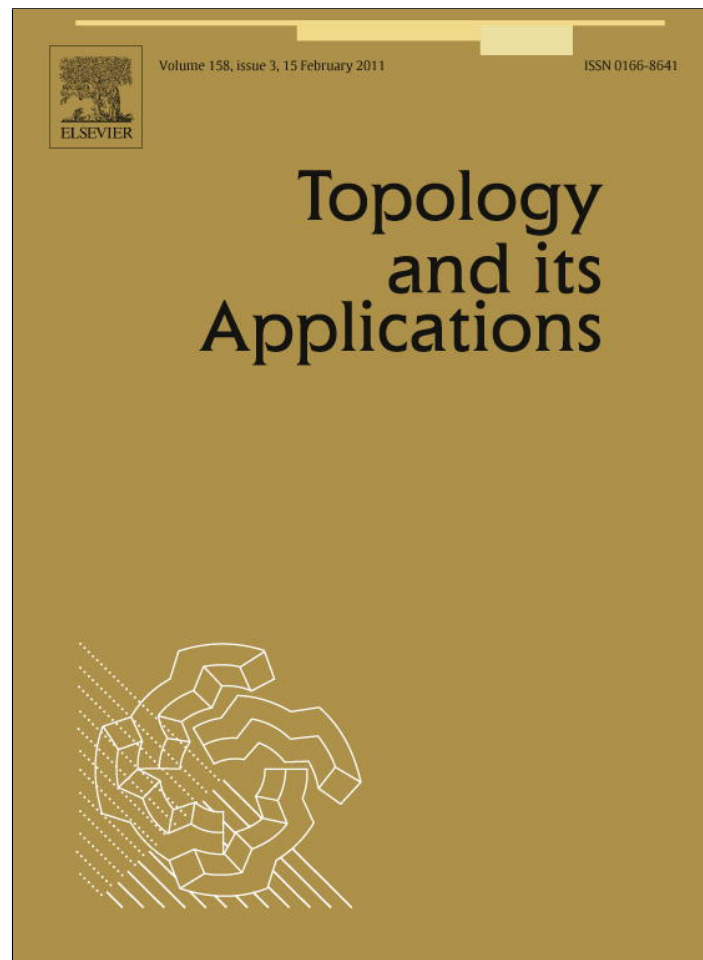


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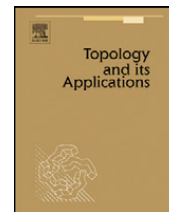
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Polish ultrametric Urysohn spaces and their isometry groups

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ABSTRACT

In this paper we give some new constructions of Polish ultrametric Urysohn spaces and investigate the universality properties of their isometry groups. It is shown that all isometry groups of Polish ultrametric Urysohn spaces, regardless of their distance sets, are embeddable into each other, and in particular universal for all isometry groups of Polish ultrametric spaces. We also consider a strengthening notion, called extensive isometric embedding, and show that any isometric embedding from a compact ultrametric space into a Polish ultrametric Urysohn space is extensive. It is shown that every isometry between two compact subsets of a Polish ultrametric Urysohn space can be extended to an isometry of the entire space. We introduce a notion of generalized trees to study Polish ultrametric spaces and prove a duality theorem between the categories of Polish ultrametric spaces and their generalized tree representations. Finally we draw some conclusions about the descriptive complexity of embeddability, biembeddability and isometry relations among Polish ultrametric spaces.

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1. Introduction

In this paper we investigate Polish ultrametric spaces and their isometry groups in the spirit of [5]. One of the main tools in this study is the analysis of Polish ultrametric Urysohn spaces. These are complete separable ultrametric spaces that are ultrahomogeneous and universal among separable ultrametric spaces with the same (or smaller) distance set. The existence of such spaces are certainly well known, but they are less studied than their countable counterpart (cf., e.g., [2,5] and the more recent [9] for a good survey). Moreover, there has not appeared any work on the properties of their isometry groups. This is in contrast with the vast amount of research done recently on the Urysohn space itself. There are numerous references containing further information on the Urysohn space; here we only mention a nice recent introduction contained in Chapter 5 of [10], and [7], proceedings of a conference devoted to the Urysohn space.

This paper is written with two purposes. The first is to give a complete survey of known constructions of Polish ultrametric Urysohn spaces and clarify their relationship with the countable ultrametric Urysohn spaces. In doing this we provide some terminology to describe the Polish ultrametric Urysohn spaces precisely. We give a total of four different constructions of the Polish ultrametric Urysohn spaces; some of them seem to be new. To be more specific, the constructions are referred to as the point-by-point construction, the Vestfrid type construction, the Katětov type construction, and the generalized tree construction. The point-by-point construction is the well known one obtained from the Fraïssé limit of the countable class of all finite metric spaces with a fixed countable distance set. The Vestfrid type construction is implicit in [13], where Vestfrid obtained a nonseparable ultrahomogeneous ultrametric space that contains an isometric copy of all

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separable ultrametric spaces. The Katětov type construction follows the general outlines of the construction by Katětov [6] for the Urysohn space \mathbb{U} . But we have to make suitable modifications in the ultrametric context. A version of the generalized tree construction was mentioned in [9]. Here we provide a slightly different construction to accommodate the proofs of some theorems.

The second purpose of the paper is to study the isometry groups of the Polish ultrametric Urysohn spaces and the isometry relation among all Polish ultrametric spaces from the point of view of descriptive set theory. We will show that all of the isometry groups of the Polish ultrametric Urysohn spaces, regardless of their distance sets, are biembeddable with the infinite permutation group S_∞ , and therefore quite complicated (but they are equally complicated in some sense). In doing this we provide a proof of the universality of the isometry groups using the Katětov type construction. We also consider some strengthening of the notion of isometry group embeddability, which we call *extensive isometric embeddings*, and show that all isometric embeddings from a compact ultrametric space into an ultrametric Urysohn space is extensive. The proof implies that any isometry between two compact subsets of a Polish ultrametric Urysohn space can be extended to an isometry of the entire space.

We also use the generalized tree construction to give a duality theorem between Polish ultrametric spaces and generalized trees. This implies some results about the relations of isometric embeddability, isometric biembeddability, and isometric equivalence. In most cases they turn out to be the most complex quasi-orders or equivalence relations they can possibly be.

The rest of the paper is organized as follows. In Section 2 we define the basic terminology and clarify some basic facts. In Section 3 we review the point-by-point construction. In Section 4 we give the Vestfrid type construction. In Section 5 we give the Katětov type construction and prove the universality of the isometry groups of the Polish R -ultrametric Urysohn space among the isometry groups of Polish R -ultrametric spaces. In Section 6 we give the generalized tree construction, prove the duality theorem, and study extensive isometric embeddings. In Section 7 we summarize our results about the S_∞ -universality for the relevant isometry groups. Finally in Section 8 we summarize our results on the notion of isometric embeddability, isometric biembeddability, and isometric equivalence.

2. Preliminaries

Let X be a set. An *ultrametric* d on X is a metric satisfying

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for any $x, y, z \in X$. If (X, d) is an ultrametric space and $x, y, z \in X$, then at least two of the distances $d(x, y)$, $d(y, z)$, $d(x, z)$ must be equal, and the third distance is no bigger. We refer to this property as *the isosceles triangle property*.

In this paper we consider only separable ultrametric spaces. It is easy to see that if X is a separable ultrametric space, then the distance set $\{d(x, y) : x \neq y \in X\}$ must be countable. In fact, it follows from the isosceles triangle property that, if $D \subseteq X$ is any dense set, then

$$\{d(x, y) : x \neq y \in D\} = \{d(x, y) : x \neq y \in X\}.$$

We will use the following notation. Let \mathbb{R}_+ denote the set of all nonnegative real numbers.

Definition 2.1. Let $R \subseteq \mathbb{R}_+$ be countable. Let X be a set. An *R -ultrametric* on X is an ultrametric d with

$$\{d(x, y) : x \neq y \in X\} \subseteq R.$$

A *Polish ultrametric space* is a separable complete ultrametric space. For any countable $R \subseteq \mathbb{R}_+$ there exists a Polish ultrametric space X whose distance set is exactly R . To see this, consider the space $X = (R, d)$ where $d(x, y) = \max\{x, y\}$.

Polish metric spaces, including Polish ultrametric spaces, were studied in [5] from the point of view of descriptive set theory. In particular, we are interested in the notion of isometric embedding and isometric equivalence among Polish (ultra)metric spaces and the notion of ultrahomogeneity and universality. To be precise, we recall these notions below. Let (X, d_X) , (Y, d_Y) be metric spaces. An *isometric embedding* from X into Y is a map $\varphi : X \rightarrow Y$ that is distance preserving, i.e., satisfying $d_Y(\varphi(x), \varphi(y)) = d_X(x, y)$ for all $x, y \in X$. An *isometry* or an *isometric equivalence* between X and Y is an isometric embedding from X onto Y . Of course, an isometry is necessarily a bijection whose inverse is also an isometry. Note that the notion of isometric embedding and isometric equivalence applies to ultrametric spaces unchanged.

We say that a metric space $X = (X, d)$ is *ultrahomogeneous* if for all finite subsets A, B of X and isometry φ between A and B , there is an isometry $\varphi^* : X \rightarrow X$ of the whole space such that $\varphi^* \upharpoonright A = \varphi$. A closely related notion is that of universality. If \mathcal{C} is a class of metric spaces and X is a metric space, we say that X is *universal* for \mathcal{C} , or *\mathcal{C} -universal*, if for every $Y \in \mathcal{C}$ there exists an isometric embedding from Y into X . A Polish metric space is *universal* if it is universal for all Polish metric spaces.

It is well known that there exists a unique Polish metric space that is both ultrahomogeneous and universal. This space was constructed by Urysohn [11], is called *the Urysohn metric space*, and denoted \mathbb{U} . In fact, Urysohn also discovered *Urysohn property*, by following which he constructed \mathbb{U} . A metric space X has *the Urysohn property* if for any finite metric space B , subspace $A \subseteq B$ and isometric embedding $\varphi : A \rightarrow X$, there is an isometric embedding $\varphi^* : B \rightarrow X$ such that $\varphi^* \upharpoonright A = \varphi$.

Urysohn [11] proved that a Polish metric space has the Urysohn property iff it is both ultrahomogeneous and universal, and that any two Polish metric spaces with the Urysohn property are isometric equivalent.

