Math 2700
Solutions To Review For Second Test.

1.) $A^2 = A \cdot A = \begin{pmatrix} 1 & -5 \\ 15 & 6 \end{pmatrix}$. $A \cdot B = \begin{pmatrix} 3 & 4 \\ 0 & 15 \end{pmatrix}$. $(BA)^T$ can be computed as either $(BA)^T$ or as $A^T B^T$. Either way, the answer is $\begin{pmatrix} 11 & 4 \\ 8 & 7 \end{pmatrix}$. $(AB)^{-1}$ can be computed as either $(AB)^{-1}$ or as $B^{-1}A^{-1}$. Either way the answer is $\begin{pmatrix} \frac{1}{3} & -\frac{4}{15} \\ 0 & \frac{1}{15} \end{pmatrix}$. Recall here the formula for the inverse of a 2×2 matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

2.) a.) The standard matrix A for T is the matrix such that $T(x) = A \cdot x$. We can just see by inspection that $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$. One can also compute A from the general formula $A = \left[\ T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n)\ \right]$ for a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

b.) Composition of transformations corresponds to multiplication of the corresponding matrices. So, the matrix corresponding to $T \circ T$ is $A \cdot A = A^2 = \begin{pmatrix} 3 & 4 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & -1 \end{pmatrix}$.

3.) a.) The standard matrix for T is given by $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. $T(x) = \vec{0}$ has a non-trivial solution iff $A \cdot x = \vec{0}$ has a non-trivial solution, that is, the homogeneous system $A \cdot x = \vec{0}$ has a non-trivial solution. This happens iff there is a column without a pivot (i.e., a free variable). Row-reducing A we see that

$$A \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

We that every column has a pivot, so there does not exist a non-trivial solution to $T(\vec{x}) = \vec{0}$. We could also say that T is a one-to-one function.

b.) T is onto iff the equation $T(x) = \vec{b}$ is solvable for all choices of $\vec{b}$. In other words the system $A \cdot x = \vec{b}$ is consistent for all $\vec{b}$. This happens iff every row of A has a pivot. Even without row-reducing we can see this is impossible, since A is a 4×3 matrix.

c.) This is equivalent to asking if the system $A \cdot x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$ is consistent. We form the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & -1 \end{array}\right)$. Row-reducing, this becomes $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right)$. We see the system in inconsistent, so the given vector is not in the range of T.

4.) a.) To write $e_1 = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we solve the system $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$, so $e_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

b.) $T(e_1) = T(\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}) = \frac{1}{2} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$.

c.) A similar computation shows $T(e_2) = T(\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}) = \frac{1}{2} \begin{pmatrix} 2 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}$. So the standard matrix of T is given by $A = [T(e_1)|T(e_2)] = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -2 \end{pmatrix}$.

5.) We have that $A = [T(e_1)|T(e_2)]$. From the given description of T we see that $T(e_1) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. So, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

6.) a.) The determinant of this matrix is $(3)(4) - (-2)(5) = 22$. Since the determinant is non-zero, this matrix is invertible. The quickest way to find the inverse of a 2×2 matrix is to use the formula mentioned in problem (1). So, $A^{-1} = \frac{1}{22} \begin{pmatrix} 4 & -5 \\ 2 & 3 \end{pmatrix}$.

b.) Here the determinant is $(-6)(-2) - (3)(4) = 0$, so the matrix is not invertible.

7.) The (square) matrix is invertible iff every column (or row) has a pivot. Row-reducing we have:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 2 & -1 & k \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & -5 & k-2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & 0 & k+2 \end{pmatrix}.$$

So, we see that A is invertible iff $k \neq -2$.

8.) We row-reduce A to I and see what these operations turn I into.

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 3 & -6 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & -2 & 3 & 1 & 1 & 0 \\ 0 & 3 & -5 & -3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & \frac{3}{2} & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & -3 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -9 & -5 \\ 0 & 1 & 0 & 4 & -5 & -3 \\ 0 & 0 & 1 & 3 & -3 & -2 \end{array}\right)$$

So, $A^{-1} = \begin{pmatrix} 7 & -9 & -5 \\ 4 & -5 & -3 \\ 3 & -3 & -2 \end{pmatrix}$.

9.) a.) Row-reducing A to I we have

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} \overset{(-3R_1+R_2)}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \overset{(\frac{1}{2}R_2)}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \overset{(-2R_2+R_1)}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, $E_{-2R_2+R_1} E_{\frac{1}{2}R_2} E_{-3R_1+R_2} A = I$.
b.) From part (a) we see that $A^{-1} = E_{-2R_2+R_1} E_{\frac{1}{2}R_2} E_{-3R_1+R_2}$, so

$$A = (E_{-2R_2+R_1} E_{\frac{1}{2}R_2} E_{-3R_1+R_2})^{-1} = E_{3R_1+R_2} E_{2R_2} E_{2R_2+R_1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

10.) a.) The null-space is the set of solutions to the homogeneous system $A \cdot x = \vec{0}$. We solve the homogebeous system by row-reduction:

$$\begin{pmatrix} 1 & 2 & -3 & 2 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 2 & 3 & -5 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & -3 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

So, the general solution to the homogeneous is given by

$$\begin{pmatrix} x_3 \\ x_3 \\ x_3 \\ 0 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

So, a basis for the null-space of A is the set $\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \}$. so, the null-space has dimension 2.

b.) We see that columns 1, 2, and 4 have pivot positions. So, going back to the original matrix, we see that $\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \}$ forms a basis for the column space of A. So, the dimension of the column space is 3.

11.) a.) If there are 3 pivot positions, then there are three pivot columns, so there are 4 columns without pivots. This is number of free variables, which in turn is the dimension of the null-space. So, the null-space of A has dimension 4.

b.) There are 3 columns with pivots, so the dimension of the column space is 3.

12.) The span of the given vectors is the same as the column space of the matrix A obtained by putting the given vectors as columns of the matrix A. Row-reducing the given matrix A we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & 3 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that columns 1, 2, and 4 have pivots, so the a basis for the column space is the set of vectors $\{\begin{pmatrix}1\\2\\5\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix}\}$.

13.) Computing the transpose of the given matrix we have $(AA^T)^T = (A^T)^T A^T = AA^T$. So, AA^T is symmetric.

14.) The elements of the subspace can be written in the form

$$a\begin{pmatrix}1\\1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + c\begin{pmatrix}-1\\3\\0\\1\end{pmatrix}.$$

So, the vectors $\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\3\\0\\1\end{pmatrix}$ span the subspace H. To be a basis, we need vectors which span the span and are also independent. To check for independence, we put these vectors into the columns of a matrix and row-reduce. We have:

$$\begin{pmatrix}1 & 1 & -1\\1 & -1 & 3\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1\\0 & 1 & -2\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}.$$

We see that not every column has a pivot, so the vectors are not independent. We see, however, that columns 1 and 2 have pivots, so we can take the corresponding columns of A. So, the vectors $\{\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\}$ form a basis for H. So, the dimension of H is 2.