Math 3410
Review For Second Test

The test will cover chapter 3 of the text. The chapter covers second order linear differential equations. Here is a summary of the main topics covered in this chaper.

Theory: The standard form for a second order linear equation is

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t).$$

In any interval I on which $p(t), q(t), g(t)$ are continuous, there will be a unique solution to this equation satisfying an initial condition of the form $y(t_0) = y_0$, $y'(t_0) = y_0'$ (where $t_0 \in I$). This is the existence and uniqueness theorem.

If the equation is homogeneous (i.e., $g(t) = 0$), then on any such interval there will be two fundamental solutions $y_1(t)$, $y_2(t)$ such that the general solution to the homogeneous equation on the interval I is of the form $y(t) = c_1y_1(t) + c_2y_2(t)$. The set of solutions on I forms a two-dimensional vector space, and y_1, y_2 are a basis (there are many possible choices for the basis).

In general, two solutions $y_1(t)$, $y_2(t)$ to the homogeneous equation on I are either dependent (i.e., one is a constant multiple of the other on I), or else for *every* $t \in I$ we have that the vectors $\begin{bmatrix} y_1(t) \\ y_1'(t) \end{bmatrix}$, $\begin{bmatrix} y_2(t) \\ y_2'(t) \end{bmatrix}$ are independent. In the latter case, y_1, y_2 form a basis for the set of solutions on I (i.e., every solution to the homogeneous equation is a linear combination of these two).

The Wronskian tells us which case we are in. It is defined by:

$$W(y_1, y_2) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}.$$

If $W(y_1, y_2) \neq 0$, then the solutions are independent, and if $W(y_1, y_2) = 0$, then they are dependent. Note that $W(y_1, y_2)$ is either always 0 or never 0 on I.

Abel's formula determines $W(y_1, y_2)$ up to a constant. The formula says:

$$W(y_1, y_2) = Ce^{-\int p(t)\, dt}.$$

Note that the equation must be in standard form to apply this formula.

For a non-homogeneous equation, the general solution is $y = y_H + y_P$, where y_H is the general solution to the homogeneous equation, and y_P is a particular solution to the non-homogeneous equation.

Constant Coefficient Homogeneous: For the equation $ay'' + by' + cy = 0$, the characteristic equation is $ar^2 + br + c = 0$.

- If the roots are distinct and real, $r_1 \neq r_2$, then the general solution to the homogeneous is $c_1e^{r_1t} + c_2e^{r_2t}$.
- If there is one repeated real root r, then the solution to the homogeneous is $c_1e^{rt} + c_2te^{rt}$.
- If the roots are complex, $r = \lambda \pm i\mu$, the the solution to the homogeneous is $c_1e^{\lambda t}\cos(\mu t) + c_2e^{\lambda t}\sin(\mu t)$.

Reduction of Order: Given one solution $y_1(t)$ to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$, we can find another by trying to find one of the form $u(t)y_1(t)$. This results in a first-order (and separable) equation for u'. You can also use the Wronskian to get the other solution.

Non-homogeneous: If we can solve the homogeneous equation, then we can solve the non-homogeneous equation $y'' + py' + qy = g$. There are two methods, undetermined coefficients and variation of parameters. The first only works in special cases but is easier. Undetermined coefficients works when g is of the form (poly) or (poly)e^{at}, or (poly)$e^{at}\cos(bt)$ (or with sin). The trial solution is of the same form (with a general polynomial of the same degree), which is then possibly multiplied by t or t^2 if necessary to guarantee than none of the terms of the trial solution is a solution to the homogeneous equation.

For variation of parameters, we try a particular solution of the form $y = u_1(t)y_1(t) + u_2(t)y_2(t)$, where y_1, y_2 are independent solutions to the homogeneous equation. This leads to the equation

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}.$$

This equation is best solved using Cramer's rule.

