

COUNTABLE ABELIAN GROUP ACTIONS AND HYPERFINITE EQUIVALENCE RELATIONS

SU GAO AND STEVE JACKSON

ABSTRACT. An equivalence relation E on a standard Borel space is hyperfinite if E is the increasing union of countably many Borel equivalence relations E_n where all E_n -equivalence classes are finite. In this article we establish the following theorem: if a countable abelian group acts on a standard Borel space in a Borel manner then the orbit equivalence relation is hyperfinite. The proof uses constructions and analysis of Borel marker sets and regions in the space $2^{\mathbb{Z}^{<\omega}}$. This technique is also applied to a problem of finding Borel chromatic numbers for invariant Borel subspaces of $2^{\mathbb{Z}^n}$.

CONTENTS

1. Preliminaries	2
2. Clopen Marker Sets	5
3. Regular Marker Regions	12
4. An Application of Regular Marker Regions	20
5. Orthogonal Marker Regions	28
6. Hyperfiniteness of $F(\mathbb{Z}^{<\omega})$	33
7. The Non-Free Part	37
8. Actions of Countable Abelian Groups	45
9. Continuous Embeddings into E_0	47
10. Open Problems and Further Remarks	53
References	54

This paper is a contribution to the study of countable Borel equivalence relations on Polish spaces. A topological space X is *Polish* if it is separable and completely metrizable. An equivalence relation E on X is *Borel* if E is a Borel subset of $X \times X$; it is *countable* if each E -equivalence class is countable. By a well known theorem of Feldman-Moore [5] any countable Borel equivalence relation E on a Polish space X is the orbit equivalence relation of a Borel action of a countable group G .

Among all countable Borel equivalence relations the class of hyperfinite equivalence relations is best understood. An equivalence relation E is *hyperfinite* if $E = \bigcup_n E_n$ where $E_n \subseteq E_{n+1}$ and each E_n is a Borel equivalence relation with all equivalence classes finite. A canonical example of hyperfinite equivalence relation

Date: October 12, 2009.

The first author acknowledges United States NSF grant DMS-0501039 for the support of his research.

is the relation E_0 defined on the Cantor space $2^{\mathbb{N}}$ as follows:

$$xE_0y \iff \exists m \forall n \geq m (x(n) = y(n)).$$

It is also well known (cf. [4]) that any hyperfinite equivalence relation is the orbit equivalence relation of a Borel action of \mathbb{Z} . In fact, any Borel \mathbb{Z} -action gives rise to a hyperfinite equivalence relation. Other characterizations of hyperfinite equivalence relations and complete abstract classifications of hyperfinite equivalence relations can also be found in [4].

However, the following has been an outstanding open problem since 1980s: For which countable group G do we have that any Borel G -action gives rise to only hyperfinite equivalence relations? Based on results in measure theory context ([3] and [10]) Weiss [12] asked if this is true for any amenable group. He also proved it for \mathbb{Z}^n . A more general result by Jackson-Kechris-Louveau [7] shows this for any finitely generated group with polynomial growth (by a famous theorem of Gromov this is exactly the class of finitely generated groups that are nilpotent-by-finite).

The main theorem of this paper is to prove that a Borel action of any countable abelian group gives rise to a hyperfinite equivalence relation. To achieve this it suffices to show the hyperfiniteness of a specific orbit equivalence relation, that is, the shift action of $\mathbb{Z}^{<\omega}$ on the space of its subsets $2^{\mathbb{Z}^{<\omega}}$. Much of the effort is to reprove Weiss' result that the shift action of \mathbb{Z}^n on $2^{\mathbb{Z}^n}$ gives rise to a hyperfinite equivalence relation. In the new proof a theory of Borel marker sets and regular Borel marker regions is developed. This allows us to prove stronger results such as that there is a continuous (even computable) reduction from the shift action of \mathbb{Z}^n into E_0 . We also present another application of this theory to a problem on Borel chromatic numbers.

1. PRELIMINARIES

In this first section we recall some general definitions and fix the notation to be used throughout the paper. More ad hoc definitions and notation will be introduced in later sections.

If E is an equivalence relation on the Polish space X , and F an equivalence relation on the Polish space Y , then we say E is *reducible* to F , written $E \leq F$, if there is a Borel function $F: X \rightarrow Y$ such that

$$\forall x, y \in X (xEy \leftrightarrow f(x)Ff(y)).$$

Thus, f gives an injection from the quotient space X/E into Y/F . We say E is *embeddable* into F , written $E \sqsubseteq F$, if there is a one-to-one $f: X \rightarrow Y$ which reduces E to F . The notion of reduction is fundamental in the theory of definable equivalence relations.

A particular case of interest is when the equivalence relation is generated by a Polish group action. Let G be a Polish group, X a Polish space and $a: G \times X \rightarrow X$ a continuous action of G on X . In this context X is called a *Polish G -space*. We use the abbreviation $g \cdot x$ to denote $a(g, x)$ if there is no danger of confusion.

The action is *free* if for any $g \in G$ with $g \neq 1_G$ and any $x \in X$, $g \cdot x \neq x$. For any Polish G -space X the *free part* of the action is the subset of X defined by

$$F_G(X) = \{x \in X : \forall g \in G (g \neq 1_G \rightarrow g \cdot x \neq x)\}.$$

In general $F_G(X)$ is an invariant $\mathbf{\Pi}_1^1$ subset of X . In case G is countable (and the action is continuous) $F_G(X)$ is an invariant G_δ subset of X , and therefore $F_G(X)$ naturally becomes a Polish G -space.

If X is a Polish G -space, the *orbit equivalence relation*, denoted by E_G^X , is defined as follows: for $x, y \in X$,

$$xE_G^X y \iff \exists g \in G (g \cdot x = y).$$

In case the action is free, E_G^X is a Borel equivalence relation (as a subset of $X \times X$). Alternatively, if G is countable (and the action is continuous), the orbit equivalence relation E_G^X is F_σ . If Y is a subset of X , then we denote the equivalence relation $E_G^X \upharpoonright (Y \times Y)$ simply by $E_G^X \upharpoonright Y$.

If X is a Polish G -space and $x \in X$, we denote the *orbit* or the *equivalence class* of x by

$$[x]_G = \{g \cdot x : g \in G\}.$$

In general, if E is an equivalence relation on X and $x \in X$, we let $[x]_E$ denote the E -equivalence class of x :

$$[x]_E = \{y \in X : yEx\}.$$

In case there is no danger of confusion we will omit all the subscripts.

Throughout the paper we will consider only countable groups G , which are Polish groups with the discrete topology. In fact, we will mostly consider a specific Polish G -space. Let 2^G denote the power set of G , or equivalently the product space $\prod_{g \in G} \{0, 1\}$ with the product topology, where $2 = \{0, 1\}$ is equipped with the discrete topology. 2^G is a Polish space. If $D \subseteq G$ is a finite subset then a function $s : D \rightarrow 2$ determines a basic (cl)open set

$$N_s = \{x \in 2^G : \forall g \in D (x(g) = s(g))\}.$$

For $x \in N_s$ we also write $s \subseteq x$. We also denote the set $\{s : D \rightarrow 2 : D \subseteq G \text{ finite}\}$ by $2^{<G}$. For $s, t \in 2^{<G}$, we say that s and t are *incompatible*, and denote by $s \perp t$, if there is $g \in \text{dom}(s) \cap \text{dom}(t)$ with $s(g) \neq t(g)$; otherwise s and t are said to be *compatible*. We write $s \subseteq t$ and say that t *extends* s if s and t are compatible and $\text{dom}(s) \subseteq \text{dom}(t)$.

Consider the natural shift action of G on 2^G : for $A \subseteq G$ and $g \in G$,

$$g \cdot A = \{gh : h \in A\}.$$

This action is continuous. For convenience we will denote the orbit equivalence relation $E_G^{2^G}$ simply by $E(G)$. We also denote by $F(G)$ the free part of this action as well as the equivalence relation $E(G) \upharpoonright F(G)$.

We will also work with the natural shift action of G on $2^{<G}$: if $g \in G$ and $s : D \rightarrow 2$ where $D \subseteq G$ is finite, then $g \cdot s : g \cdot D \rightarrow 2$ and for $h \in g \cdot D$,

$$(g \cdot s)(h) = s(g^{-1}h).$$

Most of the time we work with groups $G = \mathbb{Z}^n$ for some positive integer n . They will be expressed as additive groups as usual. When n is fixed we denote by e_1, e_2, \dots, e_n the group elements

$$\begin{aligned} e_1 &= (1, 0, \dots, 0), \\ e_2 &= (0, 1, \dots, 0), \\ &\dots\dots\dots \\ e_n &= (0, 0, \dots, 1). \end{aligned}$$

We will refer to them as *the generators* of \mathbb{Z}^n . Any $g \in \mathbb{Z}^n$ can be uniquely expressed as (g_1, g_2, \dots, g_n) or $g_1e_1 + g_2e_2 + \dots + g_n e_n$ for $g_1, g_2, \dots, g_n \in \mathbb{Z}$. We define a *norm* on \mathbb{Z}^n by

$$\|g\| = \|(g_1, g_2, \dots, g_n)\| = \max\{|g_1|, |g_2|, \dots, |g_n|\}.$$

The norm induces a *distance* function on \mathbb{Z}^n : for $g, h \in \mathbb{Z}^n$,

$$\rho(g, h) = \|g - h\|.$$

When $G = \mathbb{Z}^n$ acts on $X = F(G) = F(\mathbb{Z}^n)$ the above distance function induces further a distance function on X as follows:

$$\rho_X(x, y) = \begin{cases} \|g\|, & \text{if } g \cdot x = y, \\ \infty, & \text{if } (x, y) \notin E(G). \end{cases}$$

Note that this is well defined since the action is free. When there is no danger of confusion we will drop the subscript X in denoting ρ_X . When $M \subseteq X$ and $x \in X$, we also define, as usual,

$$\rho(x, M) = \inf\{\rho(x, y) : y \in M\}.$$

This distance function is not be confused with the genuine metric d on $2^{\mathbb{Z}^n}$ defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-\min\{\|g\| : x(g) \neq y(g)\}}, & \text{otherwise.} \end{cases}$$

Note that d is in fact an ultrametric, i.e.,

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}$$

for any $x, y, z \in 2^{\mathbb{Z}^n}$.

The set of natural numbers will be denoted as ω . We also use the interval notation: if $a, b \in \mathbb{Z}$ and $a < b$, then $[a, b]$ denotes the set $\{c \in \mathbb{Z} : a \leq c \leq b\}$.

Throughout the paper we also fix a linear order $<_n$ on \mathbb{Z}^n so that $\|g\| < \|h\|$ implies $g <_n h$. For instance, elements of \mathbb{Z} can be enumerated according to $<_1$ as

$$0, -1, 1, -2, 2, \dots, -k, k, \dots$$

The equivalence relation E_0 on 2^ω defined previously is the prototypical hyperfinite equivalence relation. All of the proofs of hyperfiniteness in this paper will proceed by showing that the equivalence relation E reduces to (or embeds into) E_0 . In this regard we note the simple fact that it does not matter if we use $E_0 = E_0(2^{\mathbb{N}})$ on $2^{\mathbb{N}}$ or the variation $E_0(\omega^\omega)$ on ω^ω , defined as the eventual equality relation on sequences of natural numbers. We state this in the following proposition.

Proposition 1.1. *There is a continuous (in fact computable or recursive) embedding from $E_0(\omega^\omega)$ to $E_0(2^{\mathbb{N}})$.*

Proof. Let $\pi: \omega \rightarrow \omega \times \omega$ be a computable bijection between ω and $\omega \times \omega$. Let $\pi(n) = (\pi_0(n), \pi_1(n))$. For every $k \in \omega$, let $((k)_0, (k)_1, \dots)$ be the base-2 expansion of k , that is, $k = \sum_i (k)_i \cdot 2^i$ (so for large enough i , $(k)_i = 0$). Define $f: \omega^\omega \rightarrow 2^{\mathbb{N}}$ by: $f(x)(n) = (x(\pi_0(n)))_{\pi_1(n)}$. f is easily an embedding, and is computable. \square

2. CLOPEN MARKER SETS

Throughout this section we fix a positive integer n .

Lemma 2.1 (Basic Clopen Marker Lemma). *Let d be a positive integer. Then there is a clopen set $S \subseteq F(\mathbb{Z}^n)$ such that*

- (i) *if $x, y \in S$ are distinct, then $\rho(x, y) > d$;*
- (ii) *for any $x \in F(\mathbb{Z}^n)$, $\rho(x, S) \leq d$.*

Proof. Let s_0, s_1, \dots enumerate all elements s of $2^{<\mathbb{Z}^n}$ such that $g \cdot N_s \cap N_s = \emptyset$ for all $g \in \mathbb{Z}^n$ with $0 < \|g\| \leq d$. Define a sequence of sets S_i by induction on i . For $i = 0$ let $S_0 = N_{s_0}$. For $i > 0$ let

$$S_i = S_{i-1} \cup \left(N_{s_i} - \bigcup_{\|g\| \leq d} g \cdot S_{i-1} \right).$$

Finally let $S = \bigcup_i S_i \cap F(\mathbb{Z}^n)$.

By induction on i we show that if $x, y \in S_i$ are distinct, then $\rho(x, y) > d$. If $i = 0$ and $x, y \in S_0$ then $x, y \in N_{s_0}$. Suppose $\rho(x, y) \leq d$. In particular there is $g \in \mathbb{Z}^n$ with $g \cdot x = y$ and $\|g\| \leq d$. This implies that $g \cdot N_{s_0} \cap N_{s_0} \neq \emptyset$, contradicting our choice of s_0 . Thus we have $\rho(x, y) > d$. Now suppose $i > 0$ and $x, y \in S_i$. A similar argument as above shows that x and y cannot be both in N_{s_i} . By induction we may also assume that x and y are not both in S_{i-1} . Thus without loss of generality we may assume that $x \in S_{i-1} - N_{s_i}$ and $y \in N_{s_i} - S_{i-1}$. By the definition of S_i we get that $x \in S_{i-1}$. Assume $\rho(x, y) \leq d$. In particular there is $g \in \mathbb{Z}^n$ with $g \cdot x = y$ and $\|g\| \leq d$. It follows that $y \in \bigcup_{\|g\| \leq d} g \cdot S_{i-1}$, contradicting $y \in S_i$. Thus again we have $\rho(x, y) > d$.

To verify clause (ii) note that for any $x \in F(\mathbb{Z}^n)$ there is s_i with $x \in N_{s_i}$. In fact, if no such s_i exists then there is $g \in \mathbb{Z}^n$ with $0 < \|g\| \leq d$ such that $g \cdot N_s \cap N_s \neq \emptyset$ for all $s \subseteq x$. The continuity of the action then implies that $g \cdot x = x$, contradicting the assumption that the action is free. Now let $x \in F(\mathbb{Z}^n)$ and let i be the smallest such that $x \in N_{s_i}$. If $x \in S_i$ then $x \in S$ and $\rho(x, S) = 0$, therefore we are done. Otherwise, assume $x \notin S_i$. By the definition of S_i it follows that $x \in \bigcup_{\|g\| \leq d} g \cdot S_{i-1}$. This is to say that there are $g \in \mathbb{Z}^n$ with $\|g\| \leq d$ and $y \in S_{i-1}$ such that $g \cdot y = x$, and thus $\rho(x, S) \leq \rho(x, y) \leq d$.

Finally we check that S is clopen in $F(\mathbb{Z}^n)$. By induction on i it is easy to see that each S_i is clopen. Thus it follows immediately from the definition that S is open in $F(\mathbb{Z}^n)$. The argument in the preceding paragraph gives that if $x \in F(\mathbb{Z}^n) - S$ then $x \in \bigcup_{0 < \|g\| \leq d} g \cdot S$. Conversely if $x \in \bigcup_{0 < \|g\| \leq d} g \cdot S$ then by (i) we have that $s \notin S$. Thus $x \in F(\mathbb{Z}^n) - S$ iff $x \in \bigcup_{0 < \|g\| \leq d} g \cdot S$. Since S is open and the action is continuous, this shows that $F(\mathbb{Z}^n) - S$ is open. \square

Remark 2.1. (1) In the above proof the set $S' = \bigcup_i S_i$ is in fact an open set in $2^{\mathbb{Z}^n}$ satisfying both clauses (i) and (ii). A closer scrutiny shows that S' is Σ_1^0 and S is Δ_1^0 in $F(\mathbb{Z}^n)$. In [2] it was already shown that there is a continuous reduction from $E(\mathbb{Z})$ to E_0 on 2^ω . Enhanced by these computability considerations one can show that there is a computable reduction from $E(\mathbb{Z})$ to E_0 . Later in Section 9 we generalize this to $E(\mathbb{Z}^n)$ for all finite $n \geq 1$.

- (2) Clause (ii) implies that S is in fact a *complete section* for the action of \mathbb{Z}^n on $F(\mathbb{Z}^n)$, that is, S meets every equivalence class (i.e., for any $x \in F(\mathbb{Z}^n)$, $S \cap [x] \neq \emptyset$). Elements of S are customarily called *markers* or *marker points*. The set S itself is called a *marker set*.
- (3) In the remainder of this article we will refer to the properties (i) and (ii) as *basic marker properties*. The integer d will be referred to as a *marker distance* and the set S the *basic marker set* for the marker distance d .

Marker arguments started from Slaman-Steel's proof of hyperfiniteness of $E(\mathbb{Z})$ in [11]. In this proof an infinite descending and vanishing sequence of Borel marker sets was constructed and then used to show that $E(\mathbb{Z})$ is an increasing union of finite Borel equivalence relations. In our context if $F(\mathbb{Z})$ admits a sequence

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_k \supseteq \cdots$$

of descending *clopen* marker sets so that $\bigcap_k S_k = \emptyset$, then the Slaman-Steel argument would give another continuous reduction from $F(\mathbb{Z})$ to E_0 (different from the one constructed in [2]). Thus it is a natural question whether such a sequence of clopen marker sets exist.

We first note that basic clopen marker sets can always be thinned down by specifying a sufficiently larger marker distance.

Lemma 2.2. *Let d be a positive integer and S_0 be a basic clopen marker set in $F(\mathbb{Z}^n)$ for the marker distance d . Let $D > 2d$ be a positive integer. Then there is a clopen set $S_1 \subseteq S_0$ such that*

- (i) *if $x, y \in S_1$ are distinct, then $\rho(x, y) > D - 2d$;*
- (ii) *for any $x \in F(\mathbb{Z}^n)$, $\rho(x, S_1) \leq D + d$.*

Proof. Let S' be a basic clopen marker set for the marker distance D . Let $<$ be a linear order of the (finite) set $B = \{g \in \mathbb{Z}^n : \|g\| \leq d\}$. Then for $x \in S_0$ let $x \in S_1$ iff there is $g \in B$ such that $g \cdot x \in S'$ and that for all $h \in B$ with $h < g$, $h^{-1}g \cdot x \notin S_0$. It is clear from the definition and the continuity of the action that S_1 is clopen. It is useful to note that for each $z \in S'$ there is exactly one $x \in S_1$ with $\rho(x, z) \leq d$. In fact, by the basic marker properties of S_0 , for each $z \in S'$ there is at least one $x \in S_0$ with $\rho(x, z) \leq d$, and thus the set

$$\{g \in B : g^{-1} \cdot z \in S_0\}$$

is nonempty. Let g be the $<$ -least element of this set and let $x = g^{-1} \cdot z$. Then for all $h \in B$ with $h < g$, $h^{-1}g \cdot x = h^{-1} \cdot z \notin S_0$. Therefore x is the unique element of S_1 with $\rho(x, z) \leq d$. Conversely, for any $x \in S_1$ there is also exactly one $z \in S'$ with $\rho(x, z) \leq d$, by a similar proof.

Now the conditions (i) and (ii) are easy to check. Let $x, y \in S_1$ be distinct and let $u, v \in S'$ be such that $\rho(x, u), \rho(y, v) \leq d$. Then by the basic marker properties of S' , $\rho(u, v) > D$. Thus $\rho(x, y) \geq \rho(u, v) - \rho(x, u) - \rho(y, v) > D - 2d$. On the other hand, let $x \in F(\mathbb{Z}^n)$ and $z \in S'$ be such that $\rho(x, z) \leq D$. Let $y \in S_1$ be such that $\rho(y, z) \leq d$. Then $\rho(x, S_1) \leq \rho(x, y) \leq \rho(x, z) + \rho(y, z) \leq D + d$. \square

Note that we cannot guarantee that S_1 has the exact basic marker properties for any distance. However, when $D \gg d$, the terms $D - 2d$ and $D + d$ are both fairly close to D . For practical purposes these approximate marker properties are good enough for our arguments. For notational convenience we will use the following

terminology. Let d be a positive integer and $0 < \epsilon < 1$ be a real number. A marker set $S \subseteq F(\mathbb{Z}^n)$ is said to satisfy the (d, ϵ) -marker properties if

- (i) if $x, y \in S$ are distinct, then $\rho(x, y) > (1 - \epsilon)d$;
- (ii) for any $x \in F(\mathbb{Z}^n)$, $\rho(x, S) \leq (1 + \epsilon)d$.

This will allow us to iterate the argument in the above proof and generate an infinite descending sequence of clopen marker sets with arbitrarily prescribed approximate marker properties.

Lemma 2.3. *Let d be a positive integer, $0 < \epsilon < 1$ a real number and S_0 a clopen marker set in $F(\mathbb{Z}^n)$ satisfying the (d, ϵ) -marker properties. Let $0 < \delta < 1$ be a real number. Then there is a positive integer D and a clopen set $S_1 \subseteq S_0$ satisfying the (D, δ) -marker properties.*

Proof. It is easy to see if we choose D so that $\delta D > 2d(1 + \epsilon)$ and S' to be a basic clopen marker set for the marker distance D , then the proof of the preceding lemma gives the desired $S_1 \subseteq S_0$ with appropriate approximate marker properties. \square

In this lemma it is obvious that S_1 will be a proper subset of S_0 if D is sufficiently large. Better yet, we can guarantee that S_1 is sparse in S_0 (on each equivalence class under an appropriately defined density notion). By iterating the lemma we can thus create an infinite descending sequence of clopen marker sets each of which is sparse in the previous set. If S is the intersection of all these clopen marker sets then for each $x \in F(\mathbb{Z}^n)$ either $S \cap [x] = \emptyset$ or $S \cap [x]$ is a singleton. We have thus proved the following fact.

Proposition 2.4. *There exists an infinite descending sequence of clopen complete sections of $F(\mathbb{Z}^n)$*

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_k \supseteq \cdots$$

such that for each $x \in F(\mathbb{Z}^n)$, $|\bigcap_k S_k \cap [x]| \leq 1$.

Note that $\bigcap_k S_k \cap [x] = \emptyset$ must hold for some $x \in F(\mathbb{Z}^n)$, since the equivalence relation $F(\mathbb{Z}^n)$ is non-smooth. The following theorem shows that $\bigcap_k S_k \cap [x] = \emptyset$ cannot hold for every $x \in F(\mathbb{Z}^n)$, in other words, $\bigcap_k S_k \cap [x] \neq \emptyset$ must also hold for some $x \in F(\mathbb{Z}^n)$. This gives a negative answer to the question following Remark 2.1, and hence a continuous reduction from $F(\mathbb{Z})$ to E_0 has to come from an argument different than Slaman-Steel's. A weaker version of the theorem, with a different proof, was proved earlier as joint work with Ben Miller.

Theorem 2.5. *Let $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_k \supseteq \cdots$ be an infinite descending sequence of closed complete sections of $F(\mathbb{Z}^n)$. Then $\bigcap_k S_k \neq \emptyset$.*

We need some combinatorial lemmas in the proof.

Lemma 2.6. *Let $s \in 2^{<\mathbb{Z}^n}$ and $g \in \mathbb{Z}^n$ with $\|g\| > 0$. Then there is $t \supseteq s$ such that $g \cdot t \perp t$.*

Proof. To avoid triviality assume $\text{dom}(s) \neq \emptyset$. Since $\text{dom}(s)$ is finite there is $h \in \text{dom}(s)$ so that $g + h \notin \text{dom}(s)$. Define t to be an extension of s so that $t(g + h) = 1 - s(h)$. Then $(g \cdot t)(g + h) = t(h) = s(h) \neq 1 - s(h) = t(g + h)$. \square

Lemma 2.7. *Let $s \in 2^{<\mathbb{Z}^n}$ and $l > 0$. Then there is $t \supseteq s$ such that for any $g \in \mathbb{Z}^n$ with $0 < \|g\| \leq l$, $g \cdot t \perp t$.*

Proof. Iteratively apply Lemma 2.6 for all g with $0 < \|g\| \leq l$. Note that if $g \cdot t \perp t$ then for any extension t' of t , $g \cdot t' \perp t'$. \square

For $s \in 2^{<\mathbb{Z}^n}$ we define \bar{s} to be

$$\bar{s}(g) = \begin{cases} 1 - s(g), & \text{if } g \in \text{dom}(s), \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We also use the following notation. For $N \in \omega$ and $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$, we write $N \mid g$ if for all $1 \leq j \leq n$, $N \mid g_j$. We say that $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$ is *positive* if for all $1 \leq j \leq n$, $g_j \geq 0$ and $\|g\| > 0$.

Lemma 2.8. *Let $N \in \omega$, $s \in 2^{[0, N-1]^n}$ and $g \in \mathbb{Z}^n$ with $\|g\| > 0$. Then there is $m > 1$ and $t \in 2^{[0, mN-1]^n}$ such that $g \cdot t \perp t$ and for any positive h with $N \mid h$ and $\|h\| \leq (m-1)N$, $h^{-1} \cdot t$ extends either s or \bar{s} .*

s	s	\bar{s}
\bar{s}	s	s
s	\bar{s}	s

t

FIGURE 1. An example of $t \supseteq s$.

Proof. Let $m > 1$ be large enough so that $\|g\| < mN$ and there are $h_0, h_1 \in [0, mN-1]^n$ with $g + h_0 = h_1$ but at least one of h_0, h_1 does not belong to $\text{dom}(s)$. We consider three cases.

