Graev ultrametrics and surjectively universal non-Archimedean Polish groups

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A B S T R A C T
We define Graev-type ultrametrics and use them to show the existence of surjectively universal non-Archimedean Polish groups. We also show the existence of surjectively universal abelian non-Archimedean Polish groups and surjectively universal non-Archimedean Polish groups admitting compatible two-sided invariant metrics.

1. Introduction

A topological group is Polish if it is separable and completely metrizable. Every Polish group admits a compatible left-invariant metric. A Polish group is called non-Archimedean if it has a neighborhood base for the identity that consists of open subgroups. It is easy to see that the class of all non-Archimedean Polish groups is closed under taking closed subgroups and continuous homomorphic images (equivalently quotients by closed normal subgroups). A prototypical example of non-Archimedean Polish group is the infinite permutation group $S_\infty$. Recall that $S_\infty$ is the group of all permutations of $\mathbb{N}$ with the composition as the group operation and with the pointwise convergence topology. It is a $G_\delta$ subset of the Baire space $\mathcal{N} = \mathbb{N}^\mathbb{N}$. The following theorem characterizes non-Archimedean Polish groups up to topological isomorphism.

**Theorem 1.1.** (Becker and Kechris [1, Theorem 1.5.1]) The following are equivalent for a Polish group $G$:

(i) $G$ is non-Archimedean;
(ii) $G$ is isomorphic to a closed subgroup of $S_\infty$;
(iii) $G$ admits a compatible left-invariant ultrametric.

It follows from this theorem that $S_\infty$ is a universal non-Archimedean Polish group. That is, for any non-Archimedean Polish group $G$, there is a topological group isomorphic embedding from $G$ into $S_\infty$. 

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Theorem 1.3. There exists a surjectively universal non-Archimedean Polish group.

We note that $S_\infty$ is far from being surjectively universal for non-Archimedean Polish groups. In fact, by a result of Schreier–Ulam (cf. [2, Theorem 4.26]), $S_\infty$ is topologically simple, i.e., it has no non-trivial closed normal subgroups. This implies that the only continuous homomorphic images of $S_\infty$ are $S_\infty$ and the trivial group.

To prove the main theorem, we construct left-invariant ultrametrics on the free group with countably infinitely many generators and take the appropriate completions of the resulting group. This construction is an adaptation of the methods used in [4] and the most recent [3], in which the existence of the surjectively universal Polish groups was proved.

A special case of the construction gives the existence of surjectively universal groups for the following classes.

Theorem 1.3.

1. There exists a surjectively universal group for the class of all abelian non-Archimedean Polish groups.
2. There exists a surjectively universal group for the class of all non-Archimedean Polish groups admitting compatible two-sided invariant metrics.

In fact, the ultrametric used in the proof of Theorem 1.3 is a simple modification of the classical Graev metric (cf. [5]) and is much easier to present than the general case. Therefore, we will present this construction first before dealing with the general case.

Theorem 1.3 can be proved by totally different methods (not involving Graev-type ultrametrics). Some of these proofs are presented in [6].

2. Graev ultrametrics

In this section we define the Graev ultrametrics and prove Theorem 1.3 using them. Much of this section is a verbatim repetition of the presentation in [5]. For the convenience of the reader we state definitions and results in full details, and only sketch the proofs.

We recall the basic notation. For a nonempty set $X$, let $X^{-1} = \{x^{-1} : x \in X\}$ be a disjoint copy of $X$ and let $e \notin X \cup X^{-1}$. Put $\overline{X} = X \cup X^{-1} \cup \{e\}$. We use the notational convention that $(x^{-1})^{-1} = x$ for $x \in X$ and $e^{-1} = e$. Let $W(X)$ be the set of words over the alphabet $\overline{X}$. A word $w \in W(X)$ is irreducible if $w = e$ or else $w = x_0 \cdots x_n$, where for any $i$, $x_i \in \overline{X}$, $x_i \neq e$ and $x_{i+1} \neq x_i^{-1}$. Let $F(X)$ be the set of irreducible words. For $w \in W(X)$, we denote by $lh(w)$ the length of $w$. Then note that an irreducible word has positive length. For each $w \in W(X)$ the reduced word for $w$, denoted $\overline{w}$, is the unique irreducible word obtained by successively replacing any occurrence of $xx^{-1}$, $x \in \overline{X}$, by $e$ and eliminating $e$ from any occurrence of the form $w_1e\overline{w}_2$, where $w_1, w_2 \in W(X)$ and at least one of $w_1$ and $w_2$ is nonempty. We can turn $F(X)$ into a group, which is called the free group on $X$, by defining $w \cdot u = (wu)'$. Note that the identity element of $F(X)$ is $e$ but not the empty word.

Assume an ultrametric $d$ on $\overline{X}$ satisfies the following conditions for all $x, y \in \overline{X}$:

(i) $d(x, y) = d(x^{-1}, y^{-1})$;
(ii) $d(x, e) = d(x^{-1}, e)$;
(iii) $d(x, y) = d(x^{-1}, y)$.

