi) = \mathrm{T}$. If $\Gamma \subseteq \mathrm{WFF}$, we say ν satisfies Γ if $\nu(\varphi) = \mathrm{T}$ for all $\varphi \in \Gamma$. We also just write, with a slight abuse of notation, $\nu(\Gamma) = \mathrm{T}$.

Exercise 2. Show that if ν_1 and ν_2 are truth value assignments, $\varphi \in \mathrm{WFF}$, and ν_1, ν_2 agree on the variables in φ, then $\nu_1(\varphi) = \nu_2(\varphi)$.

Definition 7. A formula $\varphi \in \mathrm{WFF}$ is a *tautology* if $\nu(\varphi) = \mathrm{T}$ for all tvas $\nu\colon \mathrm{Var} \to \{\mathrm{T}, \mathrm{F}\}$.

Example 8. Some tautologies are:

- $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$
- $\alpha \to (\beta \to \alpha)$
- $\neg(\alpha \to \beta) \to \alpha$

Note that we drop extra parentheses from the notation when causes no confusion to improve readability (they are officially still there, however). We also introduce another notational convention:

Definition 9. We let $(\alpha \vee \beta)$ abbreviate $((\neg\alpha) \to \beta)$ for any $\alpha, \beta \in \mathrm{WFF}$. Likewise, we let $(\alpha \wedge \beta)$ abbreviate $(\neg(\alpha \to (\neg\beta)))$.

The next definition is our semantic notion of implication, which we denote by the symbol $\models$.

Definition 10 (Logical Implication for Propositional Logic)**.** Let $\Gamma \subseteq \mathrm{WFF}$ for some language $\mathcal{L}$ for propositional logic. Let $\varphi \in \mathrm{WFF}$. Then $\Gamma \models \varphi$ if for every tva ν such that $\nu(\Gamma) = \mathrm{T}$ we have $\nu(\varphi) = \mathrm{T}$.

Note that φ is a tautology iff $\emptyset \models \varphi$

Exercise 3. Show that $\Gamma \models A$ where Γ is the following set of formulas:

(1) $(A \vee B) \to (C \vee \neg(A \to D))$
(2) $(\neg A \wedge (C \vee D)) \to (\neg C \wedge B)$
(3) $(B \vee C) \to (D \vee \neg A)$
(4) $((\neg A \wedge \neg B) \vee (C \wedge D)) \to ((\neg B \to C) \wedge (D \to \neg A))$

You can either show this using the definition of $\models$, or you can give a "proof" that from Γ you can deduce A.

We next develop the *sytactical* notion of a "proof" or *deduction*. We again have $\Gamma \subseteq \mathrm{WFF}$, $\varphi \in \mathrm{WFF}$, and we define the notion $\Gamma \vdash \varphi$ to say that there is a deduction of φ from Γ.

Definition 11 (Logical Axioms for Propositional Logic)**.** The *logical axioms* Λ of propositional logic are the wffs of the following form:

(A1) $\alpha \to (\beta \to \alpha)$ for any $\alpha, \beta \in \mathrm{WFF}$.
(A2) $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$ for any $\alpha, \beta, \gamma \in \mathrm{WFF}$.
(A3) $((\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta))$ for any $\alpha, \beta \in \mathrm{WFF}$.

Exercise 4. Show that all the logical axioms are tautologies.

Exercise 5. In *reverse Polish* notation, the connectives are written first, and no parentheses are used. More precisely, the set of wffs if defined by: $\mathrm{WFF}_0 = \mathrm{Var}$, $\mathrm{WFF}_n = \bigcup_{m<n} \mathrm{WFF}_m \cup \{\neg\alpha \colon \alpha \in \bigcup_{m<n} \mathrm{WFF}_m\} \cup \{\to \alpha\beta \colon \alpha, \beta \in \bigcup_{m<n} \mathrm{WFF}_m\}$. Show that unique readability holds for the wffs defined this way. [hint: consider the function w which assigns to each variable $A \in \mathrm{Var}$ the value $w(A) = 1$, $w(\neg) = 0$, and $w(\to) = -1$].

Definition 12. Let $\Gamma \subseteq \mathrm{WFF}$ for some language $\mathcal{L}$ of propositional logic. Let $\varphi \in \mathrm{WFF}$. A *deduction* of φ from Γ is a finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_n$ of wffs such that $\alpha_n = \varphi$, and for each $i < n$ we have one of the following:

1. (Hypothesis) $\alpha_i \in \Gamma$
2. (Logical Axiom) $\alpha_i \in \Lambda$
3. (Modus Ponens) For some $j, k < i$ we have $\alpha_k = (\alpha_j \to \alpha_i)$

We say $\Gamma \vdash \varphi$ (Γ proves φ) if there is a deduction of φ from Γ.

Lemma 13. *For every $\alpha \in$ WFF, $\emptyset \vdash (\alpha \to \alpha)$.*

Proof. We have the following deduction:

$$\emptyset$$

$$\alpha \to ((\alpha \to \alpha) \to \alpha) \quad (A1)$$
$$(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha) \to (\alpha \to \alpha)) \quad (A2)$$
$$(\alpha \to (\alpha \to \alpha) \to (\alpha \to \alpha) \quad (MP)$$
$$\alpha \to (\alpha \to \alpha) \quad (A1)$$
$$\alpha \to \alpha \quad (MP)$$

□

It becomes too tedious to write down actual deductions in many cases. So we prove *metatheorems* which tell us that deductions exist in certain cases, without having to actually write them down. Metatheorems will be important in first-order logic as well.