Case 1: $h_0 \in \text{dom}(s)$. Similar to Lemma 2.6 we would like to define the extension t so that $t(h_1) = t(g + h_0) \neq t(h_0) = s(h_0)$. This guarantees that $g \cdot t \perp t$. For this we let $h' \in \text{dom}(s)$ be such that $N \mid (h_1 - h')$. Note that there is a unique such h' . If $s(h_0) \neq s(h')$ we let $(h' - h_1) \cdot t$ extend s ; otherwise we let it extend \bar{s} . For other positive h with $N \mid h$ and $\|h\| \leq (m-1)N$ we let $h^{-1} \cdot t$ extend s or \bar{s} in an arbitrary way. This finishes the definition of t and it is as required.

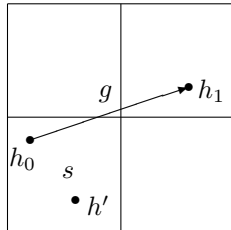


FIGURE 2. Case 1 of Lemma 2.8.

Case 2: $h_1 \in \text{dom}(s)$. Define t by following the construction of Case 1 except replacing g by g^{-1} .

Case 3: Neither $h_0 \in \text{dom}(s)$ nor $h_1 \in \text{dom}(s)$. Let $h'_0 \in \text{dom}(s)$ be such that $N \mid (h_0 - h'_0)$ and $h'_1 \in \text{dom}(s)$ be such that $N \mid (h_1 - h'_1)$. If $s(h'_0) \neq s(h'_1)$ define t so that both $(h'_0 - h_0) \cdot t$ and $((h'_1 - h_1) \cdot t$ extend s ; otherwise define t so that one of them extends s and the other extends \bar{s} . \square

The intuitive meaning of the last condition in the above lemma is that t is constructed by a tiling of s and its flip \bar{s} . In the rest of the proof all extensions are constructed this way.

Lemma 2.9. *Let $N \in \omega$, $s \in 2^{[0, N-1]^n}$ and $l > 0$. Then there is $m > 1$ and $t \in 2^{[0, mN-1]^n}$ such that*

- (i) *for any g with $0 < \|g\| \leq l$, $g \cdot t \perp t$;*
- (ii) *for any positive h with $N \mid h$ and $\|h\| \leq (m-1)N$, $h^{-1} \cdot t$ extends either s or \bar{s} .*

Proof. Again apply Lemma 2.8 repeatedly to handle all g with $0 < \|g\| \leq l$. \square

Lemma 2.10. *There is a sequence $x_0, x_1, \dots, x_k, \dots$ in $F(\mathbb{Z}^n)$ and a function $\gamma : \omega \rightarrow \omega$ with $\gamma(k) > k$ for all k such that*

$$\forall k \forall z \in [x_k] \forall g \in \mathbb{Z}^n \left(0 < \|g\| \leq k+1 \longrightarrow d(g \cdot z, z) \geq 2^{-\gamma(\|g\|-1)+1} \right).$$

Proof. By induction on k we define $x_k \in F(\mathbb{Z}^n)$, $\gamma(k) > k$ and $s_k \in 2^{D_k}$ where $D_k = [0, \gamma(k) - 1]^n$. For $k = 0$ let $\gamma(0) = 3$ and s_0 be given by

$$s_0(g) = \begin{cases} 1, & \text{if } g = (1, 1, \dots, 1), \\ 0, & \text{if } g \in [0, 2]^n \text{ but } g \neq (1, 1, \dots, 1). \end{cases}$$

It is straightforward to check that for any g with $\|g\| = 1$, $g \cdot s_0 \perp s_0$. Applying Lemma 2.9 inductively for all $l > 0$, we may obtain a sequence of extensions of s_0

$$s_0 \subseteq t_0 \subseteq t_1 \subseteq \dots \subseteq t_l \subseteq \dots$$

Let $x_0 = \bigcup_l t_l$. Then Lemma 2.9 (i) implies that $x_0 \in F(\mathbb{Z}^n)$. Now let $z \in [x_0]$, then by Lemma 2.9 (ii), there is h with $\|h\| \leq 1$ such that $h \cdot z$ extends s_0 or \bar{s}_0 . In either case, for any g with $\|g\| = 1$, there is some h' with $\|h'\| \leq 2$ such that $(g \cdot z)(h') \neq z(h')$. Hence we get that $d(g \cdot z, z) \geq 2^{-\gamma(0)+1}$ as required.

Now suppose $k \geq 0$ and assume that x_k , s_k and $\gamma(k) > k$ have been defined. Let $m > 1$ and $s_{k+1} \supseteq s_k$ be obtained by applying Lemma 2.9 to s_k and $k+2$. In particular for any g with $0 < \|g\| \leq k+2$, $g \cdot s_{k+1} \perp s_{k+1}$. Let $\gamma(k+1) = m\gamma(k)$. Then $\gamma(k+1) > k+1$. To define x_{k+1} , apply Lemma 2.9 inductively for all $l > 0$ to obtain a sequence of extensions of s_{k+1}

$$s_{k+1} \subseteq t_0^{(k+1)} \subseteq \dots \subseteq t_l^{(k+1)} \subseteq \dots$$

and let $x_{k+1} = \bigcup_l t_l^{(k+1)}$. Again Lemma 2.9 (i) implies that $x_{k+1} \in F(\mathbb{Z}^n)$. Now suppose $z \in [x_{k+1}]$ and $0 < \|g\| \leq k+2$. Consider two cases. The first is $0 < \|g\| \leq k+1$. By inductive hypothesis we have that $g \cdot s_k \perp s_k$. Moreover, note that s_{k+1} is obtained by a tiling of s_k and \bar{s}_k , and that x_{k+1} is obtained by a tiling of s_{k+1} and \bar{s}_{k+1} . As a result x_{k+1} is in fact obtained by a tiling of s_k and \bar{s}_k . By inductive hypothesis this implies that $d(g \cdot z, z) \geq 2^{-\gamma(\|g\|-1)+1}$. Finally, for the case $\|g\| = k+2$, a similar argument as that for x_0 shows that there is h' with

$\|h'\| \leq \gamma(k+1) - 1$ with $(g \cdot z)(h') \neq z(h')$, and thus $d(g \cdot z, z) \geq 2^{-\gamma(k+1)+1} = 2^{-\gamma(\|g\|-1)+1}$. \square

Proof of Theorem 2.5. Let $x_0, x_1, \dots, x_k, \dots$ be obtained from Lemma 2.10. Since S_k are complete sections of $F(\mathbb{Z}^n)$ we can find $y_k \in S_k \cap [x_k]$ for all k . By compactness of $2^{\mathbb{Z}^n}$ the set $\{y_0, y_1, \dots, y_k, \dots\}$ has an accumulation point x in $2^{\mathbb{Z}^n}$. We claim that $x \in F(\mathbb{Z}^n)$. Toward a contradiction assume $g \cdot x = x$ for some $\|g\| > 0$. Let $k \geq \|g\|$ be large enough so that

$$d(y_k, x) < 2^{-2\gamma(\|g\|-1)+1}.$$

By Lemma 2.10 we have that

$$d(g \cdot y_k, y_k) \geq 2^{-\gamma(\|g\|-1)+1}.$$

Since d is an ultrametric it follows that

$$d(g \cdot y_k, g \cdot x) = d(g \cdot y_k, x) \geq 2^{-\gamma(\|g\|-1)+1}.$$

However

$$d(g \cdot y_k, g \cdot x) \leq 2^{\|g\|} d(y_k, x) \leq 2^{\gamma(\|g\|-1)} d(y_k, x) < 2^{-\gamma(\|g\|-1)+1},$$

a contradiction. Thus we have shown that $x \in F(\mathbb{Z}^n)$.

Now for any k the set $\{y_k, y_{k+1}, y_{k+2}, \dots\} \subseteq S_k$ still has x as an accumulation point. Since S_k is closed in $F(\mathbb{Z}^n)$ we conclude that $x \in S_k$. This shows that $x \in \bigcap_k S_k$. \square

Recently, the authors and B. Seward [6] have extended theorem 2.5 to arbitrary countable groups. Recall $F(G)$ denotes the free part of 2^G under the shift action of G .

Theorem 2.11. *Let G be a countable group. Let $S_0 \supseteq S_1 \supseteq \dots \supseteq S_k \supseteq \dots$ be an infinite descending sequence of closed complete sections of $F(G)$. Then $\bigcap_k S_k \neq \emptyset$.*

The proof of theorem 2.11 involves identifying and establishing a certain *coloring property* for arbitrary countable groups. The coloring property has its origins in the proof of theorem 2.5. Specifically, we say a countable group G is *2-colorable* if there is a map $c: G \rightarrow \{0, 1\}$ such that for any $s \neq 1_G$ in G , there is a finite $T \subseteq G$ such that $\forall g \in G \exists t \in T (c(gt) \neq c(gst))$. In [6] it is shown that every countable group G has a 2-coloring, and that the existence of a 2-coloring is equivalent to the statement of theorem 2.11.

A classical tool for showing negative results is the ergodicity of measures. In fact, for any countably infinite group G , the product measure μ on 2^G is a G -invariant, non-atomic, ergodic, Borel probability measure. This implies that $E(G)$ is not smooth. The free part $F(G)$ is a G -invariant Borel subset with full measure. Thus in particular the equivalence relation $F(G)$ is non-smooth. Besides smoothness, the technique can also rule out the existence of marker sets with an extremely regular pattern, as the following proposition illustrates.

Proposition 2.12. *There is no Borel marker set $M \subseteq F(\mathbb{Z})$ such that for any $x \in F(\mathbb{Z})$ there is some $d > 1$ such that for all $y, z \in [x] \cap M$, $\rho(y, z)$ is a multiple of d .*

Proof. Let μ be the product measure on $2^{\mathbb{Z}}$. Then μ is a \mathbb{Z} -invariant, non-atomic, ergodic, Borel probability measure. Its restriction on $F(\mathbb{Z})$, which we still denote by μ , continues to be such a measure on $F(\mathbb{Z})$. Now assume M is a Borel marker set with the stated property. Then for each $x \in F(\mathbb{Z})$ let $d_x > 1$ be the largest integer such that $\rho(y, z)$ is a multiple of d_x for all $y, z \in M \cap [x]$. Let $X_d = \{x \in F(\mathbb{Z}) : d_x = d\}$ for each integer $d > 1$. Let also $X_1 = \{x \in F(\mathbb{Z}) : |[x] \cap M| = 1\}$. Then $\{X_d : d \geq 1\}$ form a countable partition of M into \mathbb{Z} -invariant Borel subsets. Since $M \cap X_1$ is a Borel transversal on X_1 , $\mu(X_1) = 0$. By ergodicity there is some $d > 1$ such that $\mu(X_d) = 1$.

Let $M_d = (d\mathbb{Z}) \cdot (M \cap X_d)$. Then M_d is a Borel complete section of $F(\mathbb{Z}) \upharpoonright X_d$ so that for any $x \in M_d$ and $y \in [x]$, $y \in M_d$ iff $\rho(x, y)$ is a multiple of d . It follows that $X_d = M_d \cup 1 \cdot M_d \cup \dots \cup (d-1) \cdot M_d$.

Now consider the action of $d\mathbb{Z}$ on X_d . Then $\mu \upharpoonright X_d$ is $d\mathbb{Z}$ -invariant, non-atomic, and ergodic. Moreover, the sets $M_d, \dots, (d-1) \cdot M_d$ are $d\mathbb{Z}$ -invariant Borel subsets with the same μ -measure. This is a contradiction since on the one hand each of them should have measure $1/d$, and on the other hand each should have measure either 0 or 1. \square

The following generalization can be established by a similar proof.

Proposition 2.13. *There is no Borel marker set $M \subseteq F(\mathbb{Z}^n)$ such that for any $x \in M$ there is a proper subgroup G of \mathbb{Z}^n such that $[x] \cap M \subseteq G \cdot x$.*

In contrast to this we note that it is possible to construct marker sets with a moderately regular pattern. This is made precise by the following result. The proof of the following lemma also provides a preview of a technique, called *big-marker-little-marker* argument, which will be important in later constructions.

Lemma 2.14. *For any integer $d > 0$ there is a clopen marker set $M_d \subseteq F(\mathbb{Z})$ such that*

- (i) *for all $z \in F(\mathbb{Z})$ there is $x \in M$ with $\rho(x, z) \leq d$, and*
- (ii) *for all $x, y \in M$, if $\rho(x, y) < 2d + 1$ then $\rho(x, y) \in \{d + 1, d + 2\}$.*

Proof. Let $D = (d + 1)^2$ and M_D be the clopen marker set obtained from applying the Basic Clopen Marker Lemma. For each $i \in [D + 1, 2D + 1]$ let

$$M_{D,i} = \{x \in M_D : i \cdot x \in M_D\}.$$

Then $\{M_{D,i} \mid i \in [D + 1, 2D + 1]\}$ form a partition of M_D into clopen sets. Now for each $i \in [D + 1, 2D + 1]$ let

$$i = q_i(d + 1) + r_i, \text{ where } q_i, r_i \in \omega \text{ and } 0 \leq r_i < d + 1.$$

Here $q_i \geq d$ since $i > D = (d + 1)^2$. Thus one can write

$$i = (q_i - r_i)(d + 1) + r_i(d + 2) = a_i(d + 1) + b_i(d + 2)$$

for some integer $a_i, b_i \geq 0$. Let

$$G_i = \{k(d + 1) \mid k \leq a_i\} \cup \{a_i(d + 1) + k'(d + 2) \mid k' < r_i\}.$$

Define

$$M_{d,i} = M_{D,i} \cup G_i \cdot M_{D,i},$$

and $M_d = \bigcup_i M_{d,i}$. Then it is easy to check that M_d is as required. \square

A generalization of this is the main theorem of the next section.

3. REGULAR MARKER REGIONS

Recall that in general a *marker set* is a Borel complete section of $F(\mathbb{Z}^n)$. In the preceding section we studied the existence of marker sets for $F(\mathbb{Z}^n)$ with various additional properties. The main new property considered is the clopenness. Marker sets constructed in earlier research have been successfully applied to show hyperfiniteness of equivalence relations. Slaman-Steel [11] constructed Borel marker sets for arbitrary countable Borel equivalence relations and used the marker sets for $F(\mathbb{Z})$ to show that $E(\mathbb{Z})$ is hyperfinite. Later several proofs of hyperfiniteness of $E(\mathbb{Z}^n)$ for $n > 1$ were found, but all of them are based on constructions of Borel marker sets for $F(\mathbb{Z}^n)$. In doing all these an important technique is to decompose the equivalence classes into finite *marker regions*. We recall two examples below.

Example 3.1. Let $M \subseteq F(\mathbb{Z})$ be a Borel marker set. For each $x \in F(\mathbb{Z})$ define

$$l_M(x) \simeq \begin{cases} (-n) \cdot x, & \text{if } n \in \omega \text{ is the least such that } (-n) \cdot x \in M, \\ \text{undefined,} & \text{if no such } n \text{ exists.} \end{cases}$$

Then define a subequivalence relation R_M of $F(\mathbb{Z})$ by

$$xR_My \iff [x] = [y] \text{ and } l_M(x) \simeq l_M(y).$$

(Recall that if f and g are partial functions, then $f(x) \simeq g(x)$ if both $f(x)$ and $g(x)$ are defined and $f(x) = g(x)$, or both $f(x)$ and $g(x)$ are undefined.) It can be easily seen that for each $x \in F(\mathbb{Z})$, the R_M -equivalence class of x is of the form $I \cdot x$ for an interval I in \mathbb{Z} (which is possibly infinite). For this reason we say that R_M gives a partition of all $F(\mathbb{Z})$ -equivalence classes into *marker intervals* (see Figure 3).



FIGURE 3. Marker intervals with marker points highlighted

Conversely, suppose R is a Borel subequivalence relation of $F(\mathbb{Z})$ so that for every $x \in F(\mathbb{Z})$, $[x]_R \neq [x]$ and $[x]_R = I \cdot x$ for some interval I in \mathbb{Z} . Then by taking all the left end points of the intervals I we recover a marker set M_R . It is also easy to see that the above procedures are inverses of each other, i.e., $R_{M_R} = R$ and $M_{R_M} = M$.

Example 3.2. Let $M \subseteq F(\mathbb{Z}^n)$ be a Borel marker set. For each $x \in F(\mathbb{Z}^n)$ define $c_M(x) = g \cdot x$, where g is the $<_n$ -least element of \mathbb{Z}^n such that $g \cdot x \in M$. Note that since M is a complete section of $F(\mathbb{Z}^n)$, c_M is always defined. Then we can again define a subequivalence relation R_M of $F(\mathbb{Z}^n)$ by

$$xR_My \iff c_M(x) = c_M(y).$$

When $n = 1$, each of the R_M -equivalence classes is still a marker interval; however they are different from the ones constructed in the preceding example (see Figure 4). When $n > 1$, the R_M -equivalence classes are called *Dirichlet regions*.

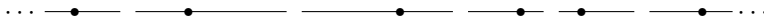


FIGURE 4. An alternative construction of marker intervals

The terminology comes from a geometric construction in \mathbb{R}^n , which we describe informally below. For this discussion, we use the standard Euclidean metric on \mathbb{R}^n and \mathbb{Z}^n (although we could use the sup norm we fixed earlier; the description is a little cleaner if we use the standard metric). Fix $x \in F(\mathbb{Z}^n)$. Write $M \cap [x] = J \cdot x$ for $J \subseteq \mathbb{Z}^n$. Regard J as a subset of \mathbb{R}^n . For each pair of distinct elements (p, q) of J find the perpendicular bisector (in the usual metric of \mathbb{R}^n) of the line segment pq , which is a hyperplane in \mathbb{R}^n . Then the Dirichlet regions in \mathbb{R}^n (see Figure 5) are polygonal regions with segments of these perpendicular bisectors as boundaries. They are characterized by the property that each region J_0 contains a unique element p_0 of J in its interior so that J_0 consists of points whose distance to p_0 is smaller than its distance to any other points in J .

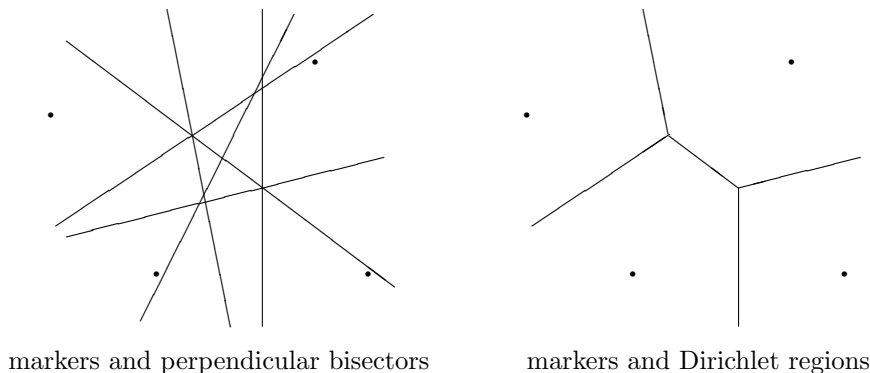


FIGURE 5. Two-dimensional Dirichlet regions

In particular let J_0 be the region containing the origin. Then $(J_0 \cap \mathbb{Z}^n) \cdot x$ is the Dirichlet region of x given by R_M . We remark that the construction is invariant. The precise meaning of this invariance is that, if we instead fix $x' = g \cdot x$ with $g \neq 0$, then the Dirichlet regions constructed in \mathbb{R}^n for x' is the result of applying a shift by g^{-1} to the Dirichlet regions constructed in \mathbb{R}^n for x . Thus by following the construction for x we will be able to recover the Dirichlet region for x . We also remark that the discussion in this paragraph only gives the main idea of the construction. The details are incomplete, because we did not address what to do if a lattice point in \mathbb{R}^n falls exactly on the boundary of a region. In this case the lattice point will be regarded as being contained in only one of the Dirichlet regions, and the conflict (among all the Dirichlet regions having this element on the boundary) is resolved according to the linear order $<_n$. We omit the precise details for now. We note that the arguments in this paper will not use Dirichlet regions, but regions which are “rectangles.” We discuss the meaning of this below.

These examples illustrate how the geometries of \mathbb{R}^n come into play in the study of hyperfiniteness of countable equivalence relations. In order to explore this connection more easily we fix some terminology to be used for the rest of the paper. In general by *marker regions* we mean the R -equivalence classes for some Borel subequivalence relation R of $F(\mathbb{Z}^n)$. Now fix an $x_0 \in F(\mathbb{Z}^n)$. From the obvious correspondence $[x_0] = \mathbb{Z}^n \cdot x_0$ the $F(\mathbb{Z}^n)$ -equivalence class $[x_0]$ can be given a geometric structure identical to that of \mathbb{Z}^n . Moreover, for any $x \in [x_0]$, there

is a unique $J \subseteq \mathbb{Z}^n$ so that $[x]_R = J \cdot x_0$, and thus the marker region of x , $[x]_R$, also inherits a geometric structure identical to that of J in \mathbb{Z}^n . If $x_1 \in [x_0]$, say $x_1 = g \cdot x_0$, then from $[x_0] = \mathbb{Z}^n \cdot x_1$ and $[x]_R = J' \cdot x_1$ we get another copy of \mathbb{Z}^n and a subset J' representing the corresponding geometric structures of $[x_0]$ and the marker region of x . However, since $J = gJ'$ the said geometric structures are identical up to a shift.

Very often the marker regions we consider come from a marker set M of $F(\mathbb{Z}^n)$. For this reason the Borel subequivalence relation is often denoted R_M . As in the example of Dirichlet regions it is often the case that R_M comes from a geometric construction on the background geometric structure of \mathbb{R}^n . And we will employ the usual terminology of solid geometry when we describe these geometric constructions in \mathbb{R}^n . For example, the Dirichlet regions are referred to as polygonal regions. Also, we say a marker region is a *rectangle* (or a *square*, or a *cube*, etc.) if it is such in \mathbb{Z}^n , or more generally it comes from such in \mathbb{R}^n in the sense of our description of this paragraph and the preceding one.

Our main objective of this section is to construct marker regions with regular geometric shapes. Our main theorem below will be a construction of rectangular marker regions. Moreover, we also restrict the choices for edge lengths of the rectangles as much as we can. By doing all these we obtain marker regions with a regular geometry. This turns out to be crucial in further constructions in hyperfiniteness proofs.

Theorem 3.1. *Let $d > 0$ be an integer. Then there is a subequivalence relation R_d^n of $F(\mathbb{Z}^n)$ such that R_d^n is clopen and the R_d^n -marker regions are n -dimensional rectangles with edge lengths either d or $d + 1$.*

Remark 3.3. When we say that R_d^n is clopen, we actually mean that $\{(x, g) \in F(\mathbb{Z}^n) \times \mathbb{Z}^n : g \cdot x R_d^n x\}$ is a clopen subset of $F(\mathbb{Z}^n) \times \mathbb{Z}^n$. Equivalently, for every $g \in \mathbb{Z}^n$, the set $\{x \in F(\mathbb{Z}^n) : g \cdot x R_d^n x\}$ is a clopen subset of $F(\mathbb{Z}^n)$. We will use this terminology throughout.

The proof of the theorem uses the big-marker-little-marker method. We will present the proof in several steps. As a first step we show that, in order to obtain the almost cubical marker regions of edge lengths approximately d , it is enough to create marker regions which are n -dimensional rectangles each of which has edge lengths *much bigger* than d . The basic idea of the proof of the following lemma is similar to that of Lemma 2.14.

Lemma 3.2. *Let $d > 0$ and $D > d^2$ be integers. Let R_D be a subequivalence relation of $F(\mathbb{Z}^n)$ so that the R_D -marker regions are n -dimensional rectangles with edge lengths greater than D . Then there is a subequivalence relation $R_d \subseteq R_D$ so that every R_D -marker region is partitioned into R_d -marker regions, which are n -dimensional rectangles with edge lengths either d or $d + 1$. Moreover, if R_D is clopen and there is $\Delta > D$ so that each R_D -marker region has edge lengths $\leq \Delta$, then R_d can also be clopen.*

Proof. As we have seen in the proof of Lemma 2.14, every integer $l > d^2$ can be written as $ad + b(d + 1)$ with $a, b \geq 0$. It follows that an n -dimensional rectangle, for instance $[0, l_1] \times \cdots \times [0, l_n]$ with $l_1, \dots, l_n > d^2$, can be tiled by n -dimensional rectangles with edge lengths either d or $d + 1$. This tiling corresponds to a finite equivalence relation R_{l_1, \dots, l_n} on the finite set $[0, l_1] \times \cdots \times [0, l_n]$.

Now for each $x \in F(\mathbb{Z}^n)$ let

$$\vec{l}(x) = (l_1(x), \dots, l_n(x))$$

where for each $1 \leq i \leq n$,

$$l_i(x) = |\{a \in \mathbb{Z} : (ae_i \cdot x)R_D x\}|.$$

Since the R_D marker regions are rectangles, the function $\vec{l}(x)$ measures the dimensions of the region containing x . It is clear that for any $xR_D y$, $\vec{l}(x) = \vec{l}(y)$. Similarly we can also measure the relative position of each element in its marker region. For each $x \in F(\mathbb{Z}^n)$ define

$$\vec{p}(x) = (p_1(x), \dots, p_n(x))$$

where for each $1 \leq i \leq n$,

$$p_i(x) = -\inf\{a \in \mathbb{Z} : (ae_i \cdot x)R_D x\}.$$

Finally we define R_d by

$$xR_d y \iff xR_D y \text{ and } \vec{p}(x)R_{\vec{l}(x)}\vec{p}(y).$$

It follows immediately that the R_d -marker regions are n -dimensional rectangles of edge lengths d or $d+1$.

Now suppose R_D is clopen and $\Delta > D$ is such that each R_D -marker region has edge lengths $\leq \Delta$. Then the values of $\vec{l}(x)$ and $\vec{p}(x)$ are all bounded by Δ . This implies that R_d as we defined above is clopen. \square

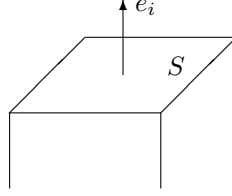
The essence of the above proof is a finitary algorithm being applied uniformly and invariantly to all equivalence classes. This illustrates a general method we will use again and again. Very often we will focus on the description of the finitary algorithm and leave the formal definitions like those in the above proof unstated.

