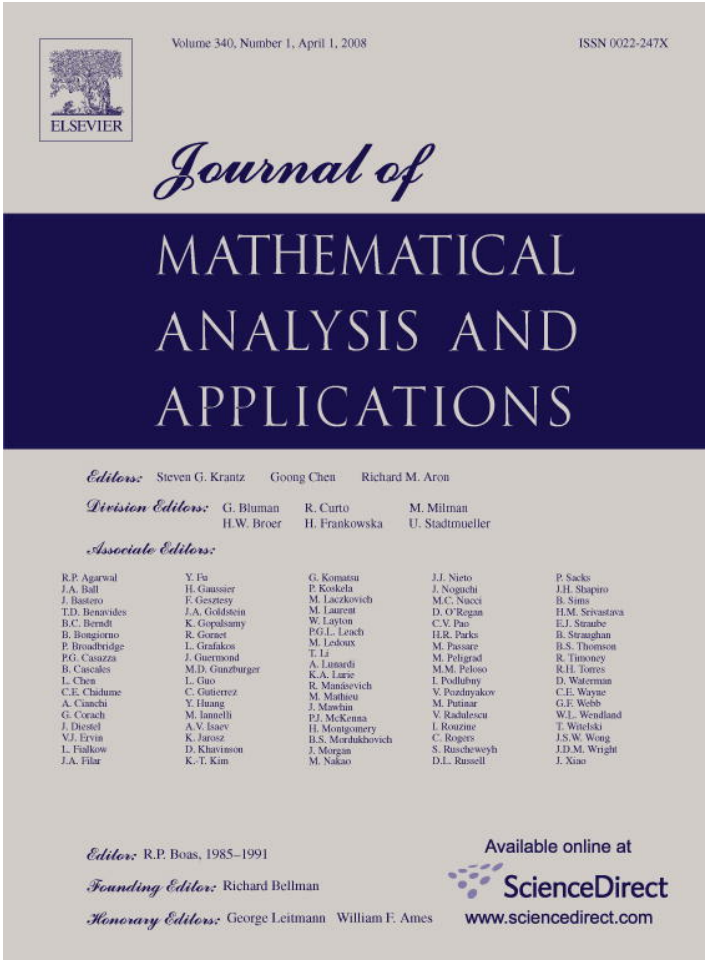

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On separable Banach subspaces

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Abstract

We show that any infinite-dimensional Banach (or more generally, Fréchet) space contains linear subspaces of arbitrarily high Borel complexity which admit separable complete norms giving rise to the inherited Borel structure.

Keywords: Separable Banach subspace; Borel structure; Polish group; Polishable subgroup; Fréchet space

1. Introduction

Let X be a topological linear space and Y a Borel linear subspace of X. We say that Y is a *separable Banach subspace*, or an *SB subspace*, of X, if there is a separable complete norm on Y which gives rise to the same Borel structure on Y as that inherited from X. For Y to be an SB subspace of X it is necessary and sufficient that there is a separable Banach space Y^* and a continuous linear operator $\varphi : Y^* \to X$ with $\varphi(Y^*) = Y$.

In this note we prove the following main theorem.

Theorem 1.1. *Let X be an infinite-dimensional Banach space and $\alpha < \omega_1$ be a countable ordinal. Then there exists an SB subspace X_α of X with $X_\alpha \notin \mathbf{\Pi}^0_\alpha$.*

We also establish the same result for infinite-dimensional Fréchet spaces. It is not clear whether the theorem is true for more general topological linear spaces.

Our interest in the subject was motivated by a result of Hjorth in [2] which states that any uncountable abelian Polish group contains Polishable subgroups of arbitrarily high Borel complexity (also see [3] for a closely related result proved with a similar method). We were asked by Sari whether an analogous result can be proved for Banach spaces, and it turned out that the answer is positive with a proof similar to Hjorth's.