Theorem 14 (Deduction Metatheorem)**.** *If $\Gamma \cup \{\varphi\} \vdash \alpha$, then $\Gamma \vdash (\varphi \to \alpha)$.*

Proof. By induction on the (shortest) length of a deduction of α from $\Gamma \cup \{\varphi\}$. Say $\alpha_0, \alpha_1, \ldots, \alpha_n = \alpha$ is a deduction of α from $\Gamma \cup \{\varphi\}$.

Case 1: $\alpha \in \Gamma$.

Then we have the following deduction of $(\varphi \to \alpha)$ from Γ:

$$\Gamma$$

$$\alpha \quad \text{(Hypothesis)}$$
$$\alpha \to (\varphi \to \alpha) \quad (A1)$$
$$\varphi \to \alpha \quad (MP)$$

Case 2: $\alpha = \varphi$

Then $\Gamma \vdash \varphi \to \varphi$ since $\emptyset \vdash \varphi \to \varphi$ by Lemma 13.

Case 3: $\alpha \in \Lambda$.
We have the following deduction of $\varphi \to \alpha$:

$$\Gamma$$

α (Logical Axiom)
$\alpha \to (\varphi \to \alpha)$ $(A1)$
$\varphi \to \alpha$ (MP)

Case 4: α obtained by Modus Ponens from β and $\beta \to \alpha$, which occur earlier in the deduction.

By induction we have $\Gamma \vdash \varphi \to \beta$ and $\Gamma \vdash \varphi \to (\beta \to \alpha)$. We have the following deduction of $\varphi \to \alpha$ from Γ:

$$\Gamma$$

$\varphi \to (\beta \to \alpha)$ (Induction)
$(\varphi \to (\beta \to \alpha)) \to ((\varphi \to \beta) \to (\varphi \to \alpha))$ $(A2)$
$(\varphi \to \beta) \to (\varphi \to \alpha)$ (MP)
$\varphi \to \beta$ (Hypothesis)
$\varphi \to \alpha$ (MP)

This completes the proof of the deduction metatheorem, Theorem 14. □

Theorem 15 (Contradiction Metatheorem)**.** *If* $\Gamma \cup \{\neg\varphi\} \vdash \alpha$ *and* $\Gamma \cup \{\neg\varphi\} \vdash \neg\alpha$*, then* $\Gamma \vdash \varphi$*.*

Proof. Since $\Gamma, \neg\varphi \vdash \alpha$, by the deduction metatheorem we have that $\Gamma \vdash (\neg\varphi) \to \alpha$. Likewise, $\Gamma \vdash (\neg\varphi \to \neg\alpha)$. But $(\neg\varphi \to \neg\alpha) \to ((\neg\varphi \to \alpha) \to \varphi)$ is a logical axiom in (A3). Applying MP twice we get that $\Gamma \vdash \varphi$. □

We will need the following two facts about double negation.

Lemma 16. $\emptyset \vdash (\neg(\neg\alpha)) \to \alpha$*.*

Proof. By the deduction metatheorem, it suffices to show that $\neg\neg\alpha \vdash \alpha$. By the contradiction metatheorem, it suffices to show that $\Gamma = \{\neg\neg\alpha, \neg\alpha\}$ is inconsistent, that is, proves β and $\neg\beta$ for some β. But $\Gamma \vdash \neg\alpha$ clearly, and also $\Gamma \vdash \neg\neg\alpha$. So Γ is inconsistent. □

Exercise 6. Give a deduction from $\emptyset$ of $\neg(\neg\alpha) \to \alpha$. [hint: use the proofs of the metatheorems.]

Lemma 17. $\emptyset \vdash (\alpha \to \neg\neg\alpha)$*.*

Proof. By the deduction metatheorem it suffices to show that $\alpha \vdash \neg\neg\alpha$. By the contradiction metatheorem it suffices to show that $\Gamma = \{\alpha, \neg\neg\neg\alpha\}$ is inconsistent. Clearly $\Gamma \vdash \alpha$, and also $\Gamma \vdash \neg\neg(\neg\alpha)$. From Lemma 16 we have that $\emptyset \vdash (\neg\neg(\neg\alpha) \to \neg\alpha$. By Modus Ponens, $\Gamma \vdash \neg\alpha$. Thus, Γ is inconsistent.

