Solutions To Review For Final

1.) a.) Row-reducing the matrix we have that $A \sim \begin{pmatrix} 0 & 1 & 0 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. A has 3 pivot positions, so $\text{rank}(A) = 3$. The dimension of the column space is equal to the rank, so $\dim(\text{col}(A)) = 3$. Since $\dim(\text{null}(A)) + \dim(\text{col}(A)) = 6$ (the number of columns of A), we have that $\dim(\text{null}(A)) = 3$ as well.

b.) A basis for the column space can be obtained by taking the columns of A corresponding to the pivot positions. So, a basis for the column space is $\mathcal{B} = \{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -1 \\ -1 \end{pmatrix}\}$. From the reduced row-echelon form of A we read off the solution to the homogeneous system. The variables x_1, x_4, x_6 are arbitrary, and the general solution is given by

$$\begin{pmatrix} x_1 \\ 7x_4 - 4x_6 \\ -2x_4 + x_6 \\ x_4 \\ x_6 \\ x_6 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 7 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

So, a basis for the null space is $\mathcal{B} = \{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}\}$.

2.) a.) The vector $\begin{pmatrix} 2 \\ 1 \\ k \end{pmatrix}$ is in $\text{null}(A)$ iff $A = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Multiplying these we get $\begin{pmatrix} 6 + 2k \\ 3 + k \\ 3 + k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. We see that $k = -3$ satisfies these equations, and is the only value which does.

b.) The vector $\begin{pmatrix} 2 \\ 1 \\ k \end{pmatrix}$ is in $\text{col}(A)$ iff the augmented system $\left(\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 3 & 1 & 1 \\ 2 & -1 & 1 & k \end{array}\right)$ is consistent. Row-reducing, this becomes $\left(\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & -1 + k \end{array}\right)$. This is consistent iff $k = 1$. So, the vector is in the column space of A iff $k = 1$.

3.) a.) Putting the vectors as columns into a matrix and row-reducing gives $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We take the vectors corresponding to the columns with pivot entries. Thus, a basis for span$\{v_1, v_2, v_3\}$ is given by $\mathcal{B} = \{v_1, v_3\}$.

b.) We append the columns corresponding to the vectors w_1, w_2, w_3, w_4 to the matrix A and row-reduce. We have

$$A_2 = \begin{pmatrix} 1 & 2 & 2 & 0 & -1 & 1 & 1 \\ 2 & 4 & 1 & 3 & 1 & 1 & -1 \\ -1 & -2 & 2 & -4 & -3 & 0 & 2 \\ 1 & 2 & 1 & 1 & 0 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, $\{v_1, v_3, w_3, w_4\}$ form a basis for $\mathbb{R}^4$.

4.) Pulling out the parameters we have that H is all vectors of the form

$$z \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}.$$

So, the vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$ span the subspace H. However, they may not be independent. We put then into the columns of a matrix amd row-reduce. We have $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ 0 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. So, we see that the vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$ are independent and span H, and so form a basis for H.

5.) a.) The vectors are independent iff every column of the corresponding matrix has a pivot. So, we form the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & -5 \end{pmatrix}$. Row-reducing (to echelon form) we have $A \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$. We see that not every column has a pivot, so the vectors are not independent. Note that A is square matrix, so we can also check independence by computing the determinant. We see that $\det(A) = 0$, which also says the vectors are dependent. The vectors span iff every row has a pivot. This is not the case here, so these vectors do not span $\mathbb{R}^3$. Note that for n vectors in $\mathbb{R}^n$ (i.e., a square matrix), the vectors are independent iff they span $\mathbb{R}^n$. To be a basis they need to span and be independent. So these vectors are not a basis.

b.) Putting u, v and x as columns into a matrix we have the matrix

$$A = \begin{pmatrix} 1 & 1 & k \\ 2 & 1 & 1 \\ -1 & 2 & k \end{pmatrix}.$$

Row-reducing we have $A \sim \begin{pmatrix} 1 & 1 & k \\ 0 & 1 & -1+2k \\ 0 & 0 & 3-4k \end{pmatrix}$. So, the vectors span $\mathbb{R}^3$ iff (i.e., every row has a pivot) $k \neq \frac{3}{4}$.

