COLLOQUIUM MATHEMATICUM

VOL. 164 2021 NO. 1

# DISJOINTNESS BETWEEN BOUNDED RANK-ONE TRANSFORMATIONS

BY

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**Abstract.** In this paper some sufficient conditions are given for when two bounded rank-one transformations are non-isomorphic and when they are disjoint. We also obtain sufficient conditions for a bounded rank-one transformation to have minimal self-joinings. For commensurate, canonically bounded rank-one transformations, isomorphism and disjointness are completely determined by simple conditions in terms of their cutting and spacer parameters.

**1. Introduction.** The research in this paper is motivated by the observation of Foreman, Rudolph, and Weiss [4], based on King's Weak Closure Theorem for rank-one transformations [12], that the isomorphism relation for rank-one transformations is a Borel equivalence relation. Our objective has been to identify a concrete algorithm to determine when two rank-one transformations are isomorphic.

The broader context of this research is the isomorphism problem in ergodic theory, originally posed by von Neumann, that asks how to determine when two (invertible) measure-preserving transformations are isomorphic.

Recall that a *measure-preserving transformation* is an automorphism of a standard Lebesgue space. Formally, it is a quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a measure space isomorphic to the unit interval with the Lebesgue measure on all Borel sets, and $T$ is a bijection from $X$ to $X$ such that $T$ and $T^{-1}$ are both $\mu$-measurable and preserve the measure $\mu$. When the algebra of measurable sets is clear, we refer to the transformation $(X, \mathcal{B}, \mu, T)$ simply by $(X, \mu, T)$.

Two measure-preserving transformations $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *isomorphic* if there is a measure isomorphism $\varphi$ from $(X, \mathcal{B}, \mu)$ to $(Y, \mathcal{C}, \nu)$ such that $\varphi \circ T = S \circ \varphi$ a.e.

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2020 *Mathematics Subject Classification*: Primary 37A05, 37A35, 37B10; Secondary 28D05, 54H20.
*Key words and phrases*: rank-one transformation, rank-one word, rank-one symbolic system, isomorphic, disjoint, commensurate, minimal self-joinings, totally ergodic, non-rigid.
Received 28 October 2019; revised 29 January 2020.
Published online 31 August 2020.

DOI: 10.4064/cm8081-2-2020  

Halmos and von Neumann showed that two ergodic measure-preserving transformations with pure point spectrum are isomorphic if and only if they have the same spectrum. Ornstein's celebrated theorem states that two Bernoulli shifts are isomorphic if and only if they have the same entropy. These are successful answers to the isomorphism problem for subclasses of measure-preserving transformations. For each of them, there is a concrete algorithm, which can be carried out at least in theory, to determine when two given measure-preserving transformations are isomorphic.

Foreman, Rudolph, and Weiss [4] showed that the isomorphism relation for ergodic measure-preserving transformations is a complete analytic equivalence relation, and in particular not Borel. Intuitively, this rules out the existence of a satisfactory answer to the original isomorphism problem of von Neumann. However, in the same paper they showed that the isomorphism relation becomes much simpler when restricted to the generic class of rank-one transformations. Although their method does not yield a concrete algorithm for the isomorphism problem for rank-one transformations, it gives hope that the isomorphism problem has a satisfactory solution for a generic class of measure-preserving transformations. Since rank-one transformations are given by their cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ (more details are given in the next section), a satisfactory solution to the isomorphism problem would correspond to a simple algorithm that yields a yes or no answer with these parameters as input.

In this paper we make some progress toward such a satisfactory solution. In Section 3, under the assumption that the cutting and spacer parameters are bounded and commensurate, we investigate the isomorphism problem and yield conditions sufficient to guarantee non-isomorphism. The basic techniques of this investigation come from the recent paper [9] by the second author. In Section 4, we investigate a stronger notion of non-isomorphism, namely, that of disjointness between measure-preserving transformations. Two measure-preserving transformations $(X, \mu, T)$ and $(Y, \nu, S)$ are *disjoint* if $\mu \times \nu$ is the only measure on $X \times Y$ that is $T \times S$-invariant and has $\mu$ and $\nu$ as marginals. A main result of this paper (Theorem 3.2) gives conditions sufficient to guarantee that two commensurate bounded rank-one transformations are disjoint. Here we follow generally the approach of del Junco, Rahe, and Swanson [10] in showing that Chacon's transformation has minimal self-joinings of all orders.

In Section 6 we apply our results about non-isomorphism and disjointness to give simple algorithms to determine isomorphism and disjointness for canonically bounded rank-one transformations that are commensurate (Corollary 5.5). The notion of canonically bounded rank-one transformations was defined in [6] and was used in [7] to characterize non-rigidity for bounded rank-one transformations. Our results on isomorphism and disjointness for canonically bounded rank-one transformations extend what was already known for a class of Chacon-like transformations. Chacon's transformation is a prototypical example of canonically bounded rank-one transformations; it can be described by the cutting parameter that is constantly equal to 3 and the spacer parameter that is constantly equal to $(0, 1, 0)$—i.e., there are no spacers inserted at the first opportunity, a single spacer inserted at the second opportunity, and no spacers inserted at the end. Given any sequence $e = (e_n : n \in \mathbb{N})$ of 0s and 1s, we can build a Chacon-like transformation $T_e$ as follows. The cutting parameter for the transformation will be constantly equal to 3 and the spacer parameter at stage $n$ will be $(0, 1, 0)$ if $e_n = 1$ and $(1, 0, 0)$ if $e_n = 0$. Fieldsteel [3] showed that the transformations $T_e$ and $T_{e'}$ that are constructed in this way are isomorphic iff $e$ and $e'$ eventually agree, i.e., there is some $N \in \mathbb{N}$ such that $e_n = e'_n$ for all $n \geq N$. It is an exercise in Rudolph's book [14] to show that if $e$ and $e'$ do not eventually agree, then $T_e$ and $T_{e'}$ are in fact disjoint.

A secondary line of investigation in this paper deals with the property of minimal self-joinings. Del Junco, Rahe, and Swanson showed that Chacon's transformation—in fact, any Chacon-like transformation—has minimal self-joinings of all orders. In Section 5 of the present paper, we extend their result by giving very general conditions which are sufficient to guarantee that a bounded rank-one transformation has minimal self-joinings of all orders (Theorem 4.1). Section 6 contains a proof of a case of Ryzhikov's theorem that a bounded rank-one transformation has minimal self-joinings if and only if it is non-rigid and totally ergodic; our proof applies to strictly bounded rank-one transformations (those for which no spacers are ever inserted at the last opportunity). We include, using characterizations of non-rigidity and total ergodicity for strictly bounded rank-one transformations (stated in [7]), a simple algorithm for determining whether a strictly bounded rank-one transformation has minimal self-joinings of all orders.

In the final section of our paper, we give some concluding remarks and explain how the main results can be generalized to the broader context of eventually commensurate constructions.

We remark that a recent paper of Danilenko [1] provides alternative proofs and generalizations of many of our results here, using a more general framework of $(C, F)$-constructions. In particular, Theorem J of [1] generalizes our Corollary 5.5, where we obtain our results for the so-called canonically bounded transformations and the class considered by Danilenko is the more general bounded transformations. Part (i) of [1, Theorem J] also extends our Theorem 3.1 and part (ii) extends our Theorem 3.2. The former extension is by providing a characterization of isomorphism rather than just a sufficient condition. In an earlier version of this paper we considered rank-one transformations that are called *adapted* by Danilenko and *strictly bounded* by us in Section 5, and Danilenko has considered general transformations that are not adapted. In the present paper, except in Section 5, we are considering non-adapted general transformations.

**2. Preliminaries.** Throughout this paper we let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$.

**2.1. Finite sequences, finite functions, and finite words.** Let $\mathcal{S}$ be the set of all finite sequences of natural numbers. We will introduce some operations and relations on $\mathcal{S}$. We view each element of $\mathcal{S}$ from three different perspectives, that is, as a finite sequence, as a function with a finite domain, and as a finite word. For each $s \in \mathcal{S}$, let $\mathrm{lh}(s)$ denote the *length* of $s$. Let $()$ denote the unique (empty) sequence of length 0. A non-empty sequence in $\mathcal{S}$ is of the form $s = (a_1, \ldots, a_n)$ where $n = \mathrm{lh}(s)$ and $a_1, \ldots, a_n \in \mathbb{N}$. We also view $()$ as the unique (empty) function with the empty domain, and view each non-empty $s \in \mathcal{S}$ as a function from $\{1, \ldots, \mathrm{lh}(s)\}$ to $\mathbb{N}$. In addition, we refer to each $s \in \mathcal{S}$ as a *word* of natural numbers. When $s \in S$, the different points of view give rise to different notations for $s$; for example, we have $s = (s(1), \ldots, s(\mathrm{lh}(s))) = s(1) \ldots s(\mathrm{lh}(s))$.

For $s \in \mathcal{S}$ and $k \leq l \in \mathrm{dom}(s)$ (i.e. $1 \leq k \leq l \leq \mathrm{lh}(s)$), define $s{\restriction}[k, l]$ to be the unique $t \in \mathcal{S}$ with $\mathrm{lh}(t) = l - k + 1$ such that for $1 \leq i \leq \mathrm{lh}(t)$, $t(i) = s(k + i - 1)$. Also define $s{\restriction}k = s{\restriction}[1, k]$ and $s{\restriction}0 = ()$.

For $s, t \in \mathcal{S}$, $t$ is a *subword* of $s$ if there are $1 \leq k \leq l \leq \mathrm{lh}(s)$ such that $t = s{\restriction}[k, l]$. When $t$ is a subword of $s$, we also say that $t$ *occurs* in $s$. If $t$ is a subword of $s$ and $1 \leq k \leq \mathrm{lh}(s)$, then we say that there is an *occurrence of $t$ in $s$ at position $k$* if $t = s{\restriction}[k, k + \mathrm{lh}(t) - 1]$. We say that $t$ is an *initial segment* of $s$, denoted $t \sqsubseteq s$, if $t = s{\restriction}[1, \mathrm{lh}(t)]$.

For $s_1, \ldots, s_n \in \mathcal{S}$, we define the *concatenation* $s_1{}^\frown \cdots {}^\frown s_n$ to be the unique word $t \in \mathcal{S}$ with length $\sum_{j=1}^{n} \mathrm{lh}(s_j)$ such that for all $1 \leq j \leq n$, $s_j = t{\restriction}[1 + \sum_{i=1}^{j-1} \mathrm{lh}(s_i), \sum_{i=1}^{j} \mathrm{lh}(s_i)]$.

For $s \in \mathcal{S}$, define $s^0 = ()$ and, if $n \in \mathbb{N}_+$, define $s^n$ to be the word $s_1{}^\frown \cdots {}^\frown s_n$ where $s_1 = \cdots = s_n = s$. Words of the form $s^n$, with $\mathrm{lh}(s) = 1$, are called *constant.*

We introduce a notion of *incompatibility* for our purpose. For $s, t \in \mathcal{S}$, we say that $s$ and $t$ are *incompatible*, denoted $s \perp t$, if both of the following hold:

- $t{\restriction}(\mathrm{lh}(t) - 1)$ is not a subword of

$$s{\restriction}(\mathrm{lh}(s) - 1)^\frown(s(\mathrm{lh}(s)) + c)^\frown s{\restriction}(\mathrm{lh}(s) - 1)$$

  for any $c \in \mathbb{N}$, and

- $s\restriction(\mathrm{lh}(s)-1)$ is not a subword of
$$t\restriction(\mathrm{lh}(t)-1)^\frown(t(\mathrm{lh}(t))+c)^\frown t\restriction(\mathrm{lh}(t)-1)$$
for any $c \in \mathbb{N}$.

Otherwise, we say that $s$ and $t$ are *compatible*.

**2.2. Infinite and bi-infinite sequences.** We will consider infinite sequences of natural numbers as well as infinite binary sequences. Again, they will be equivalently viewed as sequences, functions, and infinite words. We tacitly assume that an infinite sequence has domain $\mathbb{N}$, unless explicitly specified otherwise.

For an infinite word $V$ and natural numbers $k \leq l$, define $V\restriction[k,l]$ to be the unique $s \in \mathcal{S}$ such that $\mathrm{lh}(s) = l-k+1$ and $s(i) = V(k+i-1)$ for all $1 \leq i \leq \mathrm{lh}(s)$. Also define $V\restriction k = V\restriction[0,k]$. In the same fashion as for finite words, we may speak of when a finite word $s$ is a *subword* of $V$ or $s$ *occurs* in $V$, of there being an *occurrence of $s$ in $V$ at position $k$* for $k \in \mathbb{N}$, and of $s$ being an *initial segment* of $V$, which is denoted as $s \sqsubseteq V$.