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However, later communications with Solecki and the anonymous referee revealed that the problem had a much longer history than we had anticipated. In fact, the study of Polishable subgroups had been motivated by a rather long line of research on linear subspaces of topological linear spaces with arbitrarily high Borel complexity. As surveyed in [1], the problem of finding linear subspaces of arbitrarily high Borel complexity first appeared in [7]; its solution had been announced by Banach and Mazur, but the first published account seems to be [5] by Klee. Klee's construction does not seem to give SB subspaces.

On the other hand, Saint-Raymond [9] introduced a notion equivalent to our notion of SB subspaces, and with a nontrivial proof he essentially established that c_0 has SB subspaces of arbitrarily high Borel complexity. From this our main theorem would quickly follow (see our proof of Theorem 4.2 below).

Thus the main focus of this note is to give a self-contained, easy-to-follow proof of the main theorem.

Throughout this note let $K = \mathbb{R}$ or $\mathbb{C}$. The outline of our proof is similar to Hjorth's proof and is divided into two steps. In the first step we consider the topological linear space $K^{\mathbb{N}}$ and construct SB subspaces of arbitrarily high Borel complexity. In the second step we transplant these SB subspaces of $K^{\mathbb{N}}$ into any infinite-dimensional Banach space X by a continuous linear embedding of a subspace of $K^{\mathbb{N}}$ into X.

2. SB subspaces of $K^{\mathbb{N}}$

We first consider the topological linear space $K^{\mathbb{N}}$ and construct its SB subspaces. The following abstract lemma will be useful in the construction.

Lemma 2.1. *Let X be a separable Banach space, Y a separable topological linear space and Y_1 an SB subspace of Y. Let $T : X \to Y$ be a continuous linear operator. Then $X_1 = T^{-1}(Y_1)$ is an SB subspace of X.*

Proof. Let τ be the topology on Y_1 given by the separable complete norm. Consider the identity map $\mathrm{id}_{Y_1} : (Y_1, \tau) \to Y$. By Theorem 9.10 of [4], since (Y_1, τ) is a Baire abelian group, Y is separable, and id_{Y_1} is a group homomorphism, we have that id_{Y_1} is continuous. Let

$$G_T = \{(x, y) \in X \times (Y_1, \tau)\colon\ T(x) = \mathrm{id}_{Y_1}(y)\}.$$

Then G_T is a closed linear subspace of $X \times (Y_1, \tau)$, hence is a separable Banach space. Since the Borel structure on (Y_1, τ) is the same as that inherited from Y, the Borel structure on G_T is also the same as that inherited from $X \times Y$.

Let σ be the subspace topology on G_T inherited from $X \times Y$. Consider the map $E : X_1 \to (G_T, \sigma)$ given by $E(x) = (x, T(x))$. Then clearly E is a homeomorphism from X_1 (with the subspace topology inherited from X) onto (G_T, σ). Hence E is a Borel homomorphism from X_1 onto G_T.

A complete norm on X_1 can be defined by pulling back a complete norm on G_T along E. More explicitly, let $\|\cdot\|_X$ and $\|\cdot\|_{Y_1}$ be separable complete norms on X and Y_1, respectively. We can define a norm $\|\cdot\|$ on X_1 by

$$\|x\| = \frac{1}{2}\big(\|x\|_X + \big\|T(x)\big\|_{Y_1}\big).$$

Then $\|\cdot\|$ witnesses that X_1 is an SB subspace of X. □

Next we illustrate a general method to define SB subspaces on a product space. Let (X_n) be a sequence of separable Banach spaces with norms $\|\cdot\|_{X_n}$. The product space $\prod_n X_n$ is a linear metrizable space, but not a Banach space. Nevertheless we can define an SB subspace of $\prod_n X_n$ as follows:

$$X = \left\{(x_n) \in \prod_n X_n\colon\ \lim_n \frac{\|x_n\|_{X_n}}{n+1} = 0\right\};$$

$$\big\|(x_n)\big\|_X = \sup_n \frac{\|x_n\|_{X_n}}{n+1}.$$

Now suppose we also have a sequence of bounded linear functionals $\lambda_n : X_n \to K$ for $n \in \mathbb{N}$. Then we can define further

$$X' = \{(x_n) \in X\colon \big(\lambda_n(x_n)\big) \text{ is convergent}\};$$
$$\|(x_n)\|_{X'} = \frac{1}{2}\big(\|(x_n)\|_X + \sup_n |\lambda_n(x_n)|\big);$$
$$\lambda'\big((x_n)\big) = \lim_n \lambda_n(x_n).$$

Lemma 2.2.