6.) a.) For $A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 1 & 4 \\ -2 & 1 & -2 \end{pmatrix}$ we have, expanding along the third column, $\det(A) = (-3)\det\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} - (4)\det\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} + (-2)\det\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = -6-16-4 = -26$.

b.) The cofactor matrix of A is given by $C = \begin{pmatrix} -6 & -8 & 2 \\ -1 & -10 & -4 \\ 7 & -8 & 2 \end{pmatrix}$. So,

$$A^{-1} = \frac{1}{det(A)}C^T = -\frac{1}{26}\begin{pmatrix} -6 & -1 & 7 \\ -8 & -10 & -8 \\ 2 & -4 & 2 \end{pmatrix} = \frac{1}{26}\begin{pmatrix} 6 & 1 & -7 \\ 8 & 10 & 8 \\ -2 & 4 & -2 \end{pmatrix}.$$

7.) From the adjugate formula for the inverse we have $A^{-1}_{(3,2)} = \frac{1}{\det(A)}C_{(2,3)}$, where C is the cofactor matrix for A. We have $C_{(2,3)} = (-1)\det\begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 3 \\ 2 & -2 & 1 \end{pmatrix} = (-1)[(-1)(4-6)+(2)(-8-2)] = 18$. So, $A^{-1}_{(3,2)} = -\frac{18}{40} = -\frac{9}{20}$.

8.) We have:

$$\det\begin{pmatrix}2 & -1 & 3 & 6\\ 4 & -1 & 5 & 10\\ 1 & 3 & -2 & -3\\ 8 & -4 & 6 & 12\end{pmatrix} = \det\begin{pmatrix}2 & -1 & 3 & 6\\ 0 & 1 & -1 & -2\\ 0 & \frac{7}{2} & -\frac{7}{2} & -6\\ 0 & 0 & -6 & -12\end{pmatrix} (-2R_1+R_2, -\frac{1}{2}R_1+R_3, -4R_1+R_4)$$

$$= (2)\det\begin{pmatrix}1 & -1 & -2\\ \frac{7}{2} & -\frac{7}{2} & -6\\ 0 & -6 & -12\end{pmatrix} \text{ (cofactor along first column)}$$

$$= (-12)\det\begin{pmatrix}1 & -1 & -2\\ \frac{7}{2} & -\frac{7}{2} & -6\\ 0 & 1 & 2\end{pmatrix} \text{ (linearity along third row)}$$

$$= (-12)\det\begin{pmatrix}1 & -1 & -2\\ 0 & 0 & 1\\ 0 & 1 & 2\end{pmatrix} (-\frac{7}{2}R_1+R_2)$$

$$= (12)\det\begin{pmatrix}1 & -1 & -2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix} (R_2 \leftrightarrow R_3)$$

$$= 12.$$

9.) a.) If we break the quadrilateral into two triangles by drawing a line from p to r, then the area of $\triangle pqr = \frac{1}{2}|\det\begin{pmatrix}1 & 6\\ 10 & 9\end{pmatrix}| = 26$. The area of $\triangle rps = \frac{1}{2}|\det\begin{pmatrix}-6 & -2\\ -8 & -11\end{pmatrix}| = 25$. So, the total area of the quadrilateral is $26+25=51$.

b.) The volume of the tetrahedron is $\frac{1}{6}$ the volume of the parallelpiped. Using p as the reference point, the volume is $\frac{1}{6}|\det\begin{pmatrix}1 & 3 & -1\\ 0 & -2 & 1\\ 1 & 1 & 5\end{pmatrix}| = \frac{1}{6}|-10| = \frac{5}{3}$.

10.) The area of the annulus between the two circles is given by $A = \pi(2^2-1^2) = 3\pi$. The standard matrix corresponding to T is $A = \begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}$. The transformation T changes area by a factor of $|\det(A)| = |3| = 3$. So, the transformed region has area $(3)(3\pi) = 9\pi$.