We define an ultrametric on $F(X)$. If $w = x_1 \cdots x_n$ and $u = y_1 \cdots y_n$ are two words in $W(X)$ of the same length, then put

$$\rho_0(w, u) = \max\{d(x_i, y_i) : 1 \leq i \leq n\}.$$

Next call a word $w \in W(X)$ trivial if $w' = e$. A trivial word is also called a trivial extension of $e$. For $w \in W(X)$ with $lh(w) > 0$ but $w \neq e$, a trivial extension of $w = x_1 \cdots x_n$ is a word of the form $u_1x_1 \cdots u_nx_nu_{n+1}$, where each of $u_1, \ldots, u_{n+1}$ is either trivial or empty. In particular, if $w \in F(X)$ and $w^* \in W(X)$, then $w^*$ is a trivial extension of $w$ if and only if $(w^*)' = w$.

Definition 2.1. The Graev ultrametric $\delta_0(u, v)$ on $F(X)$ is defined as

$$\inf\{\rho_0(u^*, v^*) : u^*, v^* \in W(X), \ lh(u^*) = lh(v^*), (u^*)' = u, (v^*)' = v\}.$$
For the computation of the Graev ultrametric we recall the following definition.

**Definition 2.2.** Let \( m, n \in \mathbb{N} \) and \( m \leq n \). A bijection \( \theta \) on \( \{m, \ldots, n\} \) is a match if

1. \( \theta \circ \theta = \text{id} \); and
2. there is no \( m \leq i, j \leq n \) such that \( i < j < \theta(i) < \theta(j) \).

For any match \( \theta \) on \( \{1, \ldots, n\} \) and \( w = x_1 \cdots x_n \in W(X) \), we define

\[
x_i^\theta = \begin{cases} x_i, & \text{if } \theta(i) > i, \\ e, & \text{if } \theta(i) = i, \\ x_{\theta^{-1}(i)}, & \text{if } \theta(i) < i. \end{cases}
\]

Let \( w^\theta = x_1^\theta \cdots x_n^\theta \). Then \( w^\theta \) is trivial. The following theorem is an analog of Theorem 3.6 of [5].

**Theorem 2.3.** For any \( w \in F(X) \),

\[
\delta_u(w, e) = \min \{ \rho_u(w, w^\theta) : \theta \text{ is a match} \}.
\]

Note that for each \( w \in W(X) \) there are only finitely many matches \( \theta \) on \( \{1, \ldots, \lfloor \log(w) \rfloor \} \) and thus only finitely many trivial words of the form \( w^\theta \). Thus this theorem gives a finitary algorithm for computing the Graev ultrametric. The proof of the theorem is similar to that of Theorem 3.6 of [5]. All one needs to do is to replace each occurrence of the summation operation in the previous proof by the maximum operation. The following theorem now follows easily.

**Theorem 2.4.** Let \((X, d)\) be an ultrametric space. Then the Graev ultrametric \( \delta_u \) is a two-sided invariant ultrametric on \( F(X) \) extending \( d \). Furthermore, \( F(X) \) is a topological group in the topology induced by \( \delta_u \). If \( X \) is separable, so is \( F(X) \).

**Proof.** It is immediate from Definition 2.1 that \( \delta_u \) is two-sided invariant and for any \( D \subseteq X \) dense in \( X \), \( F(D) \) is dense in \( F(X) \). That \( \delta_u \geq 0 \) and its symmetry are also clear.

For the ultrametric inequality, by two-sided invariance it suffices to check that for any \( w, u \in F(X) \), \( \delta_u(w \cdot u, e) \leq \max(\delta_u(w, e), \delta_u(u, e)) \). By Theorem 2.3 let \( \theta, \varphi \) be matches so that \( \delta_u(w, e) = \rho_u(w, w^\theta) \) and \( \delta_u(u, e) = \rho_u(u, u^\varphi) \). Then \( \rho_u(wu, w^\theta u^\varphi) = \max(\rho_u(w, w^\theta), \rho_u(u, u^\varphi)) \). Since \( wu \) is a trivial extension of \( w \cdot u \) and \( w^\theta u^\varphi \) is trivial, \( \rho_u(w \cdot u, e) \leq \rho_u(wu, w^\theta u^\varphi) = \max(\delta_u(w, e), \delta_u(u, e)) \).

To see that \( \delta_u(w, u) = 0 \) implies \( w = u \) for \( w, u \in F(X) \), by two-sided invariance it suffices to show that \( \delta_u(w, e) = 0 \) implies \( w = e \). Suppose \( \delta_u(w, e) = 0 \) and let \( \theta \) be a match so that \( 0 = \delta_u(w, e) = \rho_u(w, w^\theta) \). Then the definition of \( \rho_u \) gives that \( w = w^\theta \), so \( w \) is trivial after all. But the only trivial element of \( F(X) \) is \( e \), hence \( w = e \).

Now if \( x, y \in X \) are distinct, then \( \delta_u(x, y) = \delta_u(y^{-1}x, e) \) and \( (y^{-1}x)^\theta \) can only result in \( y^{-1}y \) and \( ee \). Then \( y^{-1}y \) achieves the smaller value \( d(x, y) \) since \( d \) is an ultrametric. This shows that \( \delta_u \) extends \( d \).