For ultrametric spaces some of the above notions no longer apply. The notion of ultrahomogeneity continues to make sense. But that of universality and the Urysohn property need to be revised. First of all, there does not exist a single Polish ultrametric space that is universal for all Polish ultrametric spaces. This is because, for any Polish ultrametric X we can find another Polish metric space Y so that there is no isometric embedding from Y into X . To see this, note that X has a countable distance set R_0 . Fix any countable $R \supsetneq R_0$, there exists a Polish ultrametric space Y whose distance set is R (see above), and thus there is no embedding from Y into X . Thus the notion of universality and the Urysohn property become relative to a countable distance set in the ultrametric context.

Definition 2.2. Let $R \subseteq \mathbb{R}_+$ be countable. A Polish R -ultrametric space X is R -universal if for every Polish R -ultrametric space Y there is an isometric embedding from Y into X .

Definition 2.3. Let $R \subseteq \mathbb{R}_+$ be countable. An R -ultrametric space X has the R -ultrametric Urysohn property if for any finite R -ultrametric space B , subspace A and isometric embedding $\varphi : A \rightarrow X$, there is an isometric embedding $\varphi^* : B \rightarrow X$ such that $\varphi^* \upharpoonright A = \varphi$. If an R -ultrametric space X has the R -ultrametric Urysohn property, we also say that X is an R -ultrametric Urysohn space.

In the next 4 sections we will give different constructions of Polish R -ultrametric Urysohn spaces for any countable $R \subseteq \mathbb{R}_+$. Here a well-known fact is worth noting: for any $R \subseteq \mathbb{R}_+$ there exists a countable R -ultrametric Urysohn space. In fact, the following proposition shows that any countable dense subspace of an R -ultrametric Urysohn space has the R -ultrametric Urysohn property.

Proposition 2.4. Let $R \subseteq \mathbb{R}_+$ be countable. Let X be an R -ultrametric space and $Y \subseteq X$ a dense subspace. The X is an R -ultrametric Urysohn space iff Y is an R -ultrametric Urysohn space.

Proof. (\Rightarrow) Let B be a finite R -ultrametric space, $A \subseteq B$ and $\varphi : A \rightarrow Y$ be an isometric embedding. Since $Y \subseteq X$ and X has the R -ultrametric Urysohn property, there is $\varphi^* : B \rightarrow X$ such that $\varphi^* \upharpoonright A = \varphi$. Let $\epsilon < \min\{d(a, b) : a \neq b \in B\}$. For each $b \in B - A$, let $y_b \in Y$ be such that $d(y_b, \varphi^*(b)) < \epsilon$. We also let $y_b = \varphi(b)$ if $b \in A$. Then by the isosceles triangle property we have that for any $a, b \in B$, $d(\varphi^*(a), \varphi^*(b)) = d(y_a, y_b)$. This shows that the map $b \mapsto y_b$ is an isometric embedding from B into Y , and thus Y has the R -ultrametric Urysohn property.

(\Leftarrow) Suppose Y is R -ultrametric Urysohn. Let B be a finite R -ultrametric space, $A \subseteq B$ and $\varphi : A \rightarrow X$ an isometric embedding. Then by a similar construction as above we can define $\psi : A \rightarrow Y$ where for any $a \in A$, $d(\varphi(a), \psi(a)) < \epsilon < \min\{d(a, b) : a \neq b \in B\}$. It follows that ψ is an isometric embedding, and thus there is an isometric embedding $\psi^* : B \rightarrow Y$ such that $\psi^* \upharpoonright A = \psi$. Now define $\varphi^*(b) = \varphi(b)$ for $b \in A$, and $\varphi^*(b) = \psi^*(b)$ for $b \in B - A$. It is easy to check that φ^* is an isometric embedding with $\varphi^* \upharpoonright A = \varphi$. \square

By a standard back-and-forth argument one can also show the following uniqueness results.

Proposition 2.5. Let $R \subseteq \mathbb{R}_+$ be countable. Then the following hold:

- (a) Any two countable R -ultrametric Urysohn spaces are isometric;
- (b) Any two Polish R -ultrametric Urysohn spaces are isometric.

In view of the uniqueness we adopt the following notation.

Notation 2.6. Let $R \subseteq \mathbb{R}_+$ be countable. We denote the unique (up to isometry) countable R -ultrametric Urysohn space by \mathbb{K}_R^u , and the unique Polish R -ultrametric space \mathbb{U}_R^u .

It follows from Propositions 2.4 and 2.5 that \mathbb{U}_R^u is isometric with the metric completion of \mathbb{K}_R^u and that any countable dense subspace of \mathbb{U}_R^u is isometric with \mathbb{K}_R^u .

Also by the standard argument of Urysohn [11] the ultrametric Urysohn property is completely characterized by ultrahomogeneity and universality, in the following precise sense.

Proposition 2.7. Let $R \subseteq \mathbb{R}_+$ be countable. Then:

- (a) A countable R -ultrametric space is R -ultrametric Urysohn iff it is both ultrahomogeneous and universal for all countable (or finite) R -ultrametric spaces;
- (b) A Polish R -ultrametric space is R -ultrametric Urysohn iff it is both ultrahomogeneous and universal for all Polish R -ultrametric spaces.

We will be interested in the isometry groups of the ultrametric Urysohn spaces. Recall that for any Polish metric space X , the isometry group of X , denoted by $\text{Iso}(X)$, is the group of all isometries of X onto itself. Equipped with the pointwise convergence topology, $\text{Iso}(X)$ becomes a Polish group.

It is a well-known theorem of Uspenskij [12] that $\text{Iso}(\mathbb{U})$ is a *universal Polish group*, i.e., every Polish group is topologically isomorphic to a (necessarily closed) subgroup of $\text{Iso}(\mathbb{U})$. One naturally wonders about the situation with isometry groups of ultrametric Urysohn spaces. In Sections 5 we prove that $\text{Iso}(\mathbb{U}_R^u)$ is universal for all $\text{Iso}(X)$ where X are Polish R -ultrametric spaces. And then in Section 7 we show that in fact all of them are topologically isomorphic with closed subgroups of each other (as long as R is nonempty).

We also obtain results about $\text{Iso}(\mathbb{K}_R^u)$. Being dense subgroups of $\text{Iso}(\mathbb{U}_R^u)$, they are not necessarily Polish groups. However, we will show in Section 5 that they are universal for all $\text{Iso}(X)$, where X are countable R -ultrametric spaces.

3. The point-by-point construction

Throughout this section we fix a nonempty countable $R \subseteq \mathbb{R}_+$. In this section we review the well-known construction of \mathbb{K}_R^u as a Fraïssé limit of the class of all finite R -ultrametric spaces.

To begin with we need the following Basic Amalgamation Lemma.

Lemma 3.1 (*Basic Amalgamation Lemma*). *Let (A, d_A) be a finite R -ultrametric space, $x, y \notin A$, and $x \neq y$. If $(A \cup \{x\}, d_x)$ is an R -ultrametric space with $d_x \upharpoonright A = d_A$ and $(A \cup \{y\}, d_y)$ is an R -ultrametric space with $d_y \upharpoonright A = d_A$, then d_A can be extended to d such that $(A \cup \{x, y\}, d)$ is an R -ultrametric space with $d \upharpoonright (A \cup \{x\}) = d_x$ and $d \upharpoonright (A \cup \{y\}) = d_y$.*

Proof. To define d it suffices to define the value of $d(x, y)$. If there is any $a \in A$ with $d_x(x, a) \neq d_y(y, a)$, then we have to define $d(x, y)$ as $\max\{d_x(x, a), d_y(y, a)\}$ in order to make the resulting d an ultrametric. In this case it is easy to check that if $b \in A$ is such that $d_x(x, b) \neq d_y(y, b)$, then $\max\{d_x(x, a), d_y(y, a)\} = \max\{d_x(x, b), d_y(y, b)\}$. This shows that the resulting d will be an ultrametric in this case. On the other hand, if for all $a \in A$, $d_x(x, a) = d_y(y, a)$, then we can define

$$d(x, y) = \min\{d_x(x, a) : a \in A\}.$$

Then again d will be an ultrametric as required. Finally noted that resulting d is an R -ultrametric. \square

The following Intermediate Amalgamation Lemma can be proved by applying the Basic Amalgamation Lemma and a straightforward induction on the size of $B - A$.

Lemma 3.2 (*Intermediate Amalgamation Lemma*). *Let (B, d_B) be a finite R -ultrametric space, $A \subseteq B$ a subspace, $d_A = d_B \upharpoonright A$, and $x \notin B$. If $(A \cup \{x\}, d'_A)$ is an R -ultrametric space and $d'_A \upharpoonright A = d_A$, then d'_A can be extended to d'_B such that $(B \cup \{x\}, d'_B)$ is an R -ultrametric space with $d'_B \upharpoonright B = d_B$ and $d \upharpoonright (A \cup \{x\}) = d'_A$.*

Again the Intermediate Amalgamation Lemma can be repeatedly applied to obtain the following standard amalgamation property, the Amalgamation Lemma.

Lemma 3.3 (*Amalgamation Lemma*). *Let (A, d_A) be a finite R -ultrametric space. If $(A \cup B, d_B)$ is a finite R -ultrametric space with $d_B \upharpoonright A = d_A$ and $(A \cup C, d_C)$ is a finite R -ultrametric space with $d_C \upharpoonright A = d_A$, then d_A can be extended to d such that $(A \cup B \cup C, d)$ is an R -ultrametric space with $d \upharpoonright (A \cup B) = d_B$ and $d \upharpoonright (A \cup C) = d_C$.*

Now the point-by-point construction of \mathbb{K}_R^u goes as follows. Consider the collection of all finite R -ultrametric spaces. Since R is countable, this is a countable collection. Now enumerate all pairs (A, B) of finite R -ultrametric spaces where $A \subseteq B$ is a subspace of B . The n -th pair will be denoted (A_n, B_n) . Now \mathbb{K}_R^u is constructed in infinitely many stages, and at each finite stage n a finite R -ultrametric space S_n is obtained, so that $S_n \subseteq S_{n+1}$ for all n . In the end we take $\mathbb{K}_R^u = \bigcup_n S_n$. In the n -th stage of the construction, consider all isometric copies of A_m for all $m \leq n$ in the R -ultrametric space S_n . Use the Amalgamation Lemma repeatedly to arrive at S_{n+1} so that each isometric copy of A_m , $m \leq n$, in S_n can be extended to an isometric copy of B_m in S_{n+1} . The resulting space obviously has the R -ultrametric Urysohn property.

As noted before, \mathbb{U}_R^u can be constructed as the metric completion of \mathbb{K}_R^u .

4. The Vestfrid type construction

In [13] Vestfrid constructed an ultrametric space universal for all separable ultrametric space. As we noted earlier such a space cannot be separable. We use Vestfrid's idea to construct an R -ultrametric Urysohn space. Throughout the rest of this section we again fix a nonempty countable $R \subseteq \mathbb{R}_+$.