If $v_0 \sqsubseteq v_1 \sqsubseteq \cdots \sqsubseteq v_n \sqsubseteq \cdots$ is an infinite sequence of elements of $\mathcal{S}$ with $\mathrm{lh}(v_n) \to \infty$, each of which is an initial segment of the next, then there is a unique infinite sequence $V$ such that $v_n \sqsubseteq V$ for all $n \in \mathbb{N}$. We call this unique infinite sequence the *limit* of $(v_n : n \in \mathbb{N})$ and denote it by $\lim_n v_n$. Specifically, for each $n \in \mathbb{N}$ and $0 \leq i \leq \mathrm{lh}(v_n)-1$, we have $(\lim_n v_n)(i) = v_n(i+1)$. The infinite words we consider will arise as limits of such sequences of finite words.

A *bi-infinite* sequence (or word) is an element of $\mathbb{N}^{\mathbb{Z}}$. A *bi-infinite binary* sequence is an element of $\{0,1\}^{\mathbb{Z}}$. The relations of *subword* and *occurrence* can be defined similarly between finite words and bi-infinite words. With $\{0,1\}$ equipped with the discrete topology and $\{0,1\}^{\mathbb{Z}}$ equipped with the product topology, $\{0,1\}^{\mathbb{Z}}$ becomes a compact metrizable space. The *shift map* $\sigma$ on $\{0,1\}^{\mathbb{Z}}$ is defined as
$$\sigma(x)(i) = x(i+1)$$
for all $x \in \{0,1\}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. With $\{0,1\}$ equipped with any probability measure and $\{0,1\}^{\mathbb{Z}}$ equipped with the product measure of measures $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, $\sigma$ is a measure-preserving automorphism on $\{0,1\}^{\mathbb{Z}}$.

**2.3. Symbolic rank-one systems and rank-one transformations.** Both symbolic rank-one systems and rank-one transformations are constructed from the so-called cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$. The *cutting parameter* $(r_n : n \in \mathbb{N})$ is an infinite sequence of natural numbers with $r_n \geq 2$ for all $n \in \mathbb{N}$. The *spacer parameter* $(s_n : n \in \mathbb{N})$ is a sequence of finite sequences of natural numbers with $\mathrm{lh}(s_n) = r_n$ for all $n \in \mathbb{N}$.

Given cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$, a symbolic rank-one system is defined as follows. First, inductively define an infinite sequence of finite binary words $(v_n : n \in \mathbb{N})$ as

$$v_0 = 0, \quad v_{n+1} = v_n 1^{s_n(1)} v_n 1^{s_n(2)} \ldots v_n 1^{s_n(r_n-1)} v_n 1^{s_n(r_n)}.$$

We call $(v_n : n \in \mathbb{N})$ a *generating sequence.* Noting that each $v_n$ is an initial segment of $v_{n+1}$, we may define

$$V = \lim_n v_n;$$

$V$ is said to be an *infinite rank-one word.* Finally, let

$$X = X_V = \left\{x \in \{0,1\}^{\mathbb{Z}} : \text{every finite subword of } x \text{ is a subword of } V\right\}.$$

Then $X$ is a closed subspace of $\{0,1\}^{\mathbb{Z}}$ invariant under the shift map $\sigma$, i.e., $\sigma(x) \in X$ for all $x \in X$. For simplicity we still write $\sigma$ for $\sigma|_X$. We call $(X, \sigma)$ a *symbolic rank-one system.*

With cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$, one can also define a rank-one measure-preserving transformation $T$ by a cutting and stacking process as follows. First, inductively define an infinite sequence of natural numbers $(h_n : n \in \mathbb{N})$ as

$$h_0 = 1, \quad h_{n+1} = r_n h_n + \sum_{i=1}^{r_n} s_n(i). \tag{2.1}$$

Next, define sequences $(B_n : n \in \mathbb{N})$, $(B_{n,i} : n \in \mathbb{N},\, 1 \le i \le r_n)$ and $(C_{n,i,j} : n \in \mathbb{N},\, 1 \le i \le r_n,\, 1 \le j \le s_n(i))$, all of which are subsets of $[0, +\infty)$, by induction on $n$. Define $B_0 = [0,1)$ and $B_{n+1} = B_{n,1}$ for all $n \in \mathbb{N}$. Let $B_n$ be given and inductively assume that $T^k[B_n]$ are defined for $0 \le k < h_n$ so that $T^k[B_n]$, $0 \le k < h_n$, are all disjoint. Let $\{B_{n,i} : n \in \mathbb{N},\, 1 \le i \le r_n\}$ be a partition of $B_n$ into $r_n$ many sets of equal measure and let $\{C_{n,i,j} : n \in \mathbb{N},\, 1 \le i \le r_n,\, 1 \le j \le s_n(i)\}$ be disjoint sets each of which is disjoint from $B_n$ and has the same measure as $B_{n,1}$. Then define $T$ so that for all $1 \le i \le r_n$,

$$T^{h_n}[B_{n,i}] = \begin{cases} C_{n,i,1} & \text{if } s_n(i) > 0, \\ B_{n,i+1} & \text{if } s_n(i) = 0, \end{cases}$$

and for $1 \le j \le s_n(i)$,

$$T[C_{n,i,j}] = \begin{cases} C_{n,i,j+1} & \text{if } 1 \le j < s_n(i), \\ B_{n,i+1} & \text{if } j = s_n(i). \end{cases}$$

Note that the set $B_{n,r_n+1}$ is undefined, as are all $C_{n,i,j}$ if $s_n(i) = 0$. We have thus defined $T^k[B_{n+1}]$ for $0 \le k < h_{n+1}$ so that all of them are disjoint. Finally, let

$$Y = \bigcup \{T^k[B_n] : n \in \mathbb{N},\, 0 \le k < h_n\}.$$

Then $T$ is a measure-preserving automorphism of $Y$. If $Y$ has finite Lebesgue measure, or equivalently if

$$\sum_{n=0}^{\infty} \frac{h_{n+1} - h_n r_n}{h_{n+1}} < +\infty, \tag{2.2}$$

then $(Y, \lambda)$, where $\lambda$ is the normalized Lebesgue measure on $Y$, is a probability Lebesgue space. Clearly $T$ is still a measure-preserving automorphism of $(Y, \lambda)$. Such a $T$ is called a *rank-one (measure-preserving) transformation.*

Connecting the symbolic and the geometric constructions, we can see that $h_n = \mathrm{lh}(v_n)$ for all $n \in \mathbb{N}$, and each $v_n$ describes the orbit of the elements of $B_n$ in terms of their membership in the initial base of the geometric construction (here 0 stands for an element in $B_0$ and 1 stands for an element not in $B_0$). When (2.2) holds, there is a unique probability Borel measure $\mu$ on $X$, and $(X, \mu, \sigma)$ and $(Y, \lambda, T)$ are isomorphic measure-preserving transformations. In particular, the isomorphism type of $(Y, \lambda, T)$ does not depend on the numerous choices one has to make in the process of constructing $T$ (e.g. how the sets $B_{n,i}$ and $C_{n,i,j}$ are picked and how $T$ is defined on them).

Throughout the rest of the paper we tacitly assume that (2.2) is satisfied for all rank-one transformations under consideration.

For more information on the basics of rank-one transformations, particularly on the connections between the symbolic and geometric constructions, see [2], [5] and [6].

**2.4. Combinatorics of rank-one words.** Our rank-one words will always start with 0. For finite words $u$ and $v$ starting with 0 we say that $u$ is *built from* $v$, denoted $v \prec u$, if for some $n \geq 2$ there are $a_1, \ldots, a_n \in \mathbb{N}$ such that

$$u = v1^{a_1}v1^{a_2} \ldots v1^{a_{n-1}}v1^{a_n}.$$

Note that the above way to express $u$ as a concatenation of $v$ with blocks of 1s is unique. The demonstrated occurrences of $v$ in the above expression are called the *expected occurrences* of $v$ in $u$.

If $V$ is an infinite word, we also say $V$ is *built from* $v$, and write $v \prec V$, if there is an infinite sequence $(a_n : n \in \mathbb{N}_+)$ of natural numbers such that

$$V = v1^{a_1}v \ldots v1^{a_n}v \ldots.$$

Again, the above expression of $V$ as an infinite concatenation of $v$ with blocks of 1s is also unique, and the demonstrated occurrences of $v$ are also called the *expected occurrences* of $v$ in $V$. We say that $V$ is *simply built from* $v$ if $a_1 = \cdots = a_n = \cdots$. We say that $V$ is *non-degenerate* if $V$ is not simply built from any finite word.

Let $V$ be an infinite rank-one word. As in Subsection 2.3 a generating sequence $(v_n : n \in \mathbb{N})$ for $V$ is a sequence of finite words such that $v_0 = 0$,

$v_n \prec v_{n+1}$ for all $n \in \mathbb{N}$, and $V = \lim_n v_n$. It follows that $v_n \prec V$ for each $n \in \mathbb{N}$.

Throughout the rest of this paper we consider only non-degenerate infinite rank-one words.

We say that two pairs of cutting and spacer parameters, $(r_n : n \in \mathbb{N})$, $(s_n : n \in \mathbb{N})$ and $(q_n : n \in \mathbb{N})$, $(t_n : n \in \mathbb{N})$, are *commensurate* if for all $n \in \mathbb{N}$, $r_n = q_n$ and $\sum_{i=1}^{r_n} \mathrm{lh}(s_i) = \sum_{i=1}^{r_n} \mathrm{lh}(t_i)$. If $(v_n : n \in \mathbb{N})$ and $(w_n : n \in \mathbb{N})$ are the respective generating sequences for commensurate pairs of cutting and spacer parameters, then $\mathrm{lh}(v_n) = \mathrm{lh}(w_n)$ for all $n \in \mathbb{N}$.

**3. Non-isomorphism and disjointness.** In this section we give conditions for non-isomorphism and disjointness of symbolic rank-one systems in terms of their cutting and spacer parameters. We first state a theorem on non-isomorphism without proof. This is a slight generalization of [9, Proposition 2.1].

THEOREM 3.1. *Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(X, \mu, \sigma)$. Let $(r_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(Y, \nu, \sigma)$. Suppose the following hold:*

(a) *The two sets of parameters are commensurate, i.e., for all $n$,*

$$\sum_{i=1}^{r_n} s_n(i) = \sum_{i=1}^{r_n} t_n(i).$$

(b) *There is an $S \in \mathbb{N}$ such that for all $n$ and all $1 \leq i \leq r_n$,*

$$s_n(i) \leq S \quad \text{and} \quad t_n(i) \leq S.$$

(c) *There is an $R \in \mathbb{N}$ such that for infinitely many $n$,*

$$r_n \leq R \quad \text{and} \quad s_n \perp t_n.$$

*Then $(X, \mu, \sigma)$ and $(Y, \nu, \sigma)$ are not isomorphic.*

Our main theorem of this section is the following.

THEOREM 3.2. *Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(X, \mu, \sigma)$. Let $(r_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(Y, \nu, \sigma)$. Suppose the following hold:*

(a) *The two sets of parameters are commensurate, i.e., for all $n$,*

$$\sum_{i=1}^{r_n} s_n(i) = \sum_{i=1}^{r_n} t_n(i).$$

(b) *There is an* $S \in \mathbb{N}$ *such that for all* $n$ *and all* $1 \leq i \leq r_n$,

$$s_n(i) \leq S \quad \textit{and} \quad t_n(i) \leq S.$$

(c) *There is an* $R \in \mathbb{N}$ *such that for infinitely many* $n$,

$$r_n \leq R \quad \textit{and} \quad s_n \perp t_n.$$

(d) *For each* $1 < k \leq 5S$, *where* $S$ *is the bound from* (b), *either* $(X, \mu, \sigma^k)$ *or* $(Y, \nu, \sigma^k)$ *is ergodic.*

*Then* $(X, \mu, \sigma)$ *and* $(Y, \nu, \sigma)$ *are disjoint.*

The only difference between the hypotheses of Theorems 3.1 and 3.2 is condition (d) above. This condition is necessary for disjointness. In fact, if for some $k > 1$ both $(X, \mu, \sigma^k)$ and $(Y, \nu, \sigma^k)$ are not ergodic, then they have a common factor which is a cyclic permutation on an $k$-element set, and thus the two transformations are not disjoint.

The rest of this section is devoted to the proof of Theorem 3.2. We will follow the approach of del Junco, Rahe, and Swanson [10] in their proof of minimal self-joinings for Chacon's transformation, as presented by Rudolph [14, Section 6.5].