The continuity of the group operations follows from the two-sided invariance and the ultrametric inequality as usual. \( \square \)

It is worth noting that if \( d \) is a metric (rather than an ultrametric) the definitions of \( \rho_u \) and \( \delta_u \) still make sense and the resulting \( \delta_u \) is still a two-sided invariant pseudo-metric on \( F(X) \) with the ultrametric property. In this case, however, both Theorems 2.3 and 2.4 could fail, and \( \delta_u \) could fail to be a metric and might not extend \( d \). In fact, if \( \overline{X} \) has the property that for any \( x, y \in \overline{X} \) and any \( \epsilon > 0 \), there are \( x_0 = x, x_1, \ldots, x_n = y \) such that \( d(x_i, x_{i+1}) < \epsilon \) for all \( i < n \) (in particular, if \( \overline{X} \) is metrically convex), then \( \delta_u(u, u) = 0 \) for all \( w, u \in F(X) \).

Continuing with our construction, let \( \overline{F}_0(X) \) denote the completion of \((F(X), \delta_u)\), and denote the resulting ultrametric still by \( \delta_u \). Then \( \overline{F}_0(X) \) becomes a Polish group with \( \delta_u \) a compatible two-sided invariant ultrametric.

Now we can define the Graev ultrametric group on the Baire space \( \mathcal{N} = \mathbb{N}^\mathbb{N} \).

**Definition 2.5.** The Graev ultrametric group \( G_0 \) is \((F_0(\mathcal{N}), \delta_u)\), where \( \mathcal{N} = \mathbb{N}^\mathbb{N} \) is the Baire space, and the ultrametric \( d \) on \( \mathcal{N} \) is uniquely determined by

\[
d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-\min\{n : x(n) \neq y(n)\}}, & \text{if } x \neq y, \end{cases}
\]

and

\[
d(x^{-1}, y^{-1}) = d(x, y), \quad d(x, e) = d(x^{-1}, e) = d(x, y^{-1}) = 1
\]

for \( x, y \in \mathcal{N} \).
We will show that $\mathcal{G}_u$ is surjectively universal for all non-Archimedean Polish groups admitting compatible two-sided invariant metrics. For the proof we need the following result from [6]. The theorem can be seen as an analog of Theorem 1.1 for the class of non-Archimedean Polish groups admitting compatible two-sided invariant metrics.

**Theorem 2.6.** (Gao and Xuan [6]) The following are equivalent for a Polish group $G$:

(i) $G$ is non-Archimedean and admits a compatible two-sided invariant metric;
(ii) $G$ is isomorphic to a closed subgroup of a product group $\prod_{n \in \mathbb{N}} H_n$, where each $H_n$ is a countable group with the discrete topology;
(iii) $G$ admits a compatible two-sided invariant ultrametric.

We are now ready to prove Theorem 1.3.

**Theorem 2.7.** $\mathcal{G}_u$ is surjectively universal for all non-Archimedean Polish groups admitting compatible two-sided invariant metrics.

**Proof.** The proof is similar to the proof for the fact that the Graev metric group is surjectively universal for all Polish groups admitting compatible two-sided invariant metrics (cf. Theorem 2.6.8 of [7]).

Let $G$ be a non-Archimedean Polish group admitting a compatible two-sided invariant metric. By Theorem 2.6 (iii) $G$ admits a compatible two-sided invariant ultrametric $d_C$. Note that $d_C$ is complete. Without loss of generality we may assume $d_C < 1$.

Let $\phi : \mathcal{N} \to G$ be a Lipschitz map onto $G$, i.e., $d_C(\phi(x), \phi(y)) \leq d(x, y)$ for all $x, y \in \mathcal{N}$, where $d$ is the usual ultrametric on $\mathcal{N}$. Let $\hat{\phi} : F(\mathcal{N}) \to G$ be the canonical group homomorphism extending $\phi$. We show that $\hat{\phi}$ is also Lipschitz, i.e., $d_C(\hat{\phi}(u), \hat{\phi}(v)) \leq d_n(u, v)$ for all $u, v \in F(\mathcal{N})$.

By invariance and homomorphism properties, it suffices to show that $d_C(\hat{\phi}(u), 1_G) \leq d_n(u, e)$ for all $u \in F(\mathcal{N})$. For this let $v$ be a trivial word with $lh(u) = lh(v)$ and $\rho_n(u, v) = d_n(u, v)$. Assume $u = x_1 \cdots x_n$ and $v = y_1 \cdots y_n$. Then

$$
\delta_n(u, e) = \rho_n(u, v) = \max\{d(x_i, y_j) : 1 \leq i \leq n\}
\geq \max\{d_C(\hat{\phi}(x_i), \hat{\phi}(y_j)) : 1 \leq i \leq n\}
\geq d_C(\hat{\phi}(u), 1_G).
$$