(i) $\|\cdot\|_{X'}$ *witnesses that* X' *is an SB subspace of* X;
(ii) λ' *is a bounded linear functional on* X'.

Proof. Define a linear operator $T : X \to K^{\mathbb{N}}$ by $T((x_n)) = (\lambda_n(x_n))$. Then T is continuous. Note that $X' = T^{-1}(c)$, where c is the space of convergent sequences of real or complex numbers with the usual sup norm. Thus X' is an SB subspace of X by the previous lemma.

For (ii), assume that $s_m = (x_{nm}) \in X'$ converge to 0. Then $\lim_m \sup_n |\lambda_n(x_{nm})| = 0$, so $\lim_m \lambda'(s_m) = 0$. This shows that λ' is continuous. □

Now we are ready to define an ω_1-sequence of SB subspaces $(X^{(\alpha)})_{\alpha<\omega_1}$ of $K^{\mathbb{N}}$. By a transfinite induction on $\alpha < \omega_1$ we define $X^{(\alpha)}$, a norm $\|\cdot\|_\alpha$ and a functional $\lambda^{(\alpha)}$. We start with

$$X^{(0)} = \{(r, r, r, \ldots)\colon r \in K\} \simeq K;$$
$$\|(r, r, r, \ldots)\|_0 = |r|;$$
$$\lambda^{(0)}\big((r, r, r, \ldots)\big) = r.$$

Assume that $(X^{(\alpha)}, \|\cdot\|_\alpha)$ and $\lambda^{(\alpha)}$ have been defined. Note that $K^{\mathbb{N}} \simeq K^{\mathbb{N}^2}$. Let $(n, m) \mapsto \langle n, m\rangle$ be a bijection from $\mathbb{N}^2 \to \mathbb{N}$. For $s \in K^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have $s(\langle n, \cdot\rangle) \in K^{\mathbb{N}}$ and $s(\langle\cdot,\cdot\rangle) \in K^{\mathbb{N}^2}$. Define

$$X^{(\alpha+1)} = \left\{s \in K^{\mathbb{N}}\colon s\big(\langle\cdot,\cdot\rangle\big) \in \big(X^{(\alpha)}\big)^{\mathbb{N}},\ \lim_n \frac{\|s(\langle n, \cdot\rangle)\|_\alpha}{n+1} = 0,\ \big(\lambda^{(\alpha)}\big(s\big(\langle n, \cdot\rangle\big)\big)\big) \text{ is convergent}\right\};$$
$$\|s\|_{\alpha+1} = \frac{1}{2}\left(\sup_n \frac{\|s(\langle n, \cdot\rangle)\|_\alpha}{n+1} + \sup_n \big|\lambda^{(\alpha)}\big(s\big(\langle n, \cdot\rangle\big)\big)\big|\right);$$
$$\lambda^{(\alpha+1)}(s) = \lim_n \lambda^{(\alpha)}\big(s\big(\langle n, \cdot\rangle\big)\big).$$