The rectangular marker regions we need in the condition of the above lemma will come from polyhedra with faces perpendicular to the coordinate axes.

Lemma 3.3. *Let $D > 0$ be an integer. Let R_0 be a subequivalence relation of $F(\mathbb{Z}^n)$ so that the R_0 -marker regions are n -dimensional polyhedra with faces perpendicular to the coordinate axes. Suppose that for each R_0 -marker region every pair of parallel faces have a perpendicular distance greater than D . Then there is a subequivalence relation $R_1 \subseteq R_0$ so that every R_0 -marker region is partitioned into R_1 -marker regions, which are n -dimensional rectangles with edge lengths greater than D . Moreover, if R_0 is clopen and there is $\Delta > D$ so that each R_0 -marker region is contained in an n -dimensional cube of edge lengths Δ , then R_1 can also be clopen.*

Proof. It is enough to describe a finitary algorithm to decompose each n -dimensional polyhedron with the described property into n -dimensional rectangles with edge lengths greater than D . Let P be a finite polyhedral region in \mathbb{Z}^n with each of its faces perpendicular to some coordinate axis. Note that the faces of P are finite subsets of P definable from P . To be explicit, for each $1 \leq i \leq n$ the faces perpendicular to e_i are maximal finite subsets S of P such that

- (i) for any $x, y \in S$ and $g \cdot x = y$, the i -th coordinate of g is zero, and
- (ii) either $e_i \cdot S \cap P = \emptyset$ or $-e_i \cdot S \cap P = \emptyset$.

FIGURE 6. A polyhedral face S perpendicular to e_i

Let F_1, \dots, F_k be all the faces of P . For each $1 \leq j \leq k$, the face F_j partitions P into at most two parts as follows. Suppose e_i is perpendicular to F_j and without loss of generality assume $-e_i \cdot F_j \cap P = \emptyset$. By our assumption on the perpendicular distances of parallel faces, for every $0 \leq c \leq D$, $ce_i \cdot F_j \subseteq P$. We define $F_j^+ \subseteq P$ to be the set of all $x \in P$ such that for any $y \in F_j$, letting g be the unique element of \mathbb{Z}^n with $g \cdot y = x$, the i -th coordinate of g is non-negative. Let also $F_j^- = P - F_j^+$. F_j^- could be empty.

Finally define a subequivalence relation R_P on P by

$$xR_P y \iff \forall 1 \leq j \leq k (x \in F_j^+ \leftrightarrow y \in F_j^+).$$

Then the equivalence classes of R_P are rectangles whose faces are parts of linear expansions of the faces of P . These rectangles have edge lengths greater than D exactly because the parallel faces of P have perpendicular distances greater than D . This construction is illustrated in figure 7 \square

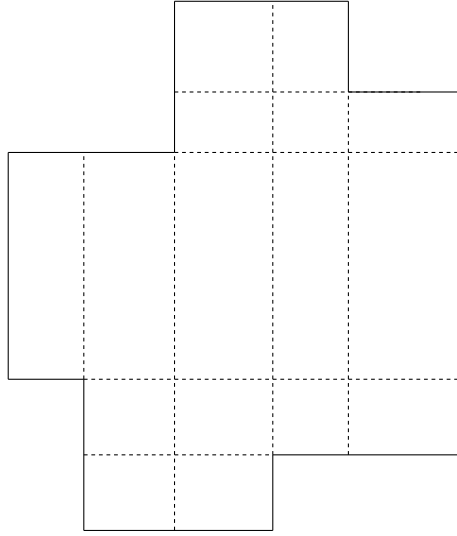


FIGURE 7. A partition of a polyhedron into rectangular regions

To finish the proof of the theorem we only need to create polyhedral marker regions with the above mentioned property. To achieve this we start with a basic clopen marker set M for a marker distance Δ much bigger than D . Consider the equivalence relation R_M defined by

$$xR_M y \iff \forall z \in M (\rho(x, z) \leq \Delta \leftrightarrow \rho(y, z) \leq \Delta).$$

The basic marker properties of M imply that R_M is in fact clopen. Note that the R_M -marker regions are already polyhedra with faces perpendicular to coordinate axes. Therefore we only need to modify these marker regions so that the perpendicular distance between parallel faces is larger than D . There are two potential difficulties. First, we need to see that there is no geometric obstacle to implement this idea, that is, we should be able to do this on any given $F(\mathbb{Z}^n)$ equivalence class. Second and more seriously, we need to be able to perform the modifications in an invariant Borel manner. To address these issues we employ again the big-marker-little-marker method.

Proof of Theorem 3.1. Let $\Delta_2 \gg \Delta_1 \gg D > d^2$. Let M_1 be a clopen marker set given by Lemma 2.1 for the marker distance Δ_1 and $M_2 \subseteq M_1$ be given by Lemma 2.2 for the marker distance Δ_2 . Let g_1, \dots, g_k enumerate all elements $g \in \mathbb{Z}^n$ with $\|g\| \leq \Delta_2 + \Delta_1$. Then $F(\mathbb{Z}^n) = \bigcup_{1 \leq i \leq k} g_i \cdot M_2$. We define a sequence A_0, A_1, \dots, A_k of subsets of M_1 as follows:

$$\begin{aligned} A_0 &= M_2, \\ A_i &= M_1 \cap (g_i \cdot M_2) - \bigcup_{j < i} A_j, \text{ for } 1 < i \leq k. \end{aligned}$$

Then A_0, A_1, \dots, A_k is a partition of M_1 into disjoint clopen sets with the property that, for any $0 \leq i \leq k$ and $x \neq y \in A_i$, $\rho(x, y) > \Delta_2 - 2\Delta_1$.

We now define R_{M_1} as promised

$$xR_{M_1} y \iff \forall z \in M_1 (\rho(x, z) \leq \Delta_1 \leftrightarrow \rho(y, z) \leq \Delta_1).$$

In order to describe the modifications we will perform on the R_{M_1} -marker regions we will use the following auxiliary notation. Let $J_1 = \{g \in \mathbb{Z}^n : \|g\| \leq \Delta_1\}$. For each $x \in M_1$ let $R_x = J_1 \cdot x$. Then R_x is the cubic region with center x and edge length $2\Delta_1 + 1$. When there is no danger of confusion, we will slightly abuse the terminology of marker regions and address each R_x also as a marker region. With this notation R_{M_1} can be equivalently expressed as

$$xR_{M_1} y \iff \forall z \in M_1 (x \in R_z \leftrightarrow y \in R_z).$$

In this sense we also say that R_{M_1} is *generated by* the collection $\{R_x \mid x \in M_1\}$ of (possibly overlapping) marker regions. To create marker regions with the desired property we will adjust the collection of cubes $\{R_x \mid x \in M_1\}$ to a collection of rectangles $\{R'_x \mid x \in M_1\}$ so that

- (i) $R_x \subseteq R'_x$ for each $x \in M_1$,
- (ii) the corresponding faces of R_x and R'_x have a perpendicular distance no more than $\frac{1}{10}\Delta_1$, and
- (iii) for each face F_1 of R'_x and any parallel face F_2 of R'_y with $\rho(x, y) \leq 3\Delta_1$, the perpendicular distance between F_1 and F_2 is at least D .

The subequivalence relation generated by the collection $\{R'_x \mid x \in M_1\}$ clearly satisfies the condition of the preceding lemma.

We define the collection $\{R'_x \mid x \in M_1\}$ by giving an inductive definition of $\{R'_x \mid x \in A_i\}$ for $0 \leq i \leq k$. For $x \in A_0$, let $R_x = R'_x$. Consider $i > 0$ and assume that for all $j < i$ and $x \in A_j$, R'_x has been defined to satisfy (i)-(iii). Now let $x \in A_i$. Let $R'_{y_1}, \dots, R'_{y_m}$ enumerate all surrounding rectangles with $y_1, \dots, y_m \in \bigcup_{j < i} A_j$ and $\rho(x, y_l) \leq 3\Delta_1$ for $1 \leq l \leq m$. Note that there is a fixed upper bound N for the number m which depends only on n . It follows that, for each face F of R_x , there are at most $2N$ many faces of $R'_{y_1}, \dots, R'_{y_m}$ which are parallel to F . Moreover these faces are the only ones the face F' of R'_x correspondent to F has to avoid being within perpendicular distance D . Therefore, as long as $\frac{1}{10}\Delta_1 > 2N(2D + 1)$, the face F can be shifted away from the center x to a suitable F' so that (ii) and (iii) are satisfied for F' . To define R'_x we need only adjust each of its $2n$ faces in turn. It is clear that (i) holds with this construction.

To recap the argument in the preceding paragraph more rigorously, we essentially gave a finitary algorithm which defines a region R' given a collection of rectangles within $[-3\Delta_1, 3\Delta_1]^n$. As before this can be turned into an invariant Borel construction of the collection $\{R'_x \mid x \in A_i\}$. Note that if $x \neq y \in A_i$, then $\rho(x, y) > \Delta_2 - 2\Delta_1 \gg \Delta_1$, thus the constructions of R'_x and R'_y do not affect each other. It follows that the resulting regions $\{R'_x \mid x \in M_1\}$ satisfy properties (i)-(iii). By the preceding lemmas the proof of the theorem is complete. \square

Note that Theorem 3.1 is a generalization of Lemma 2.14. In view of the contrast between Lemma 2.14 and Proposition 2.13, there is also limitation on how regular the higher dimensional marker regions can be. For instance, in the case $n = 2$ Theorem 3.1 guarantees the existence of Borel tilings of $F(\mathbb{Z}^2)$ by four kinds of almost square tiles, with dimensions $d \times d$, $d \times (d+1)$, $(d+1) \times d$ and $(d+1) \times (d+1)$, respectively. Proposition 2.13 implies immediately that there is no Borel tiling with only one kind of tiles used. A more tedious argument shows that there is no Borel tiling which uses only tiles of dimension $d \times d$ and $d \times (d+1)$. Similar arguments also rule out the combinations $\{d \times d, (d+1) \times d\}$, $\{d \times (d+1), (d+1) \times (d+1)\}$ and $\{(d+1) \times d, (d+1) \times (d+1)\}$. The other cases are not clear. In particular it is not clear whether there are Borel tilings which use two or three kinds of tiles as above. We conjecture that the number 4 is optimal.

We also make some comments on the role marker regions are going to play in our proof of hyperfiniteness. Since marker regions come from finite Borel equivalence relations, a natural attempt to prove hyperfiniteness is to create a sequence of such equivalence relations so that they form an increasing sequence

$$R_0 \subseteq R_1 \subseteq \dots \subseteq R_k \subseteq \dots$$

In terms of the geometry of marker regions when we require $R_k \subseteq R_{k+1}$ it is equivalent to require that the R_{k+1} -marker regions to be unions of R_k -marker regions. In this case we say that the marker regions (of different levels) *cohere* to each other. If increasing levels of coherent marker regions can be produced then the union of the equivalence relations is hyperfinite. Thus the question is whether the union is the equivalence relation $F(\mathbb{Z}^n)$.

In the Dougherty-Jackson-Kechris proof of hyperfiniteness of $F(\mathbb{Z}^n)$ increasing levels of coherent Dirichlet regions were used. However the union of the corresponding finite equivalence relations is not $F(\mathbb{Z}^n)$, but rather an equivalence relation with finite index in $F(\mathbb{Z}^n)$. Moreover, the index grows exponentially as the dimension n grows. Thus an application of this method to study $F(\mathbb{Z}^{<\omega})$ would only produce a hyperfinite subequivalence relation of $F(\mathbb{Z}^{<\omega})$ with infinite index. This does not seem to be progress since it is easy to prove abstractly that any aperiodic countable Borel equivalence relation contains a hyperfinite subequivalence relation.

In fact, if we start with clopen marker sets in Example 3.2 then the finite subequivalence relations giving the Dirichlet regions are also clopen. As a result the union of an increasing sequence of such equivalence relations is an open equivalence relation on the free part $F(\mathbb{Z}^n)$, and therefore is certainly different from the equivalence relation $F(\mathbb{Z}^n)$ (which is true F_σ). This argument also applies to the regular marker regions we have just constructed. Thus there is a simple complexity obstacle for this method to succeed.

Can we create the increasing finite equivalence relations

$$R_0 \subseteq R_1 \subseteq \dots \subseteq R_k \subseteq \dots$$

with higher complexity so that $\bigcup_k R_k = F(\mathbb{Z}^n)$? One realistic requirement is for the R_k to be relatively clopen as a subequivalence relation of $F(\mathbb{Z}^n)$ (which is in turn a subset of the space $F(\mathbb{Z}^n)^2$). It seems that clopenness is the only tool to make sure that the marker regions are finite. So the morale of the question is essentially: can we create increasing levels of coherent finite marker regions so that the unions of all levels of marker regions are the whole $F(\mathbb{Z}^n)$ classes?

Let R be a subequivalence relation of $F(\mathbb{Z}^n)$. We say that R is *nondegenerate* if for all $x \in F(\mathbb{Z}^n)$ there are $y, z \in [x]$ with $(y, z) \notin R$, that is to say, there are at least two R -marker regions in each $F(\mathbb{Z}^n)$ -equivalence class. The following theorem shows that the above question has a negative answer if we require the subequivalence relations to be nondegenerate.

Theorem 3.4. *There is no increasing sequence of nondegenerate, relatively open subequivalence relations of $F(\mathbb{Z}^n)$*

$$R_0 \subseteq R_1 \subseteq \dots \subseteq R_k \subseteq \dots$$

such that $\bigcup_k R_k = F(\mathbb{Z}^n)$.

Proof. Assume there were such a sequence. For each $k \in \omega$ let U_k be an open subset of $F(\mathbb{Z}^n)^2$ so that $R_k = U_k \cap F(\mathbb{Z}^n)$. Define the set C_k of *boundary points* of marker regions for R_k by

$$x \in C_k \iff x \in F(\mathbb{Z}^n) \text{ and } \exists g \in \mathbb{Z}^n (\|g\| = 1 \text{ and } (g \cdot x, x) \notin R_k).$$

Since R_k is nondegenerate, C_k is a complete section of $F(\mathbb{Z}^n)$. Also, since $(g \cdot x, x) \in F(\mathbb{Z}^n)$ we have that

$$x \in C_k \iff x \in F(\mathbb{Z}^n) \text{ and } \exists g \in \mathbb{Z}^n (\|g\| = 1 \text{ and } (g \cdot x, x) \notin U_k).$$

Note that there are only finitely many $g \in \mathbb{Z}^n$ with $\|g\| = 1$, hence it follows that C_k is a relatively closed subset of $F(\mathbb{Z}^n)$. Now it is clear from the definition that

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_k \supseteq \dots$$

Thus by Theorem 2.5 $\bigcap_k C_k \neq \emptyset$. Let $x \in \bigcap_k C_k$. There is $g \in \mathbb{Z}^n$ with $\|g\| = 1$ so that for infinitely many k , $(g \cdot x, x) \notin R_k$. This contradicts $\bigcup_k R_k = F(\mathbb{Z}^n)$. \square

These negative results show that the marker regions have to be used in a more sophisticated way than to serve directly as witnesses for hyperfiniteness. Also, the traditional coherent marker regions do not help and we need to go a different direction. In Section 5 we will construct *orthogonal* marker regions, which is another crucial step in our hyperfiniteness proof.

4. AN APPLICATION OF REGULAR MARKER REGIONS

In this section we present an application of regular marker regions to a Borel coloring problem. General Borel coloring problems were studied by Kechris-Solecki-Todorćević [9] and Ben Miller. The specific question we deal with below was communicated to us by Miller. Case $n = 2$ of Theorem 4.1 was joint work with Miller, again by a different proof.

We use the following notation. Let C be a set and X be an invariant subset of $2^{\mathbb{Z}^n}$. A function $c : X \rightarrow C$ is a *coloring* of X if for all $x \in X$ and $1 \leq i \leq n$, $c(e_i \cdot x) \neq c(x)$ unless $e_i \cdot x = x$. In this case we also say that c is a $|C|$ -*coloring* of X . If X is an invariant Borel subset of $2^{\mathbb{Z}^n}$, we define the *Borel chromatic number* for X , denoted by $\chi_B(X)$, to be the smallest integer $k > 1$ such that there exists a Borel k -coloring of X .

Kechris-Solecki-Todorćević [9] have shown that $\chi_B(2^{\mathbb{Z}}) = 3$ and for all $n > 1$ and any invariant Borel $X \subseteq 2^{\mathbb{Z}^n}$, $3 \leq \chi_B(X) \leq 2n + 1$. We show below that for all $n > 1$, $\chi_B(F(\mathbb{Z}^n)) \leq 4$ and $\chi_B(2^{\mathbb{Z}^n}) = 2n + 1$.

Theorem 4.1. *For each $n > 1$ there is a continuous 4-coloring of $F(\mathbb{Z}^n)$.*

Proof. Fix a $d > 2^{n+1}$ and let $R = R_d^n$ be the clopen subequivalence relation of $F(\mathbb{Z}^n)$ given by Theorem 3.1. We need to work with the geometry of R -marker regions.

To begin with, we define a new distance function on $F(\mathbb{Z}^n)$. First note that the following defines a norm on \mathbb{Z}^n : for $g = (g_1, \dots, g_n)$,

$$\|g\|_\nu = |g_1| + \dots + |g_n|.$$

This norm induces a distance function ν on $F(\mathbb{Z}^n)$, which we call the *New York distance*: for $x, y \in F(\mathbb{Z}^n)$,

$$\nu(x, y) = \begin{cases} \|g\|_\nu, & \text{if } g \cdot x = y, \\ \infty, & \text{if } (x, y) \notin F(\mathbb{Z}^n). \end{cases}$$

Two elements x and y are of New York distance 1 exactly when there is some $1 \leq i \leq n$ such that $e_i \cdot x = y$ or $e_i \cdot y = x$ (in which case $-e_i \cdot x = y$). A 4-coloring of $F(\mathbb{Z}^n)$ is a function $c : F(\mathbb{Z}^n) \rightarrow \{0, 1, 2, 3\}$ such that $c(x) \neq c(y)$ whenever $\nu(x, y) = 1$.

Consider the sets

$$\begin{aligned} B_1 &= \{x \in F(\mathbb{Z}^n) : \exists 1 \leq i \leq n (e_i \cdot x, x) \notin R \text{ or } (-e_i \cdot x, x) \notin R\}, \\ B_2 &= \{x \in F(\mathbb{Z}^n) : \exists 1 \leq i \leq n (2e_i \cdot x, x) \notin R \text{ or } (-2e_i \cdot x, x) \notin R\}. \end{aligned}$$

The set B_1 consists of the boundary points of all the R -marker regions, and B_2 consists of all points of B_1 as well as points of New York distance 1 to those in B_1 . Both B_1 and B_2 are clopen subsets of $F(\mathbb{Z}^n)$.

Continue to define

$$C_0 = \{x \in F(\mathbb{Z}^n) : \forall 1 \leq i \leq n, (e_i \cdot x, x) \notin R\}.$$

Then C_0 is a transversal for R and it consists of exactly one extremal point of each R -marker region. Furthermore, let

$$C_2 = \{x \in B_2 : \nu(x, x_0) \text{ is even, where } x_0 \in C_0 \text{ and } x_0 R x \},$$

and

$$C_1 = C_2 \cap B_1.$$

Note that $C_0 \subseteq C_1 \subseteq C_2$.

Finally, define $A_1 = B_1 - C_1$. Note that all sets defined so far are clopen.

Next define a binary relation E by

$$xEy \iff \nu(x, y) = 1 \text{ and } (x, y \in A_1 \text{ or } x, y \in C_1).$$

Let \sim be the transitive closure of E . Then \sim is an equivalence relation. The following fact will be crucial for our further construction.

Claim 1. Each \sim -equivalence class has no more than 2^{n^2} many elements.

Proof of Claim 1. First note that if xEy then $(x, y) \notin R$. To see this, assume xRy and let x_0 be the unique element in C_0 with $x_0 R x$; since $\nu(x, y) = 1$, then $\nu(x, x_0)$ differs from $\nu(y, x_0)$ by 1, and thus x and y cannot both be in A_1 or C_1 , contradicting xEy . Thus if xEy , the R -marker regions containing x and y are different. Moreover, if xEy and let R_x and R_y be the R -marker regions containing x and y respectively, then there is a face F_1 of R_x containing x and a face F_2 of R_y containing y so that F_1 and F_2 are parallel. In fact, if $\sigma \cdot x = y$, then both F_1 and F_2 are perpendicular to the vector σ . The four possible cases in dimension $n = 2$ are demonstrated in the following figure.

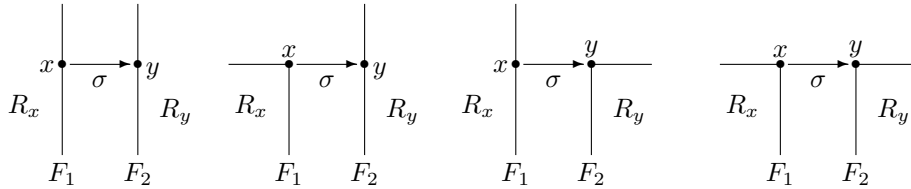


FIGURE 8. E -neighboring points and perpendicular faces

Next we show that, if $x \sim y$ and suppose $g \cdot x = y$ with $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$, then for all $1 \leq i \leq n$, $|g_i| < 2^n$. Toward a contradiction assume that for some $1 \leq i \leq n$, $|g_i| \geq 2^n$. Fix such an i . Without loss of generality assume that $g_i \geq 2^n$. Consider a chain

$$x = x_0 E x_1 E \cdots E x_k = y.$$

For each $j < k$, let $g^j = (g_1^j, \dots, g_n^j) \in \mathbb{Z}^n$ be such that $g^j \cdot x_j = x_{j+1}$. Since for each $j < k$, $\nu(x_j, x_{j+1}) = 1$, it follows that there are distinct

$$j_0, j_1, \dots, j_{2^n-1} < k$$

such that for each $l < 2^n$, $g_i^{j_l} = l$ and $g_i^{j_{l+1}} = l + 1$. It follows that for each $l < 2^n$ there are faces F_l and F'_l such that F_l contains x_{j_l} , F'_l contains $x_{j_{l+1}}$, and both F_l and F'_l are perpendicular to e_i . Also, every point in F_l is of the form $h \cdot x$ with $h_i = l$, and every point in F'_l is of the form $h \cdot x$ with $h_i = l + 1$. This implies that there are at least $2^n + 1$ faces from various R -marker regions each of which is perpendicular to e_i and their perpendicular distance is no more than 2^n . Now

each R -marker region has edge length $d > 2^{n+1}$, it follows that the $2^n + 1$ many distinct but parallel faces are all from distinct marker regions. This is impossible since there could be at most 2^n parallel faces from distinct n -dimensional cubes which form a tiling of \mathbb{Z}^n .

Now it follows immediately from the above fact that for any $x \in B_1$, the \sim -equivalence class of x is confined in a cubical region with all edges of length 2^n . Thus there are at most $(2^n)^n = 2^{n^2}$ many points in each \sim -equivalence class. \square

For any finite $S \subseteq \mathbb{Z}^n$, let π_S be the lexicographically least element of S . Note that for any finite $S \subseteq \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$, $\pi_{gS} = g\pi_S$. It follows that for any finite subset of an $F(\mathbb{Z}^n)$ -class, in general of the form $S \cdot x$ for a finite $S \subseteq \mathbb{Z}^n$, the lexicographically least element $\pi_S \cdot x$, is well defined (that is, it does not depend on the choice of x). Given each subset S of $[0, d]^n$ in \mathbb{Z}^n we fix a 2-coloring c_S of S into $\{0, 1\}$ so that $c(\pi_S) = 0$.

We are finally ready to define a continuous 4-coloring c of $F(\mathbb{Z}^n)$. Let $x \in F(\mathbb{Z}^n)$. If $x \in C_2 - C_1$, then let $c(x) = 0$. Since distinct points in $C_2 - C_1$ has ν distance > 1 , the function c defined so far is a coloring of $C_2 - C_1$. Next for $x \in C_1$, let x_0 be the lexicographically least element of $[x]_{\sim}$ and let $[x]_{\sim} = S \cdot x_0$. By the proof of the above claim, $S \subseteq [0, 2^n - 1]^n \subseteq [0, d]^n$. We define $c : [x]_{\sim} = S \cdot x_0 \rightarrow \{0, 1\}$ so that

$$c(g \cdot x_0) = c_S(g).$$

At this point note that if $x_1 \in C_1$ and $x_2 \in C_2 - C_1$ then $\nu(x_1, x_2) > 1$. Thus the function c defined so far is a coloring on C_2 .

Continuing the definition, for $x \in A_1$, we also consider $[x]_{\sim}$ and let x_0 be its lexicographically least element and $S \subseteq \mathbb{Z}^n$ so that $[x]_{\sim} = S \cdot x_0$. Define $c : [x]_{\sim} = S \cdot x_0 \rightarrow \{2, 3\}$ so that

$$c(g \cdot x_0) = c_S(g) + 2.$$

It is obvious that the function c defined so far is a coloring of $C_2 \cup A_1$.

Now we claim that if $x \notin C_2 \cup A_1$ and $\nu(x, y) = 1$, then $y \notin A_1$. To see this, let $y \in A_1$ and let $\nu(x, y) = 1$. Let $y_0 \in C_0$ be the unique point with $y_0 R y$. Then $\nu(y, y_0)$ is odd. If $(x, y) \notin R$, then $x \in B_1 = C_1 \cup A_1 \subseteq C_2 \cup A_1$. If $x R y$, then $x \in B_2$ and, since $\nu(x, y) = 1$, we have that $\nu(x, y_0)$ is even. This implies that $x \in C_2$. We have thus shown the claim.