With the Lipschitz property we can define $\varphi : F_u(\mathcal{N}) \to G$: if $(u_n)$ is a $\delta_n$-Cauchy sequence in $F(\mathcal{N})$ converging to $\sigma \in F_u(\mathcal{N})$, then define $\varphi(\sigma) = \lim_n \hat{\phi}(u_n)$. The right-hand side makes sense since $(\hat{\phi}(u_n))$ is a $d_C$-Cauchy sequence. It is now routine to check that the definition of $\varphi$ does not depend on the choice of the sequence $(u_n)$, and that $\varphi$ is still Lipschitz, and hence a continuous homomorphism from $F_u$ onto $G$. \hfill $\square$

**Theorem 2.8.** Let $K$ be the closure of the commutator subgroup $[\mathcal{G}_u, \mathcal{G}_u]$ of $\mathcal{G}_u$. Then the group $\mathcal{G}_u/K$ is surjectively universal for all abelian non-Archimedean Polish groups.

**Proof.** It is easy to see that $K$ is a closed normal subgroup of $\mathcal{G}_u$, and so $\mathcal{G}_u/K$ is a non-Archimedean Polish group. Since $[\mathcal{G}_u, \mathcal{G}_u] \leq K$, $\mathcal{G}_u/K$ is abelian.

Let $G$ be an abelian non-Archimedean Polish group. Then in particular $G$ admits a compatible two-sided invariant metric. By Theorem 2.7 there is a continuous homomorphism $\varphi$ from $\mathcal{G}_u$ onto $G$. Let $N = \ker \varphi$. Then $N$ is a closed normal subgroup of $\mathcal{G}_u$ with $[\mathcal{G}_u, \mathcal{G}_u] \leq N$. It follows that $K \leq N$.

Define $\psi : \mathcal{G}_u/K \to G$ by $\psi(gK) = \varphi(g)$. It is easy to see that the definition does not depend on the choice of $g$, and that $\psi$ is a continuous homomorphism onto $G$. \hfill $\square$

### 3. Graev ultrametrics with scales

In the rest of the paper we turn to our main theorem, Theorem 1.2. In this section we will define Graev ultrametrics with scales following the development of [4], and in the next section we will construct a surjectively universal non-Archimedean Polish group following the development of [3]. We will again give full details when we state definitions and theorems. Some of the proofs will be omitted if they involve minor modifications of proofs from the previous papers.

Let $X$ be a set and an ultrametric $d$ on $X$ be given as in the previous section. We will construct a class of left-invariant ultrametrics on $F(X)$. The key concept used in this construction is the notion of a scale as defined below.

**Definition 3.1.** Let $\mathbb{R}_+$ denote the set of non-negative real numbers. A function $\Gamma : X \times \mathbb{R}_+ \to \mathbb{R}_+$ is a scale on $X$ if the following hold for any $x \in X$ and $r \in \mathbb{R}_+$:

(i) $\Gamma(e, r) = r$; $\Gamma(x, r) \geq r$;
(ii) $\Gamma(x, r) = 0$ iff $r = 0$;
(iii) \( \Gamma(x, \cdot) \) is a monotone increasing function with respect to the second variable;
(iv) \( \lim_{r \to 0} \Gamma(x, r) = 0. \)

We also recall the following definition of a scale on a metric group.

**Definition 3.2.** Let \( G \) be a metrizable group and \( d_G \) be a compatible left-invariant metric on \( G \). Define \( \Gamma_G : G \times \mathbb{R}_+ \to \mathbb{R}_+ \) by
\[
\Gamma_G(g, r) = \max\{r, \sup\{d_G(1_G, g^{-1}hg) : d_G(1_G, h) \leq r\}\}.
\]

It is easy to see that \( \Gamma_G \) satisfies the conditions (i)–(iv) in the definition of a scale. We will also call \( \Gamma_G \) the scale on \( G \). Note that \( \Gamma_G \) is a Lipschitz function with respect to the first variable. In fact, for \( g, h \in G \) and \( r \in \mathbb{R}_+ \),
\[
|\Gamma_G(g, r) - \Gamma_G(h, r)| \leq 2d_G(g, h).
\]

We then define an ultranorm on \( W(X) \) as follows.

**Definition 3.3.** Let \( \Gamma \) be a scale on \( \overline{X} \). For \( l > 0 \), \( w \in W(X) \) with \( \text{lh}(w) = l \) and \( \theta \) a match on \( \{1, \ldots, l\} \), define \( N_u^{\Gamma, \theta}(w) \) by induction on \( l \) as follows:

