Math 3410
Review For First Test

First Order Linear Equations: $y'(t) + p(t)y(t) = g(t)$. Integrating factor $\mu(t) = e^{\int p(t)\,dt}$. General solution is $y(t) = \frac{1}{\mu(t)}[\int \mu(t)g(t)\,dt + C]$.

First Order Separable: $y'(t) = f(t)g(y)$. Solve by separating the variables: $\frac{dy}{g(y)} = f(t)\,dt$ and then integrating to get $\int \frac{dy}{g(y)} = \int f(t)\,dt + C$.

Exact Equations: $M(x,y)\,dx + N(x,y)\,dy$ is exact if $M_y = N_x$ which is equivalent to there existing a potential function $\psi(x,y)$, that is, $\psi_x = M$ and $\psi_y = N$. The general solution is $\psi(x,y) = C$. You find ψ by first integrating one of the equations for ψ and then plugging into the second equation.

- If $\frac{M_y - N_x}{N}$ is a function of x only, then equation will become exact upon multiplying by the integrating factor $\mu(x)$ which is gotten by solving $\mu'(x) = \left(\frac{M_y - N_x}{N}\right)\mu(x)$.
- If $\frac{M_y - N_x}{M}$ is a function of y only, then equation will become exact upon multiplying by the integrating factor $\mu(y)$ which is gotten by solving $\mu'(y) = -\left(\frac{M_y - N_x}{M}\right)\mu(y)$.

Autonomous First Order Equations: $y'(t) = f(y)$. This is a special case of a separable equation. However we can determine the nature of the solutions without solving the equation. We do this by finding the zeros of $f(y)$ and determining the sign of $f(y)$ in the corresponding intervals. We have the notions of stable, semi-stable, and unstable equilibrium solutions.

Miscellaneous:

- (homogeneous) The equation $\frac{dy}{dx} = F(\frac{y}{x})$ can be turned into a separable equation by the substitution $v = \frac{y}{x}$, that is, $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$. This type of equation includes those of the form $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$ where P, Q are homogeneous polynomials of the same degree.
- (Bernoulli) The equation $y'(t) + p(t)y(t) = g(t)y^n$ can be turned into a linear equation by making the substitution $v = y^{1-n}$ (for $n \neq 0, 1$).
- (second order autonomous) The second order equation $y''(t) = f(y)$ can be solved by multiplying through by y' to get $y'y'' = y'f(y)$. We then integrate both sides to get $\frac{(y')^2}{2} = \int f(y)\,dy + C$. This is a separable equation which can then be solved for y.

Theory: For a first order linear equation $y'(t) + p(t)y(t) = g(t)$, given an initial condition $y(t_0) = y_0$, there will be a unique solution in any interval I containing t_0 for which the functions $p(t), g(t)$ are continuous. For a general first order equation $y'(t) = F(t,y)$, if F and $\frac{\partial F}{\partial y}$ are continuous in a region about (t_0, y_0), then there will be a unique solution in some interval about t_0 (but we're not guaranteed how large this interval will be, and it can depend on the initial condition).

Sample Problems

1.) Find the general solution to the equation $(t+1)y'(t) + (2t+3)y(t) = t^2 + t$. Then find the particular solution satisfying $y(0) = 1$.

2.) A falling body has acceleration equal to $9.8 - \sqrt{v}$ (m/sec^2), where $v = v(t)$ is the velocity at time t. Assuming the body starts from rest, determine $v(t)$.

3.) A person buys a house for \$120,000, borrowing at a 6% annual interest rate. The person makes payments at a monthly rate of \$1,000. How long will it take to pay off the loan?

4.) Test the equation $\dfrac{dy}{dx} = \dfrac{\frac{1}{x} - 2}{\frac{1}{y} + 2y}$ for exactness, and then find the general solution.

5.) Find the general solution to the equation $y'(x) = \dfrac{3y - x}{y + x}$. Note that the right-hand side is homogeneous.

6.) Consider the first order equation $y'(t) = (y^2 - 1)(y - 1)$. Determine the equilibrium solutions, and determine if they are stable, unstable, or semi-stable. Sketch the graph of the particular solution satisfying $y(0) = 0$.

7.) Solve the equation $(6xy + y^3)\,dx + (6x^2 + 4xy^2)\,dy = 0$.

8.) Consider the equation $(3-t)y'(t) + t^2 y(t) = \frac{1}{t}$. Find the largest interval I containing $t = 2$ you can for which there is a unique solution over that interval to the differential equation which satisfies the initial condition $y(2) = 1$.

9.) A particle moves on the x-axis with position $x(t)$ at time t. Suppose $x(t)$ satisfies the differential equation $x''(t) = \cos(x(t))$. Suppose that $x(0) = 0$ and $v(0) = 0$. Find the position of the particle ($x > 0$) when the velocity is again equal to 0.