Finally for $x \notin C_2 \cup A_1$ we let $D_x = [x]_R - (C_2 \cup A_1)$. Let x_0 be the lexicographically least element of D_x and let $D_x = S \cdot x_0$. Define $c : D_x = S \cdot x_0 \rightarrow \{2, 3\}$ so that

$$c(g \cdot x_0) = c_S(g) + 2.$$

By the above claim, if $x_1 \in D_x$, $y_1 \notin D_x$ and $\nu(x_1, y_1) = 1$, then it must be that $y_1 \in C_2$; thus $c(y_1) \in \{0, 1\}$ and therefore $c(y_1) \neq c(x_1)$. This shows that c is a coloring of all of $F(\mathbb{Z}^n)$.

To see that c is a continuous coloring just note that for each coloring $\kappa \in \{0, 1, 2, 3\}$, $c^{-1}(\kappa)$ is clopen in $F(\mathbb{Z}^n)$. \square

An interesting question which remains open is whether $F(\mathbb{Z}^n)$ admits a continuous (or just Borel) 3-coloring. Investigation of this problem suggests that the answer is closely related to whether we may improve our regular marker regions so that they exhibit further regularity properties. In particular, we are interested in the following intuitive question: can we create regular marker regions so that not

only each of them is a rectangle but also they almost line up with each other? The precise question for $n = 2$ is as follows.

Question 4.1. Let $d > 0$ be an integer. Is there a Borel subequivalence relation R_d of $F(\mathbb{Z}^2)$ such that

- (a) each R_d -marker region is a rectangle with edge lengths either d or $d + 1$, and
- (b) if R_1 and R_2 are two adjacent R_d -marker regions and F_1 and F_2 are parallel edges of R_1 and R_2 respectively, then the perpendicular distance between F_1 and F_2 is either $0, 1, d$ or $d + 1$?

For the rest of the section we compute the Borel chromatic number for $2^{\mathbb{Z}^n}$. The following result was obtained independently by R. Muchnik, by a different argument (we thank A. S. Kechris for communicating this to us).

Theorem 4.2. $\chi_B(2^{\mathbb{Z}^n}) = 2n + 1$ for each $n > 1$.

To prove this theorem we need to analyze the non-free part of $2^{\mathbb{Z}^n}$. We fix some notation first. Recall that for each $x \in 2^{\mathbb{Z}^n}$ the *stabilizer* of x is the subgroup of \mathbb{Z}^n defined by

$$G_x = \{g \in \mathbb{Z}^n : g \cdot x = x\}.$$

Since \mathbb{Z}^n is abelian, we have that for $x, y \in 2^{\mathbb{Z}^n}$ with $xE(\mathbb{Z}^n)y$, $G_x = G_y$. Thus the assignment $x \mapsto G_x$ is an invariant map. Conversely, for each subgroup G of \mathbb{Z}^n , we can define

$$X_G = \{x \in 2^{\mathbb{Z}^n} : G_x = G\}.$$

Then X_G is a nonempty invariant Borel subset of $2^{\mathbb{Z}^n}$. In particular, if G is the trivial group $\{0\}$, then X_G is just the free part $F(\mathbb{Z}^n)$. Apparently,

$$2^{\mathbb{Z}^n} = \bigcup \{X_G : G \leq \mathbb{Z}^n\},$$

and thus

$$\chi_B(2^{\mathbb{Z}^n}) = \max\{\chi_B(X_G) : G \leq \mathbb{Z}^n\}.$$

To understand $\chi_B(X_G)$ we consider the quotient group \mathbb{Z}^n/G . The action of \mathbb{Z}^n on X_G naturally induces an action of \mathbb{Z}^n/G :

$$\begin{array}{ccc} \mathbb{Z}^n/G & \times & X_G & \rightarrow & X_G \\ (gG & , & x) & \mapsto & gG \cdot x = g \cdot x \end{array}$$

It is routine to see that this action is well-defined and that it is free. Now a k -coloring of X_G can be characterized as a function $c : X_G \rightarrow \{0, \dots, k-1\}$ such that $c(e_i G \cdot x) \neq c(x)$ whenever $e_i G \cdot x \neq x$.

On the other hand we can consider the Cayley graph of \mathbb{Z}^n/G given by the generators $e_1 G, \dots, e_n G$ and denote it by $\Gamma(\mathbb{Z}^n/G)$. We call the chromatic number of $\Gamma(\mathbb{Z}^n/G)$ the *chromatic number* of \mathbb{Z}^n/G and denote it by $\chi(\mathbb{Z}^n/G)$. (Note that this is not a general definition for the chromatic number of a group, and even for the groups we are considering it differs from the general definition in the literature.) More explicitly, $\chi(\mathbb{Z}^n/G)$ is the smallest $k \in \omega$ such that there is a function $c : \mathbb{Z}^n/G \rightarrow \{0, \dots, k-1\}$ such that $c(e_i gG) \neq c(gG)$ whenever $e_i G \neq G$. This number gives a lower bound for $\chi_B(X_G)$.

Lemma 4.3. $\chi_B(X_G) \geq \chi(\mathbb{Z}^n/G)$.

Proof. Let c be a Borel k -coloring of X_G . Fix $x \in X_G$. Then $c(e_i G \cdot x) \neq c(x)$ whenever $e_i G \cdot x \neq x$. Now define a k -coloring of \mathbb{Z}^n/G by

$$\hat{c}(gG) = c(gG \cdot x).$$

Assume $\hat{c}(e_i gG) = \hat{c}(gG)$. Then $c(e_i gG \cdot x) = c(gG \cdot x)$. Letting $y = gG \cdot x$, we have that $c(e_i G \cdot y) = c(y)$. Thus $e_i G \cdot y = y$. This implies that $e_i G = G$. Thus \hat{c} is a k -coloring on \mathbb{Z}^n/G . \square

In general the equality does not hold. For instance, for the trivial group G , $\chi(\mathbb{Z}^n/G) = \chi(\mathbb{Z}^n) = 2$ but $\chi_B(X_G) = \chi_B(F(\mathbb{Z}^n)) \geq 3$. However, the equality does hold when \mathbb{Z}^n/G is finite.

Lemma 4.4. *If \mathbb{Z}^n/G is finite then $\chi_B(X_G) = \chi(\mathbb{Z}^n/G)$.*

Proof. Assume that \mathbb{Z}^n/G is finite and that there is a k -coloring \hat{c} of \mathbb{Z}^n/G . Now the equivalence relation $E(\mathbb{Z}^n) \upharpoonright X_G$ is a finite Borel equivalence relation and hence it admits a Borel transversal T . We can define a Borel k -coloring c of X_G by letting $c(x) = \hat{c}(gG)$, where $gG \in \mathbb{Z}^n/G$ is the unique element such that $(gG)^{-1} \cdot x \in T$. \square

For the remainder of this proof we will focus on computing $\chi(\mathbb{Z}^n/G)$ for $G \leq \mathbb{Z}^n$ with \mathbb{Z}^n/G finite. In particular we will construct a subgroup G of \mathbb{Z}^n with $\chi(\mathbb{Z}^n/G) = 2n + 1$, which will complete the proof of Theorem 4.2. The techniques employed are algebraic and number theoretic by nature.

Let S be a subset of \mathbb{Z}^n . We say that S is \mathbb{Z} -linearly independent if for any $k_1, \dots, k_m \in \mathbb{Z}$ and distinct $g_1, \dots, g_m \in S$,

$$k_1 g_1 + \dots + k_m g_m = 0 \Rightarrow k_1 = \dots = k_m = 0.$$

Such an S can also be viewed as a subset of \mathbb{R}^n , and thus we can apply the usual linear algebra to the situation and define the following notions. Let F be a subfield of \mathbb{R} (for example, \mathbb{Q} or \mathbb{R}). We say that S is F -linearly independent if for any $r_1, \dots, r_m \in F$ and distinct $g_1, \dots, g_m \in S$,

$$r_1 g_1 + \dots + r_m g_m = 0 \Rightarrow r_1 = \dots = r_m = 0.$$

It is possible to simplify the notation substantially by noting the following fact.

Lemma 4.5. *Let $S \subseteq \mathbb{Z}^n$. Then the following are equivalent:*

- (i) S is \mathbb{Z} -linearly independent.
- (ii) S is \mathbb{Q} -linearly independent.
- (iii) S is \mathbb{R} -linearly independent.

Proof. The equivalence between (i) and (ii) is obvious. We only need to show that (ii) implies (iii). For this let $g_1, \dots, g_m \in S$ be distinct and $r_1, \dots, r_m \in \mathbb{R}$ so that $r_1 g_1 + \dots + r_m g_m = 0$. Assume it is not the case that $r_1 = \dots = r_m = 0$ and without loss of generality assume that $r_1 = 1$. Now S is \mathbb{Q} -linearly independent, thus not all of r_1, \dots, r_m are in \mathbb{Q} . Again without loss of generality we assume that $r_2 \notin \mathbb{Q}$. Now consider the \mathbb{Q} -linear subspace of \mathbb{R} generated by r_1, \dots, r_m and denote it by V . We have that $2 \leq \dim_{\mathbb{Q}}(V) \leq m$. Let $l = \dim_{\mathbb{Q}}(V)$ and without loss of generality assume r_1, \dots, r_l form a \mathbb{Q} -linearly independent set. Then for each $l < i \leq m$, r_i can be expressed as a \mathbb{Q} -linear combination of r_1, \dots, r_l . That is to say, for each $i = l + 1, \dots, m$, there are $q_{i,1}, \dots, q_{i,l} \in \mathbb{Q}$ with

$$r_i = q_{i,1} r_1 + \dots + q_{i,l} r_l.$$

Now we get that

$$\begin{aligned}
& r_1 g_1 + \cdots + r_m g_m \\
= & r_1 g_1 + \cdots + r_l g_l + r_{l+1} g_{l+1} + \cdots + r_m g_m \\
= & r_1 g_1 + \cdots + r_l g_l + (q_{l+1,1} r_1 + \cdots + q_{l+1,l} r_l) g_{l+1} + \cdots + \\
& (q_{m,1} r_1 + \cdots + q_{m,l} r_l) g_m \\
= & r_1 (g_1 + q_{l+1,1} g_{l+1} + \cdots + q_{m,1} g_m) + \cdots + r_l (g_l + q_{l+1,l} g_{l+1} + \cdots + q_{m,l} g_m) \\
= & r_1 h_1 + \cdots + r_l h_l
\end{aligned}$$

Since all entries of h_1, \dots, h_l are rationals we conclude that $h_1 = \cdots = h_l = 0$. Now let $r'_1 = 1$, $r'_i = 0$ for all $i = 2, \dots, l$, and $r'_i = q_{i,1} r_1$ for all $i = l+1, \dots, m$. Then we also have $r'_1 h_1 + \cdots + r'_l h_l = 0$ and the above computation implies that

$$r'_1 g_1 + \cdots + r'_m g_m = 0.$$

Now all of r'_1, \dots, r'_m are in \mathbb{Q} , but $r'_1 = 1 \neq 0$, contradicting the \mathbb{Q} -linear independence of g_1, \dots, g_m . \square

In view of this lemma we will drop the prefixes and speak of a subset of \mathbb{Z}^n being *linearly independent*. Next for a subset S of \mathbb{Z}^n we denote by $\langle S \rangle_{\mathbb{R}}$ the \mathbb{R} -linear subspace of \mathbb{R}^n generated by S . For a subgroup G of \mathbb{Z}^n we define the *dimension* of G , denoted $\dim(G)$, to be the dimension of $\langle G \rangle_{\mathbb{R}}$. Thus $\dim(G)$ is an integer in $[0, n]$ and $\dim(G) = 0$ iff G is the trivial group $\{0\}$. It is easy to see that, if there are linearly independent elements $g_1, \dots, g_m \in G$ generating the group G , then $\dim(G) = m$.

The next concept we need is the geometric notion of a parallelotope. Let $x_1, \dots, x_m \in \mathbb{R}^n$ be linearly independent elements of \mathbb{R}^n . Recall that the *parallelotope* generated by x_1, \dots, x_m is the set

$$\{r_1 x_1 + \cdots + r_m x_m : r_1, \dots, r_m \in [0, 1]\}.$$

Geometrically it is easier to think of it as the convex hull in \mathbb{R}^n generated by the set of *extremal points*

$$\{b_1 x_1 + \cdots + b_m x_m : b_1, \dots, b_m \in \{0, 1\}\}.$$

For our purpose we define the *fundamental parallelotope* generated by $x_1, \dots, x_m \in \mathbb{R}^n$, denoted by $L_{\mathbb{R}}(x_1, \dots, x_m)$, to be the set

$$\{r_1 x_1 + \cdots + r_m x_m : r_1, \dots, r_m \in [0, 1]\}.$$

If $g_1, \dots, g_m \in \mathbb{Z}^n$ are linearly independent, then we define the *fundamental region* generated by g_1, \dots, g_m , denoted by $L(g_1, \dots, g_m)$, to be $L_{\mathbb{R}}(g_1, \dots, g_m) \cap \mathbb{Z}^n$.

Lemma 4.6. *Let $g_1, \dots, g_n \in \mathbb{Z}^n$ be linearly independent. Then the cardinality of $L(g_1, \dots, g_n)$ is the same as the n -dimensional volume of $L_{\mathbb{R}}(g_1, \dots, g_n)$.*

Proof. Let k be the cardinality of $L(g_1, \dots, g_n)$ and v be the volume of $L_{\mathbb{R}}(g_1, \dots, g_n)$. First note that v is an integer since all the extremal points of $L_{\mathbb{R}}(g_1, \dots, g_n)$ have integer coordinates and therefore the perpendicular distances of parallel faces are all integers. For each $i = 1, \dots, n$, let N_i be the least common multiple of all the i -th coordinates of g_1, \dots, g_n . Consider the cube $C = [0, N_1] \times \cdots \times [0, N_n]$ in \mathbb{R}^n . Also consider the tiling of \mathbb{R}^n by translated copies of $L_{\mathbb{R}}(g_1, \dots, g_n)$. A moment reflection shows that partial tiles in the cube C can be combined to form full tiles, and thus there is an integer number of tiles (which are broken up and rearranged near the boundary of C) that cover exactly the cube. By considering the volume, the number of tiles should be $N_1 \dots N_n / v$. On the other hand, the number of \mathbb{Z}^n

lattice points in the region $[0, N_1) \times \dots \times [0, N_n)$ is $N_1 \dots N_n$. Thus by considering the number of lattice points, the number of tiles in C should be $N_1 \dots N_n/k$. Now they are the same number, we get that $v = k$. \square

An elegant way to compute the volume of $L_{\mathbb{R}}(g_1, \dots, g_n)$ is given by linear algebra. To do this, regard g_1, \dots, g_n as n -dimensional column vectors and form a matrix $A = A(g_1, \dots, g_n)$. Then the volume of $L_{\mathbb{R}}(g_1, \dots, g_n)$ is just $|\det(A)|$. The geometric argument for Lemma 4.6 has many interesting applications. An $n = 2$ version of it was used in the study of the Steinhaus problem in [8]. Here we will see two applications. The first one is a converse of the statement about the dimension of a subgroup of \mathbb{Z}^n .

Lemma 4.7. *Let $G \leq \mathbb{Z}^n$ and $m \leq n$. Then $\dim(G) = m$ iff there are linearly independent $g_1, \dots, g_m \in G$ generating the group G .*

Proof. As noted before one direction is obvious. For the other direction, let $\dim(G) = m$. Then there are linearly independent elements $h_1, \dots, h_m \in G$ so that $\langle G \rangle_{\mathbb{R}} = \langle h_1, \dots, h_m \rangle_{\mathbb{R}}$. Denote this linear span $\langle G \rangle_{\mathbb{R}}$ by V and let v_1, \dots, v_m be an orthonormal basis for V . Note that with respect to this new basis the elements h_1, \dots, h_m do not necessarily have integer coordinates. Nevertheless we can consider the fundamental parallelotope $L_{\mathbb{R}}(h_1, \dots, h_m)$ and let $H_0 = L_{\mathbb{R}}(h_1, \dots, h_m) \cap G$. If $H_0 = \{0\}$ we claim that in fact h_1, \dots, h_m generate the group G . To see this consider the tiling of V by translates of $L_{\mathbb{R}}(h_1, \dots, h_m)$ along \mathbb{Z} -linear combinations of h_1, \dots, h_m . If h_1, \dots, h_m do not generate the group G then some translate of $L_{\mathbb{R}}(h_1, \dots, h_m)$ contains an element of G as a non-extremal point. By translating this element back to $L_{\mathbb{R}}(h_1, \dots, h_m)$ it follows that $L_{\mathbb{R}}(h_1, \dots, h_m)$ contains an element of G other than 0, contradicting our assumption.

Now suppose H_0 contains an element $h^* \neq 0$. Then let $S_1 = \{h_1^1, \dots, h_m^1\}$ be a linearly independent m -element subset of $\{h_1, \dots, h_m, h^*\}$ so that $h^* \in S_1$. (All we need to do to find such an S_1 is to find the smallest $l \leq m$ with $h^* \in \langle h_1, \dots, h_l \rangle_{\mathbb{R}}$ and define S_1 to be $\{h_1, \dots, h_{l-1}, h^*, h_{l+1}, \dots, h_m\}$.) The above argument can be now repeated for S_1 . Namely, consider $L_{\mathbb{R}}(S_1)$ and let $H_1 = L_{\mathbb{R}}(S_1) \cap G$. If $H_1 = \{0\}$ then we again have that S_1 generates the group G . We claim that $|H_1| < |H_0|$. To see this, we define a map τ from $L_{\mathbb{R}}(S_1)$ into $L_{\mathbb{R}}(h_1, \dots, h_m)$ as follows. If $x \in L_{\mathbb{R}}(S_1) \cap L_{\mathbb{R}}(h_1, \dots, h_m)$ then let $\tau(x) = x$. If $x \in L_{\mathbb{R}}(S_1) - L_{\mathbb{R}}(h_1, \dots, h_m)$, then x is in a unique translate of $L_{\mathbb{R}}(h_1, \dots, h_m)$ and we define $\tau(x)$ to be the unique translate of x back to the fundamental parallelotope by a \mathbb{Z} -linear combination of $h_1, \dots, h_{l-1}, h_{l+1}, \dots, h_m$. The function τ is one-one. Moreover, if $x \in G$ then $\tau(x) \in G$. Therefore we have that $|H_1| \leq |H_0|$. However, since $h^* \neq \tau(x)$ for any $x \in L_{\mathbb{R}}(S_1)$, we get that $|H_1| < |H_0|$.

Repeating the above argument we get new linearly independent m -element subsets $S \subseteq G$ with the fundamental parallelotope containing less and less elements of G . In finitely many steps we obtain such a set S with the fundamental parallelotope containing no non-zero element of G . Then S generates G . \square

A further application of Lemma 4.6 concerns the size of the quotient group \mathbb{Z}^n/G when it is finite.

Lemma 4.8. *Let $g_1, \dots, g_n \in \mathbb{Z}^n$ be linearly independent elements generating a group $G \leq \mathbb{Z}^n$. Let $A = A(g_1, \dots, g_n)$ be the $n \times n$ -matrix formed by viewing g_1, \dots, g_n as column vectors. Then $|\mathbb{Z}^n/G| = |\det(A)|$.*

Proof. There is an obvious way to define a one-one correspondence between the fundamental region $L(g_1, \dots, g_n)$ and \mathbb{Z}^n/G . By Lemma 4.6 we are done. \square

Conversely, suppose \mathbb{Z}^n/G is finite. Then it is also obvious that $\dim(G) = n$. Thus by Lemma 4.7 there are linearly independent elements g_1, \dots, g_n generating the group G . Furthermore we can compute the cardinality of \mathbb{Z}^n/G by the above lemma.

Now back to our coloring problem, we turn the main problem into an algebraic one.

Lemma 4.9. *Let $G \leq \mathbb{Z}^n$. Then the following are equivalent:*

- (i) *The Cayley graph $\Gamma(\mathbb{Z}^n/G)$ is the complete graph K_{2n+1} with $2n+1$ vertices.*
- (ii) *There are $g_1, \dots, g_n \in G$ such that, letting $A = A(g_1, \dots, g_n)$ as before, we have that*
 - a) $|\det(A)| = 2n + 1$, and
 - b) *for any $i = 1, \dots, n$, $A^{-1}e_i \notin \mathbb{Z}^n$, where each e_i is regarded as a column vector, and*
 - c) *for any $i \neq j$, $i, j = 1, \dots, n$, $A^{-1}(e_i \pm e_j) \notin \mathbb{Z}^n$.*

Proof. (i) \Rightarrow (ii). Suppose $\Gamma(\mathbb{Z}^n/G) = K_{2n+1}$. Since \mathbb{Z}^n/G is finite the previous lemmas imply that there are linearly independent elements $g_1, \dots, g_n \in G$ generating the group G . Also $|\det(A)| = |\mathbb{Z}^n/G| = 2n + 1$. Now in K_{2n+1} every vertex has degree $2n$. On the other hand, in the Cayley graph $\Gamma(\mathbb{Z}^n/G)$ every vertex gG has at most $2n$ neighbors given by $\pm e_i gG$. Thus in order for $\Gamma(\mathbb{Z}^n/G)$ to be the complete graph, it is necessary that for any $g \in G$,

$$e_i gG \neq gG \text{ for any } i = 1, \dots, n, \text{ and}$$

$$e_i gG \neq \pm e_j gG \text{ for any } i \neq j, i, j = 1, \dots, n.$$

These amount to requiring that $e_i \notin G$ and $e_i \pm e_j \notin G$ for all $i \neq j$. However we claim that for any $x \in \mathbb{Z}^n$, $x \in G$ iff $A^{-1}x \in \mathbb{Z}^n$. In fact, the latter condition is equivalent to saying that there are integers k_1, \dots, k_n so that

$$x = A \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = k_1 g_1 + \dots + k_n g_n.$$

Thus we obtain condition b) and c).

The above argument can be reversed to prove (ii) \Rightarrow (i). Suppose $g_1, \dots, g_n \in G$ satisfy a)-c). Then we conclude that \mathbb{Z}^n/G has $2n + 1$ elements and each vertex of $\Gamma(\mathbb{Z}^n/G)$ has exactly degree $2n$. However such a graph has to be complete. \square

To finish the proof of Theorem 4.2 we just give an example of an $n \times n$ -matrix with integer entries so that conditions a)-c) hold. Consider the following matrix

$$A = \begin{pmatrix} 2n+1 & 2 & 3 & 4 & \cdots & n \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It is straightforward to compute $\det(A) = 2n + 1$ and A^{-1} as

$$A^{-1} = \begin{pmatrix} \frac{1}{2n+1} & -\frac{2}{2n+1} & \frac{3}{2n+1} & -\frac{4}{2n+1} & \cdots & -\frac{n}{2n+1} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It is easy enough to determine by inspection that the conditions a)-c) hold.

Now let G be the group generated by the elements given by the columns of A . Then the Cayley graph on \mathbb{Z}^n/G is K_{2n+1} , which has chromatic number $2n + 1$. This finishes the proof of Theorem 4.2.

Much of the machinery developed in the above proof is an overkill for the proof itself. However we hope that it will be relevant to the following general problem which is still of interest and unresolved.

Problem 4.2. Find the Borel chromatic number for X_G for each subgroup G of \mathbb{Z}^n .

We have not worked out the details of the computation of $\chi(\mathbb{Z}^n/G)$ even for n -dimensional groups G (equivalently for all G with \mathbb{Z}^n/G finite). A version of the problem in this context is: given an invertible $n \times n$ -matrix A with integer entries, compute the chromatic number of \mathbb{Z}^n/G where G is generated by the column vectors.

The more interesting instances of the above problem occur when $\dim(G) < n$. We have mentioned above that even when $\dim(G) = 0$ there is a nontrivial problem left open. For $0 < \dim(G) < n$ almost nothing is known.

5. ORTHOGONAL MARKER REGIONS

In §3 we constructed regular marker regions for $F(\mathbb{Z}^n)$. Recall that for each $d > 0$ we constructed a clopen subequivalence relation R_d^n of $F(\mathbb{Z}^n)$ such that on each equivalence class of $F(\mathbb{Z}^n)$, the R_d^n classes are n -dimensional rectangles with each edge length either d or $d + 1$. We refer to R_d^n as a rectangular partition of $F(\mathbb{Z}^n)$. For such a partition R_d^n , we show how to construct an *orthogonal* partition \tilde{R}_d^n . This will also be a clopen subequivalence relation of $F(\mathbb{Z}^n)$ into rectangular regions, and moreover will be orthogonal to R_d^n . By this we mean that every face of a rectangle in \tilde{R}_d^n will be fairly far away from any parallel face of a nearby rectangle in R_d^n . The precise statement is in lemma 5.2.

We adopt a running convention throughout §5 and §6. In the statements of all results involving the quantity d , we always assume d is sufficiently large so that all expressions involving d have value at least 1. For example, in the statement of lemma 5.2 it should be assumed that d is greater than $9000^n 16^{n^2}$. This avoids certain trivialities associated with small values of d .

First we need the following simple lemma.

Lemma 5.1. *Let R be a rectangular partition of \mathbb{Z}^n with each rectangle having edge lengths in $\{d, d + 1\}$. Then for any $x \in \mathbb{Z}^n$, the sphere $S(x, \frac{d}{10})$ of radius $\frac{d}{10}$ about x can intersect at most $e_n \doteq 2^n$ many parallel faces of rectangles in R .*

Proof. Without loss of generality we may assume x is the origin. Suppose that $S = S(\vec{0}, \frac{d}{10})$ intersects more than 2^n many faces of rectangles, and all of these faces are parallel to a given face \mathcal{F} . Clearly each rectangle in the partition R can have at most one face parallel to \mathcal{F} which intersects S . Thus, we have more than 2^n rectangles R_1, \dots, R_{2^n+1} , each of which has a face parallel to \mathcal{F} which intersects S . Let c_1, \dots, c_{2^n+1} be the centers of these rectangles. Since there are only 2^n “quadrants” about the origin in \mathbb{Z}^n , there must be two of these rectangles, say R_1 and R_2 such that their centers c_1 and c_2 lie in the same quadrant about $\vec{0}$. Without loss of generality, say c_1 and c_2 both have non-negative values in their coordinate representations. Say $c_1 = (a_1, a_2, \dots, a_n)$, $c_2 = (a'_1, a'_2, \dots, a'_n)$. Then, $0 \leq a_i \leq \frac{d}{10} + \frac{d}{2} = \frac{3}{5}d$ for all i . It follows that $C \doteq [\frac{3}{5}d, \frac{4}{5}d]^n \subseteq R_1$. Similarly, $C \subseteq R_2$, which contradicts the fact that R_1 and R_2 are disjoint. \square

Now we state our basic lemma on orthogonal marker regions. We will not directly use this lemma in the proof of our main result (in §6), but rather a technical strengthening of it which we give immediately afterward. Nevertheless, this lemma shows in a simpler setting the ideas involved. We note that the values of the various constants appearing in the lemma are not critical.