1. For \( l = 1 \) let \( w = x \in \overline{X} \) and define \( N_u^{\Gamma, \theta}(w) = d(e, x) \);
2. If \( l > 1 \) and \( \theta(1) = k < l \), let \( \theta_1 = \theta \mid \{1, \ldots, k\} \) and \( \theta_2 = \theta \mid \{k+1, \ldots, l\} \) and \( w = w_1 w_2 \) where \( \text{lh}(w_1) = k \) and \( \text{lh}(w_2) = l-k \);

\[
N_u^{\Gamma, \theta}(w) = \max\{N_u^{\Gamma, \theta_1}(w_1), N_u^{\Gamma, \theta_2}(w_2)\};
\]
3. For \( l > 1 \) and \( \theta(1) = l \), let \( \theta_1 = \theta \mid \{2, \ldots, l-1\} \) and \( w = x^{-1}w_1 y \) where \( x, y \in \overline{X} \); then \( \text{lh}(w_1) = l - 2 \) and \( w \) has trivial extensions \((x^{-1}w_1)(x^{-1}y)\) and \((x^{-1}y)(y^{-1}w_1 y)\). Define
\[
N_u^{\Gamma, \theta}(w) = \max\{d(x, y), \min\{\Gamma(x, N_u^{\Gamma, \theta_1}(w_1)), \Gamma(y, N_u^{\Gamma, \theta_1}(w_1))\}\}.
\]

By an easy induction on \( \text{lh}(w) \), we have the following fact.

**Lemma 3.4.** For any scale \( \Gamma \) on \( \overline{X} \), \( w \in W(X) \) and match \( \theta \),
\[
N_u^{\Gamma, \theta}(w) \geq \rho_u(w, w^\theta).
\]

**Definition 3.5.** For \( w \in F(X) \), define
\[
N_u^\Gamma(w) = \inf\{N_u^{\Gamma, \theta}(w^\theta) : w^\theta \text{ is a trivial extension of } w, \theta \text{ is a match}\}.
\]

The next lemma summarizes the basic properties of the ultranorm \( N_u^\Gamma \).

**Lemma 3.6.** Let \( \Gamma \) be a scale on \( \overline{X} \) and \( w, u \in F(X) \). Then the following hold:

(i) \( N_u^\Gamma(w) \geq d_u(w, e) \); \( N_u^\Gamma(w) = 0 \) iff \( w = e \);
(ii) \( N_u^\Gamma(w^{-1}) = N_u^\Gamma(w) \);
(iii) \( N_u^\Gamma(w \cdot u) \leq \max\{N_u^\Gamma(w), N_u^\Gamma(u)\} \);
(iv) \( N_u^\Gamma(w^{-1} \cdot u \cdot w) \to 0 \) as \( N_u^\Gamma(u) \to 0 \);
(v) \( N_u^\Gamma(x^{-1}y) = d(x, y) \) for any \( x, y \in \overline{X} \).

**Proof.** Clause (i) follows from Lemma 3.4 and Theorem 2.4. Clause (ii) follows from the definition by an induction on \( \text{lh}(w) \). Clause (iii) can be proved by a similar argument as the proof of the ultrametric property in Theorem 2.4. The proofs of (iv) and (v) do not use the ultrametric property and are verbatim repetitions of the proofs of the same clauses in [4].

The following lemma plays a key role in the proof of our main theorem in the next section.

**Lemma 3.7.** Let \( G \) be a non-Archimedean Polish group and \( d_G \) a compatible left-invariant ultrametric on \( G \). Let \( \Gamma \) be a scale on \( \overline{X} \). Let \( \varphi : \overline{X} \to G \) be a function. Suppose that for any \( x, y \in \overline{X} \) and \( r \in \mathbb{R}_+ \),

(i) \( \Gamma(x, \cdot) \) is a monotone increasing function with respect to the second variable;
(ii) \( \lim_{r \to 0} \Gamma(x, r) = 0. \)
\( \varphi(e) = 1_G; \varphi(x^{-1}) = \varphi(x)^{-1}; \)

(b) \( d_G(\varphi(x), \varphi(y)) \leq d(x, y); \) and

(c) \( \Gamma_G(\varphi(x), r) \leq \Gamma(x, r). \)

Then \( \varphi \) can be uniquely extended to a group homomorphism \( \Phi : F(X) \to G \) such that for any \( w \in F(X) \)

\[ d_G(\Phi(w), 1_G) \leq N_u^{r, \theta}(w). \]

**Proof.** For any \( w = x_1 \cdots x_l \in W(X) \) define

\[ \Phi(w) = \varphi(x_1) \cdots \varphi(x_l) \in G. \]

It follows from (a) that \( \Phi \) is a group homomorphism when restricted on \( F(X) \). Clearly \( \Phi \) is an extension of \( \varphi \) and its uniqueness is also easy to see. Thus it remains only to check the inequality in the conclusion. For this we show that, for any \( w = x_1 \cdots x_l \in W(X) \) and match \( \theta \) on \( \{1, \ldots, l\} \),

\[ d_G(\Phi(w), 1_G) \leq N_u^{r, \theta}(w). \]

This will be done by induction on \( \text{lh}(w) \).

Case (0): \( l = 1 \). Let \( w = x \in \overline{X} \). Then

\[ N_u^{r, \theta}(w) = d(x, e) \geq d_G(\varphi(x), 1_G) = d_G(\Phi(w), 1_G) \]

by (a) and (b).