Consider

$$U_R = \{(x_n) \in R^\omega : x_n \geq x_{n+1} \text{ for all } n, \text{ and } x_n \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

with the metric d_u on U_R given by

$$d_u((x_n), (y_n)) = \begin{cases} 0, & \text{if } x_n = y_n \text{ for all } n; \\ \max\{x_k, y_k\}, & \text{if } k \text{ is the least such that } x_k \neq y_k. \end{cases}$$

It is routine to check that d_u is an ultrametric. Note that the set

$$D_R = \{(x_n) \in U_R : \exists n \forall k \geq n \ x_k = 0\}$$

is countable and dense, thus U_R is separable.

Theorem 4.1. U_R is a Polish R -ultrametric Urysohn space.

Proof. We first check that d_u is a complete metric on U_R . Let $x^i = (x_n^i)$ be a d_u -Cauchy sequence. We claim that the sequence $(x_0^i)_i$ either is eventually constant or else converges to 0. For this suppose $\lim_i x_0^i \neq 0$. Then for some $\epsilon > 0$ there are infinitely many i with $x_0^i > \epsilon$. Let N be such that for all $i, j > N$, $d_u((x_n^i), (x_n^j)) < \epsilon$. Then for all $i, j > N$, if $x_0^i, x_0^j > \epsilon$ then $x_0^i = x_0^j$. Let z_0 be this common value. Since $z_0 > \epsilon$, we must have that for all $i > N$, $x_0^i = z_0$. This shows that $(x_0^i)_i$ is eventually constant.

The same argument applies to other subscripts. Thus for any k it is the case that either $(x_k^i)_i$ is eventually constant or else $\lim_i x_k^i = 0$. We note, however, that if $k < l$ and $\lim_i x_k^i = 0$ then $\lim_i x_l^i = 0$. This is because, if $\lim_i x_l^i \neq 0$, and letting $z_l = \limsup_l x_l^i > 0$, then for some i , $x_k^i < z_l = x_l^i$, contradicting the definition of U_R .

In either case we have argued that $\lim_i x_k^i$ exists for all k and that, in case $\lim_i x_k^i \neq 0$, $(x_k^i)_i$ must be eventually constant. Now we define $z_n = \lim_i x_n^i$ for all n . It is easy to check that $d_u((z_n), (x_n^i)) \rightarrow 0$ as $i \rightarrow \infty$.

This finishes the proof that d_u is a complete ultrametric on U_R . Next we check that U_R has the R -ultrametric Urysohn property.

For this let B be a finite R -ultrametric space, $A \subseteq B$ a subspace, and $\varphi : A \rightarrow U_R$ be an isometric embedding. Without loss of generality we may assume that $|B - A| = 1$. So we write $A = \{a_1, \dots, a_k\}$ and $B = A \cup \{b\}$. Let $r_i = d(b, a_i)$ for $i = 1, \dots, k$. Without loss of generality assume $0 < r_i \leq r_j$ for $i < j$. Let $x^i = \varphi(a_i)$ for $i = 1, \dots, k$, and also assume $x^i = (x_n^i)$. We need to find $(y_n) \in U_R$ so that $d_u((y_n), (x_n^i)) = r_i$ for all $i = 1, \dots, k$.

Let $l \leq k$ be the largest such that $r_1 = r_l$. For each $i = 1, \dots, l$, let p_i be the least such that $x_{p_i}^i \leq r_1$. We claim that $p_1 = \dots = p_l$. Otherwise, suppose $1 \leq i, j \leq l$ are such that $p_i \neq p_j$, and without loss of generality $p_i < p_j$. Then $x_{p_j-1}^j > r_1$ and $x_{p_j-1}^i \leq x_{p_i}^i \leq r_1$, and it follows that $d((x_n^i), (x_n^j)) > r_1$. Thus $d(a_i, a_j) > r_1$. But $d(b, a_i) = d(b, a_j) = r_1$, contradicting the isosceles triangle property. Let $p = p_1 = \dots = p_l$ be the common value.

Let $q > p$ be such that for all $i = 1, \dots, l$, $x_q^i < r_1$. Define $(y_n) \in U_R$ by letting $y_n = x_n^1$ for all $n < p$, $y_n = r_1$ for all $p \leq n \leq q$, and $y_n = 0$ for all $n > q$. We claim that (y_n) has the desired property. Once again, since $d((x_n^1), (x_n^i)) \leq r_1$ for all $i \leq l$, we must have that $x_n^i = x_n^1$ for all $i \leq l$ and $n < p$. Now fix $i = 1, \dots, l$, we have $d((y_n), (x_n^i)) = r_1$, since $y_n = x_n^i$ for all $n < p$, and $y_n \neq x_n^i$ for some $p \leq n \leq q$, and finally $y_n = r_1 \geq x_n^i$ for all $p \leq n \leq q$. Now for $i > l$, we have that $d((x_n^1), (x_n^i)) = r_i > r_1$ by the isosceles triangle property since $d((x_n^1), (x_n^1)) = d(a_i, a_1)$, $d(b, a_i) = r_i > r_1$, and $d(b, a_1) = r_1$. It follows again from the isosceles triangle property that $d((y_n), (x_n^i)) = r_i$ since $d((y_n), (x_n^1)) = r_1$ and $d((x_n^1), (x_n^i)) = r_i > r_1$. We have thus shown that (y_n) has the desired property. \square

5. The Katětov type construction

Neither of the constructions in the preceding sections is suitable for the purpose of studying their isometry groups. In this section we give yet another construction of \mathbb{U}_R^* which gives more structural information of the space as well as its isometries. This is similar to what Katětov did in [6] to construct the Urysohn metric space \mathbb{U} , whose method allowed Uspenskij to prove later in [12] that $\text{Iso}(\mathbb{U})$ is universal for all Polish groups.

Throughout this section we again fix a nonempty countable $R \subseteq \mathbb{R}_+$.

5.1. Ultrametric admissible functions

Definition 5.1. Let (X, d) be an R -ultrametric space. A function $f : X \rightarrow R$ is R -ultrametric admissible, if for any $x, y \in X$, we have

- (1) $f(x) \leq \max\{d(x, y), f(y)\}$,
- (2) $f(y) \leq \max\{d(x, y), f(x)\}$, and
- (3) $d(x, y) \leq \max\{f(x), f(y)\}$.

Notation 5.2. If X is an R -ultrametric space, we define

$$E_R(X) = \{f : X \rightarrow R : f \text{ is } R\text{-ultrametric admissible}\}$$

and d_E on $E_R(X)$ by

$$d_E(f, g) = \begin{cases} 0, & \text{if } f = g, \\ \max\{f(x), g(x)\}, & \text{where } x \in X \text{ such that } f(x) \neq g(x). \end{cases}$$

Thus $E_R(X)$ contains all possible one-point extensions of X as an R -ultrametric space and d_E is intended to be an R -ultrametric on $E_R(X)$. It follows from the Basic Amalgamation Lemma (Lemma 3.1) that d_E is well defined and that it is indeed an R -ultrametric.

Notation 5.3. If X is an R -ultrametric space, we define, for each $x \in X$,

$$f_x(y) = d(x, y), \quad \text{for all } y \in X.$$

Also, for each $\varphi \in \text{Iso}(X)$,

$$\varphi_E(f)(x) = f(\varphi^{-1}(x)), \quad \text{for all } x \in X.$$

It is easy to check that for any $x \in X$, $f_x \in E_R(X)$, and $x \mapsto f_x$ is an isometric embedding from (X, d) into $(E_R(X), d_E)$. This makes $E_R(X)$ a canonical isometric extension of X as an R -ultrametric space. Also, for any $\varphi \in \text{Iso}(X)$, $\varphi_E \in \text{Iso}(E_R(X))$, and $\varphi \mapsto \varphi_E$ is a topological group embedding (i.e., it is a homeomorphic embedding as well as a group embedding) from $\text{Iso}(X)$ into $\text{Iso}(E_R(X))$.

It is not, however, clear whether $E_R(X)$ is always separable. To be compared with the results below, we only remark that $E_R(X) - X$ is not necessarily countable. In fact, if R contains an infinite decreasing sequence converging to 0 and X is a countable dense subspace of any uncountable R -ultrametric space (a concrete example is that $X = \omega^\omega$, the Baire space), then $E_R(X) \supseteq \bar{X}$, where \bar{X} is the metric completion of X , and therefore uncountable.

5.2. Functions with finite support

To get around the issue of separability of $E_R(X)$, we use the same ideas in the Katětov construction and consider the concept of finite support for R -ultrametric admissible functions.

Definition 5.4. Let X be an R -ultrametric space, $x_1, \dots, x_n \in X$, and $f \in E_R(X)$. We say that $\{x_1, \dots, x_n\}$ is a *support* for f , if for all $y \in X$,

$$f(y) = \begin{cases} \min\{f(x_1), \dots, f(x_n)\}, & \text{if for all } i \leq n, d(x_i, y) = f(x_i), \\ f(x_i), & \text{where } i \leq n \text{ such that } d(x_i, y) < f(x_i), \\ d(x_i, y), & \text{where } i \leq n \text{ such that } d(x_i, y) > f(x_i). \end{cases}$$

We say that f has *finite support* or is *finitely supported* if some finite subset of X is a support for f .

The following theorems prove the existence of finite supported functions and some of their properties.

Theorem 5.5. Let X be an R -ultrametric space and $x_1, \dots, x_n \in X$. Then there is a function $f \in E_R(X)$ for which $\{x_1, \dots, x_n\}$ is a support. Moreover, for each $f_0 \in E_R(\{x_1, \dots, x_n\})$ there is a unique $f \in E_R(X)$ such that $\{x_1, \dots, x_n\}$ is a support of f and $f \upharpoonright \{x_1, \dots, x_n\} = f_0$.

Proof. Let $f_0 \in E_R(\{x_1, \dots, x_n\})$. Define, for all $y \in X$,

$$f(y) = \begin{cases} \min\{f_0(x_1), \dots, f_0(x_n)\}, & \text{if for all } i \leq n, d(x_i, y) = f_0(x_i), \\ f_0(x_i), & \text{where } i \leq n \text{ such that } d(x_i, y) < f_0(x_i), \\ d(x_i, y), & \text{where } i \leq n \text{ such that } d(x_i, y) > f_0(x_i). \end{cases}$$

We first check that f is well defined. For this it suffices to prove the following three claims.

