Math 3410
Solutions To Review For Final Exam

1.) The Wronskian of the three functions os given by

$$W(y_1, y_2, y_3)(t) = \det \begin{bmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{bmatrix} = 2t^3$$

At the point $t = 0$ the Wronskian is equal to 0. so, if they solutions to the linear homogeneous equation, they would have be linearly dependent on the interval containing $t = 0$ on which the coefficient functions are continuous. However, these three functions are not linearly independent, and so they cannot all be solutions to the differential equation.

2.) In standard form the equation

$$(t+1)(2+t^2)y^{(3)}(t) + (1-t)y''(t) + ty'(t) - 2y(t) = 0.$$

becomes

$$y^{(3)} + \frac{1-t}{(t+1)(t^2+2)}y''(t) + \frac{t}{(t+1)(t^2+2)}y'(t) - \frac{2}{(t+1)(t^2+2)}y(t) = 0.$$

a.) The coefficient functions have a discontinuity at $t = -1$. So, by the existence and uniqueness theorem, the solution to the equation with the given intitial conditions will exist, be unique, and be continuos on the interval $(-1, \infty)$.

b.) The coefficient functions have dincontinuities in the complex plane at $t = -1$ and $t = \pm i\sqrt{2}$. Expanding about the point $t = 1$, the power series for these functions will have radius of convergence the minimum distance to one of these points. This is $\sqrt{3}$, and the solution is guaranteed to have a radiaus of convergence at least $\sqrt{3}$.

3.) This is a constant coefficient linaer homogeneous equation with characteristic polynomial $r^3 - 3r^2 + 2r - 6 = 0$. The possible rational roots are ± 1, ± 2, ± 3, and ± 6. Testing we see that $r = 3$ is a root and then dividing we the polynomial factors as $(x-3)(x^2+2)$. So, the roots are 3 and $\pm i\sqrt{2}$. So the general solution is

$$y = c_1 e^{3x} + c_2 \cos(\sqrt{2}x) + c_3 \sin(\sqrt{2}x).$$

4.) The roots of the characteristic polynomial are $\pm 2i$ (twice) and 5 (twice). So, the general solution is

$$y = c_1 e^{5t} + c_2 t e^{5t} + c_3 \cos(2t) + c_4 \sin(2t) + c_5 t \cos(2t) + c_6 t \sin(2t).$$

5.) The homogeneous equation (which is the same as for the previous problem) has solutions e^{5t}, te^{5t}, $\cos(2t)$, $\sin(2t)$, $t\cos(2t)$, $t\sin(2t)$, Our trial particular solution is

$$y_p = t^2(At+B)\cos(2t) + t^2(Ct+D)\sin(2t) + (Et+F)e^{4t}.$$

6.) The characteristic polynomial is $r^3+r^2-2r = r(r^2+r-2) = r(r+2)(r-1)$. So, the homogeneous equation has general solution $y_h = c_1 + c_2e^t + c_3e^{-2t}$. The trial particular solution is $y_p = Ate^t$. Plugging this into the equation we get $A = 1$, and so the general solution is $y = c_1 + c_2e^t + c_2e^{-2t} + te^t$.

7.) The trial particular solution is

$$y_p = u_1(t)(e^{3t}) + u_2(t)(\cos(\sqrt{2}t)) + u_3(t)(\sin(\sqrt{2}t)).$$

The equation for the unknown functions is:

$$\begin{bmatrix} e^{3t} & \cos(\sqrt{2}t) & \sin(\sqrt{2}t) \\ 3e^{3t} & -\sqrt{2}\sin(\sqrt{2}t) & \sqrt{2}\cos(\sqrt{2}t) \\ 9e^{3t} & -2\cos(\sqrt{2}t) & -2\sin(\sqrt{2}t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t} \end{bmatrix}$$

The Wronskian is Ce^{3t}, and plugging in $t = 0$ we see that $C = 11\sqrt{2}$. So, the Wronskian is $W = 11\sqrt{2}e^{3t}$. Using Cramer's rule we have

$$u_1'(t) = \frac{1}{11\sqrt{2}e^{3t}} \det \begin{bmatrix} 0 & \cos(\sqrt{2}t) & \sin(\sqrt{2}t) \\ 0 & -\sqrt{2}\sin(\sqrt{2}t) & \sqrt{2}\cos(\sqrt{2}t) \\ \frac{1}{t} & -2\cos(\sqrt{2}t) & -2\sin(\sqrt{2}t) \end{bmatrix} = \frac{e^{-3t}}{11t}$$

Similarly,

$$u_2'(t) = \frac{1}{11\sqrt{2}e^{3t}} \det \begin{bmatrix} e^{3t} & 0 & \sin(\sqrt{2}t) \\ 3e^{3t} & 0 & \sqrt{2}\cos(\sqrt{2}t) \\ 9e^{3t} & \frac{1}{t} & -2\sin(\sqrt{2}t) \end{bmatrix} = -\frac{1}{11\sqrt{2}t}(\sqrt{2}\cos(\sqrt{2}t) - 3\sin(\sqrt{2}t))$$

Likewise $u_3' = \frac{1}{11\sqrt{2}t}(\sqrt{2}\sin(\sqrt{2}t) + 3\cos(\sqrt{2}t))$.