Case (1): \( l > 1 \) and \( \theta(1) = k < l \). Let \( \theta_1 = \theta \upharpoonright \{1, \ldots, k\} \), \( \theta_2 = \theta \upharpoonright \{k + 1, \ldots, l\} \) and \( w = w_1 w_2 \) where \( \text{lh}(w_1) = k \) and \( \text{lh}(w_2) = l - k \). Then by the inductive hypothesis and left-invariance of \( d_G \),

\[ N_u^{r, \theta}(w) = \max\{N_u^{r, \theta_1}(w_1), N_u^{r, \theta_2}(w_2)\} \]

\[ \geq \max\{d_G(\Phi(w_1), 1_G), d_G(\Phi(w_2), 1_G)\} \]

\[ = \max\{d_G(\Phi(w_1^{-1}), 1_G), d_G(\Phi(w_2), 1_G)\} \]

\[ \geq d_G(\Phi(w_1 w_2), 1_G) = d_G(\Phi(w), 1_G). \]

Case (2): \( l > 1 \) and \( \theta(1) = l \). Let \( \theta_1 = \theta \upharpoonright \{2, \ldots, l - 1\} \) and \( w = x^{-1} w_1 y \) where \( x, y \in \overline{X} \). Then by the induction hypothesis,

\[ N_u^{r, \theta_1}(w_1) \geq d_G(\Phi(w_1), 1_G) \]. Then by the definition of \( \Gamma_G \), conditions (b), (c) and the monotonicity of \( \Gamma \) we have

\[ N_u^{r, \theta}(w) = \max\{d(x, y), \min\{\Gamma(x, N_u^{r, \theta_1}(w_1)), \Gamma(y, N_u^{r, \theta_1}(w_1))\}\} \]

\[ \geq \max\{d_G(\varphi(x), \varphi(y)), \min\{\Gamma_G(\varphi(x), \Gamma(\varphi(y), d_G(\Phi(w_1), 1_G))\}\} \]

\[ \geq \max\{d_G(\Phi(x^{-1} y), 1_G), \min\{d_G(\Phi(x^{-1} w_1 x), 1_G), d_G(\Phi(y^{-1} w_1 y), 1_G)\}\} \]

\[ \geq d_G(\Phi(x^{-1} w_1 y), 1_G) = d_G(\Phi(w), 1_G). \]

as needed. \( \square \)

We are now ready to define the scaled Graev ultrametric on \( F(X) \) and extend it to a completion group.

**Definition 3.8.** Given a scale \( \Gamma \) on \( \overline{X} \) define an ultrametric \( \delta_u^{\Gamma} \) on \( F(X) \) by

\[ \delta_u^{\Gamma}(w, u) = N_u^{r}(w^{-1} \cdot u). \]

**Theorem 3.9.** For any ultrametric \( d \) on \( \overline{X} \) and scale \( \Gamma \) on \( \overline{X} \), \( \delta_u^{\Gamma} \) is a left-invariant ultrametric on \( F(X) \) extending \( d \), and \( F(X) \) is a topological group with the topology induced by \( \delta_u^{\Gamma} \). Moreover, if \( X \) is separable, then \( F(X) \) is separable.

**Proof.** Left-invariance of \( \delta_u^{\Gamma} \) is obvious. That \( \delta_u^{\Gamma} \) is an ultrametric follows from (i)–(iii) of Lemma 3.6. By Lemma 3.6 (v) \( \delta_u^{\Gamma} \) extends \( d \). The continuity of the group operations and the moreover part of the theorem follow from Lemma 3.6 (iv) and are verbatim from the proof of Theorem 3.9 of [4]. \( \square \)

It follows immediately from Lemma 3.6 and the left-invariance of \( \delta_u^{\Gamma} \) that

\[ \delta_u^{\Gamma}(w, u) \geq \delta_u(w, u) \]
for any \( w, u \in F(X) \). Thus each scaled Graev ultrametric induces a finer topology than the one induced by the Graev ultrametric.

To obtain a Polish group we follow the standard process below. Suppose \( X \) is separable and let \( \Gamma \) be a scale on \( X \). For \( w, u \in F(X) \), let

\[
\Delta_{\Gamma}^u(w, u) = \max\{\delta_{\Gamma}^u(w, u), \delta_{\Gamma}^u(w^{-1}, u^{-1})\}.
\]

Then \( \Delta_{\Gamma}^u \) is a compatible ultrametric on \( F(X) \). Let \( F_{\Gamma}^u(X) \) be the completion of \( (F(X), \Delta_{\Gamma}^u) \). Then \( F_{\Gamma}^u(X) \) is a Polish group. We denote the complete ultrametric on \( F_{\Gamma}^u(X) \) still by \( \Delta_{\Gamma}^u \). There is a unique extension of \( \delta_{\Gamma}^u \) onto \( F_{\Gamma}^u(X) \) which continues to be a compatible left-invariant ultrametric. We denote the resulting extension still by \( \delta_{\Gamma}^u \). With the compatible left-invariant ultrametric \( \delta_{\Gamma}^u, F_{\Gamma}^u(X) \) becomes a non-Archimedean Polish group.

