Solutions To Review For Final

1.) a.) Row-reducing the matrix we have that $A \sim \begin{pmatrix} 0 & 1 & 0 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. A has 3 pivot positions, so $\text{rank}(A) = 3$. The dimension of the column space is equal to the rank, so $\dim(\text{col}(A)) = 3$. Since $\dim(\text{null}(A)) + \dim(\text{col}(A)) = 6$ (the number of columns of A), we have that $\dim(\text{null}(A)) = 3$ as well.

b.) A basis for the column space can be obtained by taking the columns of A corresponding to the pivot positions. So, a basis for the column space is $\mathcal{B} = \{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -1 \\ -1 \end{pmatrix}\}$. From the reduced row-echelon form of A we read off the solutin to the homogeneous system. The variables x_1, x_4, x_6 are arbitrary, and the general solution is given by

$$\begin{pmatrix} x_1 \\ 7x_4 - 4x_6 \\ -2x_4 + x_6 \\ x_4 \\ x_6 \\ x_6 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 7 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

So, a basis for the null space is $\mathcal{B} = \{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}\}$.

2.) a.) Putting the vectors as columns into a matrix and row-reducing gives $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 1 \\ -1 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, a basis for $\text{span}\{v_1, v_2, v_3\}$ is given by $\mathcal{B} = \{v_1, v_3\}$.

b.) We append the columns corresponding to the vectors w_1, w_2, w_3, w_4 to the matrix A and row-reduce. We have

$$A_2 = \begin{pmatrix} 1 & 2 & 2 & 0 & -1 & 1 & 1 \\ 2 & 4 & 1 & 3 & 1 & 1 & -1 \\ -1 & -2 & 2 & -4 & -3 & 0 & 2 \\ 1 & 2 & 1 & 1 & 0 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, $\{v_1, v_3, w_3, w_4\}$ form a basis for $\mathbb{R}^4$.

3.) Putting the vectors as columns into a matrix gives a 4×5 matrix. This can have at most $\min\{4, 5\} = 4$ many pivot positions. So every column cannot have a pivot, which says the vectors cannot be independent.

4.) We must have

$$c_1\begin{pmatrix}1\\2\\1\end{pmatrix}+c_2\begin{pmatrix}2\\1\\-1\end{pmatrix}=c_3\begin{pmatrix}3\\1\\2\end{pmatrix}+c_4\begin{pmatrix}-5\\3\\0\end{pmatrix}.$$

Writing this homogeneous system in matrix form and row-reducing we have

$$\begin{pmatrix}1&2&-3&5\\2&1&-1&-3\\1&-1&-2&0\end{pmatrix}\sim\begin{pmatrix}1&0&0&-3\\0&1&0&1\\0&0&1&-2\end{pmatrix}$$

So, if we take $c_4=1$, then a solution is $c_1=3$, $c_2=-1$, $c_3=2$, $c_4=1$. So, a vector in both spans is $v=3\begin{pmatrix}1\\2\\1\end{pmatrix}+(-1)\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\5\\4\end{pmatrix}$.

5.) a.) We have $\det(A)=(3)\det\begin{pmatrix}0&1\\-2&1\end{pmatrix}-(4)\det\begin{pmatrix}2&1\\-2&1\end{pmatrix}+(-2)\det\begin{pmatrix}2&1\\0&1\end{pmatrix}=6-16-4=-14$.

b.) The cofactor matrix of A is given by $C=\begin{pmatrix}-6&-8&2\\5&2&-4\\1&-8&2\end{pmatrix}$. So,

$$A^{-1}=\frac{1}{det(A)}C^T=\frac{1}{14}\begin{pmatrix}6&-5&-1\\8&-2&8\\-2&4&-2\end{pmatrix}.$$

6.) From the adjugate formula for the inverse we have $A^{-1}_{(3,2)}=\frac{1}{\det(A)}C_{(2,3)}$, where C is the cofactor matrix for A. We have $C_{(2,3)}=(-1)\det\begin{pmatrix}0&1&2\\4&1&3\\2&-2&1\end{pmatrix}=(-1)[(-1)(4-6)+(2)(-8-2)]=18$. So, $A^{-1}_{(3,2)}=-\frac{18}{40}=-\frac{9}{20}$.

7.) $\det\begin{pmatrix}2&-1&3&6\\4&-1&5&10\\1&3&-2&-3\\8&-4&6&12\end{pmatrix}=\det\begin{pmatrix}2&-1&3&0\\4&-1&5&0\\1&3&-2&1\\8&-4&6&0\end{pmatrix}$ (using $-2C_3+C_4$) $=(-1)\det\begin{pmatrix}2&-1&3\\4&-1&5\\8&-4&6\end{pmatrix}=(-1)\det\begin{pmatrix}2&-1&3\\4&-1&5\\0&0&-6\end{pmatrix}$ (using $-2R_1+R_3$) $=6\det\begin{pmatrix}2&-1\\4&-1\end{pmatrix}=(6)(2)=12$.

