**Solutions To Review For Second Test**

1.) We have: $(A+2B)^T = \begin{pmatrix} 8 & -15 \\ -1 & 14 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \frac{4}{11} & -\frac{3}{11} \\ \frac{1}{11} & \frac{2}{11} \end{pmatrix}$. Also, $AB = \begin{pmatrix} -15 & 11 \\ -31 & 22 \end{pmatrix}$, and so $(AB)^{-1} = \frac{1}{11}\begin{pmatrix} 22 & -11 \\ 31 & -15 \end{pmatrix}$.

2.) From $A(A-B) = (A-B)A$ we have $A^2 - AB = A^2 - BA$, and so $AB = BA$. So, $A^2 + AB = A^2 + BA$, that is $A(A+B) = (A+B)A$.

3.) Row-reducing we have:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -3 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -5 & 1 & -3 & 1 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{5}{2} \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{5}{2} \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{5}{2} \end{bmatrix}$.

4.) Reducing $A$ to reduced row-echelon form gives $A \sim B = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.
Thus, $A$ has 3 pivot positions, so $A$ has rank 3. A basis for the column space of $A$ is the set of vectors given by the 1, 3, and 4 columns of $A$, that is the set $\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\}$. The general solution to the homogeneous system $A\vec{x} = \vec{0}$ is given by

$$\begin{pmatrix} -2x_2 - 4x_5 \\ x_2 \\ x_5 \\ 0 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

So, a basis for the null-space of $A$ is the set $\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\}$. Note that the column space has dimension 3, and the null space has dimension 2, which add to 5, the number of columns of $A$.

5.) The matrix corresponding to $T$ is a $5 \times 7$ matrix. The dimensions of the column space and null space add to 7. Since the null space has dimension 3, the column space has dimension 4. Since the column space has dimension 4, it cannot be all of $\mathbb{R}^5$, so the map cannot be onto (that is, the system $A\vec{x} = \vec{b}$ cannot be consistent for all $\vec{b}$ in $\mathbb{R}^5$).

6.) We put the given vectors into the columns of a matrix. We get $A = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ -3 & -6 & 0 & -6 \end{bmatrix}$.

Row reducing we get $A \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The span of the given set of vectors is the column space of $A$, which we see is spanned by the first and third columns of $A$. So, the vectors $\{v_1, v_3\}$ form a basis for span$\{v_1, v_2, v_3, v_4\}$.

7.) The subspace can be written as

$$a \begin{pmatrix} 4 \\ 1 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} -1 \\ 3 \\ 0 \\ -3 \end{pmatrix}.$$

We test these three vectors for independence.

$\begin{pmatrix} 4 & 3 & -1 \\ 1 & -1 & 3 \\ 0 & 0 & 0 \\ -2 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so the three vectors are independent and thus form a basis for the subspace $H$. So, the dimension of $H$ is 3.

8.) a.) The rank is 3 by definition.

b.) $A$ has $7 - 3 = 4$ columns without pivot entries, so the dimension of the null space is 4.

c.) The dimension of the column space is equal to the rank, which is 3.

9.) We row-reduce $A$ to the identity matrix by the following sequence of ERO's:

$$
\begin{aligned}
A &= \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} 1 & -\frac{1}{3} \\ 2 & 1 \end{pmatrix} \quad (\frac{1}{3}R_1) \\
&\sim \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{5}{3} \end{pmatrix} \quad (-2R_1 + R_2) \\
&\sim \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix} \quad (\frac{3}{5}R_2) \\
&\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (\frac{1}{3}R_2 + R_1)
\end{aligned}
$$

Call the corresponding elementary matrices $E_1, \ldots, E_4$. So, $E_4E_3E_2E_1A = I$, so $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}$.

10.) Row-reducing we have:

$$
\begin{pmatrix} 1 & 1 & k \\ -1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & k \\ 0 & 2 & 3+k \\ 0 & -1 & 1-2k \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & k \\ 0 & -1 & 1-2k \\ 0 & 0 & -3k+5 \end{pmatrix}
$$

So, the matrix is invertible iff $k \neq \frac{5}{3}$.

11.) Expanding along the second row we have:

$$
\det \begin{pmatrix} 1 & -1 & 2 \\ -3 & -1 & 2 \\ 1 & 4 & -1 \end{pmatrix} = -(-3)\det \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + (-1)\det \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} - (2)\det \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} = -21 + 3 - 10 = -28.
$$

12.) Call the vectors $v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ -1 \\ 2 \\ 1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}\}$. Also, let $v = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}$. Then $H = \text{span}\{v_1, v_2, v_3\} = \text{span}\{v, v_1, v_2, v_3\}$. We put these four vectors into a matrix and row-reduce, which will give us a subset of these vectors which span, and which will contain the first vector $v$. We have:

$$
\begin{pmatrix} -1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 1 \\ -3 & -1 & 2 & 3 \\ 1 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 5 & 5 & 3 \\ 0 & -7 & -7 & 0 \\ 0 & 4 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

So we see that the vectors $\{v, v_1, v_3\}$ form a basis for $H$.