If $\beta < \omega_1$ is a limit ordinal assume $(X^{(\alpha)}, \|\cdot\|_\alpha)$ and $\lambda^{(\alpha)}$ have been defined for $\alpha < \beta$. Fix an enumeration of $\{\alpha\colon \alpha < \beta\}$ as (α_n) (throughout the construction in the rest of the paper) and define

$$X^{(\beta)} = \left\{s \in K^{\mathbb{N}}\colon s\big(\langle\cdot,\cdot\rangle\big) \in \prod_n X^{(\alpha_n)}, \lim_n \frac{\|s(\langle n, \cdot\rangle)\|_{\alpha_n}}{n+1} = 0,\ \big(\lambda^{(\alpha_n)}\big(s\big(\langle n, \cdot\rangle\big)\big)\big) \text{ is convergent}\right\};$$
$$\|s\|_\beta = \frac{1}{2}\left(\sup_n \frac{\|s(\langle n, \cdot\rangle)\|_{\alpha_n}}{n+1} + \sup_n \big|\lambda^{(\alpha_n)}\big(s\big(\langle n, \cdot\rangle\big)\big)\big|\right);$$
$$\lambda^{(\beta)}(s) = \lim_n \lambda^{(\alpha_n)}\big(s\big(\langle n, \cdot\rangle\big)\big).$$

By Lemma 2.2, we can inductively prove that, for any $\alpha < \omega_1$, $X^{(\alpha)}$ is an SB subspace of $K^{\mathbb{N}}$ and $\lambda^{(\alpha)}$ is a bounded linear functional on $X^{(\alpha)}$.

3. Borel complexity

In this section we focus on Borel complexity issues. First inductively define two ω_1-sequences of subsets of the Cantor space $\mathcal{C} = 2^{\mathbb{N}}$ as follows: for $i = 0, 1$, let

$$S_i^{(0)} = \{(i, i, i, \ldots)\};$$
$$S_i^{(\alpha+1)} = \{s \in 2^{\mathbb{N}}\colon \forall n\big(s\big(\langle n, \cdot\rangle\big) \in S_0^{(\alpha)} \cup S_1^{(\alpha)}\big),\ \forall^\infty n\big(s\big(\langle n, \cdot\rangle\big) \in S_i^{(\alpha)}\big)\}.$$

For a limit ordinal $\beta < \omega_1$, note that (α_n) is an enumeration of $\{\alpha\colon \alpha < \beta\}$. Define

$$S_i^{(\beta)} = \{s \in 2^{\mathbb{N}}\colon \forall n\big(s(\langle n, \cdot\rangle) \in S_0^{(\alpha_n)} \cup S_1^{(\alpha_n)}\big),\ \forall^\infty n\big(s(\langle n, \cdot\rangle) \in S_i^{(\alpha_n)}\big)\}.$$

Let $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ denote the Baire space. The following lemma computes the Borel complexity of sets defined above.

Lemma 3.1. *For any $\alpha < \omega_1$, and any $\mathbf{\Delta}^0_{\alpha+1}$ set $E \in \mathcal{N}$, there is a continuous $f : \mathcal{N} \to \mathcal{C}$ such that*

$$x \in E \quad \Leftrightarrow \quad f(x) \in S_0^{(\alpha)}, \qquad x \notin E \quad \Leftrightarrow \quad f(x) \in S_1^{(\alpha)}.$$

Therefore, both $S_0^{(\alpha)}$ and $S_1^{(\alpha)}$ are not $\mathbf{\Pi}^0_\alpha$.

Proof. By induction on α. For $\alpha = 0$ note that, for any clopen set E in $\mathcal{N}$, the function

$$f(x) = \begin{cases} (0, 0, 0, \ldots), & x \in E, \\ (1, 1, 1, \ldots), & x \notin E, \end{cases}$$

is continuous. In general, by a theorem of Kuratowski (see [4, Exercise 22.17]), for any $E \in \mathbf{\Delta}^0_{\alpha+1}$, there is a sequence (E_n) with $E_n \in \mathbf{\Delta}^0_\alpha$ such that

$$E = \lim_n E_n = \{x \in \mathcal{N}\colon \forall^\infty n(x \in E_n)\} = \{x \in \mathcal{N}\colon \exists^\infty n(x \in E_n)\}.$$