8.) a.) Compute the area of the quadrilateral with vertices at $p=(-2,-3)$, $q=(-1,7)$, $r=(4,5)$, and $s=(2,-6)$.

b.) Compute the volume of the tetrahedron with vertices at $p=(1,1,2)$, $q=(2,1,3)$, $r=(4,-1,3)$, and $s=(0,2,7)$.

c.) If T is the linear transformation given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ -x+y \end{pmatrix}$, compute the area of $T(A)$ where A is the region between the circles of radius 1 and 2 centered at $(0,0)$.

8.) a.) If we break the quadrilateral into two triangles by drawing a line from p to r, then the area of $\triangle pqr = \frac{1}{2}|\det\begin{pmatrix} 1 & 6 \\ 10 & 9 \end{pmatrix}| = 26$. The area of $\triangle rps = \frac{1}{2}|\det\begin{pmatrix} -6 & -2 \\ -8 & -11 \end{pmatrix}| = 25$. So, the total area of the quadrilateral is $26 + 25 = 51$.

b.) The volume of the tetrahedron is $\frac{1}{6}$ the volume of the parallelpiped. Using p as the reference point, the volume is $\frac{1}{6}|\det\begin{pmatrix} 1 & 3 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 5 \end{pmatrix}| = \frac{1}{6}|-10| = \frac{5}{3}$.

c.) The area of the annulus between the two circles is given by $A = \pi(2^2 - 1^2) = 3\pi$. The standard matrix corresponding to T is $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. The transformation T changes area by a factor of $|\det(A)| = |3| = 3$. So, the transformed region has area $(3)(3\pi) = 9\pi$.

9.) a.) Since $[x]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$, $x = (-1)v_1 + (3)v_2 + (-2)v_3 = \begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$.

b.) Let $P = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & -2 & 3 \end{pmatrix}$, so P is the change of coordinates matric from $\mathcal{B}$ to std coordinates. Then $P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -8 & -3 \\ 1 & 4 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is the change of coordinates matrix from std to $\mathcal{B}$ coordinates. So, $[x]_{\mathcal{B}} = P^{-1}[x]_{\text{std}} = \frac{1}{4}\begin{pmatrix} 1 & -8 & -3 \\ 1 & 4 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{4} \\ \frac{7}{4} \\ \frac{3}{4} \end{pmatrix}$.

10.) a.) This is the matrix $P = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

b.) This is the matrix $P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$.

c.) This is the matrix $Q^{-1}P$ where $Q = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ is the change of coordinates matrix from $\mathcal{C}$ to standard coordinates. We have

$$Q^{-1}P = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$$

11.) a.) $[T]_{\mathcal{B}} = P^{-1}[T]_{\text{std}}P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

b.) $[T]_{\mathcal{C}} = (Q^{-1}P)[T]_{\mathcal{B}}(P^{-1}Q) = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

12.) The characteristic polynomial of A is $p(\lambda) = (17-\lambda)(-18-\lambda)+300 = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$. So, the eigenvalues are $\lambda=-3$ and $\lambda=2$. Since each has multiplicity 1, the matrix can be diagonalized. We find bases for the eigenspaces.

For $\lambda = 2$ we have $A - \lambda I = \begin{pmatrix} 15 & -10 \\ 30 & -20 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{pmatrix}$. So, a basis for this eigenspace is the single vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

A similar computation shows that for $\lambda=-3$ a basis consists of the single vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Let $P = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. Then $P^{-1}AP = D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$. The basis $\mathcal{B} = \{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$ is such that $[T]_{\mathcal{B}} = D$ is diagonal.

13.) We have $\begin{pmatrix} -3 & 1 & -1 \\ 3 & 2 & 1 \\ 5 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix} = \begin{pmatrix} -1-k \\ 7+k \\ -3+k \end{pmatrix}$. We must have $-1-k = \frac{7+k}{2}$, which gives $k=-3$. The eigenvalue must then be $\lambda = 2$. For $k=-3$, we see that the vector is an eigenvector for this eigenvalue. So, $k=-3$ is only value for which the vector is an eigenvector.

14.) For $\lambda = -1$ we have

$$A - \lambda I = \begin{pmatrix} 9 & -3 & -3 \\ -9 & 3 & 3 \\ 27 & -9 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The general solution to the homogeneous system is

$$\begin{pmatrix} \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2\begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$$

So, a basis for this eigenspace is $\mathcal{B} = \{\begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}\}$. This eigenspace is 2-dimensional.

A similar computation shows that for $\lambda = 2$ a basis for the eigenspace is $\mathcal{B} = \{\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}\}$. This eigenspace is 1-dimensional.

Since the dimensions of the eigenspaces add up to 3, the matrix A can be diagonalized.

Note that we did not actually have to check the $\lambda = 2$ case to see if the matrix could be diagolalized, since the dimension of this eigenspace must be exactly 1 (as the algebraic multiplicity of this eigenvalue is 1).

15.) The matrix is upper-triangular with eigenvalues $\lambda = 2, 2, 3$. The eigenspace for $\lambda = 3$ must be 1-dimensional, so we only need check the eigenspace for $\lambda = 2$. We have

$$A - 2I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has 2 pivots, so the eigenspace has dimension 1. Therefore the matrix A is not diagonalizable.