Review For Third Test

Test covers sections: 3.1, 3.2, 3.3, 5.1, 5.2

1.) Using elementary row-operations we have:

$$\begin{aligned}\det A &= \det\begin{pmatrix} 2 & 7 & -1 & 9 \\ 1 & 2 & -1 & 3 \\ 2 & -2 & 0 & 2 \\ -2 & -4 & -2 & -6 \end{pmatrix} = -\det\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 7 & -1 & 9 \\ 2 & -2 & 0 & 2 \\ -2 & -4 & -2 & -6 \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & -6 & 2 & -4 \\ 0 & 0 & -4 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & -4 & 0 \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \\ &= -24\end{aligned}$$

2.) a.) The cofactor matrix is $C = \begin{pmatrix} -7 & -11 & 1 \\ -10 & 2 & 3 \\ 4 & -7 & -5 \end{pmatrix}$.

b.) Taking the transpost of C gives the adjoint (or adjugate), which is $\begin{pmatrix} -7 & -10 & 4 \\ -11 & 2 & -7 \\ 1 & -3 & -5 \end{pmatrix}$.
If we take the dot product of the first row of the adjgate and the first column of A, we get $-7-20-4=-31$, which is the determinant.

The adjugate inverse formula is $A^{-1} = \frac{1}{\det A}C^T = -\frac{1}{31}\begin{pmatrix} -7 & -10 & 4 \\ -11 & 2 & -7 \\ 1 & -3 & -5 \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 7 & 10 & -4 \\ 11 & -2 & 7 \\ -1 & 3 & 5 \end{pmatrix}$.

3.) Taking the determinant we have $\det(A) = (-5)(2k-3)-(-1)(6-3)+k(3-k) = -k^2-7k+18 = -(k^2+7k-18) = -(k-2)(k+9)$. So, the matrix is invertble when the determiant is non-zero, which happens when k is not equal to 2 or -9.

4.) a.) The area of the triangle is one-half the area of the parallelogram. So, we have the area of the triangle is $\frac{1}{2}\left|\det\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}\right| = \frac{1}{2}(8-6) = 1$.

b.) The standard matrix A for the linear transformation T is $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$.
We have $|\det(A)| = |-3| = 3$. The area of the new triangle is $|\det(A)|$ times the area of the old triangle. So, the area of the new triangle is $(3)(1) = 3$.

5.) The quadrilateral can be divided into two triangles by a line from $(2,5)$ to $(7,4)$. The area of the triangle with vertices $(1,1)$, $(2,5)$, $(7,4)$ is $\frac{1}{2}\left|\det\begin{pmatrix}1 & 6\\ 4 & 3\end{pmatrix}\right| = \frac{21}{2}$. The area of the triangle with vertices $(2,5)$, $(6,6)$, $(7,4)$ is given by $\frac{1}{2}\left|\det\begin{pmatrix}4 & 5\\ 1 & -1\end{pmatrix}\right| = \frac{9}{2}$. So, the area of the quadrilateral is $\frac{21}{2} + \frac{9}{2} = 15$.

6.) a.) The vectors from the vertex $(1,2,-1)$ to the adjacent vertices are $\vec{u} = \begin{pmatrix}1\\1\\6\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\5\\3\end{pmatrix}$, $\vec{w} = \begin{pmatrix}2\\2\\10\end{pmatrix}$. So, the volume is given by $|\det(\vec{u}|\vec{v}|\vec{w})| = \left|\begin{pmatrix}1 & 3 & 2\\ 1 & 5 & 2\\ 6 & 3 & 10\end{pmatrix}\right| = |-4| = 4$.

b.) The volume of the tetrahedron is $\frac{1}{6}$ the volume of the corresponding tetrahedron. So, the volume is $\frac{1}{6}\left|\det\begin{pmatrix}1 & 2 & 4\\ 2 & 5 & 2\\ 1 & 4 & 1\end{pmatrix}\right| = \frac{1}{6}|9| = \frac{3}{2}$.

7.) a.) The characteristic polynomial is given by $p(\lambda) = \det(A-\lambda I) = \det\begin{pmatrix}2-\lambda & -1\\ 3 & 6-\lambda\end{pmatrix} = (2-\lambda)(6-\lambda)+3 = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3)$.

b.) The eigenvalues are the roots of the characteristic polynomial, so the eigenvalues are $\lambda = 3$ and $\lambda = 5$.

For $\lambda = 3$ we have $A - \lambda I = A = 3I = \begin{pmatrix}-1 & -1\\ 3 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$. The solution is given by $x_2\begin{pmatrix}-1\\1\end{pmatrix}$, so $\begin{pmatrix}-1\\1\end{pmatrix}$ is an eigenvector (in fact, it is a basis for the eigenspace for this eigenvalue).

For $\lambda = 5$ we have $A - \lambda I = A - 5I = \begin{pmatrix}-3 & -1\\ 3 & 1\end{pmatrix} \sim \begin{pmatrix}1 & \frac{1}{3}\\ 0 & 0\end{pmatrix}$. An eigenvector is $\begin{pmatrix}-\frac{1}{3}\\1\end{pmatrix}$. Since a multiple of an eigenvector is also an eigenvector, we can also use $\begin{pmatrix}1\\-3\end{pmatrix}$.

8.) $A - \lambda I = A - 5I = \begin{pmatrix}1 & -1 & 2\\ -1 & 1 & -2\\ 2 & -2 & 4\end{pmatrix}$. Row reducing we have $A - 5I \sim \begin{pmatrix}1 & -1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$ which has solution $x_2\begin{pmatrix}1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-2\\0\\1\end{pmatrix}$. So, a basis for the eigenspace is $\mathcal{B} = \{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\}$. The dimension of the eigenspace is 2.

9.) We have $\begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 4 \\ 3 & -3 & k \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 6+k \end{pmatrix}$. The value of k that makes the vector an eigenvector is when $6 + k = (2)(1) = 2$, or $k = -4$. The corresponding eigenvalue is $\lambda = 2$.

10.) The characteristic polynomial is clearly $(2-\lambda)^2(1-\lambda)^2 = (\lambda-1)^2(\lambda-2)^2$. The eigenvalues are $\lambda = 1$ and $\lambda = 2$.

For $\lambda = 1$ we have $A - \lambda I = A - I = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The solution is $x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the eigenspace for $\lambda = 1$ is $\mathcal{B} = \{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \}$.

For $\lambda = 2$ we have $A - \lambda I = A - 2I = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The solution is given by $x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. A basis for the eigenspace is $\mathcal{B} = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$.