The setup of the proof is standard. Let $\overline{\mu}$ be an ergodic joining of $\mu$ and $\nu$ on $X \times Y$. We need to show that $\overline{\mu} = \mu \times \nu$. By [14, Lemma 6.14] (or [10, Proposition 2]), it suffices to find some $k \geq 1$ such that $(X, \mu, \sigma^k)$ is ergodic and $\overline{\mu}$ is $(\sigma^k \times \mathrm{id})$-invariant, where id is the identity transformation on $Y$. For this, let $(x, y) \in X \times Y$ satisfy the ergodic theorem for $\overline{\mu}$, i.e., for all measurable $A \subseteq X \times Y$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(\sigma^i(x), \sigma^i(y)) = \overline{\mu}(A),$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(\sigma^{-i}(x), \sigma^{-i}(y)) = \overline{\mu}(A).$$

Such $(x, y)$ exists by the ergodicity of $\overline{\mu}$. Lemma 6.15 of [14] gives a sufficient condition to complete the proof. We state it below in our notation.

LEMMA 3.3. *Suppose there are integers* $a_n, b_n, c_n, d_n, e_n \in \mathbb{Z}$ *for all* $n \in \mathbb{N}$, *a positive integer* $k \geq 1$ *and a real number* $\alpha > 0$ *such that for all* $n \in \mathbb{N}$,

(i) $a_n \leq 0 \leq b_n$ *and* $\lim_n (b_n - a_n) = +\infty$;
(ii) $a_n \leq c_n \leq d_n \leq b_n$ *and* $a_n \leq c_n + e_n \leq d_n + e_n \leq b_n$;
(iii) $d_n - c_n \geq \alpha(b_n - a_n)$;
(iv) *for all* $c_n \leq i \leq d_n$, $x(i) = x(i + k + e_n)$ *and* $y(i) = y(i + e_n)$; *and*
(v) $(X, \mu, \sigma^k)$ *is ergodic.*

*Then* $\overline{\mu}$ *is* $(\sigma^k \times \mathrm{id})$*-invariant, and so* $\overline{\mu} = \mu \times \nu$.

Note that Lemma 3.3 has several valid variations. One variation is a symmetric version with the spaces $X$ and $Y$ switched. This version is obviously true since the setup is entirely symmetric for $X$ and $Y$. Another variation is the one in which $k \leq -1$ is a negative integer. Note that $(X, \mu, \sigma^k)$ is ergodic if and only if $(X, \mu, \sigma^{-k})$ is ergodic. This version can be obtained by applying Lemma 3.3 to $(X, \mu, \sigma^{-1})$ and $(Y, \nu, \sigma^{-1})$. Finally, we also have the variation in which both $k$ is negative and $X$ and $Y$ are switched.

Now we claim that a slightly weaker construction already suffices: it is enough to find $a_n, b_n, c_n, d_n, e_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, a positive integer $K \geq 1$ and a real number $\alpha$ such that (i)–(iii) hold and for each $n \in \mathbb{N}$, (iv) holds for some non-zero $k \in \mathbb{Z}$ with $k \in [-K, K]$. In fact, since there are only finitely many integers between $-K$ and $K$, we get some non-zero integer $k \in [-K, K]$ and infinitely many $n$ for which conditions (i)–(iv) of Lemma 3.3 are satisfied. If $k > 0$ and $(X, \mu, \sigma^k)$ is ergodic then we are done by Lemma 3.3. If $k > 0$ but $(X, \mu, \sigma^k)$ is not ergodic, then by (d), $(Y, \nu, \sigma^k)$ is ergodic. It follows that $(Y, \nu, \sigma^{-k})$ is ergodic. Now we are done by the variation of Lemma 3.3 in which both $k$ is negative and $X$ and $Y$ are switched. If $k < 0$ we similarly apply other variations of the lemma.

We now begin our construction. Let $K = 5S$ where $S$ is the bound in (b). Define

$$\alpha = \frac{1}{60(R+1)}$$

where $R$ is the bound in (c). Note that $R \geq 2$ because $r_n \geq 2$ for all $n \in \mathbb{N}$. Define

$$D = \{n \in \mathbb{N} : r_n \leq R \quad \text{and} \quad s_n \perp t_n\}.$$

Then $D$ is infinite by (c).

Let $(v_n : n \in \mathbb{N})$ be the generating sequence given by the cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$. Then for each $n \in \mathbb{N}$, $x$ is built from $v_n$. Let $(w_n : n \in \mathbb{N})$ be the generating sequence given by the cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$. Then for each $n \in \mathbb{N}$, $y$ is built from $w_n$. By the commensurability condition (a), $\mathrm{lh}(v_n) = \mathrm{lh}(w_n)$ for all $n \in \mathbb{N}$.

Fix an $n_0$ such that $\mathrm{lh}(v_{n_0}) > 10RS \geq 20S$. For any $n \in D$ with $n \geq n_0$, we define $a_n, b_n, c_n, d_n, e_n \in \mathbb{Z}$ to satisfy (i)–(iii) and (iv) with some non-zero $k_n \in [-5S, 5S]$. Define

$$a_n = -10\,\mathrm{lh}(v_{n+1}) \quad \text{and} \quad b_n = 10\,\mathrm{lh}(v_{n+1}).$$

It is clear that (i) is satisfied.

One can view the word $x$ as a collection of expected occurrences of $v_n$ with blocks of 1s separating consecutive expected occurrences of $v_n$. Note that these blocks of 1s can be equivalently described as maximal blocks of 1s that do not overlap with any expected occurrence of $v_n$. We claim that at most one of these blocks has length greater than $5S$ and overlaps the interval $[a_n, b_n]$. To see this, note that such a block must separate expected occurrences of $v_{n+6}$ and thus such blocks must be separated by an occurrence of $v_{n+6}$ which has length at least $2^5 \operatorname{lh}(v_{n+1}) > \operatorname{lh}([a_n, b_n]) = 20 \operatorname{lh}(v_{n+1}) + 1$. If there is such a block of 1s, let $i_{n,1}$ and $i_{n,2}$ denote the starting and ending positions of that block; otherwise, let $i_{n,1} = i_{n,2} = a_n$.

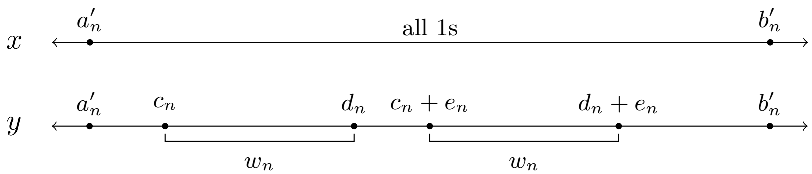
The word $y$ can also be viewed as a collection of expected occurrences of $w_n$ with blocks of 1s separating consecutive occurrences of $w_n$. Similar reasoning shows that at most one such block with length greater than $5S$ overlaps the interval $[a_n, b_n]$. If there is such a block, let $i_{n,3}$ and $i_{n,4}$ denote the starting and ending positions of that block; otherwise, let $i_{n,3} = i_{n,4} = a_n$.

Let $[a'_n, b'_n]$ be the largest subinterval of $[a_n, b_n]$ that does not include any of the positions $i_{n,1}$, $i_{n,2}$, $i_{n,3}$, or $i_{n,4}$. Note that the length of $[a'_n, b'_n]$ is at least $\frac{1}{5} \operatorname{lh}([a_n, b_n]) \geq 4 \operatorname{lh}(v_{n+1})$. We now have four possibilities.

Case 1: $x \upharpoonright [a'_n, b'_n] = y \upharpoonright [a'_n, b'_n] = 1^{b'_n - a'_n + 1}$. In this case we let $c_n = a'_n$, $d_n = c_n + \operatorname{lh}(v_n)$, $e_n = \operatorname{lh}(v_n)$, and $k_n = 1$. Of the conditions (i)–(v) above in Lemma 3.3, (v) was assumed and (i) has already been verified. It is clear based on our definitions of $c_n$, $d_n$ and $e_n$ that (ii) and (iv) also hold. It only remains to check (iii):

$$\frac{d_n - c_n}{b_n - a_n} = \frac{\operatorname{lh}(v_n)}{20 \operatorname{lh}(v_{n+1})} \geq \frac{\operatorname{lh}(v_n)}{20(R \operatorname{lh}(v_n) + RS)} \geq \frac{1}{20(R+1)} > \alpha.$$

Case 2: $x \upharpoonright [a'_n, b'_n] = 1^{b'_n - a'_n + 1}$, but in $y$ there is no block of 1s of length $5S$ that overlaps with $[a'_n, b'_n]$ but does not overlap with any expected occurrence of $w_n$. In this case we find two consecutive expected occurrences of $w_n$ in $y$ that are completely contained in $[a'_n, b'_n - 1]$. Let $c_n$ and $d_n$ be the starting and ending positions of the first of those expected occurrences of $w_n$. Let $e_n$ be such that the second of those expected occurrences of $w_n$ begins at position $c_n + e_n$. Let $k_n = 1$. It is straightforward to check that in this case conditions (i)–(v) of Lemma 3.3 are satisfied.




Case 3: $y \upharpoonright [a'_n, b'_n] = 1^{b'_n - a'_n + 1}$, but in $x$ there is no block of 1s of length $5S$ that overlaps with $[a'_n, b'_n]$ but does not overlap with any expected occurrence of $v_n$. The argument here is symmetric to Case 2.

Case 4: In $x$ (and in $y$) there is no block of 1s of length $5S$ that overlaps with $[a'_n, b'_n]$ but does not overlap with any expected occurrence of $v_n$ (or $w_n$).

The rest of this proof deals with this case. By recentering $x$ and $y$ simultaneously, we may assume without loss of generality that $a'_n \leq -2\,\mathrm{lh}(v_{n+1})$ and $b'_n \geq 2\,\mathrm{lh}(v_{n+1})$.

Before defining $c_n$, $d_n$, $e_n$ and $k_n$ we need to analyze the expected occurrences of $v_{n+1}$ in $x$ and the expected occurrences of $w_{n+1}$ in $y$. Since $y$ is built from $w_{n+1}$, by (b) the interval $[-3S, 3S]$ has a non-empty intersection with some expected occurrence of $w_{n+1}$ in $y$. Fix one such expected occurrence of $w_{n+1}$ and suppose the occurrence begins at position $l$ and finishes at position $m$. Thus $a'_n \leq -\mathrm{lh}(v_{n+1}) - 3S \leq l \leq 3S$ and $m = l + \mathrm{lh}(v_{n+1}) - 1 \leq 3S + \mathrm{lh}(v_{n+1}) \leq b'_n$.

Note that $x$ is built from $v_n$. We can then define $j \in \mathbb{Z}$ where $|j|$ is the least such that there is an expected occurrence of $v_n$ at position $l + j$. A moment of reflection shows that $|j| \leq \frac{1}{2}(\mathrm{lh}(v_n) + 6S)$. Since $\mathrm{lh}(v_n) > 20S$, the occurrence of $w_n$ at position $l$ and the occurrence of $v_n$ at position $l + j$ overlap for at least $\frac{1}{3}\,\mathrm{lh}(v_n)$ positions.