Lemma 5.2. *Let R_d^n be a clopen rectangular partition of $F(\mathbb{Z}^n)$, with each rectangle having edge lengths in $\{d, d+1\}$. Then there is a clopen subequivalence relation \tilde{R}_d^n of $F(\mathbb{Z}^n)$ satisfying:*

- (1) *Each \tilde{R}_d^n class is a rectangular region with each edge lengths between $9d$ and $12d$.*
- (2) *Every face of a region $R_1 \in \tilde{R}_d^n$ is at least $\frac{1}{9000^n 16^{n^2}}d$ from any parallel face of a region $R_2 \in R_d^n$.*

Proof. Let $s = Cd$, where $C = 110 \cdot 4 \cdot 8 \cdot 16^n$. We first let $M \subseteq F(\mathbb{Z}^n)$ be a clopen marker set for distance $\frac{s}{2}$. We may assume as in the proof of theorem 3.1 that $M = A_0 \cup A_1 \cup \dots \cup A_k$, and for any $x, y \in A_i$, $\rho(x, y) \gg s$. We let \mathcal{R} be the collection of cubes with edge length s which are centered about the points of M . As in the proof of theorem 3.1, we proceed in k steps to adjust these cubes, at step i adjusting the faces of the cubes with centers in A_i . We do this adjustment in two steps. At the end of both steps, we will have moved each face outward by no more than $\frac{s}{4}$. Figure 9 shows an initial arrangement of the rectangles before adjustment.

First, we move each face \mathcal{F} of a cube R with center x in A_i outward by no more than $\frac{s}{8}$ to avoid each parallel face \mathcal{F}' of a previously adjusted cube with $\rho(\mathcal{F}, \mathcal{F}') \leq \frac{s}{4} + \frac{3}{2}s + \frac{s}{4} = 2s$. This adjustment is illustrated in figure 10.

Since the volume of this sphere is $(8s)^n$ and each adjusted region contains a smaller cube with volume $(\frac{s}{2})^n$, there are at most $2 \cdot 16^n$ such faces. By moving these faces \mathcal{F} no more than $\frac{s}{8}$, we may avoid these previous faces by a distance of at least $\frac{s}{2 \cdot 8 \cdot 16^n}$.

Second, we now move each of these faces outward by no more than $\frac{s}{4 \cdot 8 \cdot 16^n} < \frac{s}{8}$. This will keep the distances from the previously adjusted faces considered in the first step at least $\frac{s}{4 \cdot 8 \cdot 16^n} = 110d$. We move to avoid all of the parallel faces of rectangles in R_d^n within $\frac{7}{4}s + 10d$ of the adjusted face. This adjustment is shown in figure 12.

Specifically, there are at most

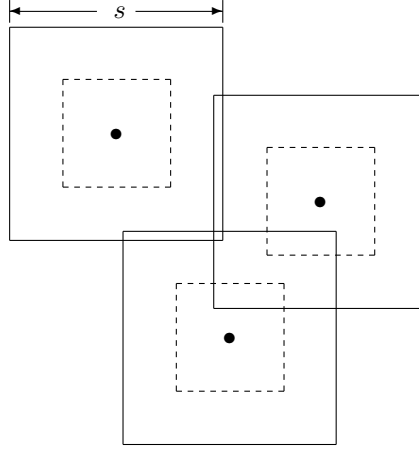


FIGURE 9. Before adjustment, the balls $S(x, s/2)$, $x \in M$, could have parallel faces that are close.

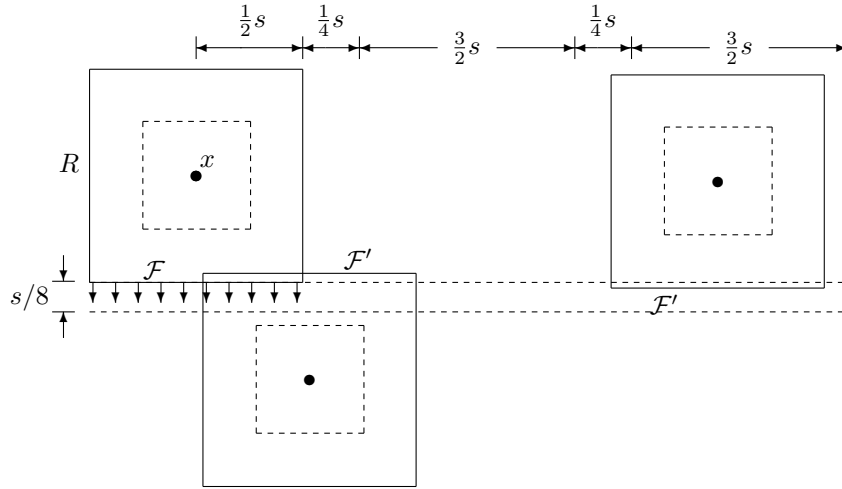


FIGURE 10. In the first step of the adjustment, the face \mathcal{F} will be adjusted no more than $s/8$ to avoid parallel faces from other previously adjusted \tilde{R}_d^n cubes.

$$(1) \quad 2 \cdot \frac{(5s + 22d)^{n-1} \cdot (\frac{s}{4 \cdot 8 \cdot 16^n} + 22d)}{d^n} \leq 2 \cdot \frac{(6s)^{n-1} \cdot \frac{s}{2 \cdot 8 \cdot 16^n}}{d^n} = \frac{1}{48} \left(\frac{3}{8}\right)^n C^n \leq 9000^n 16^{n^2}$$

faces of rectangles in R_d^n to avoid, so we may avoid them by at least

$$\frac{s}{4 \cdot 8 \cdot 16^n} \cdot 9000^{-n} \cdot 16^{-n^2} \geq \frac{1}{9000^n 16^{n^2}} d.$$

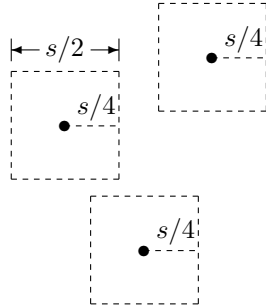


FIGURE 11. The balls $S(x, s/4)$, $x \in M$, remain unchanged and disjoint throughout the adjustment.

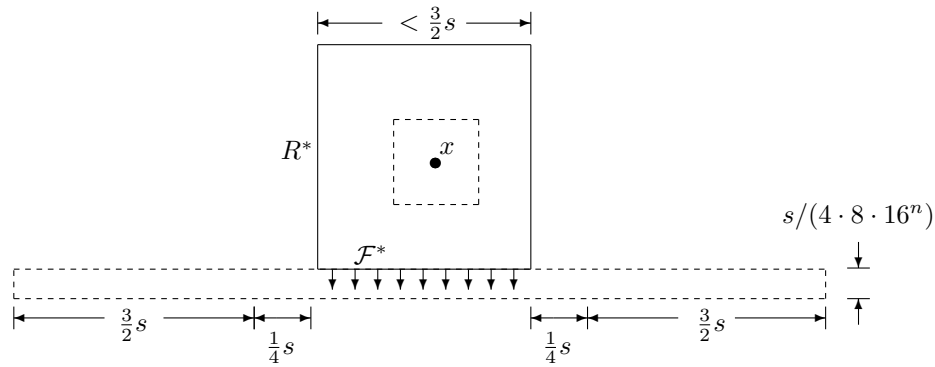


FIGURE 12. In the second step of the adjustment, the face \mathcal{F}^* will be adjusted further by no more than $s/8$ to avoid parallel faces from R_d^n cubes in a large region.

To end stage i , we then, as in the proof of theorem 3.1, extend the previously adjusted faces (of the cubes in \mathcal{R}) which intersect our new adjusted region to extend throughout the new adjusted region. This construction was described in detail in the proof of lemma 3.3. Figure 7 of that proof depicts this construction. Note that two parallel faces F_1, F_2 of previously adjusted regions which both intersect the current region must lie within $\frac{3}{2}s$ of each other. Say these faces came from regions that were adjusted at stages $i_1 < i_2 < i$. Then at stage i_2 the face F'_2 corresponding to F_2 before the adjustment was at distance at most $\frac{3}{2}s + \frac{s}{4} < 2s$ from F_1 . Thus, at stage i_2 we adjusted so that F_2 is at least $\frac{s}{4 \cdot 8 \cdot 16^n}$ away from F_1 . So, these parallel

extensions of the earlier adjusted faces throughout the new adjusted cube will be at distance at least $\frac{s}{4 \cdot 8 \cdot 16^n} = 110d$ from each other. This then partitions the adjusted region into smaller rectangles. Each such rectangle will have edge lengths at least $110d$. Similarly, each face that we extended through the new region belonged to a face of a previously adjusted region that was (before adjustment) at a distance of no more than $\frac{s}{4}$ from the new region. So, these extensions can lie at most $\frac{s}{4} + \frac{3}{2}s = \frac{7}{4}s$ from the original face. In the second stage of the adjustment, we guaranteed that these extended faces will be at least $\frac{1}{9000^n 16^{n^2}}d$ from a parallel face of R_d^n (see figure 12).

At the end of stage k , we have defined a clopen subequivalence relation R' of $F(\mathbb{Z}^n)$ such that on each $F(\mathbb{Z}^n)$ class, R' induces a partition into rectangular regions each having edge lengths between $110d$ and $\frac{3s}{2}$. Furthermore, each face of one of these rectangles is at least $9000^{-n} 16^{-n^2}d$ from a parallel face of a rectangle in R_d^n .

Finally, we subdivide each of these rectangular regions into smaller rectangular regions as follows. Divide each edge into intervals of size between $9\frac{1}{2}d$ and $11\frac{1}{2}d$ (use the fact that any positive integer ≥ 110 can be written as a positive integral linear combination of 10 and 11). Consider one of the corresponding faces (that is, perpendicular to the edge and passing through one of the endpoints of these intervals). We move it by no more than $\frac{d}{2}$ to avoid the parallel faces of rectangles in R_d^n within $10d$. There are at most

$$2 \cdot \frac{(\frac{3s}{2} + 20d)^{n-1} (\frac{d}{2} + 20d)}{d^n} \leq 2^n \cdot 21 \cdot C^{n-1}$$

faces to avoid, so we may avoid them by at least

$$\frac{\binom{d}{2}}{2^n \cdot 21 \cdot C^{n-1}} \geq \frac{1}{9000^n 16^{n^2}}d.$$

The resulting rectangular regions have edge lengths between $9d$ and $12d$. Each face of one of these regions is at least $\frac{1}{9000^n 16^{n^2}}d$ from a parallel face of a rectangle in R_d^n which is within $10d$ of the face. Thus, we have satisfied the requirements of the lemma. \square

The next lemma is the technical strengthening of lemma 5.2 that we actually need. The main difference is that we now avoid finitely many rectangular subequivalence relations instead of just one. The reader should think of the bound b which appears in the statement as a fairly small integer (its precise role will become clearer in the next section).

Lemma 5.3. *Let R_1, \dots, R_k be a sequence of clopen subequivalence relations of $F(\mathbb{Z}^n)$ satisfying the following:*

- (1) *On each $F(\mathbb{Z}^n)$ class, each R_i induces a partition into polyhedral regions R which are unions of rectangles r such that each edge length of one of the r rectangles is between d and $12d$.*
- (2) *In any ball B of radius $100,000 \cdot 16^n d$ in some $F(\mathbb{Z}^n)$ equivalence class, there are at most b integers i such that one of the regions in R_i has a face \mathcal{F} which intersects B .*

Then there is a clopen subequivalence relation $\tilde{R}_d^n \subseteq F(\mathbb{Z}^n)$ satisfying:

- (1) *Each \tilde{R}_d^n class is a rectangular region with each edge lengths between $9d$ and $12d$.*

- (2) Every face of a region $R \in \tilde{R}_d^n$ is at least $\frac{1}{9000^n 16^{n^2} b} d$ from any parallel face of a region $R' \in R_i$ (for any i).

Proof. The proof is similar to that of lemma 5.2. The only difference is that in step two of the adjustments, we must avoid all the parallel faces of not just the one family R_d^n , but of all the families R_1, \dots, R_k . However our hypothesis gives that for each rectangle of size

$$\left(\frac{5s}{2} + 22d\right) \times \dots \times \left(\frac{5s}{2} + 22d\right) \times \left(\frac{s}{4 \cdot 8 \cdot 16^n} + 22d\right) < 100,000 \cdot 16^n d$$

there are at most b many i such that some face of a region in R_i intersects this rectangle. So, there are at most

$$(2b) \cdot \left[\frac{\left(\frac{5s}{2} + 22d\right)^{n-1} \cdot \left(\frac{s}{4 \cdot 8 \cdot 16^n} + 22d\right)}{d^n} \right] \leq 9000^n 16^{n^2} b$$

many faces to avoid. Thus we may avoid them by at least

$$\frac{s}{4 \cdot 8 \cdot 16^n} \cdot 9000^{-n} \cdot 16^{-n^2} \cdot \frac{1}{b} \geq \frac{1}{9000^n 16^{n^2} b} d.$$

□

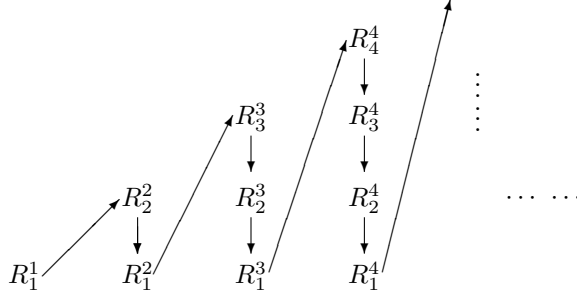
We remark that in hypothesis (1) of lemma 5.3 we use d and $12d$ rather than $9d$ and $12d$ so that formally lemma 5.3 is a strengthening of lemma 5.2.

6. HYPERFINITENESS OF $F(\mathbb{Z}^{<\omega})$

In this section we show that the free part $F \doteq F(\mathbb{Z}^{<\omega})$ of $\mathbb{Z}^{<\omega}$ with the usual action of $\mathbb{Z}^{<\omega}$ is hyperfinite. We let F_ω denote the equivalence relation on F from this action. Each subgroup \mathbb{Z}^n also acts on F and induces a subequivalence relation which we denote F_n . So, $F_1 \subseteq F_2 \subseteq \dots$, and $F_\omega = \bigcup_n F_n$. Of course, all the \mathbb{Z}^n act freely on F as well.

To show F_ω is hyperfinite, we must construct finite subequivalence relations R_i with $R_1 \subseteq R_2 \subseteq \dots$, and with $F_\omega = \bigcup_n R_n$. In fact, the R_i we will build will actually be clopen subequivalence relations. Recall this means that $\{(x, g) \in F \times \mathbb{Z}^{<\omega} : g \cdot x R_i x\}$ is clopen in $F \times \mathbb{Z}^{<\omega}$ (which is equivalent to saying that for each $g \in \mathbb{Z}^{<\omega}$, $\{x \in F : g \cdot x R_i x\}$ is clopen in F). This will then actually give a continuous embedding of F_ω into $E_0(\omega^\omega)$. The rough strategy for building the R_i is as follows. We start with a sufficiently fast growing sequence of marker distances $d_1 \ll d_2 \ll \dots$. For each i , let $R_i^i \subseteq F_i$ be a clopen subequivalence relation of F_i given by theorem 3.1 for marker distance d_i . In $i-1$ steps we will then successively modify the rectangles in R_i^i to regions $R_i^i, R_{i-1}^i, \dots, R_1^i$. Each R_j^i will be a clopen subequivalence relation of F_ω , such that each on each F_j class each region will be a finite union of rectangles with edge lengths on the order of d_j . Furthermore, the rectangles contributing to the boundary of the region will be “orthogonal” to the regions induced by $R_j^j, R_j^{j+1}, \dots, R_j^{i-1}$ (that is, we will maintain a certain distance between nearby parallel faces). The adjusted regions R_1^i will witness the hyperfiniteness of F_ω . Figure 13 illustrates the order of construction of the regions R_j^i .

As we successively adjust the regions in going from R_i^i to R_1^i , the boundaries of the regions will become increasingly “fractal like.” That is, successive adjustments will be done on increasingly small scales.

FIGURE 13. The inductive construction of R_j^i .

We note two points about our final regions R_1^i . First, they will not cohere. That is, we will not have that $R_1^i \subseteq R_1^{i+1}$. Indeed, since all the R_1^i will be clopen, this would be impossible by theorem 3.4. Second, the fact that the boundaries of the R_1^i are becoming increasingly fractal is also in some sense necessary according the following theorem we next state.

Theorem 6.1. *There does not exist a sequence of Borel subequivalence relations R_i of $F(\mathbb{Z}^2)$ such that $F_2 = \bigcup_i R_i$ where on each $F(\mathbb{Z}^2)$ class each R_i induces a partition into “bounded geometry” polygons, and such that for all $x, y \in F(\mathbb{Z}^2)$, $x F_2 y$ iff $\exists n \forall i \geq n \ x R_i y$. By “bounded geometry polygons” we mean: there are numbers $a_i < b_i$ with $\lim_{i \rightarrow \infty} a_i = \infty$ such that each edge of a polygon in R_i has length between a_i and b_i .*

So, for example, even for the finite dimensional relation F_n on $F(\mathbb{Z}^n)$, the original rectangular subequivalence relations R_d^n of theorem 3.1 can never by themselves witness the hyperfiniteness of F_n . Theorem 6.1 says that any attempt along these lines must involve modifying the R_i^i into increasingly fractal regions.

We now turn to the construction of the R_j^i . The next lemma will provide the argument for passing from R_{j+1}^i to R_j^i .

Lemma 6.2. *Let d_{j+1}, d_j be positive integers with $d_{j+1} \gg d_j$ (the exact condition necessary is specified below). Suppose R_{j+1}^i is a clopen subequivalence relation of $F_i \subseteq F_\omega$. Assume that the restriction of R_{j+1}^i to each F_{j+1} class is a union of rectangles each having edge lengths between d_{j+1} and $12d_{j+1}$. Suppose the clopen subequivalence relations $R_j^j, R_j^{j+1}, \dots, R_j^{i-1}$ of F_ω have been defined and satisfy:*

- (1) $R_j^k \subseteq F_k$ for all $j \leq k \leq i-1$.
- (2) R_j^j is a partition of each F_j class into rectangular regions with edge lengths in $\{d_j, d_j + 1\}$.
- (3) For $j < k \leq i-1$, R_j^k restricted to each F_j class gives a partition into regions R each of which is a union of rectangles having edge lengths between $9d_j$ and $12d_j$.
- (4) For each region R induced by R_j^k , there is a region R' induced by R_{j+1}^k such that each face of R is within $12d_j$ of a face of R' .
- (5) In any ball B of radius $100,000 \cdot 16^j d_j$ contained in an F_j class, there are at most $b_j \doteq j+1$ many k (where $j < k \leq i-1$) such that some region in R_j^k has a boundary face intersecting B .

- (6) For any $j < k_1 \neq k_2 \leq i - 1$, and R_1, R_2 regions induced by $R_j^{k_1}, R_j^{k_2}$ respectively, if $\mathcal{F}_1, \mathcal{F}_2$ are parallel faces of R_1, R_2 , then $\rho(\mathcal{F}_1, \mathcal{F}_2) > \frac{1}{9000 \cdot 16^{j^2} (j+1)} d_j$.

Then there is a clopen subequivalence relation $R_j^i \subseteq F_i$ satisfying the following:

- (1) On each F_j class, R_j^i induces a partition in regions R each of which is a union of rectangles with edge lengths between $9d_j$ and $12d_j$.
- (2) For each region R induced by R_j^i , there is a region R' induced by R_{j+1}^i such that each face of R is within $12d_j$ of a face of R' .
- (3) Condition (5) continues to hold, where now $j < k \leq i$.
- (4) Condition (6) continues to hold, where now $j < k \leq i$.

Proof. Let R_{j+1}^i and the $R_j^j, R_j^{j+1}, \dots, R_j^{i-1}$ be given satisfying the above. By assumption, the restriction of R_{j+1}^i to each F_{j+1} class is a union of rectangles with edge lengths between d_{j+1} and $12d_{j+1}$. When we further restrict R_{j+1}^i to F_j , note that each class is still a union of rectangles with edge lengths between d_{j+1} and $12d_{j+1}$ (a section of a rectangular region is also a rectangular region whose edge lengths are a subset of the original edge lengths). We apply lemma 5.3 as follows. Let the d of lemma 5.3 be the current d_j . Let the R_1, \dots, R_k of that lemma be the current $R_j^{j+1}, \dots, R_j^{i-1}$. Hypothesis (1) of lemma 5.3 follows from the current hypothesis (3). Hypothesis (2) of lemma 5.3 follows from the current hypothesis (5).

Let R be the clopen subequivalence relation of F_j given by lemma 5.3 (i.e., the \tilde{R}_d^n of that lemma). Using R , we define the R_j^i as required as follows. Let \mathcal{A} be the collection of all the j -dimensional rectangles induced by R . We define $xR_j^i y$ iff $x \in A_1 \in \mathcal{A}, y \in A_2 \in \mathcal{A}$, and $c(A_1)R_{j+1}^i c(A_2)$, where $c(A)$ denotes the center point of the rectangle A (defined in some reasonable manner for even edge lengths). Thus, each R_j^i class is obtained from an R_{j+1}^i class by adding or subtracting certain rectangles in \mathcal{A} , which must be near the boundary (recall that by the boundary of R_{j+1}^i we mean the set of x such that for some y with $\rho(x, y) = 1, \neg(xR_{j+1}^i y)$).

Conclusion (1) of lemma 6.2 follows from the fact that each of the rectangles in \mathcal{A} has edge lengths between $9d_j$ and $12d_j$.

Conclusion (2) is immediate from the definition of the R_j^i and the fact that each $A \in \mathcal{A}$ has edge lengths $\leq 12d_j$.

That (6) continues to hold follows immediately from conclusion (2) of lemma 5.3.

Finally we come to the key point; that condition (5) continues to hold. Suppose not, and let B be a ball in an F_j class of radius $100,000 \cdot 16^j d_j$. Suppose $j < k_1 < \dots < k_{j+2} \leq i$ are such that each $R_j^{k_j}$ has a face intersecting B . By (4), each of these boundary faces is within $12d_j$ of a boundary face of $R_{j+1}^{k_1}, \dots, R_{j+1}^{k_{j+2}}$. Since there are $j + 2$ of these faces, and we are in $j + 1$ -dimensional space now, at least two of them must be parallel. Thus, we have a face \mathcal{F}_1 of a region in $R_{j+1}^{k_a}$ and a face \mathcal{F}_2 of a region in $R_{j+1}^{k_b}$ (for some $j < a, b \leq i$) which are parallel and within $200,000 \cdot 16^j d_j + 24d_j$ of each other. However, our assumption is that distinct parallel faces from amongst the R_{j+1}^k regions are separated by at least $\frac{1}{9000 \cdot 16^{(j+1)^2} (j+2)} d_{j+1} > 200,000 \cdot 16^j d_j + 24d_j$ since $d_{j+1} \gg d_j$. This contradiction shows that (5) continues to hold and completes the proof of the lemma. \square

We are now in a position to prove the main result of this section, that the shift equivalence relation F_ω on $F(\mathbb{Z}^{<\omega})$ is hyperfinite. In fact, we prove this in a strong form which we now state.

Theorem 6.3. *There is a continuous embedding from $(F(\mathbb{Z}^{<\omega}), F_\omega)$ to E_0 .*

Proof. Fix a sufficiently fast growing sequence of natural numbers $1 \ll d_1 \ll d_2 \ll \dots$. For each i , let R_i^i be the clopen subequivalence relation of $F_i \subseteq F_\omega$ from theorem 3.1 inducing a rectangular partition of the F_i classes with edge lengths in $\{d_i, d_i + 1\}$. Inductively on i we define the clopen subequivalence relations $R_i^i, R_{i-1}^i, \dots, R_1^i$. R_i^i has already been defined, and we assume that R_{j+1}^i and all the R_l^k for $1 \leq l \leq k < i$ have been defined. In particular, all the $R_j^j, R_j^{j+1}, \dots, R_j^{i-1}$ have been defined. We assume inductively that R_{j+1}^i and the R_j^j, \dots, R_j^{i-1} satisfy the hypotheses of lemma 6.2. We then get R_j^i from lemma 6.2. From lemma 6.2, the new collection continues to satisfy these hypotheses.

So, we may assume the clopen subequivalence relations R_j^i are defined for all $i \geq 1$ and all $1 \leq j \leq i$, and they satisfy the hypotheses of lemma 6.2. Let R_i abbreviate R_1^i . We show that if $x, y \in F(\mathbb{Z}^{<\omega})$ and $xF_\omega y$, then for all large enough i we have $xR_i y$.

Fix k_0 such that $xF_{k_0} y$. Suppose that there are infinitely many i , say i_0, i_1, \dots , such that $\neg(xR_{i_m} y)$ for all m . Let $\eta = \rho(x, y)$. All $R_1^{i_m}$ must have a boundary point in the sphere of radius η about x . By thinning the sequence we may assume that for some fixed z and all m that z is a boundary point of $R_1^{i_m}$. Fix $k_1 > k_0$, and consider $k_1 + 1$ many of the i_m which are greater than k_1 , say (recycling the notation) i_0, \dots, i_{k_1+1} . From property (2) of lemma 6.2 we have that for each of these i_m that there is a point on a boundary face of $R_{k_1}^{i_m}$ which is within $12(d_1 + d_2 \cdots + d_{k_1-1})$ of z . At least two of these faces must be parallel (the faces of the regions in $R_{k_1}^{i_m}$ are $(k_1 - 1)$ -dimensional hyperplanes in k_1 dimensional space). However, conclusion (4) of lemma 6.2 implies that any two such parallel faces must be at least

$$\frac{1}{9000^{k_1} 16^{k_1^2} (k_1 + 1)} d_{k_1} > 12(d_1 + d_2 \cdots + d_{k_1-1})$$

apart, using here $d_{k_1} \gg d_{k_1-1}$. This contradiction shows that $xR_i y$ for all large enough i .