Mass Oscillators: They satisfy the equation $mu''(t) + \gamma u'(t) + ku(t) = F(t)$ where m is the mass, γ is the resistance term ($F_R = \gamma u'$), and k is the spring constant ($F_S = ku$). In the mks system mass is in kilograms, and in the English system it is in slugs (lbs. is the unit of weight, and is converted into mass by mass=(weight)/32). In this equation, $F(t)$ is the driving force. With no resistance or driving term the motion can be written as $A\cos(\omega_0 t - \delta)$ where A is the amplitude, ω_0 the natural frequency, and δ is the phase shift. To get these, we use

$$\begin{aligned} a\cos(\omega_0 t) + b\sin(\omega_0 t) &= \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(\omega_0 t) + \frac{b}{\sqrt{a^2+b^2}}\sin(\omega_0 t)\right] \\ &= A\cos(\omega_0 t - \delta) \end{aligned}$$

where $A = \sqrt{a^2+b^2}$, $\cos(\delta) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\delta) = \frac{b}{\sqrt{a^2+b^2}}$. Note that $\tan(\delta) = \frac{b}{a}$.

When there is a driving force $F = F_0\cos(\omega t)$ and the initial conditions are $u(0) = 0$, $u'(0) = 0$, then there will be a beat frequency of $\frac{|\omega - \omega_0|}{2}$. The solution in this case is of the form $u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin(\frac{\omega_0 - \omega}{2}t)\sin(\frac{\omega_0 + \omega}{2}t)$. The total amplitude $\frac{2F_0}{m|\omega_0^2 - \omega^2|}$ goes to ∞ as the driving frequency ω approaches the natural frequency ω_0.

Review Problems

1.) Can $y(t) = t^3$ be a solution to a linear homogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ on some interval I containing 0 for which $p(t)$ and $q(t)$ are continuous? Explain your answer.

2.) A certain linear homogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ has solutions $y_1(t) = t\cos(t)$ and $y_2 = e^{-t}$. Find a solution to the homogeneous equation which satisfies $y(0) = 1$, $y'(0) = 1$.

3.) Suppose y_1 and y_2 are two solutions to the equation $(1+t)y''(t) + t^2y'(t) + (2+t)y(t) = 0$ and $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$. Compute the Wronskian $W(y_1, y_2)$.

4.) Given that $y(t) = t^3$ is a solution to the differential equation $y''(t) + y(t) = t^3 + 3t^2$, find a solution $u(t)$ to this differential equation which satisfies $u(0) = -1$, $u'(0) = 1$.

5.) Find the general solutions to the following equations:
  a.) $y'' + 4y' + 3y = 0$.
  b.) $2y'' + 2y' + y = 0$.
  c.) $4y'' + 12y' + 9y = 0$.

6.) The equation $t^2y''(t) - (2t + 2t^2)y'(t) + (2 + 2t)y(t) = 0$ has $y = t$ as a solution. Find the general solution to this differential equation.

7.) Find the general solution to the equation $y''(t) - 2y'(t) + y(t) = t^{3/2}e^t$.

8.) Write down the trial particular solution according to the method of undetermined coefficients for the equation $y''(t) + 2y'(t) + 2y(t) = e^t \cos(t) + 3e^{-t} \cos(t)$. You need not solve for the coefficients.

9.) Find the general solution to the following equations:
  a.) $y'' + y' = t^3$.
  b.) $y''(t) + 4y(t) = t \sin(2t)$.

10.) A mass with a weight of 8lbs. stretches a spring a distance of 6in. Assume there is no resistance for this problem.

  a.) What is the natural frequency of the system?

  b.) Find the position function $u(t)$ assuming $u(0) = 0$, $u'(0) = 1$ft./sec. (and no driving force).

  c.) Find the equation of motion if $u(0) = 0$, $u'(0) = 0$, and there is now a driving force $F = \frac{1}{4}\cos(5t)$. What is the amplitude of the motion?