Starting from the expected occurrence of $v_n$ at position $l + j$ in $x$, we examine the next $r_n$ many consecutive expected occurrences of $v_n$ in $x$. Suppose there is an occurrence of the following word in $x$ starting at position $l + j$:

$$v_n 1^{p(1)} v_n \ldots 1^{p(r_n-1)} v_n,$$

where $p \in \mathcal{S}$ with $\mathrm{lh}(p) = r_n - 1$. Because $x$ is also built from $v_{n+1}$, and each expected occurrence of $v_{n+1}$ contains $r_n$ many expected occurrences of $v_n$, the above word is contained in an occurrence of $v_{n+1} 1^q v_{n+1}$ for some $q \in \mathbb{N}$, where each demonstrated occurrence of $v_{n+1}$ is expected. Note that

$$\begin{aligned} &v_{n+1} 1^q v_{n+1} \\ &\quad = v_n 1^{s_n(1)} v_n \ldots 1^{s_n(r_n-1)} v_n 1^{s_n(r_n)+q} v_n 1^{s_n(1)} v_n \ldots 1^{s_n(r_n-1)} v_n 1^{s_n(r_n)}. \end{aligned}$$

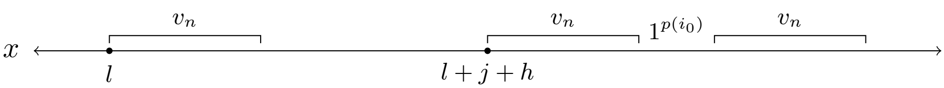
By comparison, we find that $p$ is a subword of

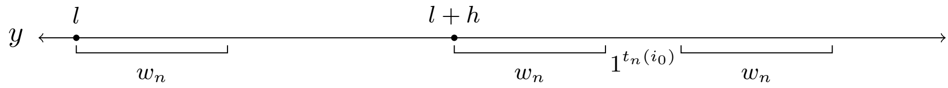
$$(s_n(1), s_n(2), \ldots, s_n(r_n - 1), s_n(r_n) + q, s_n(1), s_n(2), \ldots, s_n(r_n - 1)).$$

Since $n \in D$ and therefore $s_n \perp t_n$, we conclude that $p \neq t_n$. Let $i_0$ be the least such that $1 \leq i_0 \leq r_n - 1$ and $p(i_0) \neq t_n(i_0)$. Let

$$h = (i_0 - 1)\,\mathrm{lh}(v_n) + \sum_{i=1}^{i_0-1} t_n(i).$$

Then $l + h$ is the beginning position of an expected occurrence of $w_n$ in $y$, and $l + j + h$ is the beginning position of an expected occurrence of $v_n$ in $x$. There is an occurrence of $w_n 1^{t_n(i_0)} w_n$ in $y$ beginning at position $l + h$, and an occurrence of $v_n 1^{p(i_0)} v_n$ in $x$ beginning at position $l + j + h$.






Now we define $[c_n, d_n]$ to be the interval of overlap between the occurrence of $w_n$ in $y$ at position $l+h$ and the occurrence of $v_n$ in $x$ at position $l+j+h$. Since $|j| \leq \frac{1}{2}(\mathrm{lh}(v_n) + 6S)$ and $\mathrm{lh}(v_n) > 20S$, we get

$$d_n - c_n \geq \tfrac{1}{3}\,\mathrm{lh}(v_n).$$

Define

$$e_n = \mathrm{lh}(v_n) + t_n(i_0) \quad \text{and} \quad k_n = p(i_0) - t_n(i_0).$$

Since $[c_n, d_n]$ is contained in the occurrence of $v_n$ at position $l+j+h$, and since $k_n + e_n = \mathrm{lh}(v_n) + p(i_0)$, it follows that $x\restriction[c_n, d_n]$ and $x\restriction[c_n + k_n + e_n, d_n + k_n + e_n]$ are the same words. Similarly, $[c_n, d_n]$ is also contained in the occurrence of $w_n$ at position $l+h$, and it follows that $y\restriction[c_n, d_n]$ and $y\restriction[c_n + e_n, d_n + e_n]$ are the same words. This means that (iv) is satisfied.

Since $[c_n, d_n], [c_n + e_n, d_n + e_n] \subseteq [l, m] \subseteq [a_n', b_n'] \subseteq [a_n, b_n]$, we know that (ii) is satisfied. Finally,

$$\frac{d_n - c_n}{b_n - a_n} \geq \frac{\mathrm{lh}(v_n)}{3 \cdot 20\,\mathrm{lh}(v_{n+1})} \geq \frac{\mathrm{lh}(v_n)}{60(R\,\mathrm{lh}(v_n) + RS)} \geq \frac{1}{60(R+1)} = \alpha.$$

This shows that (iii) is satisfied. Thus, the proof of Theorem 3.2 is complete.

## 4. Minimal self-joinings

THEOREM 4.1. *Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(X, \mu, \sigma)$. Suppose the following hold:*

(a) *For some $R$ and all $n$, we have $r_n \leq R$.*
(b) *For some $S$ and all $n$ and all $1 \leq i \leq r_n$, we have $s_n(i) \leq S$.*
(c) *For all $n, c \in \mathbb{N}$ there are only two occurrences of $s_n\restriction(r_n - 1)$ in $s_n\restriction(r_n - 1)^\frown(s_n(r_n) + c)^\frown s_n\restriction(r_n - 1)$.*
(d) *$(X, \mu, \sigma)$ is totally ergodic.*

*Then $(X, \mu, \sigma)$ has minimal self-joinings of all orders.*

First we note a well-known fact that for rank-one transformations, having minimal self-joinings of order 2 implies minimal self-joinings of all orders. We thank Eli Glasner for providing us with the references and for allowing us to include the argument here for the benefit of the reader.

THEOREM 4.2. *If a rank-one transformation has minimal self-joinings of order 2, then it has minimal self-joinings of all orders.*

*Proof.* An inductive argument (cf. [8, Theorem 12.16]) shows that for any weakly mixing transformation, having minimal self-joinings of order 3 implies minimal self-joinings of all orders. A theorem of Ryzhikov [15] states that a 2-mixing measure-preserving transformation with minimal self-joinings of order 2 has minimal self-joinings of all orders. It follows that if a transformation has minimal self-joinings of order 2 but not of order 3, then it is mixing but not 2-mixing (cf. [8, Corollary 12.22]). A theorem of Kalikow [11] states that any mixing rank-one transformation is also 2-mixing (and in fact $k$-mixing for all $k > 1$). Thus one concludes that a rank-one transformation with minimal self-joinings of order 2 also has minimal self-joinings of order 3. Since having minimal self-joinings of order 2 implies weak mixing, such a transformation has minimal self-joinings of all orders. ∎

The above theorem is well-known to experts in the field and the references provided here are not meant to be exhaustive. For instance, the theorem was mentioned in [16] (without proof or further references). A weaker form of the theorem was mentioned in [13], which is sufficient for our purpose since we only consider bounded rank-one transformations, which are not mixing.

As in Theorem 3.2 and Corollary 5.5, condition (d) of Theorem 4.1 can be weakened to

(d′) For each $1 < k \leq 6S$, where $S$ is the bound from (b), $(X, \mu, \sigma^k)$ is ergodic.

This will be clear from the proof below.

The rest of this section is devoted to the proof of Theorem 4.1 for minimal self-joinings of order 2. We again follow the approach of del Junco, Rahe, and Swanson [10] in their proof of minimal self-joinings for Chacon's transformation, as presented by Rudolph [14, Section 6.5].

Let $(v_n : n \in \mathbb{N})$ be the generating sequence given by the cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$.

Lemma 4.3. *Without loss of generality, we may assume $r_n \geq 3$ for all $n \in \mathbb{N}$.*

*Proof.* Simply consider the subsequence $(v'_n : n \in \mathbb{N})$ defined as $v'_n = v_{2n}$ for all $n \in \mathbb{N}$. Then $r'_n = r_{2n} r_{2n+1} \geq 4$ is the new cutting parameter, and the new spacer parameter $s'_n$ is

$$\begin{aligned} & s_{2n} \restriction (r_{2n} - 1)^\frown (s_{2n}(r_{2n}) + s_{2n+1}(1))^\frown \\ & s_{2n} \restriction (r_{2n} - 1)^\frown (s_{2n}(r_{2n}) + s_{2n+1}(2))^\frown \cdots^\frown \\ & s_{2n} \restriction (r_{2n} - 1)^\frown (s_{2n}(r_{2n}) + s_{2n+1}(r_{2n+1})). \end{aligned} \tag{4.1}$$

If $R$ is the bound for $r_n$ in (a), then $r'_n \leq R^2$. If $S$ is the bound for all $s_n(i)$ in (b), $2S$ is a bound for all $s'_n(j)$. Since $\lim_n v_n = \lim_n v'_n$, (d) continues to hold. It remains to verify that (c) continues to hold for $s'_n$.

Towards a contradiction, suppose $s'_n\restriction(r'_n-1)$, which is in the form given by (4.1) with the last term removed, occurs in

$$s'_n\restriction(r'_n-1)^\frown(s'_n(r'_n)+c)^\frown s'_n\restriction(r'_n-1)$$

not as demonstrated. We refer to this occurrence of $s'_n\restriction(r'_n-1)$ as the *hidden occurrence*. Note that $s'_n\restriction(r'_n-1)$ starts with an occurrence of $s_{2n}\restriction(r_{2n}-1)$. Thus the hidden occurrence of $s'_n\restriction(r'_n-1)$ must start at a position where an expected occurrence of $s_{2n}\restriction(r_{2n}-1)$ in

$$s'_n\restriction(r'_n-1)^\frown(s'_n(r'_n)+c)^\frown s'_n\restriction(r'_n-1)$$

begins, because otherwise $s_{2n}\restriction(r_{2n}-1)$ occurs in some

$$s_{2n}\restriction(r_{2n}-1)^\frown(s_{2n}(r_{2n})+d)^\frown s_{2n}\restriction(r_{2n}-1)$$

not as demonstrated, contradicting (c). In other words, all expected occurrences of $s_{2n}\restriction(r_{2n}-1)$ in the hidden occurrences of $s'_n$ must be already in the form given by (4.1). By comparison, we find $s_{2n+1}\restriction(r_{2n+1}-1)$ occurs in

$$s_{2n+1}\restriction(r_{2n+1}-1)^\frown(s_{2n+1}(r_{2n+1})+c)^\frown s_{2n+1}\restriction(r_{2n+1}-1)$$

not as demonstrated, again contradicting (c). ∎

For the rest of the proof we assume that $r_n \geq 3$ for all $n \in \mathbb{N}$.

Let $E_0$ be the set of all $x \in X$ for which there is $n \in \mathbb{N}$ such that position 0 is contained in an expected occurrence of $v_n$ in $x$. Let $E = \bigcap_{k\in\mathbb{Z}} \sigma^k[E_0]$. Then $\mu(E) = 1$. In fact, by (b), $X \setminus E_0$ is finite. Thus $X \setminus E$ is at most countable.

For $n \in \mathbb{N}$, $x \in E$ and $l \in \mathbb{Z}$, let $\beta_n(x,l)$ denote the beginning position of an expected occurrence of $v_n$ in $x$ containing position $l$, if such an expected occurrence of $v_n$ exists; otherwise $\beta_n(x,l)$ is undefined. Also let $\gamma_n(x,l)$ denote the ending position of the expected occurrence of $v_n$ in $x$ containing position $l$, if such an expected occurrence of $v_n$ exists; otherwise $\gamma(x,l)$ is undefined. When both $\beta_n(x,l)$ and $\gamma_n(x,l)$ are defined, let $I_n(x,l)$ be the interval $[\beta_n(x,l), \gamma_n(x,l)]$. Then $I_n(x,l)$ corresponds to the expected occurrence of $v_n$ in $x$ containing position $l$. Note that if $I_n(x,l)$ is defined, then so is $I_{n+1}(x,l)$ and $I_n(x,l) \subseteq I_{n+1}(x,l)$.

We define a labeling function $\lambda_n : E\times\mathbb{Z} \to \{1,\ldots,r_n,\infty\}$ for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $x \in E$ and $l \in \mathbb{Z}$. If $I_n(x,l)$ is undefined, put $\lambda_n(x,l) = \infty$. Otherwise, position $l$ is contained in an expected occurrence of $v_n$ in $x$, and this expected occurrence of $v_n$ is in turn contained in an expected occurrence of $v_{n+1}$ in $x$. Since there are exactly $r_n$ many expected occurrences of $v_n$ in $v_{n+1}$, we may speak of the $i$th occurrence of $v_n$ in $v_{n+1}$ for $1 \leq i \leq r_n$. Now put $\lambda_n(x,l) = i$ if the expected occurrence of $v_n$ containing position $l$ is the $i$th occurrence of $v_n$ in the expected occurrence of $v_{n+1}$ in $x$ containing position $l$. For any $x \in E$ and $l \in \mathbb{Z}$, $\lambda_n(x,l) < \infty$ for large enough $n$.

We prove some basic facts about the labeling functions.