Finally, we define our embedding π from $(F(\mathbb{Z}^{<\omega}), F_\omega)$ to E_0 . For $x \in F(\mathbb{Z}^{<\omega})$ we let $\pi(x) = (\pi_1(x), \pi_2(x), \dots)$, where $\pi_i(x) \in \omega$ is defined as follows. Let $c_i = c_i(x)$ be the ‘‘center point’’ of the R_i class containing x . By this we mean the center of the rectangular R_i^i class closest to the R_i class of x (i.e., the unique R_i^i class within Hausdorff distance $12d_1 + \dots + 12d_{i-1}$ of the $R_i = R_1^i$ class containing x). Let $N_i = N_i(x)$ be the integer coding the neighborhood $c_i \upharpoonright \{g \in \mathbb{Z}^i \wedge \|g\| \leq 20d_i\}$ of c_i . Here we regard $\mathbb{Z}^i \subseteq \mathbb{Z}^{<\omega}$ in the natural way. Let also $h_i = h_i(x)$ be the unique element of \mathbb{Z}^i such that $h_i \cdot c_{i-1} = c_i$. Let $\pi_i(x)$ be the integer coding the pair $(N_i(x), h_i(x))$. If $k = 1$, have $\pi_k(x)$ also code the unique $h \in \mathbb{Z}$ such that $h \cdot x = c_1(x)$. π is continuous since all of the R_i regions are clopen.

If $x, y \in F$ and $xF_\omega y$, then for large enough i we have $xR_i y$, and so $c_i(x) = c_i(y)$. It follows that $\pi(x)E_0\pi(y)$. On the other hand, from the E_0 class of $\pi(x)$ we may recover the F_ω class of x using the h_i and the fact that $N_i(x)$ codes a neighborhood of size $20d_i$ about $c_i(x)$ while the distance between $c_{i-1}(x)$ and $c_i(x)$ is at most $13d_i$. Likewise, π is one-to-one since from $\pi(x)$ we can in a similar manner reconstruct x .

□

7. THE NON-FREE PART

In §6 we showed that the free part F of $X = 2^{\mathbb{Z}^{<\omega}}$ (always considered with the shift action of $\mathbb{Z}^{<\omega}$) was hyperfinite. In fact, we showed this in a strong way, namely we showed that there is a continuous embedding from F into E_0 . Here we show that the entire space X is hyperfinite. For any fixed dimension n , we can show the stronger result for $X_n = 2^{\mathbb{Z}^n}$, that is, there is a continuous embedding from X_n (with the shift action of \mathbb{Z}^n) into E_0 . We do not know if this stronger result holds for X .

Throughout this section $E = E_\omega$ denotes the equivalence relation on X generated by the shift action of $\mathbb{Z}^{<\omega}$, so $E = \bigcup_n E_n$, where E_n is the subequivalence relation generated by the action of the subgroup \mathbb{Z}^n .

For $x \in X$, let $G_x = \{g \in \mathbb{Z}^{<\omega} : g \cdot x = x\}$ be the stabilizer of x . The map $x \mapsto G_x$ is a Borel map from X to $\mathcal{P}(\mathbb{Z}^{<\omega})$ (which is essentially ω^ω). For H a subgroup of $\mathbb{Z}^{<\omega}$, let $X_H = \{x \in X : G_x = H\}$, so X_H is an invariant Borel subset of X . Let E_H denote the equivalence relation E restricted to X_H . We will define our reduction of X to E_0 by defining it on each X_H separately. More precisely, we will define a Borel map $(x, H) \mapsto \pi(x, H) \in \omega^\omega$ with domain the Borel subset of $X \times \mathcal{P}(\mathbb{Z}^{<\omega})$ consisting of those (x, H) with $G_x = H$. For each H , this map restricted to X_H (more precisely $X_H \times \{H\}$) will be a reduction of (X_H, E_H) to E_0 . This will suffice as we can then easily modify π to π' which codes H into the output as well as $\pi(x, H)$. Then π' will be a reduction of (X, E) to E_0 . Since x determines H , we will usually write $\pi(x)$ instead of $\pi(x, H)$. In the argument below we will in fact fix the subgroup H and just define π on X_H . The reader can check that the map we define is actually Borel as a function of both arguments x and H . The fact that the map $x \mapsto H_x$ is only Borel (and not continuous), however, is what prevents us from getting a continuous reduction.

We let $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, \dots , e_n, \dots be the standard set of generators for the group $\mathbb{Z}^{<\omega}$. Every element $g \in \mathbb{Z}^{<\omega}$ can be written uniquely in the form $a_1 e_1 + \dots + a_k e_k = (a_1, \dots, a_k, 0, 0, \dots)$. We say $k = \text{supp}(g)$ is the *support* of g . We view \mathbb{Z}^n as a subgroup of $\mathbb{Z}^{<\omega}$ by being the subgroup generated by e_1, \dots, e_n . We let $\vec{0}$ denote the identity element $(0, 0, \dots)$ of $\mathbb{Z}^{<\omega}$. For $g \in \mathbb{Z}^{<\omega}$ and $n \in \omega$, let $\pi_n(g) = a_n$ where $g = a_1 e_1 + \dots + a_k e_k$ (the coefficients a_l for $l > k$ are regarded as 0).

Definition 7.1. Let H be a subgroup of $\mathbb{Z}^{<\omega}$. We say that $B = \{h_1, h_2, \dots\}$ is a *normal basis* for H if the following are satisfied:

- (1) $H = \langle h_1, h_2, \dots \rangle$.
- (2) $\text{supp}(h_1) < \text{supp}(h_2) < \dots$.
- (3) For every n , $H \cap \mathbb{Z}^n = \langle h_i : \text{supp}(h_i) \leq n \rangle$.

We have the following standard algebraic fact. This is essentially the same as lemma 4.7, for which we gave a geometric proof. Here for the sake of completeness we include a simpler algebraic proof.

Lemma 7.2. *Every subgroup H of $\mathbb{Z}^{<\omega}$ has a normal basis.*

Proof. Let h_1, h_2, \dots enumerate the non- $\vec{0}$ elements of the sequence h'_1, h'_2, \dots , where h'_n is defined as follows. If $H \cap \mathbb{Z}^n = H \cap \mathbb{Z}^{n-1}$, let $h'_n = \vec{0}$. Otherwise, let

$I_n = \{\pi_n(h) : h \in H \cap \mathbb{Z}^n\}$. I_n is an ideal in \mathbb{Z} , and so is principal. Let $h'_n \in H$ be such that $I_n = \langle \pi_n(h'_n) \rangle$. It is straightforward to check that $H \cap \mathbb{Z}^n = \langle h'_1, \dots, h'_n \rangle$, and the result follows. \square

Henceforth we fix the subgroup H of $\mathbb{Z}^{<\omega}$ and a normal basis $B = \{h_1, h_2, \dots\}$ for H . It is easy to check that we may choose a normal basis in a Borel manner from H .

Definition 7.3. We say n is an *essential dimension* if $H \cap \mathbb{Z}^n = H \cap \mathbb{Z}^{n-1}$. Otherwise we say n is an *inessential dimension*.

So, n is an inessential dimension precisely when we have $n = \text{supp}(h_i)$ for some i (necessarily $i \leq n$).

For each n , we define the “size” of the n^{th} dimension, $s(n)$ to be ∞ if n is an essential dimension and otherwise if $h'_n = a_1 e_1 + \dots + a_n e_n$, then $s(n) = a_n$. Thus, $s(n)$ is finite for the inessential dimensions. The following fact records the significance of this.

Lemma 7.4. *For every $g \in \mathbb{Z}^{<\omega}$, there is a unique $g' \in \mathbb{Z}^{<\omega}$ such that $g - g' \in H$ and such that for all inessential dimensions n , $0 \leq \pi_n(g') < s(n)$.*

Proof. We first show the existence of such a g' . Let n_0 be the largest inessential $n \leq \text{supp}(g)$ such that $n(g) < 0$ or $n(g) \geq s(n)$. From the definition of $s(n)$ there is an integer a with $0 \leq \pi_{n_0}(g - ah'_{n_0}) < s(n_0)$. Also, $\pi_n(g - ah'_{n_0}) = \pi_n(g)$ for $n > n_0$. By induction, there is an $h \in H$ such that $g' = (g - ah'_{n_0}) - h$ satisfies $0 \leq \pi_n(g') < s(n)$ for all inessential n . So, $g = g' + (ah'_{n_0} + h)$ with $ah'_{n_0} + h \in H$, so we are done.

To see uniqueness, suppose g'' also satisfied the requirements, and $g'' \neq g'$. Note that $g'' - g' \in H$. Let n_0 now denote the largest n such that $\pi_n(g'') - \pi_n(g') \neq 0$. Since $g'' - g' \in H$, we must have that n_0 is inessential. However, $0 < |\pi_{n_0}(g'') - \pi_{n_0}(g')| < s(n_0)$. Without loss of generality we may assume $0 < \pi_{n_0}(g'') - \pi_{n_0}(g') < s(n_0)$. This, however, violates the definition of $s(n_0)$ since $g'' - g' \in H$ and $s(n_0)$ is the minimal values of $\pi_{n_0}(h)$ for $h \in H$ with $\text{supp}(h) \leq n_0$. \square

We say a $g \in \mathbb{Z}^{<\omega}$ is in *standard form* if for all inessential dimensions n we have $0 \leq \pi_n(g) < s(n)$. Lemma 7.4 says that the elements in standard form form a set of coset representatives for $\mathbb{Z}^{<\omega} \bmod H$. If we fix a particular $x \in X_H$, then this says that $[x]_E = \{g \cdot x : g \text{ is in standard form}\}$. So, for a fixed x we can identify $[x]_E$ with the set of $g \in \mathbb{Z}^{<\omega}$ of standard form. This identification, however, is not invariant, but depends on the particular choice of x .

We define a notion of “distance” on X_H , although it will not correspond to a true metric. Recall that for $g = a_1 e_1 + \dots + a_k e_k \in \mathbb{Z}^{<\omega}$, $\|g\| = \max\{|a_i| : i \leq n\}$.

Definition 7.5. Let $x, y \in X_H$ with $x E_n y$. Let $\rho_x^n(y) = \|g\|$, where $g \in \mathbb{Z}^n$ is the unique element in standard form such that $g \cdot x = y$.

Thus, $\rho_x^n(y)$ is the “distance” from x to y as seen in the local (i.e., centered about x) rectangular coordinate system about x . This does not give a true metric since $\rho_x^n(y) \neq \rho_y^n(x)$ in general. However, these functions are in some close to defining a metric. We make this precise in the following lemma. First we define a combinatorial function $b(n)$ which will bound the error in these functions failing to define a metric.

Definition 7.6. Fix n . For each inessential dimension $i \leq n$, let $c_i = \max_{j \leq i} \pi_j(h'_i)$. Let $s = \max\{s(i) : i \leq n\}$. Then define $b(n) = \prod_{i \leq n} (c_i + s)$.

Clearly for any inessential i with $i \leq n$ we have $s(i) \leq b(n)$.

Lemma 7.7. For any $x, y \in X_H$ with $x E_n y$ we have $|\rho_x^n(y) - \rho_y^n(x)| \leq b(n)$.

Proof. There is a $g \in \mathbb{Z}^{<\omega}$ in standard form with $\text{supp}(g) \leq n$ and $\|g\| = \rho_x^n(y)$ such that $g \cdot x = y$. Say $g = (a_1, \dots, a_n)$. Then $(-g) \cdot y = x$, and $-g = (-a_1, \dots, -a_n)$. Let $n \geq i_0 > i_1 > \dots > i_k \geq 1$ enumerate the inessential dimensions in decreasing order. To put $-g$ back into standard form we first add at most one copy of h'_{i_0} to $-g$ (we add nothing in this step if $\pi_{i_0}(-g) = 0$). This will put the coefficient of e_{i_0} in the desired range. The coefficient of e_{i_1} will increase by at most c_{i_0} , and so will be at most $c_{i_0} + s$ in absolute value after this step. To next bring the coefficient of e_{i_1} into the desired range we need add at most $c_{i_0} + s$ many copies of h'_{i_1} , which will change the coefficient of e_{i_2} by at most $c_{i_1}(c_{i_0} + s)$. So the new coefficient of e_{i_2} will be at most $s + c_{i_1}(c_{i_0} + s) \leq (c_{i_1} + s)(c_{i_0} + s)$. Continuing, after putting all the coefficients of $-g$ (on the inessential dimensions) into the desired range, the remaining coefficients are changed by at most $b(n)$. This gives the result. \square

We next extend our basic marker result to the space X_H .

Lemma 7.8. Fix n and a marker distance $d \gg b(n)$. Then there is a clopen set $M_d^n \subseteq X_H$ satisfying the following:

- (1) For every $x, y \in M_d^n$ with $x E_n y$ we have $\rho_x^n(y) \geq d - b(n)$.
- (2) For every $y \in X_H$, there is an $x \in M_d^n$ such that $\rho_x^n(y) \leq d$.

Proof. The proof is similar to that of lemma 2.1. Let s_0, s_1, \dots enumerate the elements s of $2^{<\mathbb{Z}^n}$ satisfying:

- (1) The basic neighborhood $N_s \cap X_H$ of X_H is nonempty.
- (2) For every $\vec{0} \neq g \in \mathbb{Z}^{<\omega}$ in standard form with $\|g\| \leq d + b(n)$, $g \cdot N_s \cap N_s = \emptyset$.

As in the proof of lemma 2.1 we define by induction on i a sequence of clopen sets $S_i \subseteq X_H$. Let $S_0 = N_{s_0} \cap X_H$. Note that if $x, y \in N_{s_i} \cap X_H$ and $x \neq y$, then $\rho_x^n(y), \rho_y^n(x) > d + b(n)$. In particular, (1) holds for the set S_0 . For $i > 0$ we define

$$S_i = S_{i-1} \cup (N_{s_i} - \cup\{g \cdot S_{i-1} : \|g\| \leq d \text{ and } g \text{ is in standard form}\}).$$

Let $M_d^n = \bigcup_i S_i$. If $x \neq y$ are both in S_{i-1} , then $\rho_x^n(y), \rho_y^n(x) > d$ by induction. If $x \neq y$ are both in N_{s_i} , then $\rho_x^n(y), \rho_y^n(x) \geq d + b(n)$. Finally, suppose $x \in S_{i-1}$ and $y \in N_{s_i} - \cup\{g \cdot S_{i-1} : \|g\| \leq d \text{ and } g \text{ is in standard form}\}$. Then $\rho_x^n(y) > d$. From lemma 7.7 we also have $\rho_y^n(x) > d - b(n)$. It follows that (1) holds for M_d^n .

To see (2), let $y \in X_H$. Let i be least such that $y \in N_{s_i}$. This exists since for every $\vec{0} \neq g \in \mathbb{Z}^{<\omega}$ in standard form, $g \cdot y \neq y$, and so for some neighborhood N of y , $g \cdot y \cap N = \emptyset$. If (2) failed for y , then $y \notin \cup\{g \cdot S_{i-1} : \|g\| \leq d \text{ and } g \text{ is in standard form}\}$. It follows from the definition of S_i that $y \in S_i \subseteq M_d^n$, which contradicts the failure of (2). \square

By a *rectangle* in an E_n equivalence class of X_H we mean a set of the form $R = \{g \cdot x : g \in A\}$ where A is the set of $g = (g_1, \dots, g_n) \in \mathbb{Z}^{<\omega}$ in standard form such that for all essential dimensions $i \leq n$, $a_i \leq g_i \leq b_i$ for some integers a_i, b_i . We say R is *centered about the point* x if for all essential dimensions $i \leq n$ we have $|g_i| \leq b_i$, where $\{2b_i\}$ are the *edge lengths* of the rectangle (these are explicitly

defined only for the essential dimensions; for an inessential dimension i , the implicit edge length is $s(i)$). We say x is the *center* of the rectangle R . For each essential dimension $i \leq n$ of a rectangle R in E_n as above, there are two i -faces of R , that is, the points of R of the form $\{g \cdot x: g_i = a_i\}$ and likewise for b_i . Note that if R is a rectangle centered about x , and $y \in E_n x$, then $\{g: g \cdot y \in R\}$ does not necessarily form a rectangle in \mathbb{Z}^n . However, by lemma 7.7 this set is within $b(n)$ of a rectangle in \mathbb{Z}^n . So, if the edge lengths of R are large compared with $b(n)$, then in the coordinate system centered about y , R will still be almost a rectangle of the same edge lengths. This is the key point behind the following arguments.

By an ℓ -almost rectangle in E_n of edge lengths d_1, d_2, \dots (these are specified for the essential dimensions $\leq n$) we mean a set $R \subseteq [x]_{E_n} \subseteq X_H$ such that there is a rectangle R' in E_n with edge lengths d_1, d_2, \dots and such that the ‘‘Hausdorff distance’’ between R and R' is at most ℓ . That is, $\forall x \in R \exists y \in R' \rho_x^n(y) \leq \ell$ and $\forall x \in R' \exists y \in R \rho_x^n(y) \leq \ell$. In practice, the almost rectangles we use will be regions whose boundaries are faces of actual rectangles (they won’t be actual rectangles because the different faces will come from rectangles having different centers; a ‘‘flat face’’ with respect to the coordinate system about x will not be flat in the system about another point y). Note that it still makes sense to talk about i -faces in this case. Based on this, we make the following definition.

Definition 7.9. A region R in an E_n equivalence class $[x]_{E_n}$ is said to be a \vec{b} -almost rectangle in E_n if for every $k \leq n$, and every $y \in E_n x$, $R \cap [y]_{E_k}$ is a $b(k)$ -almost rectangle in E_k .

The following geometric fact will be important for our main proof.

Lemma 7.10. *Let R be a rectangle in an E_n -class $[x]_{E_n}$. Let $y \in E_n x$. Then $\{g \cdot y: g \cdot y \in R \text{ and } g \text{ is in standard form}\}$ is a \vec{b} -almost rectangle in E_n .*

Proof. Let R be a rectangle centered about x . Say

$$R = \{g \cdot x: a_i \leq \pi_i(g) \leq b_i \text{ for all essential } i \leq n \\ \wedge 0 \leq \pi_i(g) < s(i) \text{ for inessential } i \leq n\}.$$

Let $h = (a_1, \dots, a_n)$ be such that $h \cdot y = x$, where we may assume that h is in standard form. Then $R \cap [y]_{E_m}$ is of the form:

$$\{g \cdot x: g = (\pi_1(g), \dots, \pi_m(g), -a_{m+1}, \dots, -a_n) \wedge a_i \leq \pi_i(g) \leq b_i \text{ for all essential } i \leq m \\ \wedge 0 \leq \pi_i(g) < s(i) \text{ for inessential } i \leq m\},$$

So, $R \cap [y]_{E_m}$ is also of the form:

$$\{g \cdot y: g = (a_1, \dots, a_m, 0, \dots, 0) + (\pi_1(g), \dots, \pi_m(g), 0, \dots, 0) \\ \wedge a_i \leq \pi_i(g) \leq b_i \text{ for all essential } i \leq m \\ \wedge 0 \leq \pi_i(g) < s(i) \text{ for inessential } i \leq m\}.$$

Since (a_1, \dots, a_m) is in standard form, the argument of lemma 7.7 shows that this is a $b(m)$ -almost rectangle. \square

From lemma 7.10 we have that if R_x is a rectangle centered about x in the x -coordinate system, then for any $y \in E_n x$, in the y -coordinate system R_x is a \vec{b} -almost rectangle.

We next generalize our rectangular marker region lemma to the non-free case.

Lemma 7.11. *Fix n , and let $d \gg b(n)$. There is a clopen partition \mathcal{R} of X_H such that each $R \in \mathcal{R}$ is a \vec{b} -almost rectangle in E_n with edge lengths d .*

Proof. The proof is similar to that of theorem 3.1. Fix $\Delta_2 \gg \Delta_1 \gg D > d^2$. Let $M_1 = M_{D_1}^n \subseteq X_H$ be the clopen marker set from lemma 7.8 for distance Δ_1 . Let $N \subseteq X_H$ be a clopen marker set for distance Δ_2 . For each $x \in N$, let $g_x \in \mathbb{Z}^{<\omega}$ be the least vector in standard form such that $g_x \cdot x \in M_1$ (least here refers to some fixed ordering on $\mathbb{Z}^{<\omega}$ which respects length). So, $\|g_x\| \leq \Delta_1 + b(n)$. Let $M_2 = \{g_x \cdot x : x \in N\} \subseteq M_1$. Also, M_2 is clopen. If $x \neq y$ are both in M_2 , we claim that $\rho_x^n(y) \geq \Delta_2 - (2\Delta_1 + 6b(n))$. To see this, note that there is a vector g_1 in standard form and an $x' \in N$ with $g_1 \cdot x' = x$ and $\|g_1\| \leq \Delta_1 + b(n)$. There is a vector g_2 in standard form with $g_2 \cdot x = y$ and $\|g_2\| \leq \rho_x^n(y)$. There is a g'_3 in standard form and a $y' \in N$ with $g'_3 \cdot y' = y$ and $\|g'_3\| \leq \Delta_1 + b(n)$. So, there is a g_3 in standard form with $g_3 \cdot y = y'$ and $\|g_3\| \leq \Delta_1 + 2b(n)$. Let $g = g_1 + g_2 + g_3$. Then $g \cdot x' = y'$ and $\|g\| \leq \rho_x^n(y) + 2\Delta_1 + 3b(n)$. Since g is a sum of three vectors in standard form, the argument of lemma 7.7 shows that there is g' equivalent to $g \bmod H$ with $\|g'\| \leq \|g\| + 2b(n)$. So, $\|g'\| \leq \rho_x^n(y) + 2\Delta_1 + 5b(n)$. Thus, $\rho_{x'}^n(y') \leq \rho_x^n(y) + 2\Delta_1 + 5b(n)$. Since $\rho_{x'}^n(y') \geq \Delta_2 - b(n)$, this shows $\rho_x^n(y) \geq \Delta_2 - (2\Delta_1 + 6b(n)) \gg \Delta_1$.

A similar argument shows that for every $y \in X_H$, there is an $x \in M_2$ such that $\rho_x^n(y) \leq \Delta_2 + \Delta_1 + 3b(n)$. Let g_1, \dots, g_k enumerate the elements g of $\mathbb{Z}^{<\omega}$ in standard form with $\|g\| \leq \Delta_2 + \Delta_1 + 3b(n)$. We define the sequence A_0, A_1, \dots, A_k of clopen subsets of X_H as follows. Let $A_0 = M_2$. For $i > 0$ let $A_i = M_1 \cap (g_i \cdot M_2) - \bigcup_{j < i} A_j$. So, M_1 is the disjoint union of the A_i , and for every i and every $x \neq y$ in A_i we have $\rho_x^n(y) \geq \Delta_2 - (2\Delta_1 + 6b(n))$.

For each $x \in M_1$, let $R_x \subseteq [x]_{E_n}$ be the rectangle centered about x with edge lengths $4\Delta_1$. Thus, $X_H = \bigcup_{x \in M_1} R_x$. In $k+1$ steps we adjust the rectangles R_x for x in A_0, \dots, A_k to rectangular regions R'_x . We will move each face of a rectangle R_x at each step by no more than $\frac{1}{10}\Delta_1$. At each step, as we define R'_x we also describe a partition of R'_x into smaller almost rectangular regions.

To begin, for $x \in A_0$ let $R'_x = R_x$. Suppose the adjusted rectangles R'_x have been defined for $x \in A_j$, $j < i$. Consider $x \in A_i$. We move each face of R_x to avoid parallel faces of nearby R'_y for $y \in \bigcup_{j < i} A_j$. To be precise, consider a face \mathcal{F} of R_x . It is of the form

$$\mathcal{F} = \{g \cdot x : \pi_i(g) = 2\Delta_1 \wedge \forall j \neq i (|g(j)| \leq 2\Delta_1)\},$$

where $i \leq n$ is an essential dimension, and the j 's also ranging over the essential dimensions (the case where $\pi_i(g) = -\Delta_1$ is similar). We define an integer a with $|a| \leq \frac{1}{10}\Delta_1$, and shift the face to

$$\mathcal{F}' = \{g \cdot x : \pi_i(g) = 2\Delta_1 + a \wedge \forall j \neq i (|g(j)| \leq 2\Delta_1)\}.$$

Shifting these faces will define the new rectangle R'_x . We wish to find a so that the shifted face \mathcal{F}' will satisfy: (*) for any parallel face \mathcal{G} of a rectangle in $\bigcup_{j < i} \bigcup_{y \in A_j} R_y$, and any $u \in \mathcal{F}'$, $v \in \mathcal{G}$, $\rho_u^n(v) \geq 2D + b(n)$. In saying that \mathcal{G} and \mathcal{F}' are “parallel” we naturally mean that $\mathcal{G} \subseteq \{g \cdot y : \pi_i(g) = b\}$ for some b and $y \in \bigcup_{j < i} \bigcup_{y \in A_j}$. There is clearly a bound $N = N(n)$ depending only on n , for the number of $y \in \bigcup_{j < i} A_j$ with $\rho_x^n(y) \leq 6\Delta_1$. In fact a “volume” argument shows this number to be bounded by $\frac{(12\Delta_1)^m}{(\Delta_1/2 - b(n))^m} \leq 48^m \leq 48^n$ (assuming $\frac{b(n)}{\Delta_1} < \frac{1}{4}$), where m is the number of essential dimensions $\leq n$. Note that if R'_y is a previously adjusted rectangle having

a face \mathcal{G} parallel to \mathcal{F} and such that there are $u \in \mathcal{F}$, $v \in \mathcal{G}$ with $\rho_u^n(v) \leq \Delta_1$, then $\rho_x^n(y) \leq 2\Delta_1 + 3b(n) < 6\Delta_1$. Thus, there are at most $2N$ faces \mathcal{G} to avoid. In the coordinate system about x , each of these faces consists of points whose i^{th} coordinates vary by at most $b(n)$, by lemma 7.7. The interval of possible values for a has size $\frac{2}{10}\Delta_1$, and there are $2N$ many intervals of size $b(n)$ to avoid. We may find an a in the interval of possible values which avoids all of the smaller intervals by at least $\frac{2\Delta_1}{(10)(4N)}$, given that Δ_1 is sufficiently large compared to $b(n)$ (actually we can avoid by $\frac{2\Delta_1}{(10)(2N)(1+\epsilon)}$, where ϵ goes to 0 with $\frac{b(n)}{\Delta_1}$). Let R'_x be the adjusted rectangle. We may assume that $\frac{2\Delta_1}{(10)(4N)} > 2D + b(n)$, and it follows that $(*)$ holds. This completes the definition of the R'_x for $x \in A_i$.