□

One direction of the completeness theorem is easier and called the *soundness theorem.*

Theorem 18 (Soundness Theorem for Propositional Logic)**.** *If* $\Gamma \vdash \varphi$*, then* $\Gamma \models \varphi$*.*

Proof. Suppose $\Gamma \vdash \varphi$. By induction on the length of a deduction of φ from Γ we show that $\Gamma \models \varphi$. For the first case, suppose $\varphi \in \Gamma$. In this case, $\Gamma \models \varphi$ trivially, since if $\nu(\Gamma) = \mathrm{T}$, then $\nu(\varphi) = \mathrm{T}$ as $\varphi \in \Gamma$. Next consider the case that $\varphi \in \Lambda$. then $\Gamma \models \varphi$ as all the logical axioms are tautologies. For the third case, suppose φ is obtained by MP from β and $\beta \to \varphi$ which have shorter length deductions from Γ. By induction, $\Gamma \models \beta$ and $\Gamma \models (\beta \to \varphi)$. But if ν is a tva with $\nu(\beta) = \mathrm{T}$ and $\nu(\beta \to \varphi) = \mathrm{T}$, then clearly $\nu(\varphi) = \mathrm{T}$. So, $\Gamma \models \varphi$. □

The other direction in the completeness theorem is often just called the completeness theorem.

Theorem 19 (Completeness Theorem for Propositional Logic)**.** *If* $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$*.*

Proof. Suppose $\Gamma \models \varphi$ where $\Gamma \subseteq \mathrm{WFF}_{\mathcal{L}}$ in some language $\mathcal{L}$ and $\varphi \in \mathrm{WFF}_{\mathcal{L}}$. To show $\Gamma \vdash \varphi$, it is enough, by the contradiction metatheorem, to show that $\Gamma \cup \{\neg\varphi\}$ is syntactically inconsistent (i.e., proves a contradition). Assume toward a contradiction that $\Gamma \cup \{\neg\varphi\}$ is syntactically consistent. We will define a tva ν which satisfies $\Gamma \cup \{\neg\varphi\}$, which will be a contradiction as $\Gamma \models \varphi$ and thus $\nu(\varphi) = \mathrm{T}$. More generally, we prove the following theorem. We assume the axiom of choice AC in the proof, although we actually only need a weaker form of it (we discuss this afterwards, c.f. Theorem 32).

Theorem 20. *Let* $\Gamma \subseteq WFF_{\mathcal{L}}$ *be a syntactically consistent set. Then there is a tva* $\nu\colon Var_{\mathcal{L}} \to \{T, F\}$ *which satisfies* Γ*.*

Proof. The idea is to enlarge Γ to a maximally (syntactically) consistent set of wffs. Let $\{\varphi_i\}_{i<\theta}$ be an enumeration of $\mathrm{WFF}_{\mathcal{L}}$ (we may need to use AC here to get this wellordering in the general case; the reader unfamiliar with this can consider the case where $\mathcal{L}$ is countable in which case we can take $\theta = \mathbb{N}$). Let $\Gamma_0 = \Gamma$. We define by induction on $i < \theta$ a set Γ_i. We will have that if $i < j$ then $\Gamma_i \subseteq \Gamma_j$. For $i > 0$ define

$$\Gamma_i = \begin{cases} \bigcup_{j<i} \Gamma_j \cup \{\varphi_i\} & \text{if } \bigcup_{j<i} \Gamma_j \cup \{\varphi_i\} \text{ is consistent} \\ \bigcup_{j<i} \Gamma_j & \text{otherwise} \end{cases}$$

Clearly the Γ_i are (monotonically) increasing.

Claim 21. For all $i < \theta$, Γ_i is consistent.

Proof. If not, let $i < \theta$ be least such that Γ_i is syntactically inconsistent. By assumtion, Γ_0 is consistent, so $i > 0$. From the definition of Γ_i we must have $\Gamma_i = \bigcup_{j<i} \Gamma_j$. By minimality if i we have that each Γ_j is consistent for all $j < i$. Suppose $\Gamma_i = \bigcup_{j<i} \Gamma_j \vdash \alpha, \neg\alpha$. Let $\beta_0, \dots, \beta_m$ be a deduction from Γ_i. The key point is that the deduction is a *finite* sequence. For each $k < m$, if $\beta_k \in \bigcup_{j<i} \Gamma_j$, let $j_k < i$ be such that $\beta_k \in \Gamma_{j_k}$. Let $j = \max\{j_0, \dots, j_k\}$, and so $j < i$. Note that $\vec{\beta}$ is still a deduction from Γ_j, as $\Gamma_{j_k} \subseteq \Gamma_j$ for all k. This contradicts the fact that Γ_j is consistent as $j < i$. □

The same argument in Claim 21 show that $\Gamma' = \bigcup_{i<\theta} \Gamma_i$ is consistent.

Claim 22. Γ' is a *maximally consistent* set of wffs, that is, if $\Gamma'' \supsetneq \Gamma'$, then Γ'' is inconsistent.

Proof. Suppose $\varphi \in \Gamma'' - \Gamma'$. Say $\varphi = \varphi_i$, where $i < \theta$. Since $\varphi \notin \Gamma'$, we have $\varphi \notin \Gamma_i$. From the definition of Γ_i we then have that $\bigcup_{j<i} \cup \{\varphi\}$ is inconsistent (as otherwise $\varphi \in \Gamma_i$). Thus, $\Gamma' \cup \{\varphi\}$ is also inconsistent. □

Claim 23. For evey $\varphi \in \mathrm{WFF}_{\mathcal{L}}$, either $\varphi \in \Gamma'$ or $(\neg\varphi) \in \Gamma'$.