LEMMA 4.4. *If $x, y \in E$ and $l \in \mathbb{Z}$ are such that $\lambda_n(x,l) = \lambda_n(y,l)$ for all $n \geq N$ for some $N \in \mathbb{N}$, then $x$ and $y$ are in the same $\sigma$-orbit, i.e., there is $k \in \mathbb{Z}$ such that $\sigma^k(x) = y$.*

*Proof.* We may assume without loss of generality that $\lambda_N(x,l) = \lambda_N(y,l) < \infty$. Let $k = \beta_N(x,l) - \beta_N(y,l)$. Then by an easy induction on $n \geq N$ we find that for all $n \geq N$, $k = \beta_n(x,l) - \beta_n(y,l)$. This implies that $\sigma^k(x) = y$. ■

LEMMA 4.5. *Let $x, y \in E$, $l \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. Suppose that $\lambda_{n-1}(x,l) = \lambda_{n-1}(y,l) < \infty$. Then $|I_n(x,l) \cap I_n(y,l)| \geq \mathrm{lh}(v_{n-1})$.*

*Proof.* Suppose $\lambda_{n-1}(x,l) = \lambda_{n-1}(y,l) = i$. Then the $i$th occurrence of $v_{n-1}$ in the expected occurrence of $v_n$ in $x$ containing position $l$ has a non-empty overlap with the $i$th occurrence of $v_{n-1}$ in the expected occurrence of $v_n$ in $y$ containing position $l$. This implies that for all $1 \leq j \leq r_{n-1}$, the $j$th occurrence of $v_{n-1}$ in the expected occurrence of $v_n$ in $x$ containing position $l$ has a non-empty overlap with the $j$th occurrence of $v_{n-1}$ in the expected occurrence of $v_n$ in $y$ containing position $l$. It follows that the length of $I_n(x,l) \setminus I_n(y,l)$ cannot be greater than $\mathrm{lh}(v_{n-1})$. Since $r_{n-1} \geq 2$, we have $|I_n(x,l) \cap I_n(y,l)| \geq \mathrm{lh}(v_n) - \mathrm{lh}(v_{n-1}) \geq \mathrm{lh}(v_{n-1})$. ■

Define another labeling function $\kappa_n : E \times \mathbb{Z} \to \{-1, 0, +1, \infty\}$ for all $n \in \mathbb{N}$ as follows:

$$\kappa_n(x,l) = \begin{cases} -1 & \text{if } \lambda_n(x,l) = 1, \\ 0 & \text{if } 2 \leq \lambda_n(x,l) \leq r_n - 1, \\ +1 & \text{if } \lambda_n(x,l) = r_n, \\ \infty & \text{if } \lambda_n(x,l) = \infty. \end{cases}$$

LEMMA 4.6. *For any $l \in \mathbb{Z}$ and $\mu$-a.e. $x \in X$, the set $\{n \in \mathbb{N} : \kappa_n(x,l) = 0\}$ has density at least $1/3$. In particular, for any $l \in \mathbb{Z}$ and for $\mu$-a.e. $x \in X$, there are infinitely many $n \in \mathbb{N}$ such that $\kappa_n(x,l) = 0$.*

*Proof.* Fix $l \in \mathbb{Z}$. For each $N \in \mathbb{N}_+$ let $E_N = \{x \in E : \kappa_N(x,l) < \infty\}$. Then $E_N \subseteq E_{N+1}$ for all $N \in \mathbb{N}_+$ and $E = \bigcup_{N \in \mathbb{N}_+} E_N$. For each $n \in \mathbb{N}_+$ and $\iota \in \{-1, 0, +1\}$, let $E_{n,\iota} = \{x \in E_n : \kappa_n(x,l) = \iota\}$. Then $\mu(E_{n,0}) \geq \mu(E_n)/3 \geq \mu(E_N)/3$ if $n \geq N$. Also, on each $E_N$ the functions $\kappa_N, \kappa_{N+1}, \ldots$ are independent. By the law of large numbers, for each $N \in \mathbb{N}_+$ and $\mu$-a.e. $x \in E_N$, $\{n \geq N : \kappa_n(x,l) = 0\}$ has density at least $1/3$. It follows that for $\mu$-a.e. $x \in X$, $\{n \in \mathbb{N} : \kappa_n(x,l) = 0\}$ has density at least $1/3$. ■

LEMMA 4.7. *Let $x, y \in E$, $l \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. Suppose that $\kappa_{n-1}(x,l) = 0$ and $\kappa_{n-1}(y,l) < \infty$. Then $|I_n(x,l) \cap I_n(y,l)| \geq \mathrm{lh}(v_{n-1})$.*

*Proof.* Suppose $\lambda_{n-1}(x,l) = i$. Then $1 < i < r_n$. A moment of reflection shows that, in the expected occurrence of $v_n$ in $x$ containing position $l$, either the first expected occurrence of $v_{n-1}$ overlaps the expected occurrence of $v_n$

in $y$ containing position $l$, or the last expected occurrence of $v_{n-1}$ overlaps the expected occurrence of $v_n$ in $y$ containing position $l$. This shows that $|I_n(x,l) \cap I_n(y,l)| \geq \mathrm{lh}(v_{n-1})$. ■

We now proceed to set up the proof for minimal self-joinings of order 2. Let $\overline{\mu}$ be an ergodic joining on $X \times X$ with marginals $\mu$. Suppose $\overline{\mu}$ is not an off-diagonal measure. We need to show that $\overline{\mu} = \mu \times \mu$. Again by [14, Lemma 6.14] it suffices to find some non-zero $k \in \mathbb{Z}$ such that $\overline{\mu}$ is $(\sigma^k \times \mathrm{id})$-invariant, since by (d), $(X, \mu, \sigma^k)$ is ergodic. We let $(x, y) \in X \times X$ be a $\overline{\mu}$-generic pair in the sense that the following hold:

- $(x, y)$ satisfies the ergodic theorem for $\overline{\mu}$;
- $x, y \in E$ are not in the same $\sigma$-orbit; and
- the set $\{n \in \mathbb{N} : \kappa_n(x, 0) = 0\}$ has positive density.

Each of these properties is satisfied by $\overline{\mu}$-a.e. pair in $X \times X$. It is clear that for $\overline{\mu}$-a.e. $(x, y) \in E \times E$, $x$ and $y$ are not in the same $\sigma$-orbit. By Lemma 4.6, for $\mu$-a.e. $x \in X$, the set $\{n \in \mathbb{N} : \kappa_n(x, 0) = 0\}$ has positive density.

Similar to the proof of Theorem 3.2, it suffices to find a positive integer $K \geq 1$, a positive real number $\alpha > 0$, and, for infinitely many $n \in \mathbb{N}$, $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ such that

(0) $0 < |k_n| \leq K$;
(i) $a_n < -\mathrm{lh}(v_n)$ and $b_n > \mathrm{lh}(v_n)$;
(ii) $a_n \leq c_n \leq d_n \leq b_n$ and $a_n \leq c_n + e_n \leq d_n + e_n \leq b_n$;
(iii) $d_n - c_n \geq \alpha(b_n - a_n)$;
(iv) for all $c_n \leq i \leq d_n$, $x(i) = x(i + k_n + e_n)$ and $y(i) = y(i + e_n)$.

Applications of Lemma 3.3 and its variations will show that $\overline{\mu}$ is $(\sigma^k \times \mathrm{id})$-invariant, and so $\overline{\mu} = \mu \times \mu$.

Let $K = 3S$ where $S$ is the bound in (b). Let

$$\alpha = \frac{1}{8(R+1)^2}$$

where $R$ is the bound in (a). Fix an $n_0 \in \mathbb{N}$ such that $\mathrm{lh}(v_{n_0}) > RS$ and note that for all $n \geq n_0$,

$$\frac{\mathrm{lh}(v_n)}{\mathrm{lh}(v_{n+1})} \geq \frac{\mathrm{lh}(v_n)}{R\,\mathrm{lh}(v_n) + RS} \geq \frac{1}{R+1}.$$

To finish the proof we need to show that for any $N \in \mathbb{N}$ there is some $n \geq N$ such that we can define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ satisfying conditions (0)–(iv) above. The key technical lemma below will allow us to do this.

LEMMA 4.8. *Suppose that* $n \in \mathbb{N}$ *and* $l \in \mathbb{Z}$ *are such that* $n > n_0$, $-2\,\mathrm{lh}(v_{n+1}) \leq l \leq 2\,\mathrm{lh}(v_{n+1})$, $\lambda_n(x,l) \neq \lambda_n(y,l)$, *and* $|I_n(x,l) \cap I_n(y,l)| \geq \mathrm{lh}(v_{n-1})$. *Suppose also that at least one of the following holds:*

(1) *There is no occurrence of* $1^{3S}$ *in* $x$ *that does not intersect any expected occurrence of* $v_n$ *but does overlap* $I_{n+1}(y,l)$.
(2) *There is no occurrence of* $1^{3S}$ *in* $y$ *that does not intersect any expected occurrence of* $w_n$ *but does overlap* $I_{n+1}(x,l)$.

*Then we can define* $a$, $b$, $c$, $d$, $e$, *and* $k$ *so that*

(0) $0 < |k| \le K$;
(i) $a < -\operatorname{lh}(v_n)$ *and* $b > \operatorname{lh}(v_n)$;
(ii) $a \le c \le d \le b$ *and* $a \le c+e \le d+e \le b$;
(iii) $d - c \ge \alpha(b-a)$;
(iv) *for all* $c \le i \le d$, $x(i) = x(i+k+e)$ *and* $y(i) = y(i+e)$.

*Proof.* We will assume that (1) is satisfied. The proof in the case when (2) is satisfied is similar.

Let $[c', d'] = I_n(x,l) \cap I_n(y,l)$, the interval of overlap between the expected occurrence of $v_n$ in $x$ containing position $l$ and the expected occurrence of $v_n$ in $y$ containing position $l$. Then by assumption,

$$d' - c' \ge \operatorname{lh}(v_{n-1}).$$

Define

$$a_n = -4\operatorname{lh}(v_{n+1}) \quad \text{and} \quad b_n = 4\operatorname{lh}(v_{n+1}).$$

Note that $a \le \beta_n(y,l) \le c' < d' \le \gamma_n(y,l) \le b_n$.

Let $i_y = \lambda_n(y,l)$. Then in $y$, position $l$ is contained in the $i_y$th occurrence of $v_n$ in the expected occurrence of $v_{n+1}$ from position $\beta_{n+1}(y,l)$ to position $\gamma_{n+1}(y,l)$. Correspondingly in $x$, we examine the $r_n$ many consecutive expected occurrences of $v_n$ so that position $l$ is contained in the $i_y$th occurrence of $v_n$. Suppose the following word is observed:

$$v_n 1^{p(1)} v_n 1^{p(2)} \dots v_n 1^{p(r_n-1)} v_n.$$

Since $\lambda_n(x,l) \ne \lambda_n(y,l)$, this observed word is not contained in a single expected occurrence of $v_{n+1}$. Rather, it is contained in a subword of $x$ of the form $v_{n+1}1^q v_{n+1}$, where each demonstrated occurrence of $v_{n+1}$ is expected. By comparison, we find that $p$ is a subword of

$$s_n\restriction(r_n-1)^\frown(s_n(r_n)+c)^\frown s_n\restriction(r_n-1),$$

and that $p$ does not coincide with any of the two demonstrated occurrences of $s_n\restriction(r_n-1)$. By (c), this implies that $p \ne s_n\restriction(n-1)$.

Let $i_0$ be such that $1 \le i_0 \le r_n - 1$ and $p(i_0) \ne s_n(i_0)$ and that $|i_0 - i_y|$ is the least. For definiteness first assume that $i_0 \ge i_y$. In this case let

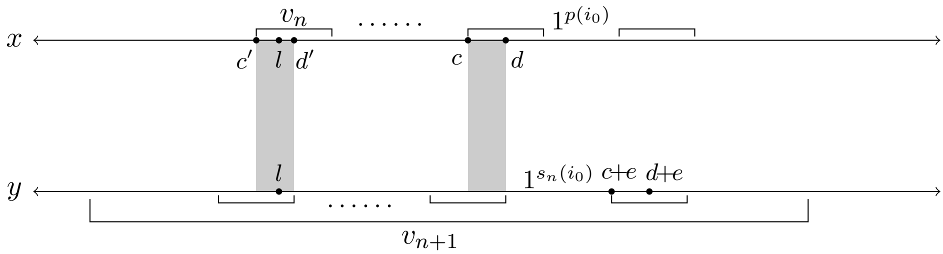
$$h = (i_0 - i_y)\operatorname{lh}(v_n) + \sum_{i=i_y}^{i_0-1} s_n(i).$$

Then in $x$ there is an occurrence of the word $v_n 1^{p(i_0)} v_n$ beginning at position $\beta_n(x,l)+h$. Similarly, in $y$ there is an occurrence of the word $v_n 1^{s_n(i_0)} v_n$ beginning at position $\beta_n(y,l)+h$. Define $[c,d]$ to be the interval of overlap between the these first demonstrated occurrences of $v_n$ in $x$ and in $y$. Then in fact $c = c' + h$ and $d = d' + h$. So

$$d - c = d' - c' \geq \mathrm{lh}(v_{n-1}).$$

Define

$$e = \mathrm{lh}(v_n) + s_n(i_0) \quad \text{and} \quad k = p(i_0) - s_n(i_0).$$




We will show condition (0) by first proving that $p(i_0) \leq 3S$. Notice that $I_{n+1}(y,l) = I_{n+1}(y,d) = I_{n+1}(y,d+e)$. Now, since $\gamma_n(x,d)+1 \leq d + \mathrm{lh}(v_n)$, we have $\gamma_n(x,d)+1 \in I_{n+1}(y,d)$. If $p(i_0) \geq 1$, then the 1 at position $\gamma_n(x,d)+1$ in $x$ is not contained in an expected occurrence of $v_n$ but is contained in $I_{n+1}(y,d)$ and thus, by (1), $p(i_0) \leq 3S$. Now, since $s_n(i_0) \leq S$ and $p(i_0) \leq 3S$, $0 < |k| \leq K$ and condition (0) is satisfied.