**Lemma 3.10.** Let \( G \) be a non-Archimedean Polish group and \( d_G \) a compatible left-invariant ultrametric on \( G \). Let \( \Gamma \) be a scale on \( X \). Let \( \psi : X \to G \) be a function with \( \psi(X) \) dense in \( G \). Assume that conditions (a)–(c) of Lemma 3.7 hold for \( \psi \). Then \( \psi \) can be uniquely extended to a continuous homomorphism from \( F_{\Gamma}^u(X) \) onto \( G \).

**Proof.** For \( g, h \in G \) let

\[
D_G(g, h) = \max\{d_G(g, h), d_G(g^{-1}, h^{-1})\}.
\]

Then \( D_G \) is a compatible complete ultrametric on \( G \). By Lemma 3.7 \( \psi \) can be uniquely extended to a group homomorphism \( \Phi : F(X) \to G \) such that for any \( w \in F(X) \),

\[
d_G(\Phi(w), 1_G) \leq N_{\Gamma}^u(w).
\]

By the left-invariance and homomorphism properties, we have that for any \( w, u \in F(X) \),

\[
d_G(\Phi(w), \Phi(u)) = d_G(\Phi(w^{-1} \cdot u), 1_G) \leq N_{\Gamma}^u(w^{-1} \cdot u) = \delta_{\Gamma}^u(w, u)
\]

and

\[
d_G(\Phi(w)^{-1}, \Phi(u^{-1})) = d_G(\Phi(w \cdot u^{-1}), 1_G) \leq N_{\Gamma}^u(w \cdot u^{-1}) = \delta_{\Gamma}^u(w^{-1}, u^{-1}).
\]

This means that

\[
D_G(\Phi(w), \Phi(u)) \leq \Delta_{\Gamma}^u(w, u).
\]

It follows that for any \( \Delta_{\Gamma}^u \)-Cauchy sequence \( (u_n), (\Phi(u_n)) \) is a \( D_G \)-Cauchy sequence. Therefore, by a standard construction \( \Phi \) can be uniquely extended to a group homomorphism \( \overline{\Phi} \) from \( \overline{F_{\Gamma}^u}(X) \) into \( G \). It is routine to check that the extended homomorphism \( \overline{\Phi} \) satisfies the Lipschitz condition, and therefore it is continuous. Now \( \overline{\Phi}(\overline{F_{\Gamma}^u}(X)) \) is a Polish subgroup of \( G \) that is dense, hence \( \overline{\Phi} \) is onto \( G \).

**4. Surjectively universal non-Archimedean Polish groups**

In this section we prove the main theorem of this paper, that there exist surjectively universal non-Archimedean Polish groups. We will show that for a suitable ultrametric space \( (X, d) \) and a scale \( \Gamma \) over \( X \), the non-Archimedean Polish group \( F_{\Gamma}^u(X) \) is such a group. The argument of this section is a direct consequence of some technical results of [3].

The following space has been considered in [3–5]. For \( n \in \mathbb{N} \), let

\[
\mathcal{N}_n = \{x \in \mathcal{N} : \forall m \geq n \ (x(m) = 0)\}.
\]

Let

\[
\mathcal{N}_\omega = \bigcup_{n \in \mathbb{N}} \mathcal{N}_n.
\]

\( \mathcal{N}_\omega \) is a countable dense subset of \( \mathcal{N} \). Let \( \mathcal{N}_\omega \) be the subspace of \( \mathcal{N} \) with the ultrametric \( d \) given in Definition 2.5.

We isolate the following technical notion from [3].

**Definition 4.1.** A scale \( \Gamma \) on \( \mathcal{N}_\omega \) is generous if for all \( x \in \mathcal{N}_\omega \) and \( r > 0 \), \( \Gamma(x, r) = \Gamma(x^{-1}, r) \), and there is \( K \in \mathbb{N} \) such that for all \( m \in \mathbb{N}_+ \) and all \( k > m + K + 5 \):

(1) for any \( x \in \mathcal{N}_m \setminus \mathcal{N}_{m-1} \) and \( r > 2^{-(m+K+5)} \), there exists \( y \in \mathcal{N}_{m-1} \) with

\[
\Gamma(x, r) \geq \min\{2^{-(K+3)}, \Gamma(y, r) + 2^{-K}d(x, y)\};
\]
(2) for all but finitely many \( x \in N_m \setminus N_{m-1} \), for all \( r \in (2^{-k}, 2^{-(m+K+5)}) \), there exists \( y \in N_{m-1} \) with

\[
\Gamma(x, r) \geq \min\left\{ 2^{-(K+3)}, \Gamma(y, r) + 2^{-K}d(x, y) \right\};
\]

(3) \( \Gamma(x, r) > 8r \) for all \( x \in N_m \setminus N_{m-1} \) and \( r \leq 2^{-(m+K+5)} \).

In [3] Ding gave several examples of generous scales. The main technical theorem of [3] is the following.

**Theorem 4.2.** (Ding [3]) Let \( \Gamma \) be a generous scale on \( N_\omega \). Then \( F_\Gamma(N_\omega) \) is a surjectively universal Polish group.

