Solutions To Review For Third Test

1.) Using elementary row-operations we have:

$$\begin{aligned}\det A &= \det\begin{pmatrix}2 & 7 & -1 & 9\\ 1 & 2 & -1 & 3\\ 2 & -2 & 0 & 2\\ -2 & -4 & -2 & -6\end{pmatrix} = -\det\begin{pmatrix}1 & 2 & -1 & 3\\ 2 & 7 & -1 & 9\\ 2 & -2 & 0 & 2\\ -2 & -4 & -2 & -6\end{pmatrix}\\ &= \det\begin{pmatrix}1 & 2 & -1 & 3\\ 0 & 3 & 1 & 3\\ 0 & -6 & 2 & -4\\ 0 & 0 & -4 & 0\end{pmatrix} = \det\begin{pmatrix}1 & 2 & -1 & 3\\ 0 & 3 & 1 & 3\\ 0 & 0 & 4 & 2\\ 0 & 0 & -4 & 0\end{pmatrix}\\ &= \det\begin{pmatrix}1 & 2 & -1 & 3\\ 0 & 3 & 1 & 3\\ 0 & 0 & 4 & 2\\ 0 & 0 & 0 & 2\end{pmatrix}\\ &= -24\end{aligned}$$

2.) a.) The cofactor matrix is $C = \begin{pmatrix}-7 & -11 & 1\\ -10 & 2 & 3\\ 4 & -7 & -5\end{pmatrix}$.

b.) Taking the transpost of C gives the adjoint (or adjugate), which is $\begin{pmatrix}-7 & -10 & 4\\ -11 & 2 & -7\\ 1 & -3 & -5\end{pmatrix}$. If we take the dot product of the first row of the adjgate and the first column of A, we get $-7-20-4=-31$, which is the determinant.

The adjugate inverse formula is $A^{-1} = \frac{1}{\det A}C^T = -\frac{1}{31}\begin{pmatrix}-7 & -10 & 4\\ -11 & 2 & -7\\ 1 & -3 & -5\end{pmatrix} = \frac{1}{31}\begin{pmatrix}7 & 10 & -4\\ 11 & -2 & 7\\ -1 & 3 & 5\end{pmatrix}$.

3.) Taking the determinant we have $\det(A) = (-5)(2k-3)-(-1)(6-3)+k(3-k) = -k^2-7k+18 = -(k^2+7k-18) = -(k-2)(k+9)$. So, the matrix is invertble when the determiant is non-zero, which happens when k is not equal to 2 or -9.

4.) a.) The area of the triangle is one-half the area of the parallelogram. So, we have the area of the triangle is $\frac{1}{2}\left|\det\begin{pmatrix}1 & 2\\ 3 & 8\end{pmatrix}\right| = \frac{1}{2}(8-6) = 1$.

b.) The standard matrix A for the linear transformation T is $A = \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$. We have $|\det(A)| = |-3| = 3$. The area of the new triangle is $|\det(A)|$ times the area of the old triangle. So, the area of the new triangle is $(3)(1) = 3$.

5.) The quadrilateral can be divided into two triangles by a line from $(2,5)$ to $(7,4)$. The area of the triangle with vertices $(1,1)$, $(2,5)$, $(7,4)$ is $\frac{1}{2}\left|\det\begin{pmatrix}1 & 6\\ 4 & 3\end{pmatrix}\right| = \frac{21}{2}$. The area of the triangle with vertices $(2,5)$, $(6,6)$, $(7,4)$ is given by $\frac{1}{2}\left|\det\begin{pmatrix}4 & 5\\ 1 & -1\end{pmatrix}\right| = \frac{9}{2}$. So, the area of the quadrilateral is $\frac{21}{2} + \frac{9}{2} = 15$.

6.) a.) The vectors from the vertex $(1,2,-1)$ to the adjacent vertices are $\vec{u} = \begin{pmatrix}1\\1\\6\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\5\\3\end{pmatrix}$, $\vec{w} = \begin{pmatrix}2\\2\\10\end{pmatrix}$. So, the volume is given by $|\det(\vec{u}|\vec{v}|\vec{w})| = \left|\begin{pmatrix}1 & 3 & 2\\ 1 & 5 & 2\\ 6 & 3 & 10\end{pmatrix}\right| = |-4| = 4$.

b.) The volume of the tetrahedron is $\frac{1}{6}$ the volume of the corresponding tetrahedron. So, the volume is $\frac{1}{6}\left|\det\begin{pmatrix}1 & 2 & 4\\ 2 & 5 & 2\\ 1 & 4 & 1\end{pmatrix}\right| = \frac{1}{6}|9| = \frac{3}{2}$.

7.) a.) The characteristic polynomial is given by $p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix}2-\lambda & -1\\ 3 & 6-\lambda\end{pmatrix} = (2-\lambda)(6-\lambda)+3 = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3)$.

b.) The eigenvalues are the roots of the characteristic polynomial, so the eigenvalues are $\lambda = 3$ and $\lambda = 5$.

