Math 3410
Solutions To Second Review

1.) No, because $y(t) = t^3$ satisfies $y(0) = 0$ and $y'(0) = 0$ as does the trivial solution $u(0) = 0$. This would violate the uniqueness theorem.

2.) Say $y = c_1 y_1 + c_2 y_2$. We must have $\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The Wronskian matrix is equal to $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. This gives $c_2 = 1$ and $c_1 = 2$, so the solution is $y = 2t\cos(t) + e^{-t}$.

3.) The Wronskian is given up to a constant by $E = Ce^{-\int p(t)\,dt}$. Putting the equation in standard form we have that $p(t) = \frac{t^2}{t+1}$. So, $-\int p(t)\,dt = -\int (t - 1 + \frac{1}{t+1}\,dt = -\frac{t^2}{2} + t - \ln(t+1)$. So, $W = \frac{C}{t+1} e^{-\frac{t^2}{2}+t}$. At $t = 0$ the Wronskian is equal to -1 and so $C = -1$. Thus, $W = -\frac{1}{t+1} e^{-\frac{t^2}{2}+t}$.

4.) The general solution to the homogeneous is given by $y_H = c_1\cos(t) + c_2\sin(t)$. So, the general solution to the equation is $y(t) = c_1\cos(t) + c_2\sin(t) + t^3$. The initial conditions give $c_1 = -1$ and $c_2 = 1$. So the solution is $y(t) = -\cos(t) + \sin(t) + t^3$.

5.) a.) The characteristic equation is $r^2 + 4r + 3 = 0$, which factors as $(r+3)(r+1)$, so we have two real roots $r_1 = -1$ and $r_2 = -3$. The general solution is $t = c_1 e^{-t} + c_2 e^{-3t}$.

b.) The characteristic equation is $2r^2 + 2r + 1 = 0$ with roots $-\frac{1}{2} \pm i(\frac{1}{2})$. The general solution is $y(t) = c_1 e^{-t/2}\cos(\frac{t}{2}) + c_2 e^{-t/2}\sin(\frac{t}{2})$.

c.) The characteristic equation is $4r^2 + 12r + 9 = 0$ which has one real root at $r = -\frac{3}{2}$. So the general solution is $y(t) = c_1 e^{-\frac{3}{2}t} + c_2 t e^{-\frac{3}{2}t}$.

6.) We use reduction of order and seek another solution of the form $y = u(t)t$. Plugging this into the equation we get $t^2(u''t + 2u') - (2t + 2t^2)(u't + u) + (2 + 2t)(ut) = 0$ which gives $(t^3)u'' + (-2t^3)u' = 0$. This is separable (as it always is) and gives $\frac{u''}{u'} = 2$, so $\ln(u') = 2t$, so $u' = e^{2t}$. Integrating we have $u = (\frac{1}{2})e^{2t}$. Since we only care up to a constant, we can just take $u = e^{2t}$. So, $y = te^{2t}$.

7.) First we solve the homogeneous. The characteristic equation is $r^2 - 2r + 1 = (r-1)^2 = 0$ with one real root $r = 1$. So, the general solution to the homogeneous is $y_H = c_1 e^t + c_2 t e^t$. We use variation of parameters to find a particular solution to the given non-homogeneous equation. We seek a solution of the form $y_p = u_1(t)e^t + u_2(t)te^t$. The Wronskian matrix is $\begin{bmatrix} e^t & te^t \\ e^t & te^t + e^t \end{bmatrix}$. So we have

$$\begin{bmatrix} e^t & te^t \\ e^t & te^t + e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^{3/2}e^t \end{bmatrix}.$$

Using Cramer's rule we have $u_1' = \dfrac{\det \begin{bmatrix} 0 & te^t \\ t^{3/2}e^t & te^t + e^t \end{bmatrix}}{\det \begin{bmatrix} e^t & te^t \\ e^t & te^t + e^t \end{bmatrix}} = \frac{-t^{5/2}e^{2t}}{e^{2t}} = -t^{5/2}$. So, $u_1 = -\frac{2}{7}t^{7/2}$. Likewise we have $u_2' = \dfrac{\det \begin{bmatrix} e^t & 0 \\ e^t & t^{3/2}e^t \end{bmatrix}}{\det \begin{bmatrix} e^t & te^t \\ e^t & te^t + e^t \end{bmatrix}} = \frac{t^{3/2}e^{2t}}{e^{2t}} = t^{3/2}$. So, $u_2 = \frac{2}{5}t^{5/2}$. So our particular solution is $y_P = -\frac{2}{7}t^{7/2}e^t + \frac{2}{5}t^{7/2}e^t = \frac{4}{35}t^{7/2}e^t$. So, the general solution is $y = c_1e^t + c_2te^t + \frac{4}{35}t^{7/2}e^t$.