Conditions (i) and (ii) are clearly satisfied. To see that (iii) is satisfied, note that

$$\frac{d-c}{b-a} \geq \frac{\mathrm{lh}(v_{n-1})}{8\,\mathrm{lh}(v_{n+1})} \geq \frac{1}{8(R+1)^2} = \alpha.$$

Finally, (iv) is satisfied because $x\upharpoonright[c,d] = x\upharpoonright[c+k+e, d+k+e]$ and $y\upharpoonright[c,d] = y\upharpoonright[c+e, d+e]$.

The alternative is the case $i_0 < i_y$. In this case we let

$$h = (i_0 - i_y + 1)\,\mathrm{lh}(v_n) - \sum_{i=i_0+1}^{i_y-1} s_n(i) \leq 0.$$

Then in $x$ there is an occurrence of the word $v_n 1^{p(i_0)} v_n$ where the beginning of the second demonstrated occurrence is at position $\beta_n(x,l)+h$. Similarly, in $y$ there is an occurrence of the word $v_n 1^{s_n(i_0)} v_n$ where the beginning of the second demonstrated occurrence is at position $\beta_n(y,l)+h$. We similarly let $[c,d]$ be the interval of overlap of these second occurrences of $v_n$ in $x$ and in $y$. Then $c = c'+h$ and $d = d'+h$. Define

$$e = -\,\mathrm{lh}(v_n) - s_n(i_0) \quad \text{and} \quad k = -p(i_0) + s_n(i_0).$$

Checking conditions (0)–(iv) is similar to the previous case. ∎

Continuing the proof, let

$$D = \{n \in \mathbb{N} : n > n_0,\ \lambda_n(x,0),\ \lambda_n(y,0) < \infty \text{ and } \lambda_n(x,0) \neq \lambda_n(y,0)\}.$$

Since $x$ and $y$ are not in the same $\sigma$-orbit, $D$ is infinite by Lemma 4.4.

LEMMA 4.9. *There is an infinite $D' \subseteq D$ such that for all $n \in D'$ either $\lambda_{n-1}(x,0) = \lambda_{n-1}(y,0) < \infty$ or both $\kappa_{n-1}(x,0) = 0$ and $\kappa_{n-1}(y,0) < \infty$.*

*Proof.* If $\mathbb{N} \setminus D$ is infinite, then

$$D' = \{n \in \mathbb{N} : n > n_0, \lambda_{n-1}(x,0) = \lambda_{n-1}(y,0) < \infty \\ \text{and } \lambda_n(x,0) \neq \lambda_n(y,0)\}$$

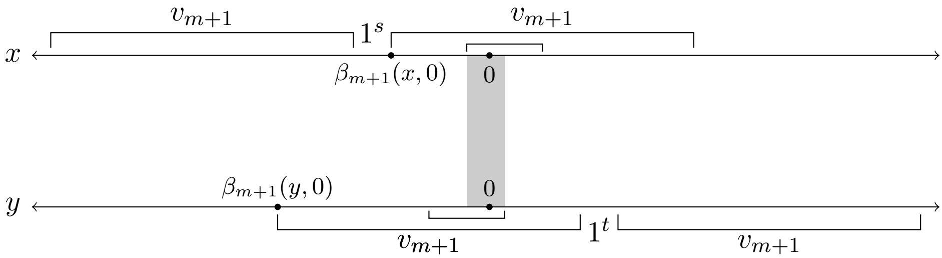
is infinite and $D' \subseteq D$. If $\mathbb{N} \setminus D$ is finite, then

$$D' = \{n \in \mathbb{N} : n > n_0,\ \kappa_{n-1}(x,0) = 0,\ \kappa_{n-1}(y,0) < \infty \\ \text{and } \lambda_n(x,0) \neq \lambda_n(y,0)\}$$

has positive density by Lemma 4.6 and therefore is infinite. ■

Fix an infinite $D' \subseteq D$ as in the above lemma and let $N \in \mathbb{N}$. Now fix $m \in D'$ with $m \geq N$ and $m \geq n_0$. We will argue below that it is possible to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ satisfying conditions (0)–(iv) above, for either $n = m$ or $n = m+1$.

First, note that by Lemmas 4.9, 4.5 and 4.7, we have $|I_m(x,0) \cap I_m(y,0)| \geq \mathrm{lh}(v_{m-1})$. Since $m \in D$, we also know that $\lambda_m(x,0) \neq \lambda_m(y,0)$. Without loss of generality we may assume $\lambda_m(x,0) < \lambda_m(y,0)$, which implies that $\beta_{m+1}(y,0) < \beta_{m+1}(x,0)$. Let $s \in \mathbb{N}$ be such that the expected occurrence of $v_{m+1}$ beginning at $\beta_{m+1}(x,0)$ in $x$ is immediately preceded by $v_{m+1}1^s$. Let $t \in \mathbb{N}$ be such that the expected occurrence of $v_{m+1}$ beginning at $\beta_{m+1}(y,0)$ in $x$ is immediately followed by $1^t v_{m+1}$.




We can easily finish the proof using Lemma 4.8 if either $s$ or $t$ is less than or equal to $3S$. If $s \leq 3S$, then there is no occurrence of $1^{3S}$ in $x$ that does not intersect any expected occurrence of $v_m$ but does overlap $I_{m+1}(y,0)$. In this case we can apply Lemma 4.8 with $n = m$ and $l = 0$ to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ which satisfy the necessary conditions. If $t \leq 3S$, then there is no occurrence of $1^{3S}$ in $y$ that does not intersect any expected occurrence of $v_m$ but does overlap $I_{m+1}(x,0)$. In this case we can also apply Lemma 4.8 with $n = m$ and $l = 0$ to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ which satisfy the necessary conditions. In either case this finishes the proof.

We now suppose that $s, t > 3S$. In this case $x$ must have an expected occurrence of $v_{m+3}$ beginning at position $\beta_{m+1}(x,0)$ and $y$ must have an expected occurrence of $v_{m+3}$ beginning at position $\gamma_{m+1}(y,0) + t + 1$.

If $t \geq \mathrm{lh}(v_{m+1})$, then in $x$ we can find two consecutive expected occurrences of $v_m$ completely contained in the interval

$$[\gamma_{m+1}(y,0) + 1, \gamma_{m+1}(y,0) + t].$$

(Note that $y$ has an occurrence of $1^{t-1}$ in that interval.) We now let $n = m$ and proceed to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$. Let $c_n$ and $d_n$ be the starting and ending positions of the first of those two expected occurrences of $v_m$. Let $e_n$ be such that the second of those expected occurrences of $v_m$ begins at $c_n + e_n - 1$. Let $k = -1$. Let $a_n = -4\,\mathrm{lh}(v_{n+1})$ and $b_n = 4\,\mathrm{lh}(v_{n+1})$. It is straightforward to check that in this case the necessary conditions are satisfied, which finishes the proof.

We now suppose that $t < \mathrm{lh}(v_{m+1})$. We know that $y$ has an occurrence of $v_{m+3}$ that begins at position $\gamma_{m+1}(y,0) + t + 1$. Consider the expected occurrence of $v_m$ in $y$ that begins at position $\gamma_{m+1}(y,0)+t+1$. It is easy to see that since $t < \mathrm{lh}(v_{m+1})$, that occurrence of $v_m$ is completely contained in the interval $I_{m+2}(x,0)$, and must have significant overlap with at least one expected occurrence of $v_m$ in $x$ in $I_{m+2}(x,0)$. To be more precise, there must be some $l_1 \leq 2\,\mathrm{lh}(v_{n+1})$ such that

(1) $I_m(y,l_1) = I_m(y, \gamma_{m+1}(y,0) + t + 1)$,
(2) $|I_m(y,l_1) \cap I_m(x,l_1)| \geq \mathrm{lh}(v_{m-1})$, and
(3) $I_{m+2}(x,l_1) = I_{m+2}(x,0)$.

We now have three cases.

Case 1: $\lambda_m(x,l_1) \neq \lambda_m(y,l_1)$. In this case there is no occurrence of $1^{3S}$ in $x$ that does not intersect any expected occurrence of $v_m$ but does overlap $I_{m+1}(y,l_1)$. We can finish the proof by applying Lemma 4.8 with $n = m$ and $l = l_1$ to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ which satisfy the necessary conditions.

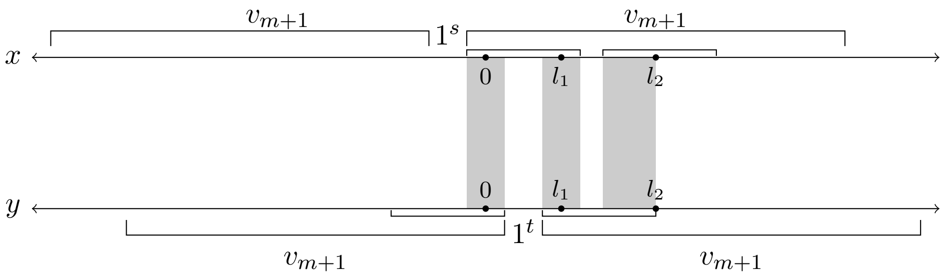
Case 2: $\lambda_m(x,l_1) = \lambda_m(y,l_1)$ but $\lambda_{m+1}(x,l_1) \neq \lambda_{m+1}(y,l_1)$. By Lemma 4.5, we know that $|I_{m+1}(x,l_1) \cap I_{m+1}(y,l_1)| \geq \mathrm{lh}(v_m)$. We also know that there is no occurrence of $1^{3S}$ in $x$ that does not intersect any expected occurrence of $v_{m+1}$ but does overlap $I_{m+2}(y,l_1)$. We can now finish the proof by applying Lemma 4.8 with $n = m+1$ and $l = l_1$ to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ which satisfy the necessary conditions.

CASE 3: $\lambda_m(x, l_1) = \lambda_m(y, l_1)$ and $\lambda_{m+1}(x, l_1) = \lambda_{m+1}(y, l_1)$. Since

$$I_m(y, l_1) = I_m(y, \gamma_{m+1}(y, 0) + t + 1)$$

and $y$ has an expected occurrence of $v_{m+3}$ beginning at $\gamma_{m+1}(y, 0)+t+1$, we must have $\lambda_m(y, l_1) = \lambda_{m+1}(y, l_1) = 1$, which implies that also $\lambda_m(x, l_1) = \lambda_{m+1}(x, l_1) = 1$. Then, since $I_{m+2}(x, l_1) = I_{m+2}(x, 0)$, we must have $I_m(x, l_1) = I_m(x, 0)$.

We can then summarize our situation as follows. We know that $x$ has an expected occurrence of $v_m$ in $I_m(x, 0) = I_m(x, l_1)$, which overlaps significantly with both the expected occurrences of $v_m$ in $y$ in $I_m(y, 0)$ and $I_m(y, l_1)$ (and those occurrences of $v_m$ must be different because they occur in different occurrences of $v_{m+1}$). We also know that $x$ has an expected occurrence of $v_{m+3}$ beginning at $\beta_m(x, 0)$ and that $y$ has an expected occurrence of $v_{m+3}$ beginning at $\beta_m(y, l_1)$.




Note that the expected occurrence of $v_m$ in $x$ in $I_m(x, l_1)$ is followed by $1^{s_m(1)}$ and then by another expected occurrence of $v_m$. It is clear that since $s_m(1) \leq S$ and $t > 3S$, position $\gamma_m(y, l_1)$ must be contained in the expected occurrence of $v_m$ in $x$ that begins at $\gamma_m(x, 0) + s_m(1)$. In fact, if we let $l_2 = \gamma_m(y, l_1)$ we have

$$|I_m(y, l_2) \cap I_m(x, l_2)| \geq |I_m(y, 0) \cap I_m(x, 0)| \geq \mathrm{lh}(v_{m-1}).$$

Notice now that $\lambda_m(x, l_2) = 2$, while $\lambda_m(y, l_2) = 1$. Also, notice that there is no occurrence of $1^{3S}$ in $x$ that does not intersect any expected occurrence of $v_m$ but does overlap $I_{m+1}(y, l_2)$. We can now finish the proof by applying Lemma 4.8 with $n = m$ and $l = l_2$ to define $a_n, b_n, c_n, d_n, e_n, k_n \in \mathbb{Z}$ which satisfy the necessary conditions.

