Math 3410
Review For Final Exam

The final exam will cover:
Chapter 2: Sections 2.1, 2.2, 2.4, 2.5, 2.6.
Chapter 3: All of chapter 3.
Chapter 4: All of chapter 4.
Chapter 5: Sections 5.1, 5.2, 5.3.
Chapter 7: Just what was covered in class (see below).
Following is a quick review of the main topics covered.

First Order Linear Equations: $y'(t) + p(t)y(t) = g(t)$. Integrating factor $\mu(t) = e^{\int p(t)\,dt}$. General solution is $y(t) = \frac{1}{\mu(t)}[\int \mu(t)g(t)\,dt + C]$.

First Order Separable: $y'(t) = f(t)g(y)$. Solve by separating the variables: $\frac{dy}{g(y)} = f(t)\,dt$ and then integrating to get $\int \frac{dy}{g(y)} = \int f(t)\,dt + C$.

Exact Equations: $M(x, y)\,dx + N(x, y)\,dy$ is exact if $M_y = N_x$ which is equivalent to there existing a potential function $\psi(x, y)$, that is, $\psi_x = M$ and $\psi_y = N$. The general solution is $\psi(x, y) = C$. You find ψ by first integrating one of the equations for ψ and then plugging into the second equation.

The Final won't cover integrating factors to make equations exact.

Autonomous First Order Equations: $y'(t) = f(y)$. This is a special case of a separable equation. However we can determine the nature of the solutions without solving the equation. We do this by finding the zeros of $f(y)$ and determining the sign of $f(y)$ in the corresponding intervals. We have the notions of stable, semi-stable, and unstable equilibrium solutions.

Theory

For a general (non-linear) first order equation $y'(t) = F(t, y)$, if F and $\frac{\partial F}{\partial y}$ are continuous in a region about (t_0, y_0), then there will be a unique solution in some interval about t_0 (but we're not guaranteed how large this interval will be, and it can depend on the initial condition).

For a linear nth order equation in standard form

$$y^{(n)}(t) + p_1(t)y_1^{(n-1)}(t) + \cdots + p_{n-1}(t)y'(t) + p_n(t)y(t) = g(t)$$

given any initial condition of the form $y(t_0) = y_0$, $y'(t_0) = y_0'$, …, $y^{(n-1)}(t_0) = y_0^{n-1}$ then there will exist a unique solution to the linear differential equation in any interval I containing t_0 for which the functions $p_1, \ldots, p_t, g$ are all continuous. This is the existence and uniqueness theorem.

If the nth order linear equation is homogeneous (i.e., $g(t) = 0$), then on any interval for which $p_1, \ldots, p_t$ are all continuous, there will exist n fundamental solutions $y_1(t), \ldots, y_n(t)$ such that on this interval the general solution is of the form $y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$, that is, the general solution is a linear combination of these fundamental solutions. In other words, the set of solutions forms a vector space of dimension n. On such an interval I, given any n solutions $y_1, \ldots, y_n$ to the linear homogeneous equation, either these functions will be linearly dependent on I or at each point $t \in I$ we will have that the column vectors

$\begin{bmatrix} y_i(t) \\ y_i'(t) \\ \vdots \\ y_i^{(n-1)}(t) \end{bmatrix}$ are linearly independent. The Wronskian matrix is the $n \times n$ matrix

$$\begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_{n-1}'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}$$

The Wronskian $W(y_1, \ldots, y_n)(t)$ is the determinant of this matrix. The Wronskian is either always 0 or never 0 in an interval I as above. It is 0 on the interval precisely when the solutions are dependent on the interval I.

Abel's formula determines $W(y_1, \ldots, y_n)(t)$ up to a constant. The formula says:

$$W(y_1, \ldots, y_n)(t) = Ce^{-\int p_1(t)\,dt}.$$

Note that the equation must be in standard form to apply this formula.

For a non-homogeneous equation, the general solution is $y = y_H + y_P$, where y_H is the general solution to the homogeneous equation, and y_P is a particular solution to the non-homogeneous equation.

We review some of the methods for solving linear equations.

Constant Coefficient Homogeneous: For the equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0,$$

the characteristic equation is $a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$.

- For each real root r repeated k times, we have the solutions e^{rt}, te^{rt}, $\ldots$, $t^{k-1}e^{rt}$.
- For each pair of complex roots $\lambda \pm \mu i$ repeated k times, we have the solutions $e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$, $\ldots$, $t^{k-1}e^{\lambda t}\cos(\mu t)$, $t^{k-1}e^{\lambda t}\sin(\mu t)$

The general solution is a linear combination of these solutions.

Reduction of Order: Given one solution $y_1(t)$ to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$, we can find another by trying to find one of the form $u(t)y_1(t)$. This results in a first-order (and separable) equation for u'.

For an nth order equation, we can also substitute $u(t)y_1(t)$ which will result in a $n-1$st order equation for u'.

Non-homogeneous: If we can solve the homogeneous linear equation, then we can solve the non-homogeneous equation $y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y_n q y = g$. There are two methods, undetermined coefficients and variation of parameters. The first only works in special cases but is easier. Undetermined coefficients works when g is of the form (poly) or (poly)e^{at}, or (poly)$e^{at}\cos(bt)$ (or with sin). The trial solution is of the same form (with a general polynomial of the same degree), which is then possibly multiplied by t^k if necessary to guarantee than none of the terms of the trial solution is a solution to the homogeneous equation.

For variation of parameters, we try a particular solution of the form $y = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$, where $y_1, \ldots, y_n$ are independent solutions to the homogeneous equation. This leads to the equation

$$\begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_{n-1}'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{bmatrix}$$

This equation is best solved using Cramer's rule.