Proof. Suppose $\varphi \notin \Gamma'$ and $\neg\varphi \notin \Gamma'$. By maximal consistency of Γ', this gives that $\Gamma' \cup \{\varphi\}$ and $\Gamma' \cup \{\neg\varphi\}$ are both inconsistent. Since $\Gamma' \cup \{\neg\varphi\}$ in inconsistent, the contradition metatheorem gives that $\Gamma' \vdash \varphi$. Since $\Gamma' \cup \{\varphi\}$ is inconsistent, it follows that Γ' is inconsistent. This contradicts the consistency of Γ'. □

So, for every wff $\varphi \in \mathrm{WFF}_{\mathcal{L}}$, exactly one of φ, $\neg\varphi$ is in Γ'.

Claim 24. If $\Gamma' \vdash \varphi$, then $\varphi \in \Gamma'$.

Proof. Suppose $\Gamma' \vdash \varphi$. If $\varphi \notin \Gamma'$, then by claim 23 we have that $\neg\varphi \in \Gamma'$. But then $\Gamma' \vdash \neg\varphi$, and so Γ' is inconsistent, a contradiction. □

We now define the tva ν. For $A \in \mathrm{Var}_{\mathcal{L}}$, define $\nu(A) = \mathrm{T}$ iff $A \in \Gamma'$. Note that $\nu(A) = \mathrm{F}$ iff $A \notin \Gamma'$ iff $(\neg A) \in \Gamma'$ by the previous line.

We show that ν satisfies Γ', and thus satisfies Γ. To show ν satisfies Γ' it suffices to show the following claim.

Claim 25. for every $\varphi \in \mathrm{WFF}_{\mathcal{L}}$, $\nu(\varphi) = T$ iff $\varphi \in \Gamma'$.

Proof. We prove the claim by induction on the wff φ. If $\varphi = A$ for some $A \in \mathrm{Var}_{\mathcal{L}}$, then the claim holds by definition of ν.

Suppose next that $\varphi = (\neg\alpha)$. Then $\nu(\varphi) = \mathrm{T}$ iff $\nu(\alpha) = \mathrm{F}$. By induction, $\nu(\alpha) = \mathrm{F}$ iff $\alpha \notin \Gamma'$. From Claim 23, $\alpha \notin \Gamma'$ iff $(\neg\alpha) = \varphi \in \Gamma'$.

Suppose next that $\varphi = (\alpha \to \beta)$. By induction the claim holds for α and β. We consider cases on $\nu(\alpha)$, $\nu(\beta)$. First assume $\nu(\alpha) = \mathrm{F}$. In this case $\nu(\varphi) = \mathrm{T}$. By induction, $\alpha \notin \Gamma'$, and so $\neg\alpha \in \Gamma'$. By Claim 24 it suffice to show that $\neg\alpha \vdash (\alpha \to \beta)$. By the deduction metatheorem, it suffices to show that $\{\alpha, \neg\alpha\} \vdash \beta$. This is immediate from the contradiction metatheorem, as $\{\alpha, \neg\alpha, \neg\beta\}$ is inconsistent.

Next assume $\nu(\alpha) = \mathrm{T}$ and $\nu(\beta) = \mathrm{T}$. In this case $\nu(\varphi) = \mathrm{T}$. By induction, $\alpha \in \Gamma'$ and $\beta \in \Gamma'$. But $\beta \vdash (\alpha \to \beta)$ since $\beta \to (\alpha \to \beta)$ is a logical axiom. So, $\Gamma' \vdash (\alpha \to \beta)$, and so by Claim 24 we have $\varphi = (\alpha \to \beta) \in \Gamma'$.

Finally, assume $\nu(\alpha) = \mathrm{T}$ and $\nu(\beta) = \mathrm{F}$. In this case $\nu(\varphi) = \mathrm{F}$. By induction, $\alpha \in \Gamma'$, and $\beta \notin \Gamma'$, and by Claim 23, $\neg\beta \in \Gamma'$. From Claim 24 it suffices to show that $\{\alpha, \neg\beta\} \vdash \neg(\alpha \to \beta)$, as then $\neg(\alpha \to \beta) \in \Gamma'$ and so $\varphi = (\alpha \to \beta) \notin \Gamma'$ by the consistency of Γ'. To see $\{\alpha, \neg\beta\} \vdash \neg(\alpha \to \beta)$, it suffices by the contradiction metatheorem to show that $\{\alpha, \neg\beta, \neg\neg(\alpha \to \beta)\}$ is inconsistent. From Lemma 16 it suffices to show that $\{\alpha, \neg\beta, \alpha \to \beta\}$ is inconsistent. This set of wffs easily proves β and $\neg\beta$, so is inconsistent.

□

This completes the proof of Theorem 20, and thus completes the proof of the completeness theorem for propositional logic. □

Exercise 7. The followinig exercise is a variant of "Einstein's Zebra Puzzle." It can be formulated formally in propositional logic, but we will not formally do so. There are five houses, each of which is a different color, has a person of different nationality living in it, they have different pets, etc. You are given the following:

(1) The Russian lives in the blue house.
(2) The German owns the dog.
(3) Water is drunk in the red house.
(4) The Norwegian drinks coffee.
(5) The red house is immediately to the right of the white house.
(6) The Ford driver owns a horse.
(7) The Dodge driver lives in a yellow house.
(8) Milk is drunk in the middle house.
(9) The Japanese person lives in the first house.
(10) The Volvo driver lives next to the person who owns the rabbit.
(11) The Dodge driver lives next to the fox owner.
(12) The Chevy driver drinks orange juice.
(13) The Englishman drives a Rolls Royce.
(14) The Japanese person lives next to the green house.

