Math 2700
Review For First Test.

1.) $A = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 1 & -2 & 2 & 0 \\ 1 & -2 & 3 & 2 \\ 2 & -4 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

For the first step we did the operations $-R_1 + R_2$, $-R_1 + R_3$, and $-2R_1 + R_3$. For the second step we did $R_2 \leftrightarrow R_3$ and $-R_2 + R_4$. For the last step we did $-3R_3 + R_2$, $R_3 + R_1$ and $-2R_2 + R_1$.

2.) a.) As a matrix equation this is: $\begin{pmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ -1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}.$

b.) Forming the augmented matrix and row-reducing we have:
$\begin{pmatrix} 1 & 3 & 1 & 2 \\ -1 & 0 & 2 & 1 \\ -1 & 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 3 & 3 & 3 \\ 0 & 4 & 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$
So the unique solution is $x = -5$, $y = 3$, $z = -2$.

3.) This matrix is already in reduced row-echelon form. x_2 and x_5 are arbitrary.
The general solution is given by: $\begin{pmatrix} x_2 + 2x_5 + 1 \\ x_2 \\ -3x_5 + 2 \\ -2x_5 + 3 \\ x_5 \\ 4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 0 \\ 4 \end{pmatrix}.$

4.) a.) The vector $\begin{pmatrix} 1 \\ 2 \\ h \end{pmatrix}$ is in the span of $\{v_1, v_2, v_3\}$ iff the system $c_1v_1 + c_2v_2 + c_3v_3 = \begin{pmatrix} 1 \\ 2 \\ h \end{pmatrix}$ is consistent. We form the corresponding augmented matrix and row-reduce:
$\begin{pmatrix} 2 & -1 & 3 & 1 \\ -1 & 1 & -2 & 2 \\ 1 & 1 & 0 & h \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -2 & 2 \\ 0 & 1 & -1 & 5 \\ 0 & 2 & -2 & h+2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & h-8 \end{pmatrix}$
which is now in echelon form. The system is consistent iff $h = 8$.

b.) The span is the set of vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ such that the system $\begin{pmatrix} 2 & -1 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 1 & 1 & 0 & b_3 \end{pmatrix}$ is consistent. Row-reducing this becomes $\begin{pmatrix} 1 & -1 & 2 & -b_2 \\ 0 & 1 & -1 & b_1 + 2b_2 \\ 0 & 0 & 0 & -2b_1 - 3b_2 + b_3 \end{pmatrix}$. So, the vector is in the span iff $-2b_1 - 3b_2 + b_3 = 0$ (this defines a plane through the origin).

5.) a.) They span $\mathbb{R}^3$ iff for all $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$ the system $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \vec{b}$ is consistent. This will happen iff every row in the echelon form of the matrix $\begin{pmatrix} 1 & -1 & -3 & 2 \\ 2 & 3 & 4 & 1 \\ -1 & 1 & 3 & 1 \end{pmatrix}$ has a pivot entry. Row-reducing this becomes $\begin{pmatrix} 1 & -1 & -3 & 2 \\ 0 & 5 & 10 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ which is in echelon form, and we see that every row has a pivot entry. Thus, $\{v_1, v_2, v_3, v_4\}$ span $\mathbb{R}^3$.

b.) The vectors are independent iff the system $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = \vec{0}$ has only the trivial solution. That is, iff every column in the echelon form has a pivot entry. Even without row-reducing we know this is impossible since the matrix is a 3×4 (i.e., more columns than rows).

c.) If we omit the third column from the above matrix, the previous computation shows that the new matrix is row-equivalent to $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & 3 \end{pmatrix}$. Since every column now has a pivot entry, the vectors $\{v_1, v_2, v_4\}$ are independent.

6.) a.) We form the matrix A by insering the given vectors as colunmns of the matrix. This gives the matrix $A = \begin{pmatrix} -2 & 2 & -3 & 5 \\ 1 & -1 & 2 & 3 \\ 1 & -1 & 2 & 4 \\ -2 & 2 & -3 & 2 \end{pmatrix}$. Reducing to echelon form we have:

$$A \sim \begin{pmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & -3 & 5 \\ 1 & -1 & 2 & 4 \\ -2 & 2 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that there is not a pivot in every column, so the given vectora are not independent.

b.) There is also not a pivot in every row, so the vectors do not span $\mathbb{R}^4$.

7.) a.) We form the matrix whose columns are the given vectors, and then row-reduce. $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & k & -1 \\ 1 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & k & -3 \\ 0 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & k & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & k & -3 \end{pmatrix} \sim$
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & -3-\frac{2}{3}k \end{pmatrix}$. The vectors are independent iff every column of A has a pivot. This happens iff $k \neq -\frac{9}{2}$.

b.) The vectors span $\mathbb{R}^3$ iff every row of A has a pivot. Since this is a square matrix, it is the same answer as for (a).