We have thus shown that $(X, \mu, \sigma)$ has minimal self-joinings of order 2, and therefore minimal self-joinings of all orders.

**5. Applications to canonically bounded transformations.** In this section we present some applications of our main results to the class of canonically bounded rank-one transformations. We give combinatorial criteria for isomorphism, disjointness, and minimal self-joinings for these transformations in terms of their cutting and spacer parameters. These criteria are, in principal, easy to check.

**5.1. Canonical generating sequences.** The notion of the canonical generating sequence was developed in [7] in the study of topological conjugacy of symbolic rank-one systems. We first recall some definitions and basic results related to this notion.

Let $\mathcal{F}$ be the set of all binary words that both start and end with 0. Recall that if $u, v \in \mathcal{F}$ and $v \prec u$, then for some $n \geq 1$ and $a_1, \ldots, a_n \in \mathbb{N}$ we have

$$u = v1^{a_1}v \ldots v1^{a_n}v.$$

If in addition $a_1 = \cdots = a_n$, then we say that $u$ is *simply built from* $v$, and denote $v \prec_s u$. Note that unlike $\prec$, $\prec_s$ is not a transitive relation.

It is easy to see that every infinite rank-one word $V$ allows a generating sequence $(v_n : n \in \mathbb{N})$ in which every $v_n$ is an element of $\mathcal{F}$. If $V$ is an infinite rank-one word, the *canonical generating sequence* of $V$ is defined as the sequence enumerating in increasing $\prec$-order the set of all $v \in \mathcal{F}$ such that there do not exist $u, w \in \mathcal{F}$ satisfying $u \prec v \prec w \prec V$ and $u \prec_s w$. By definition, the canonical generating sequence, if it exists, is unique. In [7] it was shown that if $V$ is non-degenerate, the canonical generating sequence of $V$ exists and is in fact infinite.

Given any non-degenerate infinite rank-one word, the *canonical cutting and spacer parameters* are those giving rise to the canonical generating sequence.

A rank-one transformation $T$ is *bounded* if some cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ giving rise to $T$ are bounded, i.e., there is $B > 0$ such that $r_n \leq B$ and $s_n(i) \leq B$ for all $n \in \mathbb{N}$ and $1 \leq i \leq r_n$. We say that $T$ is *strictly bounded* if some bounded cutting and spacer parameters for $T$ give rise to a generating sequence $(v_n : n \in \mathbb{N})$ in which all $v_n$ are elements of $\mathcal{F}$. Alternatively, $T$ is strictly bounded if and only if there are some bounded cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ such that $s_n(r_n) = 0$ for all $n \in \mathbb{N}$. Also, we say that $T$ is *canonically bounded* if some canonical cutting and spacer parameters giving rise to $T$ are bounded. A canonically bounded rank-one transformation is necessarily strictly bounded, and a strictly bounded rank-one transformation is necessarily bounded, but the converse are not true. The following theorem characterizes exactly which strictly bounded rank-one transformations are canonically bounded.

THEOREM 5.1 ([6]). *Let $T$ be a strictly bounded rank-one transformation. Then $T$ is non-rigid, i.e. $T$ has trivial centralizer, if and only if $T$ is canonically bounded.*

**5.2. Replacement schemes and topological conjugacy.** Given infinite rank-one words $V$ and $W$, a *replacement scheme* is a pair $(v, w)$ of finite binary words such that $v \prec V$, $w \prec W$, and for all $k \in \mathbb{N}$ there is an expected occurrence of $v$ in $V$ at position $k$ if and only if there is an expected occurrence of $w$ in $W$ at position $k$. This notion is closely related to the topological conjugacy between symbolic rank-one systems.

In fact, if $v \prec V$, then every $x \in X_V$ can be uniquely expressed as

$$x = \cdots v1^{a_{-i}}v \cdots v1^{a_0}v \cdots v1^{a_i}v \cdots$$

for $\ldots a_{-i}, \ldots, a_0, \ldots, a_i, \ldots \in \mathbb{N}$. We say that $x$ is *built from* $v$. The demonstrated occurrences of $v$ are again said to be *expected.* When $(v, w)$ is a replacement scheme for $V$ and $W$, we may define a map $\phi : X_V \to X_W$ so that

$$\phi(x) = \cdots w1^{b_{-i}}w \cdots w1^{b_0}w \cdots w1^{b_i}w \cdots$$

i.e., $\phi(x)$ is built from $w$, and so that for all $k \in \mathbb{Z}$, there is an expected occurrence of $v$ in $x$ at position $k$ if and only if there is an expected occurrence of $w$ in $\phi(x)$ at position $k$. Intuitively, $\phi(x)$ is obtained from $x$ by replacing every expected occurrence of $v$ in $x$ by $w$, adding or deleting 1s as necessary. It is easy to see that $\phi$ is a topological conjugacy between $X_V$ and $X_W$. We showed in [7] that all topological conjugacies essentially arise this way.

THEOREM 5.2 ([7]). *Let $V$ and $W$ be non-degenerate infinite rank-one words. Then $(X_V, \sigma)$ and $(X_W, \sigma)$ are topologically conjugate if and only if there exists a replacement scheme for $V$ and $W$.*

For the subject of this paper it is important to note that $\phi$ is also a measure-preserving isomorphism. This follows from the unique ergodicity of symbolic rank-one systems. Thus the existence of replacement schemes is a sufficient condition for two symbolic rank-one systems to be isomorphic.

In the case of commensurate parameters, there is a straightforward way to identify replacement schemes and therefore it is easy to determine topological conjugacy.

COROLLARY 5.3. *Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a non-degenerate infinite rank-one word $V$. Let $(r_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a non-degenerate infinite rank-one word $W$. Suppose the two sets of parameters are commensurate. Then $(X_V, \sigma)$ and $(X_W, \sigma)$ are topologically conjugate if and only if there is $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n = t_n$.*

As mentioned above, this also gives an explicit sufficient condition for two symbolic rank-one systems to be measure-theoretically isomorphic.

**5.3. Isomorphism and disjointness of canonically bounded transformations.** Our objective is to give combinatorial criteria for isomorphism and disjointness for canonically bounded rank-one transformations in terms of their cutting and spacer parameters. Ideally, these criteria will be easy to check.

Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be the cutting and spacer parameters for a rank-one transformation $T$. Assume $s_n(r_n) = 0$ for all $n \in \mathbb{N}$. For an integer $d > 1$, consider the statement

$$(\mathrm{E}_d) \qquad \forall N \in \mathbb{N}\, \exists n, i \in \mathbb{N}\, [n \geq N,\ 1 \leq i \leq r_n - 1, \text{ and } h_N + s_n(i) \not\equiv 0 \bmod d]$$

where $(h_n : n \in \mathbb{N})$ is the sequence defined in (2.1).

The following fact has been proved in [7].

THEOREM 5.4 ([7]). *Let $T$ be a strictly bounded rank-one transformation with cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$. Then for any integer $d > 1$, $T^d$ is ergodic if and only if* $(\mathrm{E}_d)$ *holds.*

We can now state our main result about commensurate, canonically bounded rank-one transformations.

COROLLARY 5.5. *Let $T$ be a rank-one transformation with bounded canonical cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$. Let $S$ be a rank-one transformation with bounded canonical cutting and spacer parameters $(q_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$. Suppose the parameters for $T$ and $S$ are commensurate. Then the following hold:*

(1) *$T$ and $S$ are isomorphic if and only if there is $N \in \mathbb{N}$ such that $s_n = t_n$ for all $n \geq N$.*
(2) *$T$ and $S$ are disjoint if and only if $s_n \neq t_n$ for infinitely many $n \in \mathbb{N}$ and for every integer $d > 1$, either $T^d$ or $S^d$ is ergodic.*

As in Theorem 3.2, if $D$ is an upper bound for the sequences $(s_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$, then (2) can be strengthened to

(2′) $T$ and $S$ are disjoint if and only if $s_n \neq t_n$ for infinitely many $n \in \mathbb{N}$ and for every integer $1 < d \leq 5D$, either $T^d$ or $S^d$ is ergodic.

The rest of this subsection is devoted to the proof of Corollary 5.5.

Let $(v_n : n \in \mathbb{N})$ be the canonical generating sequence given by the canonical cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$. Let $V = \lim_n v_n$. Then $T$ is isomorphic to the symbolic rank-one system $(X_V, \mu, \sigma)$ for a uniquely ergodic Borel probability measure $\mu$. So we will assume that $T$ is $(X, \mu, \sigma)$. Let $(w_n : n \in \mathbb{N})$ be the canonical generating sequence given by the canonical cutting and spacer parameters $(q_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$. Let $W = \lim_n w_n$. We will similarly assume that $S$ is the symbolic rank-one system $(X_W, \nu, \sigma)$ for a suitable measure $\nu$. By commensurability, for all $n \in \mathbb{N}$, $q_n = r_n$ and $\mathrm{lh}(v_n) = \mathrm{lh}(w_n)$.

First consider isomorphism. The condition is sufficient since it gives a replacement scheme, which in turn gives rise to a topological conjugacy which is also a measure-theoretic isomorphism. More specifically, if $s_n = t_n$ for all $n \geq N$, then $(v_N, w_N)$ is a replacement scheme, and the topological conjugacy it induces is an isomorphism between $T$ and $S$.

For the necessity, assume that $s_n \neq t_n$ for infinitely many $n \in \mathbb{N}$. Before proceeding we prove a basic fact about compatibility.

LEMMA 5.6. *Let $s, t, s', t' \in \mathcal{S}$. Suppose $s \neq t$, $\mathrm{lh}(s) = \mathrm{lh}(t) = l > 0$, and $\mathrm{lh}(s') = \mathrm{lh}(t') = m > 0$. Assume the following two words are compatible:*

$$s^\frown(s'(1))^\frown s^\frown(s'(2))^\frown s^\frown\cdots{}^\frown s^\frown(s'(m))^\frown s^\frown(0), \tag{5.1}$$

$$t^\frown(t'(1))^\frown t^\frown(t'(2))^\frown s^\frown\ldots{}^\frown t^\frown(t'(m))^\frown t^\frown(0). \tag{5.2}$$

*Then $s'$ and $t'$ are both constant words.*

*Proof.* Let $u^\frown(0)$ be the word in (5.1) and $z^\frown(0)$ be the word in (5.2). Without loss of generality suppose $z$ is a subword of $u^\frown(c)^\frown u$ for some $c \in \mathbb{N}$. Since $s \neq t$, the first occurrence of $t$ in $z$ cannot line up with any occurrence of $s$ in $u$, i.e., in the occurrence of $z$ in $u^\frown(c)^\frown u$, the starting position of the first occurrence of $t$ is not the same as the starting position of any demonstrated occurrence of $s$. Since $\mathrm{lh}(s) = \mathrm{lh}(t) = l > 0$, this implies that there is $1 \leq j \leq l$ such that $t'(1) = s(j)$. But then $t'(2) = \cdots = t'(m) = s(j)$, thus $t'$ is constant. By symmetry, $s'$ is also constant. ∎

Now return to the proof of Corollary 5.5(1). We have assumed that there are infinitely many $n \in \mathbb{N}$ with $s_n \neq t_n$. We inductively define an infinite sequence $(n_k : k \in \mathbb{N})$ of natural numbers as follows. Define $n_0 = 0$. In general, assume $n_k$, $k \geq 0$, has been defined. Define $n_{k+1} = n_k + 1$ if $s_{n_k} = t_{n_k}$. Otherwise, $s_{n_k} \neq t_{n_k}$, and we define $n_{k+1} = n_k + 2$ if $s_{n_k+1}\upharpoonright(r_{n_k+1} - 1)$ is not constant, and $n_{k+1} = n_k + 3$ otherwise. Let $v'_k = v_{n_k}$ and $w'_k = w_{n_k}$ for all $k \in \mathbb{N}$. Then $(v'_n : n \in \mathbb{N})$ is a subsequence of $(v_n : n \in \mathbb{N})$ giving rise to $T$ and $(w'_n : n \in \mathbb{N})$ is a subsequence of $(w_n : n \in \mathbb{N})$ giving rise to $S$. Let $(r'_n : n \in \mathbb{N})$ and $(s'_n : n \in \mathbb{N})$ be the cutting and spacer parameters corresponding to $(v'_n : n \in \mathbb{N})$. Since each $v'_n$ is in $\mathcal{F}$, $s'_n(r'_n) = 0$. Let $(q'_n : n \in \mathbb{N})$ and $(t'_n : n \in \mathbb{N})$ be the cutting and spacer parameters corresponding to $(w'_n : n \in \mathbb{N})$. Similarly, $t'_n(q'_n) = 0$. It is clear that the newly defined parameters are commensurate. We claim that the newly defined parameters for $T$ and $S$ satisfy all the other hypotheses of Theorem 3.1. Thus $T$ and $S$ are not isomorphic.