By the inductive hypothesis, there are $f_n : \mathcal{N} \to \mathcal{C}$ such that

$$x \in E_n \quad \Leftrightarrow \quad f_n(x) \in S_0^{(\alpha)}, \qquad x \notin E_n \quad \Leftrightarrow \quad f_n(x) \in S_1^{(\alpha)}.$$

Define $f : \mathcal{N} \to \mathcal{C}$ by letting

$$f(x)\big(\langle n, m\rangle\big) = f_n(x)(m).$$

Then it is easy to see that f is continuous. Hence for any n, $f(x)(\langle n, \cdot\rangle) = f_n(x) \in S_0^{(\alpha)} \cup S_1^{(\alpha)}$ and

$$x \in E \quad \Leftrightarrow \quad \forall^\infty n \quad (x \in E_n) \quad \Leftrightarrow \quad \forall^\infty n \quad \big(f_n(x) \in S_0^{(\alpha)}\big) \quad \Leftrightarrow \quad f(x) \in S_0^{(\alpha+1)}.$$

Similarly

$$x \notin E \quad \Leftrightarrow \quad f(x) \in S_1^{(\alpha+1)}.$$

Now suppose $\beta < \omega_1$ is a limit ordinal. By the same result of Kuratowski, for any $E \in \mathbf{\Delta}^0_{\beta+1}$, there is a sequence (E_n) with $E_n \in \bigcup_{\alpha<\beta} \mathbf{\Delta}^0_\alpha$ such that

$$E = \lim_n E_n = \{x \in \mathcal{N}\colon \forall^\infty n(x \in E_n)\} = \{x \in \mathcal{N}\colon \exists^\infty n(x \in E_n)\}.$$

Then in a similar fashion as above we can define a continuous function $f : \mathcal{N} \to \mathcal{C}$ such that

$$x \in E \quad \Leftrightarrow \quad f(x) \in S_0^{(\beta)}, \qquad x \notin E \quad \Leftrightarrow \quad f(x) \in S_1^{(\beta)}. \qquad \square$$

Lemma 3.2. *For any $\alpha < \omega_1$ and $i = 0, 1$, we have $S_i^{(\alpha)} \subseteq X^{(\alpha)}$ and for any $s \in S_i^{(\alpha)}$, we have $\|s\|_\alpha \leqslant 1$ and $\lambda^{(\alpha)}(s) = i$.*

Proof. This is a routine induction on α. $\square$

We have thus obtained SB subspaces of $K^{\mathbb{N}}$ of arbitrarily high Borel complexity, as the following theorem shows.

Theorem 3.3. *For any $\alpha < \omega_1$, $\ker(\lambda^{(\alpha)})$ is an SB subspace of $K^{\mathbb{N}}$ and is not $\mathbf{\Pi}^0_\alpha$.*

Proof. $\ker(\lambda^{(\alpha)})$ is a closed subspace of $X^{(\alpha)}$, therefore is an SB subspace of $K^{\mathbb{N}}$. Note that $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to K^{\mathbb{N}}$ is a continuous embedding and $\mathrm{id}_{\mathcal{C}}^{-1}(\ker(\lambda^{(\alpha)})) = S_0^{(\alpha)}$. Since $S_0^{(\alpha)}$ is not $\mathbf{\Pi}^0_\alpha$, neither is $\ker(\lambda^{(\alpha)})$. $\square$

4. SB subspaces of Banach spaces

To transplant the SB subspaces of $K^{\mathbb{N}}$ into any infinite-dimensional Banach space we need to deal with one more point of technicality.