Finally, we partition each new adjusted rectangle R'_x . For each essential dimension $i \leq n$, let $\mathcal{G}_0^i, \dots, \mathcal{G}_{k_i}^i$ enumerate the faces of the R'_y , $y \in \bigcup_{j < i} A_j$, which intersect R'_x . Each of these faces \mathcal{G}_j^i is a rectangular face in the coordinate system of some y with yE_nx . That is, $\mathcal{G}_j^i \subseteq \mathcal{H}_j^i = \{g \cdot y : \pi_i(g) = c_j\}$ for some c_j . The $\{\mathcal{H}_j^i\}$ partition R'_x into smaller regions, each of which is a \vec{b} -almost rectangle in E_n with edge lengths at least $2D$. Note that a finite intersection of \vec{b} -almost rectangles is a \vec{b} -almost rectangle. From this observation it follows that all of the regions produced are \vec{b} -almost rectangles. The set $S = R'_x - \bigcup_{j < i} \bigcup_{y \in A_i} R'_y$ is a union of a subset of these \vec{b} -almost rectangular regions.

At the end of step k , we have partitioned each E_n class into \vec{b} -almost rectangular regions, each having edge lengths of at least $2D - 2b(n)$, that is, for any u, v on parallel faces of the almost rectangular subregion, $\rho_x^n(y) \geq 2D - 2b(n)$. Finally, since $2D - 2b(n) > D > d^2$, it is straightforward to divide each such \vec{b} -almost rectangular region T into \vec{b} -almost rectangular subregions with edge lengths almost d , that is, for u, v on parallel faces we have $d - b(n) - 1 \leq \rho_x^n(y) \leq d + b(n) + 1$. Namely, work in the x -coordinate system. For each essential dimension $i \leq n$, let $[a_i, b_i]$ be an interval in \mathbb{Z} approximating within $b(n)$ the values of $\pi_i(g)$ for the set of g such that $g \cdot x \in T$. Divide each such interval into subintervals each of length d or $d + 1$. This defines the partition of T into \vec{b} almost rectangles of edge length d . \square

We next give the analog of lemma 5.2 in the non-free case.

Lemma 7.12. *Fix n and $d \gg b(n)$. Suppose R_d^n is a clopen partition of (X_H, E_n) into \vec{b} -almost rectangular regions of edge length d . Then there is a clopen subequivalence relation \tilde{R}_d^n on (X_H, E_n) satisfying:*

- (1) *Each \tilde{R}_d^n class is a \vec{b} -almost rectangle with edge lengths between $9d$ and $12d$.*
- (2) *If \mathcal{F}_1 is a face of a region $R_1 \in \tilde{R}_d^n$, and \mathcal{F}_2 a parallel face of a region $R_2 \in R_d^n$, then for any $u \in \mathcal{F}_1$, $v \in \mathcal{F}_2$, $\rho_x^n(y) > \frac{1}{9000 \cdot 16^{n^2}} d$.*

Proof. First note that if d is sufficiently large compared to $b(n)$ then the estimate of lemma 5.1 holds for \vec{b} -almost rectangles of edge lengths d (the same proof will lead to $[\frac{3}{5}d + b(n), \frac{4}{5}d - b(n)]^n \subseteq R_1 \cap R_2$, a contradiction). As in lemma 5.2, let $s = Cd$ where $C = 110 \cdot 4 \cdot 8 \cdot 16^n$. As in lemma 7.11, there is a clopen set $M = M_0 \cup M_1 \cup \dots \cup M_k$ such that the set \mathcal{R} of rectangles of edge lengths s centered at the points of M covers X_H , and for each l , and any $x \neq y$ in M_l , $\rho_x^n(y) \gg s$. As in the proof of lemma 5.2, we proceed in k steps. At step l we

adjust the rectangles with centers in M_l . As in that proof we do each adjustment in two steps. The set we are adjusting is a rectangle R_x in the coordinate system about the center point x . For each essential dimension i , we adjust each of the two faces of R_x . The precise notion of “face” and “adjustment” is as in the proof of lemma 7.11. As in lemma 5.2, in the first step we adjust by no more than $\frac{s}{8}$, and in the second step by no more than $\frac{s}{4 \cdot 8 \cdot 16^n} < \frac{s}{8}$. The various faces to be avoided are now no longer true faces as seen in the x coordinate system, but within $b(n)$ of a face (as in the proof of lemma 7.11). The only point is that as long as d is sufficiently large compared to $b(n)$, then the inequalities in the proof of lemma 5.2 continue to hold.

Consider, for example the second step of the adjustment. Each of the surrounding almost rectangles to be avoided can be approximated within $b(n)$ by a rectangle with edge lengths d (for the essential dimensions). Say there are $m \leq n$ essential dimensions $i \leq n$. Then the estimate for the left-hand side of equation (1) from lemma 5.2 is modified by replacing occurrences of d in the numerator by $d + b(n)$, and in the denominator by $d - b(n)$. So, we have:

$$2 \cdot \frac{\left(\frac{3s}{2} + 20(d + b(n))\right)^{m-1} \cdot \left(\frac{s}{4 \cdot 8 \cdot 16^n} + 20(d + b(n))\right)}{(d - b(n))^m}$$

as the upperbound, which is still less than $9000^n \cdot 16^{n^2}$ if d is sufficiently large compared to $b(n)$. As in the proof of lemma 7.11, we can avoid these almost rectangles by

$$\frac{s}{4 \cdot 8 \cdot 16^n \cdot (1 + \epsilon)} 9000^{-n} \cdot 16^{-n^2} \geq \frac{1}{9000^n \cdot 16^{n^2}} d,$$

for ϵ small enough, where ϵ goes to 0 with $\frac{b(n)}{d}$. Thus, we have the same estimate as in lemma 5.2. Similar reasoning allows us to finish the argument as in lemma 5.2. \square

In an exactly similar fashion we also have the following analog of lemma 5.3.

Lemma 7.13. *Fix n , and $d \gg b(n)$. Let R_1, \dots, R_k be a sequence of clopen subequivalence relations of (X_H, E_n) satisfying the following:*

- (1) *On each E_n class, each R_l induces a partition into regions R which are unions of \vec{b} -almost rectangles r such that each edge length of one of the r rectangles is between d and $12d$.*
- (2) *In any spherical region S of radius $100,000 \cdot 16^n d$ in some E_n equivalence class, there are at most b integers l such that one of the regions in R_l has a boundary face \mathcal{F} which intersects S and is parallel to some given face.*

Then there is a clopen subequivalence relation $\tilde{R}_d^n \subseteq E_n$ satisfying:

- (1) *Each \tilde{R}_d^n class is a \vec{b} -rectangular region with each edge lengths between $9d$ and $12d$.*
- (2) *Every boundary face of a region $R \in \tilde{R}_d^n$ is at least $\frac{1}{9000^n 16^{n^2} b} d$ from any parallel boundary face of a region $R' \in R_l$ (for any l).*

We now state our main result for the non-free part.

Theorem 7.14. *Let H be a subgroup of $\mathbb{Z}^{<\omega}$ and $X_H = \{x \in X : G_x = H\}$, where $G_x = \{g \in \mathbb{Z}^{<\omega} : g \cdot x = x\}$. Let E_H be the equivalence relation on X_H from the shift action of $\mathbb{Z}^{<\omega}$. Then there is a continuous embedding from (X_H, E_H) into E_0 .*

To prove theorem 7.14, we need the analog of lemma 6.2 which we now state.

Lemma 7.15. *Let d_{j+1}, d_j be positive integers with $d_{j+1} \gg d_j$, $d_j \gg b(j)$, and $d_{j+1} \gg b(j+1)$. Suppose R_{j+1}^i is a clopen subequivalence relation of E_i . Assume that the restriction of R_{j+1}^i to each E_{j+1} class is a union of \vec{b} -almost rectangles having edge lengths between d_{j+1} and $12d_{j+1}$. Suppose the clopen subequivalence relations $R_j^j, R_j^{j+1}, \dots, R_j^{i-1}$ of E_ω have been defined and satisfy:*

- (1) $R_j^k \subseteq E_k$ for all $j \leq k \leq i-1$.
- (2) R_j^j is a partition of each E_j class into \vec{b} -almost rectangular regions with edge lengths d_j .
- (3) For $j < k \leq i-1$, R_j^k restricted to each E_j class gives a partition into regions R each of which is a union of \vec{b} -almost rectangles having edge lengths between $9d_j$ and $12d_j$.
- (4) For each region R induced by R_j^k , there is a region R' induced by R_{j+1}^k such that the Hausdorff distance between R and R' is at most $12d_j$.
- (5) In any spherical region S of radius $100,000 \cdot 16^j d_j$ contained in an E_j class, and for each essential dimension $i \leq j$ there are at most $f_j \doteq j+1$ many k (where $j < k \leq i-1$) such that some region in R_j^k has an i boundary face intersecting S .
- (6) For any $j < k_1 \neq k_2 \leq i-1$, and R_1, R_2 regions induced by $R_j^{k_1}, R_j^{k_2}$ respectively, if $\mathcal{F}_1, \mathcal{F}_2$ are parallel faces of R_1, R_2 , then for any $u \in \mathcal{F}_1, v \in \mathcal{F}_2$, $\rho_x^j(y) > \frac{1}{9000^j 16^{j^2(j+1)}} d_j$.

Then there is a clopen subequivalence relation $R_j^i \subseteq E_i$ satisfying the following:

- (1) On each E_j class, R_j^i induces a partition in regions R each of which is a union of \vec{b} -almost rectangles with edge lengths between $9d_j$ and $12d_j$.
- (2) For each region R induced by R_j^i , there is a region R' induced by R_{j+1}^i such that the Hausdorff distance between R and R' is at most $12d_j$.
- (3) Condition (5) continues to hold, where now $j < k \leq i$.
- (4) Condition (6) continues to hold, where now $j < k \leq i$.

Proof. The proof is almost identical to that of lemma 6.2. We first apply lemma 7.13 to produce the partition into \vec{b} -almost rectangular regions \tilde{R} of edge lengths d_j . The R_1, \dots, R_k of that lemma are the current R_j^j, \dots, R_j^{i-1} . The hypotheses of lemma 7.13 are guaranteed by assumption (5) above and $d_j \gg b(j)$. Our assumption is that the restriction of R_{j+1}^i to each E_{j+1} class is a union of \vec{b} -almost rectangles of edge lengths at least $d_{j+1} \gg d_j$. When we further restrict each R_{j+1}^i region to each E_j class, we still have regions which are finite unions of \vec{b} -almost rectangles of edge lengths between d_{j+1} and $12d_{j+1}$. This follows from the definition of a \vec{b} -almost rectangle. In particular, these restricted regions are unions of $b(j)$ -almost rectangles (this is the main point; when restricted to E_j the error is bounded by $b(j)$, rather than $b(j+1)$).

We next use R_{j+1}^i and \tilde{R} to define R_j^i . As in lemma 6.2, let \mathcal{A} denote the collection j -dimensional \vec{b} -almost rectangles of edge length d_j induced by \tilde{R} . In particular, each of these regions is a $b(j)$ -almost rectangle. Each almost rectangle $R \in \mathcal{A}$ has a center $c(R) \in R$. We define $xR_j^i y$ iff $x \in R_1 \in \mathcal{A}, y \in R_2 \in \mathcal{A}$, and $c(R_1)R_{j+1}^i c(R_2)$. Clearly each R_j^i class is a union of \vec{b} -almost rectangles of

edge lengths d_j . Also, each region of R_j^i is within $12d_j$ of a region in R_{j+1}^i . Note that since each R_j^i region is a union of $b(j)$ -almost rectangles, every point of the boundary of a region in R_j^i is within $12d_j + 2b(j)$ of a point of the boundary of R_{j+1}^i (and this implies it is within $12d_j + 3b(j)$ of a boundary face of an approximating rectangle for one of the $b(j)$ almost rectangular regions comprising R_{j+1}^i).

The proof that property (6) continues to hold is as before. In view of the comments at the end of the previous paragraph, this is because the inequalities used in the proof of lemma 6.2 continue to hold if $d_{j+1} \gg d_j \gg b(j)$. \square

Finally, we turn to the proof of theorem 7.14. The proof is now similar to that of theorem 6.3. We start with a sufficiently fast growing sequence $1 \ll d_1 \ll d_2 \ll \dots$, where $d_i \gg b(i)$ for all i . For each i , let R_i^i be the clopen subequivalence relation of E_i whose classes are \vec{b} -almost rectangles given by lemma 7.11. We define inductively the sequence $R_i^i, R_{i-1}^i, \dots, R_1^i$. Assume R_{j+1}^i has been defined as have all the R_j^k for $j \leq k < i$. Assume also that each R_{j+1}^i region restricted to each E_{j+1} class is a union of \vec{b} -almost rectangles of edge lengths between d_{j+1} and $12d_{j+1}$, and that the hypotheses of lemma 7.15 are satisfied. Lemma 7.15 then gives R_j^i and maintains the hypotheses. Thus, we define all the R_j^i . Let R_i again abbreviate R_1^i .

Suppose xEy , and fix k_0 so that $xE_{k_0}y$. Let $\eta = \rho_x^{k_0}(y)$. Suppose $i_0 < i_1 < \dots$ are infinitely many i such that $\neg xR_i y$. All $R_1^{i_m}$ must have a boundary point in the sphere in E_{k_0} of radius η about x (in the x -coordinate system), so we may assume that some fixed point z lies on all their boundaries. From the remarks we made in the proof of lemma 7.15, the point z is within

$$(12d_1 + 3b(1)) + (12d_2 + 3b(2)) + \dots + (12d_{k_0-1} + 3b(k_0 - 1))$$

of a point on a boundary face of an approximating rectangle in $R_{k_0}^i$ for each i . Since $b(i) \ll d_i$ and $d_{k_0-1} \ll d_{k_0}$, we still contradict property (6) of lemma 7.15 as before.

This completes the proof of theorem 7.14.

8. ACTIONS OF COUNTABLE ABELIAN GROUPS

In this section we summarize the results obtained in the previous two sections and deduce some corollaries. First we have the following basic theorem, which is a restatement of theorem 7.14.

Theorem 8.1. *Let E be the orbit equivalence relation induced by the shift action of $\mathbb{Z}^{<\omega}$ on $2^{\mathbb{Z}^{<\omega}}$. Then E is hyperfinite.*

This has the following immediate corollary.

Corollary 8.2. *Let G be a countable abelian group acting in a Borel manner on a standard Borel space and let E be the induced orbit equivalence relation. Then E is hyperfinite.*

Proof. Since every countable abelian group is a homomorphic image of $\mathbb{Z}^{<\omega}$, any action of G is also an action of $\mathbb{Z}^{<\omega}$. By a result of Becker-Kechris ([1, Theorem 3.5.3]), the shift action of $\mathbb{Z}^{<\omega}$ is universal among all Borel actions of $\mathbb{Z}^{<\omega}$. For the reader unfamiliar with this result, a complete proof is contained in the proof of theorem 8.5 below. \square

This (partially) answers a question of Weiss (c.f. [4]) and also gives a positive answer to the following question of Kechris. Consider the equivalence relation \sim_P on the set \mathbb{R}_+ of positive real numbers defined by

$$x \sim_P y \iff x/y \in \mathbb{Q}.$$

This is known as the *commensurability* equivalence relation or the *Pythagorean* equivalence relation. Kechris asked if it is hyperfinite. Since the equivalence relation is induced by the product action of the multiplicative group of the positive rational numbers, we have the affirmative answer.

Corollary 8.3. *The commensurability equivalence relation \sim_P is hyperfinite.*

In contrast, recall that the *Vitali* equivalence relation \sim_V on \mathbb{R} is defined by

$$x \sim_V y \iff x - y \in \mathbb{Q}.$$

It is much easier and straightforward to see that \sim_V is hyperfinite. A token of its simplicity is that the usual reduction used to show that $\sim_{V \leq B} E_0$ is a Baire class 1 function and in fact continuous on the set $\mathbb{R} - \mathbb{Q}$ of irrationals.

In the rest of this section we give some further corollaries of our main theorem. First note that the proof of the main theorem also shows the following slight variation.

Theorem 8.4. *Let $2 \leq k \leq \omega$. Let E be the orbit equivalence relation induced by the shift action of $\mathbb{Z}^{<\omega}$ on $X = k^{\mathbb{Z}^{<\omega}}$. Then E is hyperfinite.*

Moreover, letting H be a subgroup of $\mathbb{Z}^{<\omega}$ and $X_H = \{x \in X : G_x = H\}$, then there is a continuous embedding of $(X_H, E \upharpoonright X_H)$ into E_0 .

Next we consider continuous free actions of countable abelian groups. In the special case where the underlying space is 0-dimensional we always have continuous embeddings into E_0 .

Theorem 8.5. *Let G be a countable abelian group acting continuously and freely on a 0-dimensional Polish space X . Let E be the induced orbit equivalence relation. Then there is a continuous embedding of E into E_0 .*

Proof. This follows from the techniques used in [4], specifically those used in the proofs of Propositions 1.2, 1.4 and 1.6 of [4], which we recall below. First, fixing a countable clopen base $\{U_i\}$ for the topology of X , we can define a reduction $f_0 : X \rightarrow (2^{\mathbb{N}})^G$ by

$$f_0(x)(g)(i) = 1 \iff g^{-1} \cdot x \in U_i.$$

f_0 is indeed an injective continuous G -map, i.e., $g \cdot f_0(x) = f_0(g \cdot x)$. It follows then that the image of f_0 is an invariant Borel subset of the free part of $(2^{\mathbb{N}})^G$. Next, via any bijection between \mathbb{N} and $\mathbb{Z} - \{0\}$ we identify the latter space with $(2^{\mathbb{Z} - \{0\}})^G$. Then f_0 is essentially a continuous reduction from X to $(2^{\mathbb{Z} - \{0\}})^G$. In the third step define a further reduction $f_1 : (2^{\mathbb{Z} - \{0\}})^G \rightarrow 3^{G \times \mathbb{Z}}$ by

$$f_1(p)(g, n) = \begin{cases} p(g)(n), & \text{if } n \neq 0, \\ 2, & \text{if } n = 0. \end{cases}$$

Then f_1 is still injective and continuous, and sends the free part of $(2^{\mathbb{Z} - \{0\}})^G$ into the free part of $3^{G \times \mathbb{Z}}$ (by the proof of [4, Proposition 1.6]). In the fourth step we

fix an onto homomorphism $\pi : \mathbb{Z}^{<\omega} \rightarrow G \times \mathbb{Z}$ and note that the trivial reduction $f_2 : 3^{G \times \mathbb{Z}} \rightarrow 3^{\mathbb{Z}^{<\omega}}$ defined by

$$f_2(p)(g) = p(\pi(g))$$

is injective and continuous and sends the free part of $3^{G \times \mathbb{Z}}$ into X_H , where $H = \ker(\pi)$. Now by theorem 8.4 the last equivalence relation is continuously embeddable into E_0 . \square

The 0-dimensionality condition on X is of course necessary. In case X is connected every continuous function from X into $2^{\mathbb{N}}$ is constant. Regarding the commensurability equivalence relation we have the following corollary.

Corollary 8.6. *There is a Baire class 1 reduction from \sim_P into E_0 , which is continuous on the set of positive irrational numbers.*

Proof. Repeat the above proof verbatim assuming $\{U_i\}$ to be the usual countable open base for \mathbb{R}_+ , i.e., each U_i is an open interval with rational endpoints. Then f_0 is Baire class 1 and is continuous on the set of positive irrational numbers. The proof then gives a Baire class 1 reduction of \sim_P to E_0 , which is continuous on the irrational part. \square

In general, without assuming that the underlying space is 0-dimensional the following theorem is all we know.

Theorem 8.7. *Let G be a countable abelian group acting continuously on a Polish space X . Let E be the induced orbit equivalence relation. Let E_ω^* be the orbit equivalence relation induced by the shift action of $\mathbb{Z}^{<\omega}$ on $\mathbb{R}^{\mathbb{Z}^{<\omega}}$. Then there is a continuous embedding of E into E_ω^* .*

Proof. This is again achieved by composing several reductions. First we define a reduction $f_0^* : X \rightarrow X^G$ by

$$f_0^*(x)(g) = g \cdot x.$$

Then f_0^* is injective and continuous, and it is a reduction of the orbit equivalence relation on X to that on X^G . Next we note that any Polish space is homeomorphic to a G_δ subset of $[0, 1]^{\mathbb{N}}$. Thus by identifying X with its homeomorphic copy in $[0, 1]^{\mathbb{N}}$ we may regard f_0^* to be a reduction from X to $([0, 1]^{\mathbb{N}})^G$. Now the reduction functions f_1 and f_2 in the preceding proof can be similarly redefined to get reductions $f_1^* : ([0, 1]^{\mathbb{N}})^G \rightarrow \mathbb{R}^{G \times \mathbb{Z}}$ and $f_2^* : \mathbb{R}^{G \times \mathbb{Z}} \rightarrow \mathbb{R}^{\mathbb{Z}^{<\omega}}$. The composition $f_2^* \circ f_1^* \circ f_0^*$ is a continuous embedding of E into E_ω^* . \square

9. CONTINUOUS EMBEDDINGS INTO E_0

In this section we show that there is a continuous, in fact computable (or recursive), embedding from $X = 2^{\mathbb{Z}^n}$ with the shift map π to E_0 . Let $n \geq 1$ be fixed throughout this section. We let E denote the shift equivalence relation on X , and F the shift equivalence relation on $F(X)$, where $F(X)$ denotes the free part of X .

Theorem 9.1. *For each $n \geq 1$, there is a computable one-to-one $f : 2^{\mathbb{Z}^n} \rightarrow 2^\omega$ such that for all $x, y \in 2^{\mathbb{Z}^n}$ we have xEy iff $f(x)E_0f(y)$.*

Proof. The proof of theorem 9.1 will use the ideas of theorem 7.14 along with those of [2] where it was shown that there is a continuous embedding from $2^{\mathbb{Z}}$ into E_0 . For the rest of this section the dimension n is fixed.

Suppose that $G = \langle g_1, \dots, g_p \rangle$ is a subgroup of \mathbb{Z}^n , with g_1, \dots, g_p a normal basis (see definition 7.1) for G . Let $\tilde{X}_G = \{x \in X : \forall g \in G \ g \cdot x = x\}$. In particular, $X = \tilde{X}_{G_0}$, where $G_0 = \{e\}$ is the identity subgroup of \mathbb{Z}^n . Recall that X_G denotes those $x \in X$ such that $H_x = G$, where $H_x = \{h \in G : h \cdot x = x\}$. Thus, $X_G \subseteq \tilde{X}_G$. Note that X_G is the “free part” of \tilde{X}_G , that is, the part of \tilde{X}_G whose stabilizer group is exactly equal to G . We write F_G for $E \upharpoonright X_G$.

For each such G (and associated normal basis) we define a computable embedding π_G from (\tilde{X}_G, E) into E_0 . By this we mean that there is a partial recursive function from X to 2^ω , which we also denote by π_G , such that $\pi_G \upharpoonright \tilde{X}_G$ is total and is an embedding from (\tilde{X}_G, E) into E_0 . π_{G_0} is then the desired embedding for theorem 9.1. We define π_G by reverse induction on G . That is, we show first that the result holds for all G of dimension n , and then prove it for G assuming it holds for all $H \supsetneq G$. This is a legitimate induction as there is no infinite strictly increasing sequence of ideals in \mathbb{Z}^n .

Consider first the case $\dim(G) = p = n$. In this case, each E class of \tilde{X}_G is finite. Let h be a computable function so that for $x \in \tilde{X}_G$, $h(x)$ is the lexicographically least member of $[x]_E$. h is a (total) computable function which reduces (\tilde{X}_G, E) to the equality relation on 2^ω . There is easily a computable function h' from 2^ω to 2^ω which embeds $(2^\omega, =)$ into $(2^\omega, E_0)$. Let $g \in \mathbb{Z}^n$ be least such that $g \cdot x = h(x)$. Then let $\pi_G(x) = g \frown h'(h(x))$. π_G is easily an embedding from (\tilde{X}_G, E) into E_0 .

Assume henceforth that $p < n$, and the result is true for all strictly larger subgroups $H \supsetneq G$. We proceed to define π_G . All of the constructions below depend on the normal basis (g_1, \dots, g_p) for G , but we suppress mentioning this, and consider this basis to be fixed.

To begin, fix a sufficiently fast growing sequence $d_1 \ll d_2 \ll \dots$ of marker distances. We may assume that the map $i \mapsto d_i$ is computable. We will use the terminology from §7. In particular, let b denote the value of $b(n)$ from definition 7.6. Here, of course, n is fixed, so this is just a single number. A b -almost rectangle will be as in definition 7.9, using now the constant function $b(k) = b$. We may assume that d_1 is sufficiently large compared to b . We now repeat the proofs of lemma 7.11, 7.12, and 7.13 in this context, that is, for n fixed. To fix notation, we summarize the main steps of these proofs.

