Math 3410
Solutions To First Review

1.) Writing the equation in standard form we have $y'(t) + (2 + \frac{1}{t+1})y(t) = t$. The integrating factor is $\mu = e^{2t+\ln(t+1)} = (t+1)e^{2t}$. The general solution is

$$\begin{aligned} y(t) &= \frac{e^{-2t}}{t+1}\left[\int e^{2t}(t+1)t\,dt + C\right] \\ &= \frac{e^{-2t}}{t+1}\left[t^2\frac{e^{2t}}{2} + C\right] \\ &= \frac{1}{2}\frac{t^2}{t+1} + C\frac{e^{-2t}}{t+1} \end{aligned}$$

If $y(0) = 1$ then $C = 1$, so the particular solution is $y(t) = \frac{1}{2}\frac{t^2}{t+1} + \frac{e^{-2t}}{t+1}$.

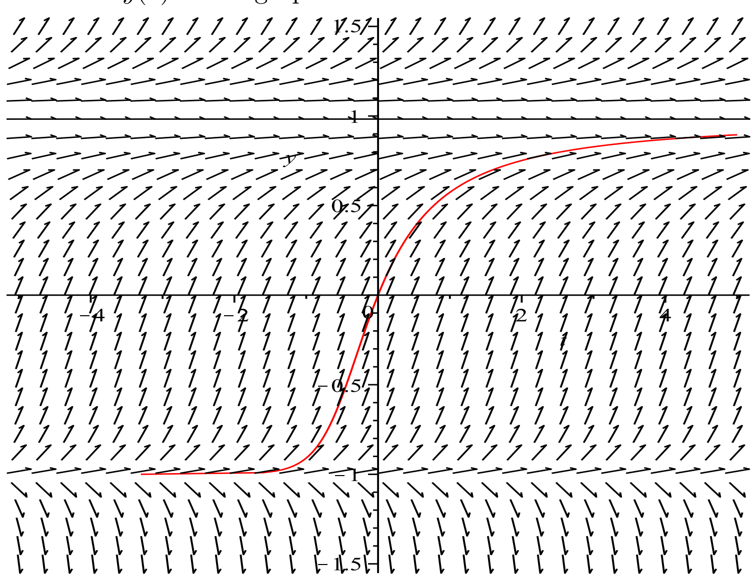
2.) The differential equation is $v'(t) = 9.8 - \sqrt{v}$. This is a separable equation. we have $\frac{dv}{9.8-\sqrt{v}} = dt$, so $\int \frac{dv}{9.8-\sqrt{v}} = t + C$. Do the integral by the substitution $v = u^2$. It becomes $\int \frac{2u}{9.8-u}\,du = \int \left(-2 + \frac{19.6}{9.8-u}\right) du = -2u - 19.6\ln(9.8-u) = -2\sqrt{v} - 19.6\ln(9.8-\sqrt{v})$. So, the general solution is $-2\sqrt{v} - 19.6\ln(9.8-\sqrt{v}) = t + C$. The initial condition is $v(0) = 0$. So, $C = -19.6\ln(9.8)$. So, the solution is $t = 19.6\ln(9.8) - 2\sqrt{v} - 19.6\ln(9.8-\sqrt{v})$.

3.) The differential equation is $y'(t) = 0.06y(t) - 12000$, where $y(t)$ is the amount owed at time t. This is a first order linear, and also a separable equation. Solving as a first order linear we get $y(t) = e^{0.06t}[\int e^{-0.06t}(-12000)\,dt + C] = Ce^{0.06t} + \frac{12000}{0.06} = 200000 + Ce^{0.06t}$. The initial condition is $y(0) = 120000$, so $C = -80000$. So, $y(t) = 200000 - 80000e^{0.06t}$. The loan will be paid off when $y(t) = 0$, so $200000 = 80000e^{0.06t}$, or $e^{0.06t} = \frac{5}{2}$. So, $t = \frac{1}{0.06}\ln(\frac{5}{2}) = \frac{50}{3}\ln(\frac{5}{2}) \approx 15.2715$ years.

4.) Written in the standard form the equation is $(\frac{1}{x} - 2)\,dx + (-\frac{1}{y} - 2y)\,dy = 0$. So, $M = \frac{1}{x} - 2$ and $N = -\frac{1}{y} - 2y$. We have $M_y = 0$, $N_x = 0$, so the equation is exact (note that it is also separable, but we solve it here as an exact equation). Starting first with $\psi_x = M = (\frac{1}{x} - 2)$ we get $\psi = \ln(x) - 2x + h(y)$. Using then $\psi_y = N = -\frac{1}{y} - 2y$ we get $h'(y) = -\frac{1}{y} - 2y$ so $h(y) = -\ln(y) - y^2 + C$. So, $\psi = \ln(x) - 2x - \ln(y) - y^2$, and the general solution is $\ln(x) - 2x - \ln(y) - y^2 = C$.