Question: Who owns the Zebra?

□

The following corollary of the completeness theorem is called the *compactness theorem* for propositional logic.

Corollary 26. Let $\Gamma \subseteq$ WFF. If every finite subset of Γ is satisfiable, then Γ is satisfiable.

Proof. Suppose every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable. If Γ is not satisfiable then $\Gamma \models (A \wedge \neg A)$ (trivially, as Γ is not satisfiable). By the completeness theorem, $\Gamma \vdash (A \wedge \neg A)$. Since proofs are finite, $\Gamma_0 \vdash (A \wedge \neg A)$ for some finite $\Gamma_0 \subseteq \Gamma$. By the soundness theorem, $\Gamma_0 \models (A \wedge \neg A)$, a contradtion as Γ_0 is satisfiable and $(A \wedge \neg A)$ is not. □

Remark 27. One can prove the compactness theorem directly, without referring to the notion of $\vdash$, by a proof similar to that of the completeness theorem. Starting with a Γ which is finitely satisfiable (every finite subset is satisfiable), one enlarges Γ to a Γ' which is a maximal finitely satisfiable set. From Γ' one reads off a tva ν which satisfies Γ' and hence Γ, as in the proof of the completeness theorem.

As an application of the completeness/compactness theorem, we prove a result about graph colorings. Recall a graph (G, E) is a set G with a binary relation E (the edge relation) which is irreflexive and symmetric. If $G_0 \subseteq G$, the induced subgraph on G_0 is $(G_0, E \cap G_0 \times G_0)$. A chromatic k-coloring of the graph is a map $c \colon G \to \{1, 2, \ldots, k\}$ such that if $x, y \in G$ and $(x, y) \in E$, then $c(x) \neq c(y)$.

Fact 28. Let (G, E) be a graph. If every finite induced subgraph $G_0 \subseteq G$ can be chromatically k-colored, then G can be chromatically k-colored.

Proof. Let $\mathcal{L} = \{A_{g,i} \colon g \in G, 1 \leq i \leq k\}$ be a language for propositional logic. Intuitively, we think of $A_{g,i}$ as being the assertion that $c(g) = i$. We let Γ be the union of following set of wffs, which attempt to assert that we have k-coloring of G.

(1) For each $g \in G$ we have the wff $A_{g,1} \vee A_{g,2} \vee \cdots \vee A_{g,k}$.
(2) For each $g \in G$, and each $1 \leq i, j \leq k$ with $i \neq j$, the wff $A_{g,i} \to \neg A_{g,j}$.
(3) For each $g, h \in G$ with $(x, y) \in E$, and each $1 \leq i \leq k$, the wff $A_{g,i} \to \neg A_{h,i}$.

Then Γ is finitely satisfiable as all induced subgraphs are k-colorable. Namely, a finite $\Gamma_0 \subseteq \Gamma$ only mentions finitely many $A_{g,j}$. Let G_0 be the $g \in G$ such that $A_{g,j}$ appears in Γ_0 for some j. The induced subgraph $(G_0, E \cap G_0 \times G_0)$ has a k-coloring c_0 which defines a tva ν_0 satisfying Γ_0. That is, let $\nu_0(A_{g,i}) = \mathrm{T}$ iff $c_0(g) = i$.

By the compactness theorem, Γ is satisfiable, say by the tva ν. This defines the map c by $c(g) = i$ iff $\nu(A_{g,i}) = \mathrm{T}$. From the definition of Γ we easily see that c is a chromatic k-coloring. $\square$

The next exercise asks you to use the compactness theorem for propositional logic to prove a result known as *König's lemma*. Bt a *tree* on a set X we mean a set $T \subseteq X^{<\omega}$ closed under initial segment, that is, if $s \in T$ then $s \restriction k \in T$ for all $k < |s|$ (the length of s). A tree T is *finitely splitting* if every $s \in T$ has only finitely many immediate successors in T (an immediate successor of $s \in T$ is a $t \in T$ with $|t| = |s| + 1$ and $s = t \restriction (|t| - 1)$.

Exercise 8. Show that if T is a finitely splitting tree on a set X, and T is infinite, then there is a *branch* b through T. This means $b \in X^\omega$ and $\forall n \in \mathbb{N}\ b \restriction n \in T$. [hint: Let $\mathcal{L}$ be the language with variables A_s for each $s \in T$. Think of A_s as asserting "s is in the branch b." Write a Γ such that a tva satisfying Γ gives a branch b through the tree T.]

The proof of Theorem 20 we gave used AC in the general case (uncountably many propositional variables). The amount of choice needed is actually not full AC, but a somewhat weaker principle. The completeness theorem for propositional logic is equivalent (in ZF, that is, not assuming choice) the *prime ideal theorem* which is equivalent to the statement that every filter on a set can be extended to an ultrafilter (the actual staement of the prime ideal theorrem is that every filter on any boolean algebra can be extended to an ultrafilter). This is also equivalent to Tychnoff's theorem for Hausdorff spaces.