For each i , let R_i^z be the subequivalence relation of F_G obtained from lemma 7.11 for edge length d_i . Each R_i^z class is a b -almost rectangle with edge lengths d_i . Let $C_i \subseteq X_G$ denote the set of centers of these b -almost rectangular regions. The proofs of lemmas 7.8, 7.11 show that C_i is relatively clopen in X_G . In fact, the C_i are uniformly Δ_1^0 . That is, there are Σ_1^0 relations $C, D \subseteq 2^{\mathbb{Z}^n} \times \omega$ such that for all $x \in X_G$ we have $C(x, i) \leftrightarrow \neg D(x, i) \leftrightarrow x \in C_i$. To see this, note that the d_i and the sequences s_n (defined for each i) as in the proof of lemma 2.1 are all given by computable functions. The rest of the proof of theorem 3.1 (and lemma 7.11) involves effective operations on these sets.

The proof of theorem 3.1 (and so also lemma 7.11) then shows that the relations R_i^z are uniformly Δ_1^0 on X_G . By this we mean that there are Σ_1^0 relations $A, B \subseteq 2^{\mathbb{Z}^n} \times \mathbb{Z}^n \times \omega$ such that for $x \in X_G$, $g \in \mathbb{Z}^b$, and $i \in \omega$ we have $R_i^z(x, g \cdot x) \leftrightarrow A(x, g, i) \leftrightarrow \neg B(x, g, i)$. Note that the sets A and B are actually disjoint on $\tilde{X}_G \times \mathbb{Z}^n \times \omega$ since if $(x, g, i) \in A \cap B$ with $x \in \tilde{X}_G$, then there would be an $x' \in X_G$ with $(x', g, i) \in A \cap B$ (since A, B are open and X_G is dense in \tilde{X}_G).

As in the proof of lemma 7.15 and theorem 7.14 (which in turn are modifications of the proof of theorem 6.3), we proceed in $i - 1$ steps to modify the subequivalence relations R_i^i to relations $R_{i-1}^i, R_{i-2}^i, \dots, R_1^i$, all of which are subequivalence relations of F_G . Thus, the R_j^i are subequivalence relations of F_G and satisfy the following:

- (1) On each F_G class, R_j^i induces a partition into regions R each of which is a union of b -almost rectangles with edge lengths between $9d_j$ and $12d_j$.
- (2) For each region R induced by R_j^i , there is a region R' induced by R_{j+1}^i such that the Hausdorff distance between R and R' is at most $12d_j$.
- (3) For any $j < k_1 \neq k_2 \leq i - 1$, and R_1, R_2 regions induced by $R_j^{k_1}, R_j^{k_2}$ respectively, if $\mathcal{F}_1, \mathcal{F}_2$ are parallel boundary faces of R_1, R_2 , then $\rho(\mathcal{F}_1, \mathcal{F}_2) > \frac{1}{9000^j 16^{j^2(j+1)}} d_j$.

Let $R_i = R_1^i$. The proof of lemma 7.15 is effective, and thus the R_i are uniformly Δ_1^0 on X_G . Again, by this we mean that there are Σ_1^0 relations $A, B \subseteq X \times \mathbb{Z}^n \times \omega$ such that for all $x \in X_G, g \in \mathbb{Z}^n$, and $i \in \omega$, $A(x, g, i) \leftrightarrow B(x, g, i) \leftrightarrow xR_i g \cdot x$. The proof of theorem 7.14 shows that the R_i give a continuous embedding of (X_G, F_G) into E_0 , and by our remarks this embedding is actually computable on X_G . We must show, however, that there is a computable embedding from all of \tilde{X}_G into E_0 , that is, the embedding must work on the “non-free” part of \tilde{X}_G as well.

For each i and rectangular region R in R_j^i , let $c(R)$ be the unique point of C_i in R . For $x \in X_G$ and $i \in \omega$ let $c_i(x) = c(R)$ where R is the unique R_j^i class in $[x]_E$ such that $\rho(R, [x]_{R_i}) \leq 12(d_1 + \dots + d_i)$. The uniqueness of this R follows from (2) above and the fact that the d_j are sufficiently fast growing. Similarly, it is easy to see that $c_i(x)R_i x$, that is, x and $c_i(x)$ lie in the same R_i region.

We now extend these notions to all of \tilde{X}_G . The relation $C(x, i) \leftrightarrow x \in C_i$ is Σ_1^0 on $X_G \times \omega$. There is a computable total function f which enumerates basic open sets $f(m) = U_m \subseteq 2^{\mathbb{Z}^n} \times \omega$ such that $C = \bigcup_m (U_m \cap (X_G \times \omega))$. Let $\tilde{C} \subseteq \tilde{X}_G \times \omega$ be the canonical Σ_1^0 relation with $\tilde{C} \cap (\tilde{X}_G \times \omega) = C$. That is, $\tilde{C} = \bigcup_m (U_m \cap (\tilde{X}_G \times \omega))$. For each m , let $\tilde{C}^m = \bigcup_{j \leq m} (U_j \cap (\tilde{X}_G \times \omega))$. So, $\tilde{C} = \bigcup_m \tilde{C}^m$. Note that each \tilde{C}^m is Δ_1^0 , in particular it is clopen. Let $\tilde{C}_i = \{x \in \tilde{X}_G : (x, i) \in \tilde{C}\}$, and let $\tilde{C}_i^m = \{x \in \tilde{X}_G : (x, i) \in \tilde{C}^m\}$.

Similarly, let $\tilde{A} \subseteq \tilde{X}_G \times \mathbb{Z}^n \times \omega$ be the canonical Σ_1^0 extension to $\tilde{X}_G \times \mathbb{Z}^n \times \omega$ of A , so $\tilde{A} \cap (X_G \times \mathbb{Z}^n \times \omega) = A$, and let \tilde{A}^m be the Δ_1^0 approximating sets so that $\tilde{A} = \bigcup_m \tilde{A}^m$. Likewise, let \tilde{B} be the canonical Σ_1^0 extension of B to \tilde{X}_G , and \tilde{B}^m the approximating Δ_1^0 relations. Define $\tilde{R}_i \subseteq \tilde{X}_G \times \tilde{X}_G$ by

$$\begin{aligned} x\tilde{R}_i y \leftrightarrow \exists z \in \tilde{X}_G \exists g_1, g_2 \in \mathbb{Z}^n (z \in \tilde{C}_i \wedge \|g_1\|, \|g_2\| < 20d_i \\ \wedge g_1 \cdot z = x \wedge g_2 \cdot z = y \wedge \tilde{A}(z, g_1, i) \wedge \tilde{A}(z, g_2, i)). \end{aligned}$$

The \tilde{R}_i are uniformly Σ_1^0 on $\tilde{X}_G \times \tilde{X}_G$. Also, $\tilde{R}_i \cap (X_G \times X_G) = R_i$. Each \tilde{R}_i is easily a symmetric, transitive relation on \tilde{X}_G (but not necessarily reflexive). For example, to see \tilde{R}_i is transitive, suppose $x\tilde{R}_i y$ and $y\tilde{R}_i w$. Let z_1, g_1, g_2 witness $x\tilde{R}_i y$, and z_2, g_3, g_4 witness $y\tilde{R}_i w$. Since R_i is an equivalence relation on X_G and X_G is dense in \tilde{X}_G , it follows that $z_1 = z_2$ and hence z_1, g_1, g_4 witness that $x\tilde{R}_i w$. [To see that $z_1 = z_2$, note that otherwise there would be $z'_1, z'_2, y' \in X_G$ with $z'_1 \neq z'_2, z'_1, z'_2 \in C_i$ and $R_i(z'_1, y'), R_i(z'_2, y')$. Thus, $z'_1 R_i z'_2$, a contradiction.]

So, on each E class $[x]_E$, for $x \in \tilde{X}_G$, each \tilde{R}_i gives a partition of a subset of $[x]_E$.

For each $x \in 2^{\mathbb{Z}^n}$ and $m \geq 1$, let $N_m(x)$ denote the basic open set in \tilde{X}_G :

$$N_m(x) = \{y \in \tilde{X}_G : \forall g \in \mathbb{Z}^n (||g|| \leq m \rightarrow y(g) = x(g))\}.$$

We may identify $N_m(x)$ with $x \upharpoonright \{g : ||g|| \leq m\}$.

The following definition will be important for our construction.

Definition 9.2. For $x \in \tilde{X}_G$ and $i, m \in \omega$, we say x is i -determined by depth m if there is a $g \in \mathbb{Z}^n$ with $||g|| < 20d_i$ such that $z \doteq g^{-1} \cdot x$ satisfies the following:

- (1) $N_m(z) \subseteq \tilde{C}_i^m$
- (2) $N_m(z) \times \{g\} \times \{i\} \subseteq \tilde{A}^m$.
- (3) For every $h \in \mathbb{Z}^n$ with $||h|| < 20d_i$ we have $N_m(z) \times \{h\} \times \{i\} \subseteq \tilde{A}^m$ or $N_m(z) \times \{h\} \times \{i\} \subseteq \tilde{B}^m$.

Roughly speaking, this says that there is some neighborhood of depth m about z which the m^{th} stage of the computation forces to be contained in $z \in \tilde{C}_i$, and also forces $x = g \cdot z$ to be \tilde{R}_i equivalent to z and also forces for all $||h|| < 20d_i$ whether or not $h \cdot z$ is \tilde{R}_i equivalent to z (thus, this neighborhood which is determined by the m^{th} stage of the computation completely determines the points which are \tilde{R}_i equivalent to z).

Note that if $x \in X_G$, then for each i there is a large enough m so that x is i -determined by depth m . On the other hand, if $x \in \tilde{X}_G - X_G$, then for any given i there may or may not be an m such that x is i -determined by depth m . Note also that the set of (x, i, m) such that x is i -determined by depth m is a Δ_1^0 subset of $\tilde{X}_G \times \omega \times \omega$. Finally, note that for any $x, y \in \tilde{X}_G$ and $i, m \in \omega$ that if x is i -determined by depth m and $x\tilde{R}_iy$, then y is also i -determined by depth m . This follows from property (3) of definition 9.2 and the fact that if $z\tilde{R}_iy$ then $y = h \cdot z$ for some $h \in \mathbb{Z}^n$ with $||h|| < 20d_i$.

We now proceed to the definition of the computable embedding from E into E_0 . For $x \in \tilde{X}_G$, we define $\pi(x) \in \omega^\omega$ in stages. At each stage $k \in \omega$ we define an output $\pi_k(x) \in \omega$. We will then take $\pi(x)$ to be the sequence $(\pi_1(x), \pi_2(x), \dots)$. The output $\pi_k(x)$ will be effectively computable from x, k , and the relation $D(x, i, m) \leftrightarrow x$ is i -determined by depth m . This will give that π is computable.

For $x \in \tilde{X}_G$, and $i, m \geq 1$, if x is i -determined by depth m , we let x_m^i be the unique point in \tilde{C}_i with $x\tilde{R}_ix_m^i$. Note that this point, if it exists, does not depend on m . That is, if x_m^i is defined then so is x_p^i for all $p \geq m$ and $x_p^i = x_m^i$. If x is not i -determined by depth m , then we leave x_m^i undefined. For $x \in \tilde{X}_G$ still and $k \in \omega$, let $i_k(x)$ be the largest $i \leq k$ such that x is i -determined by depth k , if such an i exists. If no such i exists set $i_k(x) = 0$.

We say x is *active* at stage k if $i_k(x) > i_{k-1}(x)$, and otherwise say x is *passive* at stage k . Fix now $x \in \tilde{X}_G$ and assume $\pi_1(x), \dots, \pi_{k-1}(x)$ have been defined. To define $\pi_k(x)$ we consider the following cases. The idea is that when x is active at stage k we “guess” that x is the “free” part X_G and define $\pi_k(x)$ accordingly. When x is passive at stage k , we “guess” that x is in the “non-free” part $\tilde{X}_G - X_G$ and define $\pi_k(x)$ accordingly. The key points are that if $x \in \tilde{X}_G - X_G$ then we will eventually guess correctly, while if $x \in X_G$ then we guess correctly infinitely often.

Case I x is active at stage k .

In this case $i_k(x) > i_{k-1}(x)$. By definition, x is $i_k(x)$ -determined by depth k . Let z_k be the unique point of $\tilde{C}_{i_k(x)}$ such that $x\tilde{R}_{i_k(x)}z_k$. Similarly define z_{k-1} . Let $g_k \in \mathbb{Z}^n$ be the unique vector g in standard form with $g \cdot z_{k-1} = z_k$ (if z_{k-1} is not defined, set g_k to be the unique g in standard form such that $g \cdot x = z_k$). Let $N_k = N_{20d_{i_k}}(z_k) = z_k \upharpoonright \{g: \|g\| \leq 20d_{i_k}\}$. We let $\pi_k(x)$ be an integer coding (g_k, N_k) .

Case II x is passive at stage k .

Let z_k be as in the previous case, with the provision that if $i_k = 0$ then set $z_k = x$. Let h_k be the least element h of $\mathbb{Z}^n - G$ (in some reasonable enumeration of \mathbb{Z}^n) such that $h \cdot z_k(g) = z_k(g)$ for all $g \in \mathbb{Z}^n$ with $\|g\|$ and $\|h + g\|$ less than $20k$. It is easy to see that there is always such an h using the fact that $\mathbb{Z}^n - G$ is infinite. Let $G_k = \langle G, h_k \rangle$. Let $\pi_k(x)$ code G_k (that is, a normal basis for G_k), and $\pi_k^{G_k}(z_k)$, where $\pi_k^{G_k}$ is the stage k output for the map π^{G_k} , which is defined by induction. Finally, if $k-1$ was an active stage or if the value of h_{k-1} is different from h_k then we code also $\pi_0^{G_k}(z_k), \dots, \pi_{k-1}^{G_k}(z_k)$ into $\pi_k(x)$.

This completes the definition of the map $\pi = \pi^G$. As we remarked before, π is a computable map from \tilde{X}_G to ω^ω . We must show that π is a reduction of (\tilde{X}_G, E) to (ω^ω, E_0) , and that π is one-to-one.

Lemma 9.3. *Suppose x, y are in \tilde{X}_G and xEy . If $x \in X_G$ (and so also $y \in X_G$) then for infinitely many k we have that x and y are active at stage k , and for large enough k , x is active at stage k iff y is active at stage k . Furthermore, if z_k, w_k are as in the definition of π_k above for x and y respectively, then for large enough active k we have that $z_k = w_k$. If $x \in \tilde{X}_G - X_G$ (and so also $y \in \tilde{X}_G - X_G$) then for all large enough k we have that x and y are passive at stage k .*

Proof. Suppose first that $x \in X_G$. Then for all i there is an m such that x is i -determined by depth m . For $k \geq m$ we have $i_k \geq i$, and so the i_k tend to infinity. Thus, there are infinitely many k such that x is active at stage k . Since the \tilde{R}_i witness hyperfiniteness on X_G , there is an i_0 such that for all $i \geq i_0$, $x\tilde{R}_i y$. For such i and any m it follows that x is i -determined by depth m iff y is i -determined by depth m . From this and the fact that the i_k tend to infinity it follows that for all large enough k that $i_k(x) = i_k(y)$, where $i_k(x)$ refers to the value for the point x and likewise for $i_k(y)$. Hence, for large enough k we have that x is active at stage k iff y is active at stage k . For these large k (i.e., with $i_k \geq i_0$) we also have that $z_k = w_k$ since $x\tilde{R}_{i_k} y$.

Suppose now that $x, y \in \tilde{X}_G - X_G$. Thus there is an $h \in \mathbb{Z}^n - G$ such that $h \cdot x = x$ (and also $h \cdot y = y$). Since $h \notin G$, there is an $h' \in h + G$ with h' in standard form, and $h' \neq \vec{0}$. From the marker construction of lemma 7.8 it follows that for any i with $d_i > \|h'\|$ that $\tilde{C}_i \cap [x]_E = \emptyset$. In particular, for any such i , x and y will never be i -determined by depth m (for any m). Say this happens for all $i \geq i_0$. Let k_0 be large enough so that for any $i < i_0$, if x or y is i -determined by depth m , for some m , then that x or y is i -determined by depth k_0 . Thus, for all $k \geq k_0$ the values of $i_k(x)$ and $i_k(y)$ are stabilized, that is, x and y are passive at stage k . \square

Lemma 9.4. *π is a reduction from (\tilde{X}_G, E) to (ω^ω, E_0) .*

Proof. Suppose first that $x, y \in X_G$ and xEy . From lemma 9.3 there are infinitely many k such that x and y are both active at stage k , and $z_k = w_k$ (these are defined

in the statement of lemma 9.3). In case I, the output consists of g_k together with N_k . Note that N_k depends only on z_k (w_k for $\pi_k(y)$). Thus, $N_k(x) = N_k(y)$. Since $g_k = g_k(x)$ is the unique vector in standard form such that $g_k \cdot z_{k-1} = z_k$, and likewise $g_k(y)$ is the unique vector in standard form such that $g_k(y) \cdot w_{k-1} = w_k$, we have $g_k(x) = g_k(y)$. Thus, $\pi_k(x) = \pi_k(y)$ for large enough active k . In case II, the values of $G_k = \langle G, h_k \rangle$ and $\pi_k^{G_k}(z_k)$ being output also depend only on z_k , and so will be the same for x and y . Next suppose that $x, y \in \tilde{X}_g - X_g$ and xEy . From lemma 9.3, for large enough k , both x and y will be passive at stage k so we are in Case II in the definition of π_k for both x and y . For large enough k , $h_k(x) = h_k(y)$ will be the least element h of $\mathbb{Z}^n - G$ such that $h \cdot x = x$ (equivalently $h \cdot y = y$). Thus, for large enough k the value of h_k and thus G_k will be the same for both x and y , and will be equal to the constant value $\langle G, h \rangle$. By induction $\pi_k^{G_k}(x) = \pi_k^{G_k}(y)$ for large enough k . So, for large enough k the k -stage outputs will be the same for x and y .

Suppose next that $x, y \in \tilde{X}_G$ and $\pi(x)E_0\pi(y)$. First note that either x, y are both in X_G or both in $\tilde{X}_G - X_G$. This follows from lemma 9.3 and the fact that we distinguish the outputs in the active versus the passive cases. So, assume first that $x, y \in X_G$. By considering the output values $\pi_k(x)$ for the infinitely many k where x is active, we see that the E_0 class of $\pi(x)$ determines a tail of a sequence (g_k, N_k) as in Case I. Since $\|g_k\| < 2d_{i_k} + 2d_{i_{k-1}} < 20d_{i_k}$ and the definition of N_k involves all h with $\|h\| < 20d_{i_k}$, it follows that from the E_0 class of $\pi(x)$ we can reconstruct the E class of x . Suppose next that $x, y \in \tilde{X}_G - X_G$. From lemma 9.3 there is a limiting value z of the z_k (that is, the z_k are eventually equal to z), and also a limiting value w of the w_k . Of course, zEx and wEy . The outputs $\pi_k(x)$ are eventually of the form $(H, \pi^H(z))$, where $H = \langle G, h \rangle$ and h is the least element of $\mathbb{Z}^n - G$ such that $h \cdot x = x$. Since $\pi(x)E_0\pi(y)$, h is also the least element of $\mathbb{Z}^n - G$ such that $h \cdot y = y$. It follows that $\pi^H(z)E_0\pi^H(w)$. If $H \subsetneq \mathbb{Z}^n$, then by induction it follows that zEw , and thus xEy . If $H = \mathbb{Z}^n$, then the E_0 class of $\pi(x)$ determines the E_0 class of $h' \circ h(x)$, where h, h' are as defined in the $p = n$ case. From the properties of h' , this determines $h(x)$, which is the lexicographically least element of the finitely many elements of $[x]_E$. So, $\pi(y)$ determines also this same element, and it follows that $[x]_E = [y]_E$. \square

Finally, we show that π is one-to-one.

Lemma 9.5. π is one-to-one from \tilde{X}_G to ω^ω .

Proof. Suppose that $\pi(x) = \pi(y)$. From lemma 9.4 we know that xEy . Suppose first that $x, y \in X_G$. Consider the sequence $k_0 < k_1 < \dots$ of k which are active for x (and hence also y , since this determined from the output $\pi(x) = \pi(y)$). For each k_q , the output $\pi_{k_q}(x)$ encodes the unique g_k in standard form (relative to G) such that $g_k \cdot z_{k_q-1} = z_{k_q}$. Thus, the sequence of z_{k_q} is determined from the output $\pi(x)$. From this it follows as in lemma 9.4 that x is also determined from $\pi(x)$, and so $x = y$.

Suppose next that $x, y \in \tilde{X}_G - X_G$. Let z, w again denote the limiting values of the z_k and w_k as in lemma 9.1. The value of z_k only changes for active k , and the output $\pi_k(x)$ records the change, that is, the g_k such that $g_k \cdot z_{k-1} = z_k$. Also, the output $\pi(x)$ determines the $g \in \mathbb{Z}^n$ such that $g \cdot x = z_{k_0}$, where k_0 is least so that $i_{k_0} > 0$. Since $\pi(x) = \pi(y)$, we also have $g \cdot y = w$. Let $h \in \mathbb{Z}^n$ be the least element of $\mathbb{Z}^n - G$ such that $h \cdot x = y$ (and thus for y as well). Let $H = \langle G, h \rangle$.

Let \bar{k} be the least stage such that for all $k \geq \bar{k}$, x is k is passive at stage k and $g_k = h$. Then $\pi_k(x)$ encodes not only $\pi_{\bar{k}}^H(x)$ but also $\pi_0^H(x), \dots, \pi_{\bar{k}-1}^H(x)$ as well. Thus, the output $\pi(x)$ determines completely $\pi^H(z)$. If $H \subsetneq \mathbb{Z}^n$, then by induction z is determined from $\pi(x)$, and thus $x = g^{-1} \cdot z$ as well. If $H = \mathbb{Z}^n$, then we use the fact that π_H is one-to-one in this case. \square

This completes the proof of theorem 9.1. \square

10. OPEN PROBLEMS AND FURTHER REMARKS

In this final section we summarize the problems left open by this article for future research. First of all the main result of this paper is a small step toward the solutions of the following hierarchy of fascinating open questions in the field.

Problem 10.1 (The Increasing Union Problem). Let $E = \bigcup_n E_n$ where $E_n \subseteq E_{n+1}$ and each E_n is a hyperfinite equivalence relation. Is E hyperfinite?

Problem 10.2 (Weiss). Is every amenable equivalence relation hyperfinite? In particular, is every orbit equivalence relation induced by an action of a countable amenable group hyperfinite?

The following is a less ambitious problem along the line. However, a positive solution would already surpass all known results to date.

Problem 10.3. Is every orbit equivalence relation of a countable nilpotent group action hyperfinite?

We do not know if the method used in this article will continue to play a role in these problems. Nevertheless, some of our results seem to be applicable in a more general context. For instance the basic marker lemmas are all adaptable to the case of general countable groups.

Another problem on which we made some initial progress in this article but is still open is the computation of the Borel chromatic numbers. We restate problem 4.2 below.

Problem 10.4. What is the Borel chromatic number of $X_G = \{x \in 2^{\mathbb{Z}^n} : G_x = G\}$ for G a subgroup of \mathbb{Z}^n , $n \geq 2$?

This question has not been completely answered even for the free part (corresponding to the trivial G). Our theorem 4.1 showed that 4 colors suffice but it is not clear if it is optimal. Recycling an old piece of terminology we pose the following question.

Problem 10.5 (The Borel Four-Color Problem). Is there a Borel 3-coloring of $F(\mathbb{Z}^n)$ for $n \geq 2$?

We also conjecture that the answer is uniform for all $n \geq 2$. Closely related to this problem is the intuitive question of how regular we can make the Borel marker regions. We already formulated a precise question as question 4.1. Here we pose a general vague question.

Problem 10.6. Do there exist Borel marker regions for $F(\mathbb{Z}^n)$ which are almost cubes and are almost lined up?

Finally we reiterate that in contrast to our results in the preceding section we do not know if E_ω (the shift equivalence relation on $2^{\mathbb{Z}^{<\omega}}$) continuously embeds into E_0 . In view of the proof of Theorem 8.5 this is equivalent to the following problem.

Problem 10.7. Let G be a countable abelian group acting continuously on a 0-dimensional Polish space X . Is E_G^X continuously reducible to E_0 ?

REFERENCES

- [1] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Mathematical Society Lecture Note Series 232, Cambridge University Press, 1996.
- [2] C. Boykin and S. Jackson, *Some applications of regular markers*, Logic Colloquium '03, Lecture Notes in Logic vol. 24, Association for Symbolic Logic, La Jolla, CA (2006), 38–55.
- [3] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory and Dynamical Systems 1 (1981), 430–450.
- [4] R. Dougherty, S. Jackson and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Transactions of the American Mathematical Society 341 (1994), no. 1, 193–225.
- [5] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I.*, Transactions of the American Mathematical Society 234 (1977), 289–324.
- [6] S. Gao, S. Jackson, and B. Seward, *A coloring property for countable groups*, to appear.
- [7] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, Journal of Mathematical Logic 2 (2002), no. 1, 1–80.
- [8] S. Jackson and R. D. Mauldin, *On a lattice problem of H. Steinhaus*, Journal of the American Mathematical Society 15 (2002), no. 4, 817–856.
- [9] A. S. Kechris, S. Solecki, and S. Todorcevic, *Borel chromatic numbers*, Advances in Mathematics 141 (1999), no. 1, 1–44.
- [10] D. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I. The Rohlin lemma*, Bulletin of the American Mathematical Society 2 (1980), 161–164.
- [11] T. Slaman and J. Steel, *Definable functions on degrees*, Cabel Seminar 81–85, 37–55. Lecture Notes in Mathematics 1333, Springer-Verlag, 1988.
- [12] B. Weiss, *Measurable dynamics*, Conference in Modern Analysis and Probability (R. Beals et al. eds.), 395–421. Contemporary Mathematics 26 (1984), American Mathematical Society, RI, 1984.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, PO BOX 311430, DENTON, TX 76210
E-mail address: `sgao@unt.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, PO BOX 311430, DENTON, TX 76210
E-mail address: `jackson@unt.edu`