Series Solutions Given a second order linear homogeneous equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

we say the point $x = x_0$ is an ordinary point if $P(x_0) \neq 0$ (and all the functions are continuous at x_0). If x_0 is an ordinary point and P, Q, R are analytic at x_0 (that is, they are given by a power series) with radius of convergence ρ, then given any initial condition $y(x_0) = y_0$, $y'(x_0) = y_0'$, there will be a unique solution $y(x)$ satisfying this initial condition which will be analytic at x_0 and will have radius of convergence at least ρ. The solution will be a power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The first two coefficients a_0, a_1 are arbitrary and the rest are determined from them. Two fundamental solutions y_1, y_2 can be given by letting (for y_1) $a_0 = 1$, $a_1 = 0$, and (for y_2) $a_0 = 0$, $a_1 = 1$.

We determine the recurrence relation for a_n by substituting the power series above into the given differential equation. The best strategy is to get all the summation terms of the form $\sum(\cdots)(x - x_0)^n$ and then combine the terms.

Systems of Linear Equations Given a system of linear differential equations (we'll restrict our discussion here to two by two systems) of the form

$$y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t)$$
$$y_2'(t) = a_{21}y_1(t) + a_{2,2}y_2(t)$$

we write this in matrix form as

$$Y'(t) = AY(t)$$

where $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We assume here that A can be diagonalized (this will always be the case if A has distinct eigenvalues or if A is symmetric). So, let P be a matrix whose columns are linearly independent eigenvectors. Then $P^{-1}AP = D$, where D is a diagonal matrix with the eigenvalues of A on the diagonal (in the same order corresponding to the columns of P). Note that $A = PDP^{-1}$. Substituting in $Z = P^{-1}Y$, the system becomes

$$Z'(t) = DZ.$$

This is a diagonal system and is easily solved. We then then get the solution to the original system as $Y = PZ$.

We can use the same method for a system of second order equations:

$$y_1''(t) = a_{11}y_1(t) + a_{12}y_2(t)$$
$$y_2''(t) = a_{21}y_1(t) + a_{2,2}y_2(t)$$

Application: Mass Oscillators. A single mass oscillator of mass m and spring constant k will satisfy the equation

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

where γ is the resistance constant and $F(t)$ is the driving force. Recall that the spring constant and resistance constant are given by $f_d = ku$, $f_e = \gamma u'$ (u is the displacement and u' the velocity).

For a system of several mass oscillators, we have a system of equations. Given two oscillators of masses m_1 and m_2 and spring constants k_1 and k_2 (and no resistance or driving force) we have the system:

$$\begin{aligned} m_1 u_1''(t) &= -k_1 u_1 + k_2(u_2 - u_1) \\ m_2 u_2''(t) &= -k_2(u_2 - u_1) \end{aligned}$$

In matrix form this is given by (we assume $m_1 = m_2 = 1$)

$$\begin{bmatrix} u_1''(t) \\ u_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

This is solved by diagonalizing the matrix as discussed above.

Sample Problems

1.) Can the functions $y_1 = t$, $y_2 = t^2$, $y_3 = t^3$ all be solutions to a third order linear homogeneous equation $y^{(3)} + p(t)y'' + q(t)y' + r(t)y = 0$ where p, q, r are continuous in an interval about $t = 0$?

2.) Consider the third order equation

$$(t+1)(2+t^2)y^{(3)}(t) + (1-t)y''(t) + ty(t) - 2y(t) = 0.$$

a.) In what interval about the point $t = 1$ is the solution to this equation with the initial values $y(1) = 0$, $y'(1) = 1$, $y''(1) = 2$ guaranteed to be continuous?

b.) In what interval about $t = 1$ is this solution guaranteed to be analytic?

3.) Find the general solution to the equation $y^{(3)}(x) - 3y''(x) + 2y'(x) - 6y(x) = 0$.

4.) Write down the general solution to the 6th order homogeneous equation with constant coefficients whose characteristic polynomial is $(x^2+4)^2(x-5)^2$.

5.) Write down the trial particular solution for the non-homogeneous equation

$$y^{(6)} - 10y^{(5)} + 33y^{(4)} - 80y^{(3)} + 216y'' - 160y' + 400y = t\cos(2t) + 3te^{4t}$$

You may use that $x^6-10x^5+33x^4-80x^3+216x^2-160x+400=(x^2+4)^2(x-5)^2$. You do not need to solve for the coefficients.

6.) Find the general solution to the equation $y^{(3)}+y''-2y'=3e^t$.

7.) Use the method of variation of parameters to write down a trial particular solution to the equation (compare with problem 3):

$$y^{(3)}(x)-3y''(x)+2y'(x)-6y(x)=\frac{1}{t}.$$

Determine the values of u_1', u_2', u_3', but you do not need to integrate these.

8.) A two mass oscillator has $m_1=m_2=1$ and spring constants $k_1=8$, $k_2=3$.

a.) Write down the system of differential equations for the position functions $u_1(t)$, $u_2(t)$.

b.) Write this system in matrix form.

9.) Solve the system from problem (8) and determine the fundamental frequencies of the system.

10.) Consider the equation $(1+x)y''(x)+y'(x)+x^2y(x)=0$.

a.) If we find a solution to this equation as a power series about the point $x=0$, how large is the radius of convergence guaranteed to be?

b.) Find the recurrence relation for the coefficients of the power series.

c.) Find the first six terms of the power series for the two fundamental solutions y_1, y_2 about the point $x=0$ (that is, compute $a_0,\dots,a_5$).

11.) Find the first five terms (that is, $a_0,\dots,a_4$) of the power series solution for the equation $y''(x)+\frac{1}{1-x}y'(x)+(1+x)y(x)=0$ (about the point $x=0$) satisfying the initial conditions $y(0)=1$, $y'(0)=1$.