The setup of the proof is as follows. Let \( G \) be a Polish group, and let \( (U_n)_{n \in \mathbb{N}} \) be a nbhd base of \( 1_G \) such that \( U_0 = G \), \( U_n = U_{n-1}^{-1} \), and \( U_{n+1} \subseteq U_n \) for all \( n \in \mathbb{N} \).

Let \( K \) be a constant witnessing that \( \Gamma \) is a generous scale. In view of condition (2) of the definition of generosity, for \( m \in \mathbb{N}_+ \) and \( k > m + K + 5 \), let \( E_m^k \) be the finite subset of \( x \in N_m \setminus N_{m-1} \) such that there exists \( r \in (2^{-k}, 2^{-(m+K+5)}) \) so that for all \( y \in N_{m-1} \),

\[
\Gamma(x, r) < \min\left\{ 2^{-(K+3)}, \Gamma(y, r) + 2^{-K}d(x, y) \right\}.
\]

Also for each \( x \in N_\omega \) and \( n \in \mathbb{N} \), let

\[
\pi_n(x) = \begin{cases} x(i), & \text{if } i < n, \\ 0, & \text{otherwise}. \end{cases}
\]

Define a strictly increasing function \( f : \mathbb{N} \to \mathbb{N} \) and \( \varphi : N_\omega \to G \) by induction as follows.

(i) Define \( f(0) = 0 \) and \( f(1) = 1 \). For the unique element \( 0 = (0, 0, \ldots) \) of \( N_0 \), define \( \varphi(0) = 1_G \).

(ii) For \( n \geq 1 \), suppose \( f(n) \) has been defined and \( \varphi \) has been defined on \( N_{n-1} \). Extend the definition of \( \varphi \) onto \( N_n \) such that

\[(iia) \text{ for all } x \in N_n, \varphi(x) \in \varphi(\pi_{n-1}(x))U_{f(n-1)} \cap U_{f(n-1)}\varphi(\pi_{n-1}(x)); \quad \text{and}
\]

\[(iib) \text{ for all } y \in N_{n-1}, \varphi(y)U_{f(n-1)} \cap U_{f(n-1)}\varphi(y) \text{ is covered by the collection}
\]

\[
\left\{ \varphi(x)U_{f(n)} \cap U_{f(n)}\varphi(x) : x \in N_n, \pi_{n-1}(x) = y \right\}.
\]

To define \( f(n+1) \), consider the open nbhd of \( 1_G \) defined by

\[
B_n = \bigcap \left\{ \varphi(x)U_{f(n)}\varphi(x)^{-1} \cap \varphi(x)^{-1}U_{f(n)}\varphi(x) : x \in E_m^{n+K+6}, 1 \leq m \leq n \right\}.
\]

Define \( f(n+1) \) to be the least integer \( N > f(n) \) such that \( U_N \subseteq B_n \).

Define a new nbhd base \( (V_n)_{n \in \mathbb{N}} \) by letting \( V_n = U_{f(n)} \) for all \( n \in \mathbb{N} \). Follow the proof of the Birkhoff–Kakutani theorem (cf. [7, Theorem 2.2.1]) to define a compatible left-invariant metric on \( G \) as follows. First define

\[
\rho(g, h) = \min\left\{ 2^{-(n+K+3)}, g^{-1}h \in V_n \right\}.
\]

Then let

\[
d_G(g, h) = \inf\left\{ \sum_{i=0}^{l} \rho(g_i, g_{i+1}) : g_0 = g, g_{l+1} = h, g_1, \ldots, g_l \in G, l \in \mathbb{N} \right\}.
\]

Then \( d_G \) is a compatible left-invariant metric on \( G \) and for all \( g, h \in G \),

\[
\frac{1}{2}\rho(g, h) \leq d_G(g, h) \leq \rho(g, h) \leq 2^{-(K+3)}.
\]

The rest of the proof verifies that conditions (a)–(c) of Lemma 3.7 are satisfied with \( \varphi \) and \( d_G \).

With the appropriate modifications we have the following theorem in our context.

**Theorem 4.3.** Let \( \Gamma' \) be a generous scale on \( N_\omega \). Then \( F_{\Gamma'}(X) \) is a surjectively universal non-Archimedean Polish group.

**Proof.** Let \( G \) be a non-Archimedean Polish group, and let \( (U_n)_{n \in \mathbb{N}} \) be a nbhd base for \( 1_G \) consisting of decreasing open subgroups. Repeat the above definitions of the functions \( f \) and \( \varphi \). Again let \( V_n = U_{f(n)} \) for all \( n \in \mathbb{N} \). Now note that the distance function \( \rho \) defined above is already a compatible left-invariant ultrametric on \( G \). Thus instead of following Birkhoff–Kakutani we just let \( d_G = \rho \).

The rest of Ding’s proof of Theorem 4.2 shows that conditions (a)–(c) of Lemma 3.7 are also satisfied by \( \varphi \) and \( d_G \). Conditions (iia) and (iib) in the above construction of \( \varphi \) guarantee that \( \varphi(N_\omega) \) is dense in \( G \). Thus by Lemma 3.10 there is a continuous homomorphism from \( F_{\Gamma'}(N_\omega) \) onto \( G \). \( \Box \)
References