Let

$$L = \left\{ s \in K^{\mathbb{N}} \colon \lim_n \frac{|s(n)|}{n+1} = 0 \right\}; \qquad \|s\|_L = \sup_n \frac{|s(n)|}{n+1}.$$

Then $(L, \|\cdot\|_L)$ is a separable Banach space. For $\alpha < \omega_1$, we let

$$L^{(\alpha)} = L \cap \ker\bigl(\lambda^{(\alpha)}\bigr).$$

Lemma 4.1. *For $\alpha < \omega_1$, $L^{(\alpha)}$ is SB subspace of L and is not $\mathbf{\Pi}^0_\alpha$ in L.*

Proof. Note that $\mathrm{id}_L : L \to K^{\mathbb{N}}$ is continuous. Thus by Lemma 2.1, $L^{(\alpha)}$ is an SB subspace of L for any $\alpha < \omega_1$.

Also note that $\mathcal{C} \subseteq L$ and $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to L$ is a continuous embedding. Thus $S_0^{(\alpha)} = \mathrm{id}_{\mathcal{C}}^{-1}(L^{(\alpha)})$. By Lemma 3.1, we know that $L^{(\alpha)}$ is not $\mathbf{\Pi}^0_\alpha$ in L. □

Now we are ready for the main theorem.

Theorem 4.2. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space and $\alpha < \omega_1$. Then there exists an SB subspace X_α of X so that $X_\alpha \notin \mathbf{\Pi}^0_\alpha$.*

Proof. Let (x_n) be a sequence in X such that, for any $s \in K^{\mathbb{N}}$,

$$\sum_n s(n)x_n = 0 \quad \Leftrightarrow \quad s = 0.$$

(For instance (x_n) can be taken to be a basic sequence in X.) Without loss the generality, we may assume that $\|x_n\| \leqslant \frac{1}{(n+1)2^{n+1}}$.

For $s \in L$ and $n < m$, we have

$$\left\| \sum_{k=n}^m s(k)x_k \right\| \leqslant \sum_{k=n}^m \bigl|s(k)\bigr| \|x_k\| \leqslant \|s\|_L \sum_{k=n}^m \frac{k+1}{(k+1)2^{k+1}} \leqslant \frac{\|s\|_L}{2^n}.$$

So $\sum_n s(k)x_k$ is convergent in X. Thus we can define a linear operator $F : L \to X$ as follows:

$$F(s) = \sum_n s(k)x_k.$$

By the choice of (x_n), F is an injection. Note that

$$\bigl\|F(s)\bigr\| \leqslant \sum_n \bigl|s(k)\bigr| \|x_k\| \leqslant \|s\|_L \sum_n \frac{n+1}{(n+1)2^{n+1}} \leqslant \|s\|_L,$$

thus F is continuous. In particular F is a Borel embedding.

For $\alpha < \omega_1$, the norm on $L^{(\alpha)}$ induces a norm on $F(L^{(\alpha)})$ which witness that it is an SB subspace of $F(L)$. Therefore $F(L^{(\alpha)})$ is an SB subspace of X. Finally $F(L^{(\alpha)})$ is not $\mathbf{\Pi}^0_\alpha$ because $L^{(\alpha)}$ is not. □

We remark that the proof also works for infinite-dimensional Fréchet spaces. Recall that a *Fréchet space* is a completely metrizable locally convex linear space. In fact, let X be a Fréchet space and $\|\cdot\|$ be a complete (F)-norm on X. (For the definition and existence of (F)-norms, see, e.g. [6, §15.11].) By the Hahn–Banach Theorem (see, e.g. [6, §17.2]) we can find a sequence of closed subspaces (L_n) of X such that, for any n, $L_n \supset L_{n+1}$, $L_n \neq L_{n+1}$. Let $x_n \in L_n \setminus L_{n+1}$. Then for any $s \in K^{\mathbb{N}}$,

$$\sum_n s(n)x_n = 0 \quad \Leftrightarrow \quad s = 0.$$

From then on the proof of the above theorem works verbatim for X.

For general completely metrizable linear spaces our problem seems to be connected with the following open problem [8, Problem 4.2.5]: Does every infinite-dimensional (F)-normed space contain an infinite-dimensional closed subspace with a non-trivial continuous linear functional? If this problem has a positive answer, then our theorem can be generalized to all completely metrizable linear spaces.

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