11.) λ is an eigenvalue of A iff $\det(A-\lambda I) = 0$. For $\lambda = 3$ we have $A - \lambda I = \begin{pmatrix}2 & -1 & 3\\ 1 & 2 & 5\\ 2 & -6 & -4\end{pmatrix}$. We have $A - eI \sim \begin{pmatrix}1 & 2 & 5\\ 0 & -5 & -7\\ 0 & -10 & -14\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 5\\ 0 & 1 & \frac{7}{5}\\ 0 & 0 & 0\end{pmatrix}$. Since we have non-trivial solutions to the homogeneous (i.e., not every column has a pivot) we see that 3 is an eigenvalue for the matrix. Continuing to reduced row-echelon form we have $A-3I \sim \begin{pmatrix} 1 & 0 & \frac{11}{5} \\ 0 & 1 & \frac{7}{5} \\ 0 & 0 & 0 \end{pmatrix}$. The general solution is $\begin{pmatrix} -\frac{11}{5}x_3 \\ -\frac{7}{5}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -\frac{11}{5} \\ -\frac{7}{5} \\ 1 \end{pmatrix}$. So, we may take $\begin{pmatrix} -11 \\ -7 \\ 5 \end{pmatrix}$ as an eigenvector. Notice the eigenspace here has dimension 1.

12.) To be an eigenvector we must have $\begin{pmatrix} -3 & 1 & -1 \\ 3 & 2 & 1 \\ 5 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix} = \begin{pmatrix} -1-k \\ 7+k \\ -3+k \end{pmatrix}$ is equal to $\lambda \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}$ for some (eigenvalue) λ. We must have $7+k = 2(-1-k)$ and so $k=-3$. Plugging in $k=-3$ we have $\begin{pmatrix} -3 & 1 & -1 \\ 3 & 2 & 1 \\ 5 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$. We see that for $k=-3$ the vector is an eigenvector (with eigenvalue 2).

13.) The eigenvalues are $\lambda = -1$ and $\lambda = 2$. The eigenspace for $\lambda = 2$ must have dimension 1. The eigenspace for $\lambda = -1$ may have dimension 1 or 2 (we must row-reduce to determine).

For $\lambda = -1$ we have $A - \lambda I = \begin{pmatrix} 9 & -3 & -3 \\ -9 & 3 & 3 \\ 27 & -9 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The general solution is $\begin{pmatrix} \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$. So, the eigenspace for $\lambda = -1$ has dimendion 2. We may take as a basis $\mathcal{B} = \{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \}$. At this point we know the matrix can be diagonalized, as eaqch eigenspace has the maximum possible dimension.

For $\lambda = 2$ we have $A - 2I = \begin{pmatrix} 6 & -3 & -3 \\ -9 & 0 & 3 \\ 27 & -9 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{9}{2} & -\frac{3}{2} \\ 0 & -9 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}$. The general solution is $\begin{pmatrix} \frac{1}{3}x_3 \\ -\frac{1}{3}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$. We can take as a basis for the eigenspace $\mathcal{B} = \{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \}$.

14.) The characteristic polynomial of A is $p(\lambda) = (5-\lambda)(-4-\lambda) - (-18) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. So, the eigenvalues are $\lambda = -1$ and $\lambda = 2$. Since each has multiplicity 1, the matrix can be diagonalized. We find bases for the eigenspaces.

For $\lambda = 2$ we have $A - \lambda I = \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$. The general solution is $\begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So, a basis for this eigenspace is the single vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

A similar computation shows that for $\lambda = -1$ a basis consists of the single vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. So, $P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$. Then $P^{-1}AP = D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

15.) a.) The matrix is upper-triangular with eigenvalues $\lambda = 2, 2, 3$. The eigenspace for $\lambda = 3$ must be 1-dimensional, so we only need check the eigenspace for $\lambda = 2$. We have

$$A - 2I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has 2 pivots, so the eigenspace has dimension 1. Therefore the matrix A is not diagonalizable.

b.) This matrix is also upper-triangular with characteristic polynomial $p(\lambda) = -(\lambda - 2)^2(\lambda - 1)$. The eigenvalues are $\lambda = 2$ (with multiplicity 2) and $\lambda = 1$ (with multiplicity 1). We only need check $\lambda = 2$. We have $A - 2I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have two rows of zeros, so two columns without pivots, so the dimension of the eigenspace for $\lambda = 2$ is 2. So, the matrix can be diagonalized.

A basis for the $\lambda = 2$ eigenspace is $\mathcal{B} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$. A basis for $\lambda = 1$ eigenspace is $\mathcal{B} = \{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \}$. So, the matrix $P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ will diagonalize A. That is $P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.