The following exercise shows directly that the completeness theorem for propositional logic implies products of the two element space (with the discrete topology) are compact.

Exercise 9. Let $\mathcal{I}$ be an arbitrary index set. Let X_α, for $\alpha \in \mathcal{I}$, be the space $X_\alpha = \{0, 1\}$ with the discrete topology. Show that the completeness theorem of propositional logic implies that $X = \prod_{\alpha \in \mathcal{I}} X_\alpha$ is compact (without assuming any form of choice). [hint: Let $\mathcal{F}$ be a collection of closed subsets of X with the finite intersection property. Let $\mathcal{L}$ be the language with variables A_α for each $\alpha \in \mathcal{I}$. For each $F \in \mathcal{F}$, let B_F be the collection of all basic open sets contained in $X - F$, so $X - F = \cup B_F$. For each basic open set B, say $B = \pi_{\alpha_1}^{-1}(i_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(i_n)$, let φ_B be the wff $\neg(\psi_1(A_{\alpha_1}) \wedge \cdots \wedge \psi_n(A_{\alpha_n}))$ where $\psi(A_{\alpha_k}) = A_{\alpha_k}$ if $i_k = 1$ and $\psi(A_{\alpha_k}) = \neg A_{\alpha_k}$ if $i_k = 0$. Let $W = \{\varphi_B \colon \exists F \in \mathcal{F}\ (B \in B_F)\}$. Show that W is finitely satisfiable, and apply the compactness theorem.]

We show that the completeness theorem implies the (equivalent form of) the prime ideal theorem.

Fact 29. (ZF) Assume the completeness theorem for propositional logic. Then every filter $\mathcal{F}$ on a set X can be extended to an ultrafilter on X.

Proof. Let $\mathcal{F}$ be a filter on the set X. Let $\mathcal{L}$ be the language with variables A_Y for each $Y \in \mathcal{P}(X)$. Let W be the following collection of wffs:

(1) $\neg A_\emptyset$
(2) A_X
(3) $A_Y \to A_Z$ for all $Y \subseteq Z$ in $\mathcal{P}$
(4) $(A_Y \wedge A_Z) \to A_{Y \cap Z}$ for all $Y, Z \in \mathcal{P}$
(5) A_Y for all $Y \in \mathcal{F}$
(6) $A_Y \vee A_{X-Y}$ for all $Y \in \mathcal{P}$

W is finitely satisfiable. To see this, let $W_0 \subseteq W$ be finite. Then there are only finitely many $Y \in \mathcal{P}$ such that A_Y appears in a wff of W_0. Say $Y_1, \dots, Y_n$ are these elements of $\mathcal{F}$. Let B be the finite boolean algebra generated by $\{Y_1, \dots, Y_n\}$ (this has size at most 2^n). Let $A_0 = \cap\{Y_i \colon Y_i \in \mathcal{F}\}$. Note that $A_0 \neq \emptyset$ as $\mathcal{F}$ is a filter. Define $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n$ inductively by: $A_{k+1} = A_k \cap Y_{k+1}$ if this intersection is non-empty, and $A_{K+1} = A_k$ otherwise. Let $A = A_n$. We clearly have that $A \neq \emptyset$, and $A \subseteq Y_i$ if $Y_i \in \mathcal{F}$. This defines an ultrafilter $\mathcal{U}_0$ on B, namely, $Z \in \mathcal{U}_0$ iff $A \subseteq Z$. Also, for each Y_i we have that either $A \subseteq Y_i$ or $A \subseteq X - Y_i$. It follows that for every $Z \in B$ that either $A \subseteq B$ or $A \subseteq X - Z$, and so $\mathcal{U}_0$ is an ultrafilter on B.

Then $\mathcal{U}_0$ defines a tva ν_0 satisfying W_0, namely $\nu_0(A_Z) = 1$ iff $Z \in \mathcal{U}_0$ for all variables A_Z in W_0. Note that (6) is satisfied as $\mathcal{U}_0$ is an ultrafilter on B.

By the compactness theorem, there is a tva ν satisfying W. Let $\mathcal{U} = \{Y \subseteq X \colon \nu(A_Y) = \mathrm{T}\}$. (1)-(4) guarantees that $\mathcal{U}$ is a filter on X. Property (5) gives that $\mathcal{F} \subseteq \mathcal{U}$. Property (6) gives that $\mathcal{U}$ is an ultrafilter.

□

Remark 30. The proof of Fact 29 can easily be modified to show that any filter $\mathcal{F}$ on a boolean algebra B (not necessarily $\mathcal{P}(X)$) can be extended to an ultrafilter on B. Use variables A_b for each $b \in B$, and write down the analogs of (1)-(6).

Fact 31. (ZF) Assume every filter on a set can be extended to an ultrafilter. Then the completeness theorem for propositional logic holds.