Claim 1. If $d(x_i, y) < f_0(x_i)$ and $d(x_j, y) < f_0(x_j)$, then $f_0(x_i) = f_0(x_j)$. Assume not, and without loss of generality assume $f_0(x_i) < f_0(x_j)$. Then since $f \in E_R(\{x_1, \dots, x_n\})$, we have from the isosceles triangle property that $d(x_i, x_j) = f_0(x_j)$. Now

$$d(x_i, x_j) \leq \max\{d(x_i, y), d(x_j, y)\} < \max\{f_0(x_i), f_0(x_j)\} = f_0(x_j),$$

a contradiction.

Claim 2. If $d(x_i, y) < f_0(x_i)$ and $d(x_j, y) > f_0(x_j)$, then $f_0(x_i) = d(x_j, y)$. Assume not, and further assume $f_0(x_i) > d(x_j, y)$ first. Then $f_0(x_i) > f_0(x_j)$, and from $f \in E_R(\{x_1, \dots, x_n\})$ and the isosceles triangle property we have that $d(x_i, x_j) = f_0(x_i)$. By the isosceles triangle property for x_i, x_j, y in X we also obtain that $d(x_j, y) = f_0(x_i)$, contradicting our further assumption. Now we consider the subcase $f_0(x_i) < d(x_j, y)$. Now $d(x_i, y) < f_0(x_i) < d(x_j, y)$, and so the isosceles triangle property gives $d(x_i, x_j) = d(x_j, y)$. Now we have that $d(x_i, x_j) > f_0(x_j)$ as well as $d(x_i, x_j) > f_0(x_i)$, contradicting the isosceles triangle property derived from the assumption $f \in E_R(\{x_1, \dots, x_n\})$.

Claim 3. If $d(x_i, y) > f_0(x_i)$ and $d(x_j, y) > f_0(x_j)$, then $d(x_i, y) = d(x_j, y)$. Assume not, and without loss of generality assume $d(x_i, y) < d(x_j, y)$. Then $d(x_i, x_j) = d(x_j, y) > f_0(x_j)$ and $d(x_i, x_j) > d(x_i, y) > f_0(x_i)$, contradicting the assumption $f \in E_R(\{x_1, \dots, x_n\})$.

It is obvious that $f \upharpoonright \{x_1, \dots, x_n\} = f_0$ and that if $f \in E_R(X)$ then $\{x_1, \dots, x_n\}$ is a support for f . It thus remains only to check that f is R -ultrametric admissible. For $x \in \{x_1, \dots, x_n\}$ and $y \in X$ this follows easily from the definition of f . For arbitrary $x, y \in X$ a routine check of all possible cases similar to the argument above confirms the admissibility of f .

It is clear from the definition of finite support that f is uniquely determined by f_0 . \square

The kind of arguments used in the above proof can also be used to prove the following theorems, which we state without proof.

Theorem 5.6. *Let X be an R -ultrametric space and $f \in E_R(X)$. If $F_1 \subseteq X$ is a finite support for f and $F_1 \subseteq F_2$, where F_2 is finite, then F_2 is also a finite support for f .*

Theorem 5.7. *Let X be an R -ultrametric space, $x_1, \dots, x_n \in X$ is finite, and $f \in E_R(X)$. Then $\{x_1, \dots, x_n\}$ is a finite support for f iff for all $y \in X$,*

$$f(y) = \min\{\max\{f(x_i), d(y, x_i)\} : i = 1, \dots, n\}.$$

We will use the following notation.

Notation 5.8. Let X be an R -ultrametric space and $A \subseteq X$. For each $n \in \omega$, define

$$E_R(X, A, n) = \{f \in E_R(X) : \exists x_1, \dots, x_n \in A, \{x_1, \dots, x_n\} \text{ is a support for } f\}$$

and

$$E_R(X, A, \omega) = \bigcup_{n \in \omega} E_R(X, A, n).$$

We also let $E_R(X, n) = E_R(X, X, n)$ and $E_R(X, \omega) = E_R(X, X, \omega)$.

Note that $E(X, A, n) \subseteq E(X, A, n + 1)$ for all n . Thus the union in the definition of $E_R(X, A, \omega)$ is an increasing union. Apparently $E_R(X, A, \omega)$ is the set of all functions in $E_R(X)$ with a finite support in A . Each of the sets defined above is a subset of $E_R(X)$, and is therefore an R -ultrametric space with the appropriate restriction of d_E . For notational simplicity we denote all these restrictions also by d_E .

Also note that for every $x \in X$, f_x has $\{x\}$ as a finite support, and therefore X can be viewed naturally as a subset of $E_R(X, 1)$, in particular a subset of $E_R(X, \omega)$.

5.3. The separability of $E_R(X, \omega)$ and $E_R(X)$

Now suppose $D \subseteq X$ is a countable dense subset of X . Then there are only countably many finite subsets F of D ; for each finite $F \subseteq D$, there are only countably many functions in $E_R(F)$; and finally by Theorem 5.5 every function in $E_R(F)$ uniquely determines a function $f \in E_R(X, D, \omega)$. This shows that $E_R(X, D, \omega)$ is countable. Somewhat surprisingly, we have that $E_R(X, \omega) - X$ is also countable, as the following simple proposition shows.

Proposition 5.9. *Let X be an R -ultrametric space and $D \subseteq X$ a dense subset. Then $E_R(X, \omega) - X \subseteq E_R(X, D, \omega)$. In particular, if X is separable then $E_R(X, \omega) - X$ is countable, and $E_R(X, \omega)$ is separable.*

Proof. We only show that $E_R(X, \omega) - X \subseteq E_R(X, D, \omega)$. Suppose $f \in E_R(X, \omega) - X$ has support $\{x_1, x_2, \dots, x_n\} \subseteq X$. Thus $f(x_i) > 0$ for all $i = 1, \dots, n$, and we may let $\epsilon = \min\{f(x_i) : i = 1, \dots, n\}$, and $\epsilon > 0$. Since D is a dense subset of X , for each $i = 1, \dots, n$ we may find $y_i \in D$ such that $d(x_i, y_i) < \epsilon$. Then it is easy to see that $\{y_1, y_2, \dots, y_n\}$ is a support for f . Thus $f \in E_R(X, D, \omega)$. \square

This is of course in contrast to the fact we noted earlier that $E_R(X) - X$ need not be countable. We give below a description of $E_R(X)$ as well as a criterion for its separability.

Theorem 5.10. *Let X be a separable R -ultrametric space. Then the following hold:*

(a) $E_R(X) = \bar{X} \cup E_R^+(X)$, where \bar{X} is the metric completion of X , and

$$E_R^+(X) = \{f \in E_R(X) : \inf\{f(x) : x \in X\} > 0\}.$$

In particular, if X is a Polish R -ultrametric space, then $E_R(X) = X \cup E_R^+(X)$.

(b) $E_R(X)$ is separable iff $E_R^+(X)$ is countable.

Proof. (a) Consider

$$E_R^0(X) = \{f \in E_R(X) : \inf\{f(x) : x \in X\} = 0\}.$$

Obviously $E_R(X) = E_R^0(X) \cup E_R^+(X)$. It suffices to note that $E_R^0(X)$ is naturally isometric to \bar{X} . The natural isometry between $E_R^0(X)$ and \bar{X} is obvious: for each $f \in E_R^0(X)$ there exists a sequence $\{x_n\} \subseteq X$ such that $f(x_n) \rightarrow 0$ as $n \rightarrow \infty$. Then $\{x_n\}$ is Cauchy. Conversely, every Cauchy sequence in X gives rise to a unique element in \bar{X} , which in turn gives rise to an element of $E_R^0(X)$. The details are routine to check.

(b) (\Leftarrow) Trivial.

(\Rightarrow) Assume that $E_R^+(X)$ is uncountable. Then for some $\epsilon > 0$ the set

$$\{f \in E_R(X) : \inf\{f(x) : x \in X\} > \epsilon\}$$

is uncountable. But for any two distinct elements f, g of this set, $d_E(f, g) = \max\{f(x), g(x)\}$ for any $x \in X$ with $f(x) \neq g(x)$, and therefore $d_E(f, g) > \epsilon$. This shows that the set is uncountable and discrete. \square

We do not have any example where $E_R^+(X)$ is uncountable. In the following we consider the question: Is $E_R(X, \omega)$ dense in $E_R(X)$? A positive answer would prove that $E_R(X)$ is always separable. However, we have the following example in which $E_R^+(X)$ is countable (and therefore $E_R(X)$ is separable) but $E_R(X, \omega)$ is not dense in $E_R(X)$.

Example 5.11. Consider $X = (\frac{1}{2}, +\infty) \cap \mathbb{Q}$ with $d(x, y) = \max\{x, y\}$ for $x, y \in X$, and $R = X$.

To see that $E_R(X, \omega)$ is not dense in $E_R(X)$, consider $f(x) = x$ for all $x \in X$. It is easy to check that $f \in E_R(X)$. However, we note that for any $g \in E_R(X, \omega)$, $d_E(f, g) > \frac{1}{2}$. To see this, suppose $x_1 < \dots < x_n$ is a support for g . Then

$$g(x) = \begin{cases} x_1, & \text{if } x \leq x_1, \\ x, & \text{if } x > x_1. \end{cases}$$

Now $d_E(f, g) = x_1 > \frac{1}{2}$. Thus $f \notin \overline{E_R(X, \omega)}$.

On the other hand, we have in fact that $E_R(X)$ is countable. To see this let $f \in E_R^+(X)$. We consider two cases. Case 1: There is $x_0 \in X$ with $f(x_0) < x_0$. In this case we claim that $f(x) = x_0$ for all $x < x_0$ and $f(x) = x$ for all $x > x_0$. In fact, if $x < x_0$ then the isosceles triangle property involving x, x_0 and f gives that $f(x) = x_0$ since $d(x, x_0) = x_0$ and $f(x_0) < x_0$. Similarly, if $x > x_0$ then $d(x, x_0) = x > x_0 > f(x_0)$ and hence $f(x) = x$ by the isosceles triangle property. Case 2: For every $x \in X$, $f(x) \geq x$. In this case if there is $x_0 \in X$ such that $f(x_0) > x_0$ then we claim that $f(x) = f(x_0)$ for all $x \leq x_0$ and $f(x) = x$ for all $x > x_0$. This again follows easily from the isosceles triangle property.

To summarize, either there is $x_0 \in X$ such that $f(x_0) < x_0$, in which case the pair $(x_0, f(x_0)) \in R^2$ completely determines the function f , giving only countably many possibilities for f , or else f is the identity function on X , or else there is $x_0 \in X$ such that $f(x_0) > x_0$, in which case the function f is completely determined by the value $f(x_0) \in R$, giving again only countably many possibilities for f . Overall there are only countably many possibilities for such f , and hence $E_R^+(X)$ is countable.