8.) The characteristic equation for the homogeneous equation is $r^2 + 2r + 2 = 0$ which has roots $t = -1 \pm i$. So, the homogeneous equation has the general solution $y_H = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t)$. Our trial solution is then $y_P = Ae^t\cos(t) + Be^t\sin(t) + Cte^{-t}\cos(t) + Dte^{-t}\sin(t)$. Note that we had to multiply by t for the part corresponding to the second term to avoid solutions to the homogeneous equation.

9.) We use the method of undetermined coefficients.

a.) The characteristic equation for the homogeneous is $r^2 + r = r(r+1) = 0$ which has roots $r = 0, -1$. So, the general solution to the homogeneous is $y(t) = c_1 + c_2e^{-t}$. The trial solution is $y_P = t(At^3 + Bt^2 + Ct + D)$. Substituting this in we get $12At^2 + 6Bt + 2C + 4At^3 + 3Bt^2 + 2Ct + D = t^3$. This gives $A = \frac{1}{4}$, $B = -1$, $C = -\frac{3}{4}$, $C = 3$, and $D = -6$. So, the particular solution is $Y_P = \frac{1}{4}t^4 - t^3 + 3t^2 - 6t$. The general solution to the given equation is $y(t) = c_1 + c_2e^{-t} + \frac{1}{4}t^4 - t^3 + 3t^2 - 6t$.

b.) The characteristic equation for the homogeneous is $r^2 + 4 = 0$ which has roots $r = \pm 2i$. So, the general solution to the homogeneous is $y(t) = c_1\cos(2t) + c_2\sin(2t)$. The trial solution is $y_P = t(At + B)\cos(2t) + t(Ct + D)\sin(2t)$. Plugging this in we get $(2A - 4At^2 - 4Bt + 8Ct + 4D + 4At^2 + 4Bt)\cos(2t) + (2C - 4Ct^2 - 4Dt - 8At - 4B + 4Ct^2 + 4Dt)(\sin(2t) = t\sin(2t) = (2A + 8Ct + 4D)\cos(2t) + (2C - 8At - 4B)\sin(2t) = t\sin(2t)$. This gives $C = 0$, $2A + 4D = 0$, $2C - 4B = 0$, $A = -\frac{1}{8}$. So, $B = 0$ and $D = \frac{1}{16}$. The general solution is $y(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{16}t\sin(2t) - \frac{1}{8}t^2\cos(2t)$.

10.) The object has a mass of $\frac{8}{32} = \frac{1}{4}$ slugs. The spring constant is given by $8 = k(\frac{1}{2})$, so $k = 16$. The equation of motion is $mu'' + ku = F$, or $(\frac{1}{4})u''(t) + 16u(t) = F$.

a.) The natural frequency is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{64} = 8$.

b.) With no driving force the equation of motion is $u''(t) + 64u(t) = 0$. The general solution to this is $u(t) = c_1 \cos(8t) + c_2 \sin(8t)$. The initial conditions give $c_1 = 0$, $c_2 = \frac{1}{8}$, so the solution is $u(t) = \frac{1}{8}\sin(2t)$.

c.)The equation of motion is now $u''(t)+64u(t) = \cos(5t)$. We use undetermined coefficients to get a particular solution. We try $y_P = A\cos(5t) + B\sin(5t)$. Substituting in we have $(-25A+64A)\cos(5t)+(-25G+64B)\sin(5t) = \cos(5t)$, so $A = \frac{1}{39}$, $B = 0$, so $y_P = \frac{1}{39}\cos(5t)$. The general solution is $y(t) = c_1 \cos(8t) + c_2 \sin(8t) + \frac{1}{39}\cos(5t)$. The initial conditions now give $c_1 = -\frac{1}{39}$, $c_2 = 0$, so the motion is given by $u(t) = -\frac{1}{39}\cos(8t) + \frac{1}{39}\cos(5t)$. This is equal to $-\frac{1}{39}(\cos(8t) - \cos(5t)) = -\frac{1}{39}(-2\sin(\frac{3}{2}t)\sin(\frac{13}{2}t)) = \frac{2}{39}\sin(\frac{3}{2}t)\sin(\frac{13}{2}t)$. So the amplitude is $\frac{2}{39}$.