Math 2700
Solutions to Review For First Test.

1.) $$A=\begin{bmatrix}1&-2&2&-1\\1&-2&2&0\\1&-2&3&2\\2&-4&5&1\end{bmatrix}\sim\begin{bmatrix}1&-2&2&-1\\0&0&0&1\\0&0&1&3\\0&0&1&3\end{bmatrix}\sim\begin{bmatrix}1&-2&2&-1\\0&0&1&3\\0&0&0&1\\0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-2&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}.$$

2.) a.) As a matrix equation this is: $\begin{bmatrix}1&3&1\\-1&0&2\\-1&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\4\end{bmatrix}$.

b.) Forming the augmented matrix and row-reducing we have:
$$\begin{bmatrix}1&3&1&2\\-1&0&2&1\\-1&1&2&4\end{bmatrix}\sim\begin{bmatrix}1&3&1&2\\0&3&3&3\\0&4&3&6\end{bmatrix}\sim\begin{bmatrix}1&3&1&2\\0&1&1&1\\0&0&-1&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&-5\\0&1&0&3\\0&0&1&-2\end{bmatrix}.$$
So the unique solution is $x=-5$, $y=3$, $z=-2$.

3.) This matrix is already in reduced row-echelon form. x_2 and x_5 are arbitrary.

The general solution is given by: $$\begin{bmatrix}x_2+2x_5+1\\x_2\\-3x_5+2\\-2x_5+3\\x_5\\4\end{bmatrix}=x_2\begin{bmatrix}1\\1\\0\\0\\0\\0\end{bmatrix}+x_5\begin{bmatrix}2\\0\\-3\\-2\\1\\0\end{bmatrix}+\begin{bmatrix}1\\0\\2\\3\\0\\4\end{bmatrix}.$$

4.) a.) The vector $\begin{bmatrix}1\\2\\h\end{bmatrix}$ is in the span of $\{v_1,v_2,v_3\}$ iff the system $x_1v_1+x_2v_2+x_3v_3=\begin{bmatrix}1\\2\\h\end{bmatrix}$ is consistent. We form the corresponding augmented matrix and row-reduce:
$$\begin{bmatrix}2&-1&3&1\\-1&1&-2&2\\1&1&0&h\end{bmatrix}\sim\begin{bmatrix}-1&1&-2&2\\0&1&-1&5\\0&2&-2&h+2\end{bmatrix}\sim\begin{bmatrix}1&-1&2&-2\\0&1&-1&5\\0&0&0&h-8\end{bmatrix}$$
which is now in row-echelon form. The system is consistent iff $h=8$.

b.) The span is the set of vectors $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}\in\mathbb{R}^3$ such that the system $\begin{bmatrix}2&-1&3&b_1\\-1&1&-2&b_2\\1&1&0&b_3\end{bmatrix}$ is consistent. Row-reducing this becomes $\begin{bmatrix}1&-1&2&-b_2\\0&1&-1&b_1+2b_2\\0&0&0&-2b_1-3b_2+b_3\end{bmatrix}$. So, the vector is in the span iff $-2b_1 - 3b_2 + b_3 = 0$ (this defines a plane through the origin).

5.) a.) They span $\mathbb{R}^3$ iff for all $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$ the system $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = \vec{b}$ is consistent. This will happen iff every row in the echelon form of the matrix $\begin{bmatrix} 1 & -1 & -3 & 2 \\ 2 & 3 & 4 & 1 \\ -1 & 1 & 3 & 1 \end{bmatrix}$ has a pivot entry. Row-reducing this becomes $\begin{bmatrix} 1 & -1 & -3 & 2 \\ 0 & 5 & 10 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ which is in echelon form, and we see that every row has a pivot entry. Thus, $\{v_1, v_2, v_3, v_4\}$ span $\mathbb{R}^3$.

b.) The vectors are independent iff the system $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4$ has only the trivial solution. That is, iff every column in the row-echelon form has a pivot entry. Even without row-reducing we know this is impossible since the matrix is a 3×4 (i.e., more columns than rows).

c.) If we omit the third column from the above matrix, the previous computation shows that the new matrix is row-equivalent to $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -3 \\ 0 & 0 & 3 \end{bmatrix}$. Since every column now has a pivot entry, the vectors $\{v_1, v_2, v_4\}$ are independent.

6.) a.) We solve the homogeneous system by row-reduction:
$\begin{bmatrix} 2 & 1 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & 3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & -2 & 0 \\ 0 & 2 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. x_3 is arbitrary, the general solution is $x_3 \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$.

b.) Reducing the augmented system to row-echelon form we have:
$\begin{bmatrix} 2 & 1 & 2 & b_1 \\ 3 & 2 & 1 & b_2 \\ 2 & 3 & -6 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 1 & \frac{b_1}{2} \\ 0 & \frac{1}{2} & -2 & -\frac{3}{2}b_1 + b_2 \\ 0 & 0 & 0 & 5b_1 - 4b_2 + b_3 \end{bmatrix}$. So, the system has a solution iff $5b_1 - 4b_2 + b_3 = 0$.

7.) If the columns span $\mathbb{R}^m$, then every row in the row-echelon form of A has a pivot entry, so A has m pivot entries. For this to be possible, we must have $n \geq m$. In other words, fewer than m vectors cannot span $\mathbb{R}^m$.

8.) One choice would be $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

9.) T corresponds to a 4×3 matrix, namely, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Row-reducing, this matrix becomes $A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

a.) T is one-to-one since every column has a pivot entry, that is, the system $A \cdot \vec{x} = \vec{0}$ has only the trivial solution.

b.) T is not onto since not every row has a pivot entry (so the system $A \cdot \vec{x} = \vec{b}$ will not be consistent for all choices of $\vec{b}$). Note that we don't need to row-reduce to see this, we just need the fact that $4 > 3$.

c.) Row-reducing the augmented system he have:

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

so the system is inconsistent, and thus the vector is not in the range of T.

10.) We already did this, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

11.) a.) To write $e_1 = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ we solve the system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$, so $e_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b.) $T(e_1) = T(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \frac{1}{2} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$.

c.) A similar computation shows $T(e_2) = T(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \frac{1}{2} \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -2 \end{bmatrix}$. So the standard matrix of T is given by $A = [T(e_1)|T(e_2)] = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -2 \end{bmatrix}$.

12.) We have that $A = [T(e_1)|T(e_2)]$. From the given description of T we see that $T(e_1) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. So, $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.