5.4. The Katětov type construction

Let X be a separable R -ultrametric space. We define

$$\begin{aligned} X_0 &= X, \\ X_1 &= E_R(X_0, \omega), \\ &\dots \\ X_{n+1} &= E_R(X_n, \omega), \end{aligned}$$

and let

$$X_\omega = \bigcup_{n \in \omega} X_n.$$

The union in the definition of X_ω is an increasing union via the natural isometric embeddings from X_n into $X_{n+1} = E_R(X_n, \omega)$.

Now each X_n is separable, and so is X_ω . Moreover, X_ω obviously has the R -ultrametric Urysohn property. By Proposition 2.4 the metric completion of X_ω , denoted $\overline{X_\omega}$ is a Polish R -ultrametric Urysohn space. This finishes our Katětov type construction of \mathbb{U}_R^u .

The following theorem gives the universality of $\text{Iso}(\mathbb{U}_R^u)$ by an argument similar to Uspenskij [12].

Theorem 5.12. $\text{Iso}(\mathbb{U}_R^u)$ is universal for all $\text{Iso}(X)$, where X are separable R -ultrametric spaces.

Proof. We first verify that if $\varphi \in \text{Iso}(X)$, then $\varphi_E \upharpoonright E_R(X, \omega) \in \text{Iso}(E_R(X, \omega))$. That is, if $\{x_1, \dots, x_n\}$ is a finite support for f , then $\varphi_E(f)$ is also finitely supported. This is now obvious, since $\{\varphi^{-1}(x_1), \dots, \varphi^{-1}(x_n)\}$ is a support for $\varphi_E(f)$ by definition. Moreover, it is still the case that $\varphi_E \upharpoonright X = \varphi$.

Next we show that for any $\varphi \in \text{Iso}(X)$ there is a canonical isometry $\varphi^* \in \text{Iso}(\overline{X_\omega})$ such that $\varphi^* \upharpoonright X = \varphi$. For this we make the following definition given $\varphi \in \text{Iso}(X)$:

$$\begin{aligned} \varphi_0 &= \varphi, \\ \varphi_1 &= (\varphi_0)_E \upharpoonright X_1, \\ &\dots \\ \varphi_{n+1} &= (\varphi_n)_E \upharpoonright X_{n+1}, \end{aligned}$$

and let

$$\varphi_\omega = \bigcup_{n \in \omega} \varphi_n.$$

Again the union in the definition of φ_ω is an increasing union, since each φ_{n+1} is a natural extension of φ_n . Now $\varphi_\omega \in \text{Iso}(X_\omega)$. We can let $\varphi^* = \overline{\varphi_\omega}$ be the unique extension of φ_ω to $\overline{X_\omega}$. This finishes the definition of φ^* .

It is routine to check that $\varphi \mapsto \varphi^*$ is a topological group embedding from $\text{Iso}(X)$ into $\text{Iso}(\overline{X_\omega})$. Since $\overline{X_\omega}$ and \mathbb{U}_R^u are isometric, we have thus shown that $\text{Iso}(\mathbb{U}_R^u)$ is universal for all $\text{Iso}(X)$, where X are separable R -ultrametric spaces. \square

For $\text{Iso}(\mathbb{K}_R^u)$ we also obtain the following universality result.

Theorem 5.13. $\text{Iso}(\mathbb{K}_R^u)$ is universal for all $\text{Iso}(X)$, where X are countable R -ultrametric spaces.

Proof. If X is countable then $E_R(X, \omega)$ is countable by Proposition 5.9. It follows that X_ω is countable. Since it has the R -ultrametric Urysohn property, X_ω is isometric with \mathbb{K}_R^u . Since $\text{Iso}(X)$ topologically embeds into $\text{Iso}(X_\omega)$, we have the desired universality property for $\text{Iso}(\mathbb{K}_R^u)$. \square

6. The generalized tree construction

In this section we give the fourth construction of ultrametric Urysohn spaces. As noted in e.g. [9], the countable R -ultrametric Urysohn space \mathbb{K}_R^u can be realized as the space of all finitely supported elements of \mathbb{Q}^R with the metric

$$d(x, y) = \max\{r \in R: x(r) \neq y(r)\}$$

when $x \neq y$. One can then take \mathbb{U}_R^u as the completion of \mathbb{K}_R^u .

We will take an approach here that is slightly different in details. In particular, we will realize \mathbb{U}_R^u directly as the space of branches of some generalized trees. By exploring the notion of these generalized trees we can realize all Polish ultrametric spaces the same way. Our trees generalize the usual descriptive set theoretic trees on ω .

Let R' denote the set of all limit points of R (the Cantor–Bendixson derivative of R). Note that when $0 \notin R'$ the spaces \mathbb{K}_R^u and \mathbb{U}_R^u coincide. Thus in the rest of this section we will focus on the case $0 \in R'$.

Throughout this section we fix a countable $R \subseteq \mathbb{R}_+$ with $0 \in R'$.

6.1. Branches of R -trees

Definition 6.1. Let $\omega^{<R}$ denote the set of all functions

$$u: [a, +\infty) \cap R \rightarrow \omega$$

where $a \in R$, such that the set

$$\{b \in R \cap [a, +\infty): u(b) \neq 0\}$$

is finite. If $u \in \omega^{<R}$ and $b \in \text{dom}(u)$, then we denote by $u \upharpoonright b$ the function $u \upharpoonright ([b, +\infty) \cap R)$, which is also an element of $\omega^{<R}$. For $u, v \in \omega^{<R}$, we say that u is an *initial segment* of v , or v *extends* u , and denote by $u \subseteq v$ or $v \supseteq u$, if there is $b \in \text{dom}(v)$ such that $u = v \upharpoonright b$. We call $(\omega^{<R}, \subseteq)$ the *full R -tree*.

The full R -tree $\omega^{<R}$ is a generalized tree in the sense that for any $u \in \omega^{<R}$ the set of initial segments of u is linearly ordered by \subseteq . This is in contrast with the usual notion of trees in set theory where the initial segments of every element are well ordered.

Definition 6.2. For every $u \in \omega^{<R}$, the level of u is defined by

$$l(u) = \text{inf dom}(u) = \text{min dom}(u).$$

With this notion we have that $u \subseteq v$ iff $u = v \upharpoonright l(u)$.

Definition 6.3. A subset T of $\omega^{<R}$ is called an R -tree if it is closed under taking initial segments, i.e., if $u \subseteq v$ and $v \in T$ then $u \in T$. An R -tree T is *pruned* if for every $u \in T$ and $a \in R$ with $a < l(u)$ there is $v \in T$ with $u \subseteq v$.

Definition 6.4. Let T be an R -tree. A branch of T is a function $f \in \omega^R$ such that for all $a \in R$, $f \upharpoonright a \in T$, where $f \upharpoonright a$ denotes $f \upharpoonright ([a, +\infty) \cap R)$. If $u = f \upharpoonright a$ we also write $u \subseteq f$ and say that u is an *initial segment* of f . The set of all branches of T is denoted by $[T]$.

We mention the following easy observations without proof.

Lemma 6.5. Let T be an R -tree. Then:

- (i) For every $f \in [T]$ the set $\{a \in R: f(a) \neq 0\}$ is either finite or a decreasing sequence converging to 0.
- (ii) If $f \neq g \in [T]$, then the set $\{a \in R: f(a) \neq g(a)\}$ has a maximum element.

Clause (ii) in the above lemma allows us to define an R -ultrametric on $[T]$ for any R -tree T .

Definition 6.6. Let T be an R -tree. We define a metric on $[T]$ by

$$d(f, g) = \begin{cases} 0, & \text{if } f = g, \\ \max\{a \in R: f(a) \neq g(a)\}, & \text{otherwise.} \end{cases}$$

We denote by X_T the space $([T], d)$. Also, we denote by X_R the space $([\omega^{<R}], d)$.

It is easy to check that d is an R -ultrametric. Note that the definition of d is independent of the specific R -tree T : all of these metrics are simply the restrictions to $[T]$ the corresponding metric defined for $[\omega^{<R}]$. In particular X_T has the subspace topology inherited from X_R .

Notation 6.7. For any $u \in \omega^{<R}$, we define

$$N_u = \{f \in [\omega^{<R}]: u \subseteq f\}.$$

Obviously for any $f \in N_u$, $N_u = \{g \in [\omega^{<R}]: d(f, g) < l(u)\}$. The collection of all N_u for $u \in \omega^{<R}$ forms a countable base of clopen sets for X_R . This shows that X_R is second countable. It follows that all X_T are second countable since they are topological subspaces of X_R .

Lemma 6.8. If T is an R -tree, then X_T is a Polish R -ultrametric space.

Proof. It suffices to check that d is complete. For this let $\{f_n\}$ be a d -Cauchy sequence in X_T . Since $0 \in R'$ we may fix a decreasing sequence $\{a_k\}$ of elements of R converging to 0. For each k there is N_k such that for all $m, n \geq N_k$, $d(f_n, f_m) < a_k$. Without loss of generality we may fix such N_k so that $N_k < N_{k+1}$ for all k .

For all $n, m \geq N_k$, since $d(f_n, f_m) < a_k$, we have $f_n \upharpoonright a_k = f_m \upharpoonright a_k$. In view of this we define $f \in [\omega^{<R}]$ by letting $f \upharpoonright a_k = f_{N_k} \upharpoonright a_k$ for all k . Then for all $n \geq N_k$, $d(f_n, f) < a_k$. Thus $\lim_n f_n = f$.

It remains to verify that $f \in [T]$. For this consider an arbitrary $a \in R$, and let k be large enough such that $a_k < a$. Then $f \upharpoonright a \subseteq f \upharpoonright a_k = f_{N_k} \upharpoonright a_k$, and since $f_{N_k} \in [T]$ and T is an R -tree, we have that $f \upharpoonright a \in T$ as required. \square

In fact, the same argument gives the following analog of a classical fact in descriptive set theory.

Proposition 6.9. A subset C of X_R is closed iff there is an R -tree T such that $C = [T]$. Moreover, for every closed $C \subseteq X_R$ there is a unique pruned R -tree T with $C = [T]$.

Proof. (\Leftarrow) The previous proof also shows that $[T]$ is a closed subset of X_R .

(\Rightarrow) Suppose $C \subseteq X_R$ is closed. Define $T = \{u \in \omega^{<R} : \exists f \in C \ u \subseteq f\}$. It is obvious that T is a pruned R -tree and $C \subseteq [T]$. To see $[T] \subseteq C$, fix again a decreasing sequence $\{a_k\}$ of elements of R converging to 0. Let $f \in [T]$ and $g_k \in C$ be such that $g_k \upharpoonright a_k = f \upharpoonright a_k$. Then $\lim_k g_k = f$ and therefore $f \in C$.