Proof. It is enough to prove Theorem 20 under the current assumption. So, let Γ be a syntactically consistent set of wffs in a language $\mathcal{L} = \{A_\alpha\}_{\alpha \in \mathcal{I}}$. Let $\mathcal{F}$ be the collection of all $F \subseteq \mathrm{TVA}_{\mathcal{L}}$ such that for some finite $\Gamma_0 \subseteq \Gamma$ we have that $F \supseteq \{\nu \colon \nu(\Gamma_0) = \mathrm{T}\}$. We claim that $\mathcal{F}$ is a filter on $\mathrm{TVA}_{\mathcal{L}}$. Clearly if $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$ then $F_2 \in \mathcal{F}$. Also we clearly have that if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$. We need to show that $\mathcal{F}$ is non-trivial, that is, $\emptyset \notin \mathcal{F}$. This is equivalent to saying that if $\Gamma_0 \subseteq \Gamma$ is finite, then there is a tva ν which satisfies Γ. If $\Gamma_0 \subseteq \Gamma$ is finite, then Γ_0 is syntactically consistent since Γ is. Now, the proof of the completeness theorem for finite sets Γ_0 does not require any form of AC (we extend Γ_0 to a maximal Γ_0' in finitely many steps, as there are only finitely many wffs that use the varibales in Γ_0). Thus, $\mathcal{F}$ is a non-trivial filter on $\mathrm{TVA}_{\mathcal{L}}$. By assumption, we can extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ on $\mathrm{TVA}_{\mathcal{L}}$. For each sentence variable A_α, let $s(A_\alpha) = \{\nu \in \mathrm{TVA}_{\mathcal{L}} \colon \nu(A_\alpha) = \mathrm{T}\}$. More generally, for any wff φ, let $s(\varphi)$ be the set of tvas ν such that $\nu(\varphi) = \mathrm{T}$. Note that the complement of $s(A_\alpha)$ is the set of tvas that make A_α false, that is, $\mathrm{TVA}_{\mathcal{L}} - s(A_\alpha) = s(\neg A_\alpha)$. Since $\mathcal{U}$ is an ultrafilter, one (and exactly one) of the sets $s(A_\alpha)$, $s(\neg A_\alpha) \in \mathcal{U}$. So, we define the tva $\nu_{\mathcal{U}}$ by

$$\nu_{\mathcal{U}}(A_\alpha) = \begin{cases} \mathrm{T} & \text{if } s(A_\alpha) \in \mathcal{U} \\ \mathrm{F} & \text{if } s(\neg A_\alpha) \in \mathcal{U} \end{cases}$$

We show that $\nu_{\mathcal{U}}$ satisfies Γ. It suffices to show the claim that that for all wffs φ that $\nu_{\mathcal{U}}(\varphi) = \mathrm{T}$ iff $s(\varphi) \in \mathcal{U}$, since if $\varphi \in \Gamma$ then $s(\varphi) \in \mathcal{F} \subseteq \mathcal{U}$ by the definition of $\mathcal{F}$. We prove this claim by induction on φ.

If $\varphi = A_\alpha$, then by definition of $\nu_{\mathcal{U}}$, $\nu_{\mathcal{U}}(A_\alpha) = \mathrm{T}$ iff $s(A_\alpha) \in \mathcal{U}$.

If $\varphi = \neg\alpha$, then $\nu_{\mathcal{U}}(\varphi) = \mathrm{T}$ iff $\nu_{\mathcal{U}}(\alpha) = \mathrm{F}$. By induction, $\nu_{\mathcal{U}}(\alpha) = \mathrm{T}$ iff $s(\alpha) \in \mathcal{U}$. So, $\nu_{\mathcal{U}}(\alpha) = \mathrm{F}$ iff $\neg(s(\alpha) \in \mathcal{U})$ iff $\mathrm{TVA}_{\mathcal{L}} - s(\alpha) \in \mathcal{U}$ (as $\mathcal{U}$ is an ultrafilter). Since $s(\varphi) = s(\neg\alpha) = \mathrm{TVA}_{\mathcal{L}} - s(\alpha)$, this holds iff $s(\varphi) \in \mathcal{U}$.

If $\varphi = (\alpha \to \beta)$, then $\nu_{\mathcal{L}}(\varphi) = \mathrm{T}$ iff $\nu_{\mathcal{L}}(\alpha) = \mathrm{F}$ or $\nu_{\mathcal{L}}(\beta) = T$. By induction, this happens iff $\neg(s(\alpha) \in \mathcal{U})$ or $s(\beta) \in \mathcal{U}$. As $\mathcal{U}$ is an ultrafilter this happens iff $\mathrm{TVA}_{\mathcal{L}} - s(\alpha) \in \mathcal{U}$ or $s(\beta) \in \mathcal{U}$. This happens iff $(\mathrm{TVA}_{\mathcal{L}} - s(\alpha)) \cup s(\beta) \in \mathcal{U}$ (the union of two sets is in an ultrafilter iff at least one of the two sets is in). But $s(\varphi) = s(\alpha \to \beta) = (\mathrm{TVA}_{\mathcal{L}} - s(\alpha)) \cup s(\beta)$ by definition of the truth value of $\alpha \to \beta$. So, $\nu_{\mathcal{L}}(\varphi) = \mathrm{T}$ iff $s(\varphi) \in \mathcal{U}$. □

Putting these results together we get the following.