For $\lambda = 3$ we have $A - \lambda I = A = 3I = \begin{pmatrix}-1 & -1\\ 3 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$. The solution is given by $x_2\begin{pmatrix}-1\\1\end{pmatrix}$, so $\begin{pmatrix}-1\\1\end{pmatrix}$ is an eigenvector (in fact, it is a basis for the eigenspace for this eigenvalue).

For $\lambda = 5$ we have $A - \lambda I = A - 5I = \begin{pmatrix}-3 & -1\\ 3 & 1\end{pmatrix} \sim \begin{pmatrix}1 & \frac{1}{3}\\ 0 & 0\end{pmatrix}$. An eigenvector is $\begin{pmatrix}-\frac{1}{3}\\1\end{pmatrix}$. Since a multiple of an eigenvector is also an eigenvector, we can also use $\begin{pmatrix}1\\-3\end{pmatrix}$.

8.) $A - \lambda I = A - 5I = \begin{pmatrix}1 & -1 & 2\\ -1 & 1 & -2\\ 2 & -2 & 4\end{pmatrix}$. Row reducing we have $A - 5I \sim \begin{pmatrix}1 & -1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ which has solution $x_2\begin{pmatrix}1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-2\\0\\1\end{pmatrix}$. So, a basis for the eigenspace is $\mathcal{B} = \{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\}$. The dimension of the eigenspace is 2.

9.) We have $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 4 \\ 3 & -3 & k \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 6+k \end{pmatrix}$. The value of k that makes the vector an eigenvector is when $6+k=(2)(1)=2$, or $k=-4$. The corresponding eigenvalue is $\lambda=2$.

10.) The characteristic polynomial is clearly $(2-\lambda)^2(1-\lambda)^2=(\lambda-1)^2(\lambda-2)^2$. The eigenvalues are $\lambda=1$ and $\lambda=2$.

For $\lambda=1$ we have $A-\lambda I=A-I=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The solution is $x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the eigenspace for $\lambda=1$ is $\mathcal{B}=\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}\}$.

For $\lambda=2$ we have $A-\lambda I=A-2I=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The solution is given by $x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. A basis for the eigenspace is $\mathcal{B}=\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$.

11.) Here is one way. First use the row-operation $-R_1+R_4$ to get $\det(A) = \det \begin{pmatrix} 3 & 1 & 2 & 1 & -1 \\ 2 & -1 & 1 & 0 & 4 \\ -2 & 1 & 3 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 6 \end{pmatrix}$. Then use cofactor expansion along the 4th column to get $\det(A)=(-1)\det \begin{pmatrix} 2 & -1 & 1 & 4 \\ -2 & 1 & 3 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 2 & 1 & 6 \end{pmatrix}$. Then use cofactor expansion along the 3rd row to get $\det(A)=(-1)(-1)\det \begin{pmatrix} 2 & -1 & 4 \\ -2 & 1 & 1 \\ -1 & 2 & 6 \end{pmatrix}$. Using the row-operation R_1+R_2 we then have $\det(A)=(-1)(-1)\det \begin{pmatrix} 2 & -1 & 4 \\ 0 & 0 & 5 \\ -1 & 2 & 6 \end{pmatrix}$. Cofactor expansion along the second row then gives $\det(A) = (-1)(-1)(-5)\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = (-5)(4-1) = -15$.

12.) a.) You can either use row-operations or linearity of the determinant in each row (or column). Using linearity we get $\det(M) = \det\begin{pmatrix} d & e & f \\ a & b & c \\ a & b & c \end{pmatrix} + \det\begin{pmatrix} d & e & f \\ 2g & 2h & 2i \\ a & b & c \end{pmatrix}$. The first determinant is 0, so we get $\det(M) = 2\det\begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$. Two row-interchanges give $\det(M) = (2)(-1)(-1)\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (2)(3) = 6$.

b.) This time we will use row operations. Using $R_2 + R_3$ we get $\det(M) = \det\begin{pmatrix} 2a-3d & 2b-3e & 2c-3f \\ a+g & b+h & c+i \\ 2a & 2b & 2c \end{pmatrix} = 2\det\begin{pmatrix} 2a-3d & 2b-3e & 2c-3f \\ a+g & b+h & c+i \\ a & b & c \end{pmatrix}$. Using $-R_3 + R_2$ and $-2R_3 + R_1$ we have $\det(M) = 2\det\begin{pmatrix} -3d & -3e & -3f \\ g & h & i \\ a & b & c \end{pmatrix}$ Using linearity in the first row we have $\det(M) = (-6)\det\begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$. Using two row-interchanges this gives $\det(M) = (-6)(-1)(-1)(3) = -18$.

13.) Suppose λ is an eigenvalue, so $Ax = \lambda x$ for some eigenvector x. Then $A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$. But $A^2 = I$, so $x = \lambda^2 x$, so $(1-\lambda^2)x = \vec{0}$. Since $x \neq \vec{0}$, we have $\lambda^2 = 1$, so $\lambda = \pm 1$.