Finally for the uniqueness of a pruned R -tree T with $[T] = C$ just note that if T, S are both pruned R -trees and $T \neq S$, then $[T] \neq [S]$. \square

We are now ready to show that X_R is a Polish R -ultrametric Urysohn space.

Theorem 6.10. X_R is a Polish R -ultrametric Urysohn space.

Proof. Let B be a finite R -ultrametric space, $A \subseteq B$, and $\varphi : A \rightarrow X_R$ an isometric embedding. Without loss of generality we assume that $|B - A| = 1$. Assume in fact $A = \{a_1, \dots, a_n\}$ and $B = A \cup \{b\}$. Let $f_i = \varphi(a_i)$ for all $i = 1, \dots, n$. Let $r_i = d(b, a_i)$ for all $i = 1, \dots, n$. Without loss of generality assume that $r_1 \leq \dots \leq r_n$. Also let l be the largest such that $r_1 = r_l$.

We define a $g \in X_R$ so that $d(g, f_i) = r_i$ for all $i = 1, \dots, n$. For this note that for all $i, j = 1, \dots, l$, $d(a_i, a_j) \leq \max\{d(b, a_i), d(b, a_j)\} = r_1$. Hence for all $i, j \leq l$, $d(f_i, f_j) \leq r_1$, and therefore $f_i \upharpoonright (r_1, +\infty) \cap R = f_j \upharpoonright (r_1, +\infty) \cap R$.

Pick any $m \in \omega - \{f_1(r_1), \dots, f_l(r_1)\}$. Define $g \in X_R$ by

$$g(x) = \begin{cases} f_1(x), & \text{if } x > r_1, \\ m, & \text{if } x = r_1, \\ 0, & \text{if } x < r_1. \end{cases}$$

Then it is clear that $d(g, f_i) = r_i = r_1$ for all $i = 1, \dots, l$. For $l < i \leq n$, $d(b, a_i) = r_i > r_1 = d(b, a_1)$, which implies that $d(a_i, a_1) = r_i > r_1$ and so $d(f_i, f_1) = r_i > r_1$. Since $g \upharpoonright (r_1, +\infty) \cap R = f_1 \upharpoonright (r_1, +\infty) \cap R$, we have that $d(f_i, g) = r_i$. Thus g is as required. \square

Note that the Baire space ω^ω can be viewed as $[\omega^{<R}]$ with $R = \{2^{-n} : n \in \omega\}$. It follows that the Baire space is R -ultrametric Urysohn.

6.2. The duality theorem

One of the advantages of the current approach is the possibility to explore a duality between Polish ultrametric spaces and generalized trees. An immediate corollary of Proposition 6.9 and Theorem 6.10 is that the two categories of objects are intrinsically correspondent.

Proposition 6.11. For any Polish R -ultrametric space X there is an R -tree T such that X is isometric to X_T .

Proof. By Theorem 6.10 X_R is a Polish R -ultrametric Urysohn space, and it follows from Proposition 2.7(b) that X_R is universal for all Polish R -ultrametric spaces (up to isometric embedding). Thus for any Polish R -ultrametric space X there is an isometric copy of X in X_R as a subspace. Such a subspace must be closed, and therefore by Proposition 6.9 it is of the form X_T for some (pruned) R -tree T . \square

We next investigate the notion of isomorphic embedding and isomorphism between R -trees.

Definition 6.12. Let T, S be R -trees. An *isomorphic embedding* (or simply an *embedding*) from T into S is a map $\varphi : T \rightarrow S$ such that, for all $u, v \in T$,

- (i) $l(\varphi(u)) = l(u)$;
- (ii) $u = v$ iff $\varphi(u) = \varphi(v)$;
- (iii) $u \subseteq v$ iff $\varphi(u) \subseteq \varphi(v)$.

We write $\varphi : T \hookrightarrow S$ if φ is an isomorphic embedding from T into S . We say that T is (*isomorphically*) *embeddable* into S , and simply denote $T \hookrightarrow S$, if there is an isomorphic embedding from T into S . An *isomorphism* between T and S is an isomorphic embedding that is onto. If φ is an isomorphism between T and S , then we write $\varphi : T \cong S$. We say that T and S are *isomorphic*, and denote $T \cong S$, if there is an isomorphism between T and S .

The following duality theorem is the main result of this subsection.

Theorem 6.13. Let T, S be pruned R -trees. Then the following hold:

- (1) $T \hookrightarrow S$ iff there is an isometric embedding from X_T into X_S ;
- (2) $T \cong S$ iff X_T and X_S are isometric.

Proof. (1) (\Rightarrow) Let $\varphi : T \hookrightarrow S$ be an isomorphism embedding. Define $\psi : X_T \rightarrow X_S$ by letting

$$\psi(f) = \bigcup_{a \in R} \varphi(f \upharpoonright a).$$

For $a < b \in R$, we have that $f \upharpoonright b \subseteq f \upharpoonright a$ and therefore $\varphi(f \upharpoonright b) \subseteq \varphi(f \upharpoonright a)$. This implies that $\psi(f)$ is well defined. It is straightforward to check that ψ is an isometric embedding.

(\Leftarrow) Let $\psi : X_T \rightarrow X_S$ be an isometric embedding. We define $\varphi : T \hookrightarrow S$ as follows. To each $u \in T$ we associate some $f_u \in [T]$ with $u \subseteq f_u$. Such branches exist since T is pruned. Define

$$\varphi(u) = \psi(f_u) \upharpoonright l(u).$$

We first check that $\varphi(u)$ is independent of the choice of f_u . For this let $g_u \in [T]$ with $u \subseteq g_u$. Then $d(f_u, g_u) < l(u)$. Since ψ is an isometric embedding, we have that $d(\psi(f_u), \psi(g_u)) = d(f_u, g_u) < l(u)$. Hence $\psi(f_u) \upharpoonright l(u) = \psi(g_u) \upharpoonright l(u)$. Thus $\varphi(u)$ is well defined.

It is clear that $\varphi(u) \in S$ for all $u \in T$, since $\psi(f_u) \in X_S = [S]$. It remains to check that $\varphi : T \hookrightarrow S$. It is immediate from the definition of φ that $l(\varphi(u)) = l(u)$. If $u \neq v \in T$ and $l(u) \neq l(v)$, then of course $\varphi(u) \neq \varphi(v)$ since they have different levels. Suppose $u \neq v \in T$ are such that $l(u) = l(v)$. Then for some $a \in R$ with $a > l(u)$, $u(a) \neq v(a)$. It follows that $d(f_u, f_v) \geq a$, and therefore $d(\psi(f_u), \psi(f_v)) \geq a$. This implies that $\varphi(u) \neq \varphi(v)$. To verify that $u \subseteq v$ iff $\varphi(u) \subseteq \varphi(v)$ we first suppose $u \subseteq v$. Then $l(u) \geq l(v)$ and $d(f_u, f_v) < l(u)$. It follows that $d(\psi(f_u), \psi(f_v)) < l(u)$ and therefore $\psi(f_v) \upharpoonright l(u) = \psi(f_u) \upharpoonright l(u) = \varphi(u)$. This in turn implies that $\varphi(v) \upharpoonright l(u) = \varphi(u)$, and thus $\varphi(u) \subseteq \varphi(v)$. Conversely, suppose $\varphi(u) \subseteq \varphi(v)$. Then $\psi(f_v) \upharpoonright l(u) = \psi(f_u)$. This implies that $d(\psi(f_v), \psi(f_u)) < l(u)$, and thus $d(f_v, f_u) < l(u)$. It follows that $v \upharpoonright l(u) = u$, and therefore $u \subseteq v$. We have thus shown that $T \hookrightarrow S$.

(2) Note that if $\varphi : T \cong S$ then the isometric embedding ψ defined in the (\Rightarrow) direction of (1) is in fact onto. In fact we verified that $\psi(X_T)$ is a closed subspace of X_S . If $g \in X_S - \psi(X_T)$ then there is $w \in S$ such that $g \in N_w \cap S$ and $N_w \cap X_T = \emptyset$. Since $\varphi : T \rightarrow S$ is onto there is $u \in T$ with $\varphi(u) = w$. Now let $f \in [T]$ be such that $u \subseteq f$, then $\varphi(f) \in N_w \cap \varphi(X_T)$, contradicting our assumption.

Conversely, if $\psi : X_T \rightarrow X_S$ is an isometry, then the isomorphic embedding $\varphi : T \rightarrow S$ defined in the (\Leftarrow) direction of (1) is also onto. \square

An R -tree T can be naturally viewed as a countable relational structure with the following language

$$L = \{\subseteq\} \cup \{V_a : a \in R\},$$

where the interpretations of the binary relation symbol \subseteq is obvious, and that of the unary relation symbols V_a , $a \in R$, is given by

$$V_a(u) \iff l(u) = a.$$

Each isomorphism of T (onto itself) is just an automorphism of the countable L -structure T . The group of automorphisms, denoted $\text{Aut}(T)$, coincide with the group of all isomorphisms of T . The group $\text{Aut}(T)$ is isomorphic to a closed subgroup of S_∞ , the group of all permutations of ω , with the subspace topology inherited from the Baire space ω^ω (cf., e.g., [4, Theorem 2.4.4]).

As a corollary of the duality theorem we note that $\text{Aut}(T)$ and $\text{Iso}(X_T)$ are isomorphic as topological groups.

Proposition 6.14. *Let T be a pruned R -tree. Then $\text{Aut}(T)$ and $\text{Iso}(X_T)$ are isomorphic topological groups.*

Proof. It is straightforward to check that the correspondences $\varphi \mapsto \psi$ and $\psi \mapsto \varphi$ defined in the first part of the proof of Theorem 6.13 are inverses of each other. When $S = T$ the defined correspondences give a bijection between $\text{Aut}(T)$ and $\text{Iso}(X_T)$. It is easy to check that the correspondence $\varphi \mapsto \psi$ is a group homomorphism and a homeomorphism. \square

6.3. Extensive isometric embeddings

In this subsection we consider the following strong notion of isometry group embeddability.

Definition 6.15. Let X, Y be metric spaces. An isometric embedding φ from X to Y is *extensive* if there is a topological group embedding $\Phi : \text{Iso}(X) \rightarrow \text{Iso}(Y)$ such that for every $\alpha \in \text{Iso}(X)$, $\Phi(\alpha) \upharpoonright \varphi(X) = \varphi \circ \alpha \circ \varphi^{-1}$.