Theorem 32. *(*ZF*) The following are equivalent:*

1. *The completeness theorem for propositional logic.*
2. *The prime ideal theorem (every filter on a boolean algebra can be extended to an ultrafilter).*
3. *Every filter on a set can be extended to an ultrafilter.*
4. *Tychnoff's theorem for Hausforff spaces (a product of compact Hausdorff spaces is compact).*

Remark 33. The full Tychnoff theorem (an arbitrary product of compact spaces–not necessarily Hausdorff–is compact) is equivalent to AC, the full axiom of choice. This is known to be strictly stronger than the prime ideal theorem.

Proof. (1) implies (2) by Remark 30, and (2) implies (3) as $\mathcal{P}(X)$ is a boolean algebra for every set X. (3) imlies (1) by Fact 31. Thus, (1), (2), and (3) are equivalent.

(3) implies (4) by one of the usual proofs of Tychnoff's theorem. Namely, let X_α, $\alpha \in \mathcal{I}$, be compact Hausdorff spaces. Let $X = \prod_{\alpha \in \mathcal{I}} X_\alpha$ with the product topology. A topological space is compact iff every filter $\mathcal{F}$ on X has a cluster point (a point $x \in X$ such that every (open) neighborhood of x is $\mathcal{F}$-positive, that is, meets every element $F \in \mathcal{F}$). So, let $\mathcal{F}$ be a filter on X. By (3), let $\mathcal{U} \supseteq \mathcal{F}$ be an ultrafilter on X. For each $\alpha \in \mathcal{I}$, let $\mathcal{U}_\alpha = \pi_\alpha(\mathcal{U})$, where $\pi_\alpha \colon X \to X_\alpha$ is the projection map from X to X_α (projection on the αth coordinate). Since $\mathcal{U}$ is an ultrafilter, so is each $\mathcal{U}_\alpha$. Since each X_α is compact, each $\mathcal{U}_\alpha$ has a cluster point. Since $\mathcal{U}_\alpha$ is an ultrafilter, any cluster point y of $\mathcal{U}_\alpha$ is actually a limit point of $\mathcal{U}_\alpha$ (i.e., each neighborhood of y is in $\mathcal{U}_\alpha$). Since X_α is Hausdorff, limit points of filters are unique. Let $x_\alpha \in X_\alpha$ be the unique limit point of $\mathcal{U}_\alpha$ (the key point is that the limit point is unique, so we don't need AC to pick one out). Let $x \in X$ be the point defined by the x_α, that is, $\pi_\alpha(x) = x_\alpha$ for all $\alpha \in \mathcal{I}$. Then x is a limit point of $\mathcal{U}$ [If $B = \pi^{-1}_{\alpha_1}(V_{\alpha_1}) \cap \cdots \cap \pi^{-1}_{\alpha_n}(V_{\alpha_n})$ is a basic open set in X about x (so $x_{\alpha_i} \in V_{\alpha_i}$ for each i), then each $V_{\alpha_i} \in \mathcal{U}_{\alpha_i} = \pi_{\alpha_i}(\mathcal{U})$, that is, $\pi^{-1}_{\alpha_i}(V_{\alpha_i}) \in \mathcal{U}$ for each i, so we have $B \in \mathcal{U}$ as well.] As x is a limit point of $\mathcal{U}$ and $\mathcal{F} \subseteq \mathcal{U}$, it follows that x is a cluster point of $\mathcal{F}$.

Finally, we show (4) implies (1). Let $\mathcal{L}$ be a language of propositional logic with variables $\mathrm{Var}_{\mathcal{L}} = \{A_\alpha\}_{\alpha \in \mathcal{I}}$. It is enough to show, as in the proof of Theorem 19, that if $\Gamma \subseteq \mathrm{WFF}_{\mathcal{L}}$ is syntactically consistent, then there is a tva ν with $\nu(\Gamma) = \mathrm{T}$. For each $\alpha \in \mathcal{I}$, let $X_\alpha = \{0, 1\}$ with the discrete topology (of course, this is compact Hausdorff space). Let $X = \prod_{\alpha \in \mathcal{I}} X_\alpha$. By (4), X is compact (it is also Hausdorff, as any product of Hausdorff spaces is Hausdorff). For each finite $\Gamma_0 \subseteq \Gamma$, let $F_{\Gamma_0} \subseteq \mathrm{TVA}_{\mathcal{L}} = X$ be the set of all tvas ν such that $\nu(\Gamma_0) = \mathrm{T}$. since Γ_0 is finite and syntactically consistent, this set is non-empty by the finite version of the completeness theorem. Each F_{Γ_0} is a closed (in fact, clopen) subset of X, and $\mathcal{F} = \{F_{\Gamma_0} \colon \Gamma_0 \subseteq \Gamma \text{ is finite}\}$ has the finite intersection property. Since X is compact, $\cap \mathcal{F} \neq \emptyset$. If $\nu \in \cap \mathcal{F}$, then $\nu(\Gamma_0) = \mathrm{T}$ for all finite $\Gamma_0 \subseteq \Gamma$. In particular, $\nu(\varphi) = \mathrm{T}$ for all $\varphi \in \Gamma$, and so ν satisfies Γ. $\square$