8.) a.) The system of equations is:

$$\begin{aligned} u_1''(t) &= -8u_1 + 3(u_2 - u_1) \\ u_2''(t) &= -3(u_2 - u_1) \end{aligned}$$

b.) In matrix form this is

$$\begin{bmatrix} u_1'' \\ u_2'' \end{bmatrix} = \begin{bmatrix} -11 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

We can also write this $U'' = AU$ where $A = \begin{bmatrix} -11 & 3 \\ 3 & -3 \end{bmatrix}$ and $U = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$.

9.) We first diagonalize the matrix A. The characteristic polynomial of the matrix A is $\det(A-\lambda I) = \lambda^2 + 14\lambda + 24 = (\lambda+2)(\lambda+12)$. So, the eigenvalues of A are $\lambda = -2$ and $\lambda = -12$. Sovling for the eigenvectors we get $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Let $P = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$. Then $P^{-1}AP = D$ where D is the diagonal matrix $D = \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix}$. Also, $A = PDP^{-1}$. We make the change of variable $Z = P^{-1}U$, and the system becomes the diagonal system $Z'' = DZ$. Solving for Z we get $z_1 = A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$ and $z_2 = C\cos(\sqrt{12}t) + D\sin(\sqrt{12}t)$. So, $Y = PZ$. We get

$$\begin{aligned} y_1 =& A\cos(\sqrt{2}t) - 3C\cos(\sqrt{12}t) + B\sin(\sqrt{2}t) - 3D\sin(\sqrt{12}t) \\ y_2 =& 3A\cos(\sqrt{2}t) + C\cos(\sqrt{12}t) + 3B\sin(\sqrt{2}t) + D\sin(\sqrt{12}t) \end{aligned}$$

10.) a.) $P(z) = 1 + x$ which has a root at $x = -1$ (remember we must consider complex zeros in general). So, $\rho \geq 1$.

b.) We try a power series of the form $y = \sum_{n=0}^{\infty} a_x x^n$. Substituting this into the given equation $(1+x)y''(x) + y'(x) + x^2 y(x) = 0$ we get

$$(1+x)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Multiplying through we get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} + \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Changing the sums to get a common tern of x^n in each we write this as:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_{n+1}(n+1)(n)x^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

Combining the sums this gives:

$$(a_1+2a_2)+(4a_2+6a_3)x+\sum_{n=2}^{\infty}[a_{n+2}(n+2)(n+1)+a_{n+1}(n+1)(n)+a_{n+1}(n+1)+a_{n-2}]x^n = 0$$

or

$$(a_1+2a_2)+(4a_2+6a_3)x+\sum_{n=2}^{\infty}[a_{n+2}(n+2)(n+1)+a_{n+1}(n+1)^2+a_{n-2}]x^n = 0$$

So, we have $a_1 + 2a_2 = 0$, $2a_2 + 3a_3 = 0$, and $a_{n+2} = \dfrac{-a_{n-2} - a_{n+1}(n+1)^2}{(n+1)(n+2)}$.

c.) Using the recurrence relation we find that

$$\begin{aligned} y_1 =& 1 - \frac{1}{12}x^4 + \frac{1}{15}x^5 + \cdots \\ y_2 =& x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{3}{20}x^5 + \cdots \end{aligned}$$

11.) We substitute $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$ into the equation. We have

$$\begin{aligned}&(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots)\\&+ (1 + x + x^2 + x^3 + \cdots)(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)\\&+ (1 + x)(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = 0\end{aligned}$$

We are computing the solution with $y(0) = 1$, $y'(0) = 1$, so $a_0 = 1$, $a_1 = 1$.

The constant coefficient of the above equation is $a_2 + a_1 + a_0$. Setting this to 0 we get $a_2 = -1$. The coefficient of x is $6a_3 + 2a_2 + a_1 + a + 1 + a_0 = 6a_3 + 2a_2 + 2a_1 + a_0$. Setting this equal to 0 we get $a_3 = -\frac{1}{6}$. The coefficient of x^2 is $12a_4 + 3a_3 + 2a_2 + a_1 + a_2 + a_1$. Setting this equal to 0 we get $a_4 = \frac{1}{8}$. The coefficient of x^3 is $20a_5 + a + 1 + 2a_2 + 3a_3 + 4a_4 + a_3 + a_2$. Setting this equal to 0 we get $a_5 = \frac{13}{120}$.

So, the first six terms of the power series for y are

$$y(x) = 1 + x - x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{13}{120}x^5 + \cdots$$