If there is an extensive isometric embedding from X into Y then in particular $\text{Iso}(X)$ is isomorphic to a topological subgroup of $\text{Iso}(Y)$. The proof of Theorem 5.12 has shown the following result.

Theorem 6.16. *For every Polish R -ultrametric space X there is an extensive isometric embedding from X into \mathbb{U}_R^u .*

Note that if φ is an extensive isometric embedding from X into Y , then every isometry of $\varphi(X)$ can be extended to an isometry of Y . We remark that not every isometric embedding from a Polish R -ultrametric space into \mathbb{U}_R^u is extensive. To see this consider the Baire space ω^ω , which is a Polish R -ultrametric Urysohn space for a suitable R . In fact not every isometric embedding from ω^ω into itself is extensive. For example consider an embedding $\varphi: \omega^\omega \rightarrow \omega^\omega$ so that the range of φ is

$$Y = \{x \in \omega^\omega: x(0) \neq 0 \text{ or } x(1) \neq 0\}.$$

If α is any isometry of Y sending $Y \cap N_{(0)}$ to $Y \cap N_{(1)} = N_{(1)}$ then α cannot be extended to any isometry of ω^ω .

It is easy to see that a composition of extensive isometric embeddings is extensive.

The following is the main theorem of this subsection.

Theorem 6.17. *If X is a compact R -ultrametric space then every isometry embedding from X into X_R is extensive.*

Note that even in the case of finite X the conclusion of the theorem does not follow abstractly from the ultrahomogeneity of X_R . Our proof will also yield as a corollary the following ultrahomogeneity property for compact sets.

Corollary 6.18. *Any isometry between two compact subsets of X_R can be extended to an isometry of X_R .*

This result is similar to Bogatyĭ's theorem [1] that any isometry between two compact subsets of \mathbb{U} can be extended to an isometry of \mathbb{U} .

The rest of this subsection is devoted to a proof of Theorem 6.17. In the process we prove a number of general lemmas.

Lemma 6.19. *Let X be a compact R -ultrametric space. Then the set*

$$R_0 = \{d(x, y): x \neq y \in X\}$$

is either finite or a decreasing sequence converging to 0.

Proof. If X is finite then R_0 is of course finite. If X is infinite then R_0 cannot be finite, since then (e.g. by Ramsey's theorem) there would exist infinitely many elements in X whose pairwise distances are the same, contradicting the compactness assumption. In general we show that R_0 does not contain any infinite (strictly) increasing sequence. Otherwise, suppose $\{a_k\}$ is such a sequence. Let $x_k, y_k \in X$ be such that $d(x_k, y_k) = a_k$. By compactness we may assume that both $\{x_k\}$ and $\{y_k\}$ converge, say $\lim_k x_k = x$ and $\lim_k y_k = y$. Then $d(x, y) = \lim_k a_k > a_k$ for all k . Let k be large enough so that $d(x_k, x), d(y_k, y) < a_k$. Then by the isosceles triangle property we get that $d(x_k, y_k) = \lim_k a_k > a_k$, contradiction. Now it follows that (R_0, \geq) is a well order. By a similar argument one can show that R_0 does not contain any infinite descending sequence with a nonzero limit. Thus the order type of (R_0, \geq) has to be ω , which means that R_0 is a single decreasing sequence. Moreover, its limit has to be 0. \square

In the remainder we fix a compact R -ultrametric space X . We also fix an enumeration $r_n, n < N$, for the elements of R_0 in decreasing order, where $N \in \omega \cup \{+\infty\}$.

Fix an isometric embedding φ from X into X_R . In order to show that φ is extensive, it suffices to define a topological group embedding $\psi \mapsto \psi^*$ from $\text{Iso}(\varphi(X))$ into $\text{Iso}(X_R)$ such that $\psi^* \upharpoonright X_T = \psi$. Let T be the unique pruned R -tree with $\varphi(X) = [T]$. By the duality theorem (Theorem 6.13) and Proposition 6.14 it suffices to define a topological group embedding $\phi \mapsto \phi^*$ from $\text{Aut}(T)$ into $\text{Aut}(\omega^{<R})$ such that $\phi^* \upharpoonright T = \phi$.

Lemma 6.20. *If T is a pruned R -tree with $[T]$ compact, then for every $r \in R$ there are finitely many $u \in T$ with $l(u) = r$.*

Proof. If $u \neq v \in T$ with $l(u) = l(v) = r$ then $d(f, g) \geq r$ for any $f \in N_u$ and $g \in N_v$. If there are infinitely many distinct $u \in T$ with $l(u) = r$ then there exists an infinite discrete subset of $[T]$, contradicting compactness. \square

Lemma 6.21. *Let T be a nonempty R -tree. If there is $u \in \omega^{<R} - T$ such that for all $a \in \text{dom}(u)$, $u \upharpoonright a \notin T$, then R has a maximum element.*

Proof. Otherwise, suppose $\{b_n\}$ is a strictly increasing sequence of elements of $\text{dom}(u)$ with $\lim_n b_n = \sup R$ (could be $+\infty$). Let $v \in T$ be arbitrary. Since $\{a \in \text{dom}(v): v(a) \neq 0\}$ is finite, there must exist N such that for all $n > N$, $b_n \in \text{dom}(v)$ and $v \upharpoonright b_n = \bar{0}$, where $\bar{0}$ denotes the constant 0 function with an appropriate domain. The same argument applies to u , and therefore there is $N' \geq N$ such that for all $n > N'$, $u \upharpoonright b_n = \bar{0}$. Thus for any $n > N'$, $u \upharpoonright b_n = v \upharpoonright b_n \in T$, contradicting our assumption. \square

Lemma 6.22. Suppose R has a maximum element, and let $m = \max R$. Let T be a pruned R -tree. Let

$$S_m = \{u \notin T : l(u) = m\}, \quad T_m = \{v \in T : l(v) = m\},$$

and

$$S = \{u \notin T : \forall a \in \text{dom}(u) \ u \upharpoonright a \notin T\}.$$

Then S and $\omega^{<R} - S$ are both pruned R -trees, and the natural inclusion embedding from $[\omega^{<R} - S]$ into X_R is extensive.

Proof. Note that S_m and T_m form a partition of all nodes of $\omega^{<R}$ of level m , and

$$S = \{u \in \omega^{<R} : \exists u' \in S_m \ u' \subseteq u\},$$

$$\omega^{<R} - S = \{v \in \omega^{<R} : \exists v' \in T_m \ v' \subseteq v\}.$$

This shows that both S and $\omega^{<R} - S$ are R -trees, and $[S]$ and $[\omega^{<R} - S]$ form a partition of X_R . Let ϵ be the identity on S . Given any $\alpha \in \text{Aut}(\omega^{<R} - S)$, $\epsilon \cup \alpha$ is obviously an automorphism of $\omega^{<R}$. Then $\alpha \mapsto \epsilon \cup \alpha$ is the desired topological group embedding from $\text{Aut}(\omega^{<R} - S)$ into $\text{Aut}(\omega^{<R})$. \square

In view of the preceding two lemmas we assume without loss of generality for the rest of the proof that

$$\forall u \notin T \ \exists a \in \text{dom}(u) \ u \upharpoonright a \in T. \tag{*}$$

Lemma 6.23. Let T be a pruned R -tree satisfying $(*)$. For $u \notin T$, define

$$t(u) = \bigcup \{v \subseteq u : v \in T\}.$$

Then

- (i) $t(u) \in \omega^{<R}$ iff $t(u) \in T$ iff $\text{dom}(t(u))$ has a minimum;
- (ii) if $R - \text{dom}(t(u))$ has no maximum, then for any $w \in \omega^{<R}$ with $t(u) \subseteq w$ and $l(w) < \text{inf dom}(t(u))$, $w \notin T$.

Proof. (i) By $(*)$ $t(u)$ is nonempty. In general $t(u)$ need not be an element of $\omega^{<R}$, since it is a partial function whose domain need not have a minimum. It is clear that $t(u) \in T$ iff $\text{dom}(t(u))$ has a minimum iff $t(u) \in \omega^{<R}$.

(ii) Suppose $R - \text{dom}(t(u))$ has no maximum. We can find an infinite increasing sequence b_k of elements of $\text{dom}(u)$ such that $\lim_k b_k = \sup(R - \text{dom}(t(u))) \leq \text{inf dom}(t(u))$. If $w \in T$, $t(u) \subseteq w$ and $l(w) < \text{inf dom}(t(u))$, then $l(w) < \lim_k b_k$, and therefore there is k such that $w \upharpoonright b_k \in T$. This means that $b_k \in \text{dom}(t(u))$, contradicting our assumption that $b_k < \text{inf dom}(t(u))$. \square

Lemma 6.24. Let T be a pruned R -tree with $[T]$ compact and satisfying $(*)$. Then for all $u \notin T$, $t(u) \in T$ and $R - \text{dom}(t(u))$ has a maximum.

Proof. To show $t(u) \in T$ it suffices to argue that $\text{dom}(t(u))$ has a minimum.

Assume not, and let $\{a_k\}$ be an infinite decreasing sequence with $\lim_k a_k = \text{inf dom}(t(u))$. Toward a contradiction to compactness, we obtain an infinite sequence f_n in $[T]$ so that $d(f_n, f_m) > l(u)$ for all $n \neq m$. Define f_n by induction. To begin with, fix any $g \in X_R$ with $u \subseteq g$. Let $k_1 = 1$, and $f_1 \in [T]$ be arbitrary such that $u \upharpoonright a_1 \subseteq f_1$. Then $\text{inf dom}(t(u)) < d(f_1, g) \leq a_1$. Let k_2 be the least with $a_{k_2} < d(f_1, g)$. Let $f_2 \in [T]$ be arbitrary such that $u \upharpoonright a_{k_2} \subseteq f_2$. Then $\text{inf dom}(t(u)) < d(f_2, g) \leq a_{k_2}$. Continuing indefinitely, we obtain infinite sequences

$$a_{k_1} > a_{k_2} > \dots > a_{k_n} > \dots > \text{inf dom}(t(u))$$

and $f_1, f_2, \dots, f_n, \dots$ such that $d(f_{n+1}, g) \leq a_{k_{n+1}} < d(f_n, g) \leq a_{k_n}$ for all n . Thus for $n < m$, $d(f_n, f_m) = d(f_n, g) > a_{k_{n+1}} > \text{inf dom}(t(u)) > l(u)$ as promised.

Since T is pruned, there exists $w \in T$ with $l(w) < l(t(u))$ and $t(u) \subseteq w$. By (ii) of the preceding lemma, $R - \text{dom}(t(u))$ must have a maximum. \square

In our context T is a pruned R -tree with $[T]$ compact and satisfying $(*)$, therefore in view of the preceding lemma we may define, for any $u \notin T$,

$$m(u) = \max(R - \text{dom}(t(u))) \quad \text{and} \quad s(u) = u \upharpoonright m(u).$$

Note that $s(u) \notin T$.