8.) a.) We determine which vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ are the span of the given two vectors by determining when the system $c_1 v_1 + c_2 v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is consistent. This gives the augmented matrix $\begin{pmatrix} 1 & -2 & x \\ 2 & 1 & y \\ -3 & 1 & z \end{pmatrix}$. Row reducing we get $\begin{pmatrix} 1 & -2 & x \\ 2 & 1 & y \\ -3 & 1 & z \end{pmatrix} \sim$
$\begin{pmatrix} 1 & -2 & x \\ 0 & 5 & -2x+y \\ 0 & -5 & 3x+z \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & x \\ 0 & 5 & -2x+y \\ 0 & 0 & x+y+z \end{pmatrix}$. The system is consistent iff $x+y+z=0$. So, the span of the two vectors is the plane in $\mathbb{R}^3$ with equation $x+y+z=0$.

9.) a.) This gives a homogeneous system with coefficient matrix $A = \begin{pmatrix} 2 & 1 & 2 & b_1 \\ 3 & 2 & 1 & b_2 \\ 2 & 3 & -6 & b_3 \end{pmatrix}$.
Row-reducing we get $A \sim \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$. The general solution is given by $\begin{pmatrix} -3x_3 \\ 4x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$.

b.) We form the augmented matrix corresponding the matrix equation $A \cdot \vec{x} = \vec{b}$. This gives $\begin{pmatrix} 2 & 1 & 2 & b_1 \\ 3 & 2 & 1 & b_2 \\ 2 & 3 & -6 & b_3 \end{pmatrix}$. Row-reducing we get $\begin{pmatrix} 2 & 1 & 2 & b_1 \\ 3 & 2 & 1 & b_2 \\ 2 & 3 & -6 & b_3 \end{pmatrix} \sim$
$\begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2}b_1 \\ 3 & 2 & 1 & b_2 \\ 2 & 3 & -6 & b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2}b_1 \\ 0 & \frac{1}{2} & -2 & -\frac{3}{2}b_1 + b_2 \\ 0 & 2 & -8 & -b_1 + b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2}b_1 \\ 0 & 1 & -4 & -3b_1 + 2b_2 \\ 0 & 2 & -8 & -b_1 + b_3 \end{pmatrix} \sim$
$\begin{pmatrix} 1 & \frac{1}{2} & 1 & \frac{1}{2}b_1 \\ 0 & 1 & -4 & -3b_1 + 2b_2 \\ 0 & 0 & 0 & 5b_1 - 4b_2 + b_3 \end{pmatrix}$. So, the system is consistent iff $5b_1 - 4b_2 + b_3 = 0$.

10.) For the columns to span $\mathbb{R}^m$, every row must have a pivot entry, so there must by m pivot entries. This means we must have $n \geq m$.

11.) We want the matrix equation $A \cdot \vec{x} = \vec{0}$ to be equivalent to the system $x_1 + x_2 + x_3 = 0$. One way to do this is to take that equation and add two trivial equations $0 = 0$. The matrix would be $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

12.) a.) $A \cdot u = \begin{pmatrix} -8 \\ 10 \end{pmatrix}$.

b.) $a \cdot (3u - 2v) = A \cdot \begin{pmatrix} -1 & 15 & -4 \end{pmatrix} = \begin{pmatrix} -46 \\ 68 \end{pmatrix}$.

c.) This is the set of solutions to the homogeneous system. Row-reducing we get $A = \begin{pmatrix} 4 & -2 & 3 \\ -1 & 5 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & -2 \\ 0 & 18 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & -5 & -2 \\ 0 & 1 & \frac{11}{18} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{19}{18} \\ 0 & 1 & \frac{11}{18} \end{pmatrix}$. The general solution in parametric form is $\begin{pmatrix} -\frac{19}{19}x_3 \\ -\frac{11}{18}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{19}{18} \\ -\frac{11}{18} \\ 1 \end{pmatrix}$. Since x_3 is arbitrary, we could also write this as $x_3 \begin{pmatrix} -19 \\ -11 \\ 18 \end{pmatrix}$. This is a line through the origin in $\mathbb{R}^3$.

13.) This is a homgeneous system, so it must be consistent. We solve by rowreduction: $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & -1 & 4 & 3 \\ 1 & 0 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We see that x_3 is a free variable. The general solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

14.) a.) If we call the points $p = (2, -1, 4)$ and $q = (3, 1, 6)$, then the line through the points is given by $p + t(q - p)$. This gives $\begin{pmatrix} 2+t \\ -1+2t \\ 4+2t \end{pmatrix}$. Writing this in parametric form we have $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

b.) Doing the same for the second pair of points gives the line $\begin{pmatrix} 1+2u \\ 2+u \\ -1+3u \end{pmatrix}$. To see if these lines intersect, we see if we can solve the system

$$\begin{aligned} 2 + t &= 1 + 2u \\ -1 + 2t &= 2 + u \\ 4 + 2t &= -1 + 3u \end{aligned}$$

The augmented matrix for this system is given by $\begin{pmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 2 & -3 & -5 \end{pmatrix}$. Row-reducing, this becomes $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 14 \end{pmatrix}$. So, this system is inconsistent, which means the two lines do not intersect.