5.) The numerator and denominator of the right-hand side are homogeneous polynomials of degree 1. So, we can write the equation as $\frac{dy}{dx} = \frac{3\frac{y}{x} - 1}{\frac{y}{x} + 1}$. Let $v = \frac{y}{x}$, so $y = xv$. Thus, $\frac{dy}{dx} = x\frac{dv}{dx} + v$. So, the equation becomes $\frac{dy}{dx} = x\frac{dv}{dx} + v = \frac{3v-1}{v+1}$. So, $x\frac{dv}{dx} = \frac{3v-1}{v+1} - v = \frac{-v^2+2v-1}{v+1}$. This is a separable equation which gives $\frac{v+1}{-v^2+2v-1}\,dv = \frac{1}{x}\,dx$. So, $-\int \frac{v+1}{(v-1)^2}\,dv = \ln(x) + C$. Do the first integral by partial fractions. It becomes $-\left[\int \frac{1}{v-1}\,dv + \int \frac{2}{(v-1)^2}\,dv\right] = -\ln(v-1) + \frac{2}{v-1} = \ln(x) + C$. Substituting back in we get $-\ln(\frac{y}{x} - 1) + \frac{2x}{y-x} = \ln(x) + C$.

6.) We have $y'(t) = (y-1)^2(y+1)$. The equilibrium solutions are $y = -1$ and $y = 1$. Testing the sign of the right-hand side in the three intervals determined by the zeros we get: for $y < -1$, $y' < 0$, for $-1 < y < 1$, $y' > 0$, and for $y > 1$, $y' > 0$. This gives that $y = -1$ is unstable and $y = 1$ is semi-stable. The particular solution with $y(0) = 0$ is graphed below.



7.) For the equation $(6xy + y^3)\,dx + (6x^2 + 4xy^2)\,dy = 0$ we have $M = 6xy + y^3$ and $N = 6x^2 + 4xy^2$. So, $M_y = 6x + 3y^2$ and $N_x = 12x + 4y^2$, so the equation is not exact. We next look for an integrating factor. $M_y - N_x = -6x - y^2$. We see that $\frac{M_y - N_x}{M} = \frac{1}{y}$ is a function of y only. So, we can get an integrating factor by solving the equation $\mu'(y) = -\left(\frac{M_y - N_x}{M}\right)\mu(y)$. That is, $\mu'(y) = \frac{1}{y}\mu$. This gives $\ln(\mu) = \ln(y)$, so we can take $\mu(y) = y$ (we ignore the constant of integration in finding integrating factors). Multiplying the original equation through by y we get $(6xy^2 + y^4)\,dx + (6x^2y + 4xy^3)\,dy = 0$. Starting over with this equation we have $M_y = 12xy + 4y^3$ and $N_x = 12xy + 4y^3$ so this new equation is exact. We first solve $\psi_x = M = 6xy^2 + y^4$ so $\psi = 3x^2y^2 + y^4x + h(y)$. Plugging this to the equation $\psi_y = N = 6x^2y + 4xy^3$ we have $6x^2y + 4xy^3 + h'(y) = 6x^2y + 4xy^3$, so $h'(y) = 0$, so $h(y) = C$. Thus, the general solution is $3x^2y^2 + xy^4 = C$.

8.) The standard form for the equation is $y'(t) + \left(\frac{t^2}{3-t}\right) y(t) = \frac{1}{t(3-t)}$. So, the $p(t)$ and $g(t)$ functions are continuous on the interval $(0, 3)$. So, the existence and uniqueness theorem for first order linear equations says that there will be a unique solution on the interval $(0, 3)$ satisfying $y(2) = 1$.

9.) To solve the differential equation $x''(t) = \cos(x(t))$ we multiply both sides of the equation by $x'(t)$ to get $x'(t)x''(t) = \cos(x(t))x'(t)$. Integrating both sides with respect to t we get: $\frac{(x'(t))^2}{2} = \sin(x(t)) + C$. When $t = 0$, both x and x' are equal to 0, so $C = 0$. Thus, $\frac{(x'(t))^2}{2} = \sin(x(t))$. The velocity is again equal to 0 when $x = \pi$,