Now for every $v \in T$ such that $R - \text{dom}(v)$ has a maximum, we let $n(v) = \max(R - \text{dom}(v))$ and define

$$S_v = \{w \notin T : v \subseteq w \text{ and } l(w) = n(v)\}.$$

By Lemma 6.20 each S_v is infinite.

We use the following easy combinatorial lemma.

Lemma 6.25. *Let S_n , $n \in \omega$, be an infinite sequence of sets of the same cardinality. Then for all $i, j \in \omega$ there exist bijections $\sigma_{i,j}$ from S_i onto S_j such that, for all $i, j, k \in \omega$,*

$$\sigma_{i,k} = \sigma_{j,k} \circ \sigma_{i,j}.$$

Proof. For all $i \in \omega$ let $\sigma_{i,i+1}$ be arbitrarily fixed bijections from S_i onto S_{i+1} . Then for any $i < j$ define $\sigma_{i,j} = \sigma_{j-1,j} \circ \dots \circ \sigma_{i,i+1}$ and $\sigma_{j,i} = \sigma_{i,j}^{-1}$. It is routine to check by induction that this definition works. \square

Now for any $l \in R$ consider the collection of all S_v , where $v \in T$ and $n(v) = l$. Note that all such $v \in T$ have the same level. By applying the preceding lemma to this collection we obtain bijections σ_{v_1,v_2} from S_{v_1} to S_{v_2} in this collection.

We are now ready to define, given an automorphism ϕ of T , the extension ϕ^* . For $u \in T$ we let $\phi^*(u) = \phi(u)$. If $u \notin T$, then with the above notation we let

$$\phi^*(u)(a) = \begin{cases} \phi(t(u))(a), & \text{if } a \in \text{dom}(t(u)), \\ \sigma_{t(u),\phi(t(u))}(s(u))(a), & \text{if } a = m(u), \\ u(a), & \text{if } a < m(u). \end{cases}$$

We check that ϕ^* is an automorphism of $\omega^{<R}$. First note that $l(\phi^*(u)) = l(u)$ for all $u \in \omega^{<R}$. Also note that $u \in T$ iff $\phi^*(u) \in T$. It is also straightforward to check that ϕ^* is one-to-one and that it preserves initial segments. Note that the definition also ensures $(\phi^{-1})^* = (\phi^*)^{-1}$, from which it follows that ϕ^* is onto. This shows that ϕ^* is an automorphism of $\omega^{<R}$.

It is also obvious from the definition that $\phi^* \upharpoonright T = \phi$. It follows that $\phi \mapsto \phi^*$ is an open embedding from $\text{Aut}(T)$ into $\text{Aut}(\omega^{<R})$. It is straightforward to check that it is a group homomorphism, using the property of maps given in Lemma 6.25.

Finally it remains to verify that $\phi \mapsto \phi^*$ is continuous. For this let $u, v \in \omega^{<R}$ and consider all ϕ^* with $\phi^*(u) = v$. By our definition either both $u, v \in T$ or $u, v \notin T$. In the first case, there is nothing to prove: the set of all ϕ with $\phi(u) = v$ coincides with the set of all ϕ^* with $\phi^*(u) = v$, and hence is subbasic open. If $u, v \notin T$, then note that $\phi^*(u) = v$ iff $\phi(t(u)) = t(v)$ and $\sigma_{t(u),\phi(t(u))}(s(u)) = s(v)$. This shows that the set of all ϕ with $\phi^*(u) = v$ is either subbasic clopen or empty, hence it is open.

We have thus finished the proof of Theorem 6.17.

7. S_∞ -universality of isometry groups

In this section we collect some results about the universality of isometry groups of ultrametric Urysohn spaces. First we note the following easy observation about extensive embeddings.

Theorem 7.1. *Let $R_1 \subseteq R_2$ be subsets of \mathbb{R}_+ . Then the natural embedding j from $\omega^{<R_1}$ into $\omega^{<R_2}$ defined by*

$$j(u)(a) = \begin{cases} u(a), & \text{if } a \in R_1, \\ 0, & \text{otherwise,} \end{cases}$$

induces an extensive isometric embedding from X_{R_1} into X_{R_2} .

Proof. Given an automorphism ϕ of $j(\omega^{<R_1})$ the extension ϕ^* is defined as follows. For $u \in \omega^{<R_2}$ let

$$p(u) = \bigcup \{j(v) \subseteq u : v \in \omega^{<R_1}\}$$

and

$$q(u) = u \upharpoonright (R_2 - \text{dom}(p(u))).$$

Then $u = p(u) \cup q(u)$. Let

$$\phi^*(u) = \bigcup \{\phi(j(v)) : v \in \omega^{<R_1} \text{ and } j(v) \subseteq u\}.$$

Then it is easy to check that $\phi \mapsto \phi^*$ is a topological group embedding as required. \square

In particular $\text{Iso}(X_{R_1})$ topologically embeds into $\text{Iso}(X_{R_2})$.

We consider the following notion of universality for topological groups.

Definition 7.2. Let G be a topological group isomorphic to a closed subgroup of S_∞ . We say that G is S_∞ -universal if S_∞ is also isomorphic to a closed subgroup of G .

In particular S_∞ -universal groups are biembeddable with S_∞ and are therefore equally complicated with S_∞ in some sense.

Theorem 7.3. *Let R be a nonempty countable subset of \mathbb{R}_+ . Then $\text{Iso}(\mathbb{U}_R^u)$ is S_∞ -universal.*

Proof. $\text{Iso}(\mathbb{U}_R^u)$ is isomorphic to a closed subgroup of S_∞ (by e.g. Proposition 6.14). Let $r > 0$ be any element of R , and let $R_0 = \{r\}$. Then $\mathbb{U}_{R_0}^u$ is a countable discrete space with the trivial metric with value r . Thus $\text{Iso}(\mathbb{U}_{R_0}^u)$ is isomorphic to S_∞ . Since $\mathbb{U}_{R_0}^u$ is in particular an R -ultrametric space, it follows from Theorem 5.12 (or Theorem 7.1 above) that $\text{Iso}(\mathbb{U}_R^u)$ topologically embeds into $\text{Iso}(\mathbb{U}_{R_0}^u)$. This shows that S_∞ embeds into $\text{Iso}(\mathbb{U}_R^u)$ and therefore $\text{Iso}(\mathbb{U}_R^u)$ is S_∞ -universal. \square

In particular, for any two nonempty countable $R_1, R_2 \subseteq \mathbb{R}_+$, $\text{Iso}(\mathbb{U}_{R_1}^u)$ and $\text{Iso}(\mathbb{U}_{R_2}^u)$ are biembeddable into each other, and therefore are equally complicated in some sense.

8. Notions of classification for Polish ultrametric spaces

In this final section we collect some results for various notions of classification for Polish ultrametric spaces. The notions we consider include the quasi-order of isometric embeddability and the equivalence relations of isometric biembeddability and isometry. Undefined terminology in this section can be found, e.g., in [4].

Notation 8.1. Let R be a countable subset of \mathbb{R}_+ . We denote by, respectively, \hookrightarrow_R^i the isometric embeddability, \equiv_R^i the isometric biembeddability, and \cong_R^i the isometry for all Polish R -ultrametric spaces.

Louveau and Rosendal [8] have shown that the isometric embeddability for general Polish ultrametric spaces is a complete analytic quasi-order, and it follows from an abstract argument they gave that the isometric biembeddability is a complete analytic equivalence relation. The distance set their spaces have is $\{2^{-n} : n \in \omega\}$.

Another relevant result is a theorem of Friedman–Stanley [3] that isomorphism of trees on ω is Borel bireducible with countable graph isomorphism. It is also well known that this equivalence relation is most complex among all orbit equivalence relations on standard Borel spaces induced by a Borel action of S_∞ (cf., e.g., [4, Chapter 13]). For this reason we also refer to any equivalence relation Borel bireducible with countable graph isomorphism as S_∞ -universal.

The following theorem summarizes some easy observations.

Theorem 8.2. *Let R_1 and R_2 be countable subsets of \mathbb{R}_+ . If there exists an order preserving injection $\rho : R_1 \cup \{0\} \rightarrow R_2 \cup \{0\}$ with $\rho(0) = 0$ that is continuous at 0, then $\hookrightarrow_{R_1}^i$ (or $\equiv_{R_1}^i, \cong_{R_1}^i$) is Borel reducible to $\hookrightarrow_{R_2}^i$ (or $\equiv_{R_2}^i, \cong_{R_2}^i$, respectively). Consequently, if there is an order preserving bijection $\rho : R_1 \cup \{0\} \rightarrow R_2 \cup \{0\}$ with $\rho(0) = 0$ that is both open and continuous at 0, then the corresponding relations for the two classes are Borel bireducible.*

Proof. Using the order preserving injection ρ we can naturally turn any Polish R_1 -ultrametric space into a Polish R_2 -ultrametric space. This procedure preserves all the relations of interest. \square

For distance sets with 0 as a limit point we can now completely determine the complexity of the relations.

Theorem 8.3. *If R is a countable subset of \mathbb{R}_+ with $0 \in R'$, then \hookrightarrow_R^i is a complete analytic quasi-order, \equiv_R^i is a complete analytic equivalence relation, and \cong_R^i is S_∞ -universal.*

Proof. The duality theorem (Theorem 6.13) gives the upper bounds, and the preceding theorem gives the lower bounds via Louveau–Rosendal theorem and the Friedman–Stanley theorem. \square

For the case $0 \notin R'$ we note the following results.

Theorem 8.4. *If R is a finite subset of \mathbb{R}_+ , then $\omega^{<R}$ is a full infinite splitting tree of height $|R|$. Consequently, $\hookrightarrow_R^i, \equiv_R^i$ and \cong_R^i are Borel bireducible with the corresponding relations for countable trees of height $|R|$.*

The case of $0 \notin R'$ but R being infinite has been briefly discussed in [5, Section 8C]. The same argument there gives the following lower bound for \cong_R^i .

Theorem 8.5. *Let R be a countably subset of \mathbb{R}_+ with $0 \in R'$. Then the isomorphism for all countable trees with countably many branches is Borel reducible to \cong_R^i .*

As discussed in [5], the isomorphism for all countable trees with countably many branches is known to be absolutely Δ_2^1 bireducible with countable graph isomorphism, but is not known to be Borel bireducible with countable graph isomorphism. Thus in the case of $0 \in R'$ the exact complexity of the relations has not been determined.

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