To verify the claim, first note that $n_k < n_{k+1} \leq n_k + 3$ for all $k \in \mathbb{N}$. This implies boundedness of the newly defined cutting and spacer parameters. In fact, if $R$ is a bound for $(r_n : n \in \mathbb{N})$, then $R^3$ is a bound for $(r'_n : n \in \mathbb{N})$. If $S$ is a bound for $(s_n : n \in \mathbb{N})$, then $S$ is still a bound for $(s'_n : n \in \mathbb{N})$.

It remains to verify that for infinitely many $k \in \mathbb{N}$, $s'_k \perp t'_k$. By our construction of the sequence $(n_k : k \in \mathbb{N})$, there are infinitely many $k$ such that either $n_{k+1} = n_k + 2$ or $n_{k+1} = n_k + 3$. We claim that for each of these $k$ we have $s'_k \perp t'_k$. First suppose $k$ is such that $n_{k+1} = n_k + 2$. By our construction this means that $s_{n_k} \neq t_{n_k}$ and $s_{n_k+1}\restriction(r_{n_k+1}-1)$ is not constant. In this case, we have

$$s'_k = s_{n_k}\restriction(r_{n_k}-1)^\frown(s_{n_k+1}(1))^\frown\cdots^\frown(s_{n_k+1}(r_{n_k+1}-1))^\frown s_{n_k}\restriction(r_{n_k}-1)^\frown(0),$$

$$t'_k = t_{n_k}\restriction(r_{n_k}-1)^\frown(t_{n_k+1}(1))^\frown\cdots^\frown(t_{n_k+1}(r_{n_k+1}-1))^\frown t_{n_k}\restriction(r_{n_k}-1)^\frown(0).$$

By Lemma 5.6, $s'_k \perp t'_k$. Next suppose $k$ is such that $n_{k+1} = n_k + 3$. By our construction this means that $s_{n_k} \neq t_{n_k}$ and $s_{n_{k+1}}\restriction(r_{n_k+1}-1)$ is constant. A similar application of Lemma 5.6 will complete the proof, provided that we verify that the word

$$s_{n_k+1}\restriction(r_{n_k+1}-1)^\frown(s_{n_k+2}(1))^\frown\cdots^\frown(s_{n_k+2}(r_{n_k+2}-1))^\frown s_{n_k+1}\restriction(r_{n_k+1}-1)$$

is not constant. Assume it is. Note that this sequence corresponds to the way $v_{n_k+3}$ is built from $v_{n_k+1}$. Thus $v_{n_k+1} \prec_s v_{n_k+3}$ and $v_{n_k+2}$ is not in the canonical generating sequence. This contradicts our assumption that $(v_n : n \in \mathbb{N})$ is a canonical generating sequence.

We have thus shown Corollary 5.5(1). For Corollary 5.5(2), the necessity of the condition is clear (cf. the remarks after the statement of Theorem 3.2). For the sufficiency, it is enough to construct new pairs of cutting and spacer parameters as above, and apply Theorem 3.2.

**5.4. A case of Ryzhikov's theorem.** Ryzhikov [16] announced the following theorem on minimal self-joinings for non-rigid, totally ergodic, bounded rank-one transformations.

THEOREM 5.7 (Ryzhikov [16]). *Let $T$ be a bounded rank-one transformation. Then $T$ has minimal self-joinings of all orders if and only if $T$ is non-rigid and totally ergodic.*

As a corollary to Theorem 4.1, we obtain the theorem in the case of strictly bounded rank-one transformations.

It is easy to verify that having minimal self-joinings implies mild mixing (having no rigid factors), which in turn implies non-rigidity. Having minimal self-joinings also implies weak mixing, which in turn implies total ergodicity. Thus the two conditions are necessary.

For the sufficiency, let $T$ be a bounded rank-one transformation with cutting and spacer parameters $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ with $s_n(r_n) = 0$ for all $n \in \mathbb{N}$. Assume that $T$ is non-rigid and totally ergodic. By Theorem 5.1, $T$ is canonically bounded. Thus we may assume without loss of generality that $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ are canonical cutting and spacer parameters, which are also bounded. Let $(v_n : n \in \mathbb{N})$ be the canonical generating sequence given by $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$. We inductively define an infinite sequence $(n_k : k \in \mathbb{N})$ of natural numbers as follows. Define $n_0 = 0$. In general, assume $n_k$, $k \geq 0$, has been defined. Define $n_{k+1} = n_k + 2$ if $s_{n_k+1}\restriction(r_{n_k+1}-1)$ is not constant, and define $n_{k+1} = n_k + 3$ otherwise. Let

$v'_k = v_{n_k}$ for all $k \in \mathbb{N}$. Then $(v'_n : n \in \mathbb{N})$ is a subsequence of $(v_n : n \in \mathbb{N})$, which still generates $T$. Let $(r'_n : n \in \mathbb{N})$ and $(s'_n : n \in \mathbb{N})$ be the cutting and spacer parameters corresponding to $(v'_n : n \in \mathbb{N})$. Since $n_k < n_{k+1} \leq n_k + 3$ for all $k \in \mathbb{N}$, these newly defined cutting and spacer parameters are still bounded.

To prove the corollary, we will apply Theorem 4.1 to $(r'_n : n \in \mathbb{N})$ and $(s'_n : n \in \mathbb{N})$. The only condition to verify is (c), that is, for all $n \in \mathbb{N}$ and $c \in \mathbb{N}$, there are only two occurrences of $s'_n\restriction(r_n - 1)$ in $s'_n\restriction(r_n-1)^\frown(c)^\frown s'_n\restriction(r_n-1)$. Note that for every $k > 0$, $s'_k$ is of the form

$$s_{n_k}\restriction(r_{n_k}-1)^\frown(u(1))^\frown s_{n_k}\restriction(r_{n_k}-1)^\frown\cdots{}^\frown(u(m))^\frown s_{n_k}\restriction(r_{n_k}-1)^\frown(0)$$

where $u$ is either $s_{n_k+1}\restriction(r_{n_k+1}-1)$ or

$$s_{n_k+1}\restriction(r_{n_k+1}-1)^\frown(s_{n_k+2}(1))^\frown\cdots{}^\frown(s_{n_k+2}(r_{n_k+2}-1))^\frown s_{n_k+1}\restriction(r_{n_k+1}-1).$$

As in the proof of Corollary 5.5, $u$ is not constant in either case: in the former case $s_{n_k+1}\restriction(r_{n_k+1}-1)$ is assumed not to be constant, and in the latter case $u$ corresponds to the way $v_{n_k+2}$ is built from $v_{n_k}$, and therefore is not constant since $v_{n_k+1}$ is assumed to be on the canonical generating sequence. Now if there is $c \in \mathbb{N}$ such that $s'_n\restriction(r_n-1)$ occurs in $s'_n\restriction(r_n-1)^\frown(c)^\frown s'_n\restriction(r_n-1)$ not as demonstrated, then by a similar argument to the proof of Lemma 5.6, it would follow that $u$ is constant, a contradiction.

This concludes the proof of Ryzhikov's theorem in the case of strictly bounded rank-one transformations.

It is worth noting here that in [7] simple algorithms are given to determine whether a strictly bounded rank-one transformation is non-rigid and is totally ergodic. Combining these gives a simple algorithm for determining whether a strictly bounded rank-one transformation has minimal self-joinings of all orders. We include, in the language of this paper, that simple algorithm.

THEOREM 5.8. *Let $(X, \mu, \sigma)$ be a strictly bounded rank-one transformation. Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be bounded cutting and spacer parameters giving rise to $(X, \mu, \sigma)$, with $s_n(r_n) = 0$ for all $n$. Let $(v_n : n \in \mathbb{N})$ be the generating sequence of the corresponding to those cutting and spacer parameters and, for all $N \in \mathbb{N}$, let $h_n = |v_n|$. (One can also describe $h_n$ as the height of the stage-$n$ tower in the cutting and stacking construction corresponding to cutting and spacer parameters.) Then $(X, \mu, \sigma)$ has minimal self-joinings of all orders if and only if both of the following conditions hold:*

(1) *There exists $k \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exist $n, m, i, j$, with $N \leq n, m \leq N + k$, $0 < i < r_n$ and $0 < j < r_m$, such that $s_n(i) \neq s_m(j)$.*
(2) *For all $N \in \mathbb{N}$ and $d > 1$ there exist $n \geq N$ and $0 < i < r_n$ such that $d \nmid h_n + s_n(i)$.*

In fact, under the assumptions of the theorem above, (1) holds if and only if $(X, \mu, \sigma)$ is non-rigid, and (2) holds if and only if $(X, \mu, \sigma)$ is total ergodic (see [7, Theorems 1.2 and 1.3]).

**6. Concluding remarks.** Some results of this paper are applicable in a broader context than stated. We have noted that Theorems 3.2, 4.1 and Corollary 5.5 can be strengthened with "partial total ergodicity" assumptions replacing the total ergodicity assumptions, which we denoted by (d′) and (2′) respectively. Here we note that Theorems 3.1, 3.2 and Corollary 5.5 can be further strengthened with an "eventual commensurability" assumption replacing the commensurability assumption. For instance, Theorem 3.1 can be strengthened as follows.

THEOREM 6.1. *Let $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(X, \mu, \sigma)$. Let $(v_n : n \in \mathbb{N})$ be the generating sequence given by $(r_n : n \in \mathbb{N})$ and $(s_n : n \in \mathbb{N})$.*

*Let $(q_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$ be cutting and spacer parameters giving rise to a symbolic rank-one system $(Y, \nu, \sigma)$. Let $(w_n : n \in \mathbb{N})$ be the generating sequence given by $(q_n : n \in \mathbb{N})$ and $(t_n : n \in \mathbb{N})$.*

*Suppose the following hold:*

(a) *The two sets of parameters are "eventually commensurate", i.e., there are $N, M \in \mathbb{N}$ such that $\mathrm{lh}(v_N) = \mathrm{lh}(w_M)$ and for all $n \in \mathbb{N}$, $r_{N+n} = q_{M+n}$ and*

$$\sum_{i=1}^{r_{N+n}-1} s_{N+n}(i) = \sum_{i=1}^{q_{M+n}-1} t_{M+n}(i).$$

(b) *There is an $S \in \mathbb{N}$ such that for all $n$ and all $1 \leq i \leq r_n - 1$,*

$$s_n(i) \leq S \quad \textit{and} \quad t_n(i) \leq S.$$

(c) *There is an $R \in \mathbb{N}$ such that for infinitely many $n$,*

$$r_n \leq R \quad \textit{and} \quad s_n \perp t_n.$$

*Then $(X, \mu, \sigma)$ and $(Y, \nu, \sigma)$ are not isomorphic.*

Theorem 3.2 and Corollary 5.5 allow similar generalizations. It should be clear that the proofs of these generalizations are identical to the proofs given in [9] and in this paper.

It is, however, not clear how to determine whether two rank-one transformations allow eventually commensurate cutting and spacer parameters. Of course, if two rank-one transformations do not allow such parameters, then they are not isomorphic. We conjecture that there is a Borel procedure for this determination.

**Acknowledgments.** We would like to thank Eli Glasner for useful discussions on the topics of the paper and for providing us with the references related to Theorem 4.2. We also benefited from discussions with Matt Foreman, Cesar Silva, and Benjy Weiss as a part of a SQuaRE program at the American Institute of Mathematics (AIM) focusing on the isomorphism problem of rank-one transformations. Last but not least, we thank the anonymous referee for helpful comments that resulted in improvements of the paper.

The first author acknowledges the US NSF grants DMS-1201290 and DMS-1800323 for the support of his research. He also acknowledges the support of the Issac Newton Institute (INI) for Mathematical Sciences at the University of Cambridge for a research visit during which a substantial part of this paper was written. He was a Visiting Fellow to the Mathematical, Foundational and Computational Aspects of the Higher Infinite (HIF) program at the INI, and he thanks the organizers of the program and the Scientific Advisory Committee for this opportunity.

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