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BOREL COMPLEXITY OF ISOMORPHISM BETWEEN QUOTIENT BOOLEAN ALGEBRAS

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§1. Introduction and nomenclature.

1.1. History of the question. In response to a question of Farah, "How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?" ([Far04]), one of us (Oliver) proved that there are continuum-many nonisomorphic Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ with $\mathcal{I}$ a Borel ideal on the natural numbers, and in fact that this result could be improved simultaneously in two directions:

i) "Borel ideal" may be improved to "analytic P-ideal"
ii) "continuum-many" may be improved to "E_0-many"; that is, E_0 is Borel reducible to the isomorphism relation on quotients by analytic P-ideals.

See [Oli04].

In [AdKech00], Adams and Kechris showed that the relation of equality on Borel sets (and therefore, any Borel equivalence relation whatsoever) is Borel reducible to the equivalence relation of Borel bireducibility. (In somewhat finer terms, they showed that the partial order of inclusion on Borel sets is Borel reducible to the quasi-order of Borel reducibility.) Their technique was to find a collection of, in some sense, strongly mutually ergodic equivalence relations, indexed by reals, and then assign to each Borel set B a sort of "direct sum" of the equivalence relations corresponding to the reals in B. Then if $B_1 \subseteq B_2$ it was easy to see that the equivalence relation thus induced by B_1 was Borel reducible to the one induced by B_2, whereas in the opposite case, taking x to be some element of $B_1 \setminus B_2$, it was possible to show that the equivalence relation corresponding to x, which was part of the equivalence relation induced by B_1, was not Borel reducible to the equivalence relation corresponding to B_2.

The purpose of the current work is to show that every Borel equivalence relation is reducible to the isomorphism relation on quotients by Borel ideals, and we shall follow approximately the same general plan that was used by Adams and Kechris. However there are a couple of significant differences.

First, note that B will in general be uncountable, so the "direct sum" is over uncountably many objects. For Adams and Kechris this was not a problem; they could consider a Polish space in "two dimensions", letting $\langle x_0, x_1\rangle$ be equivalent

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to $\langle y_0, y_1\rangle$ just in case $x_0 = x_1$ and $y_0 \, E_{x_0} \, y_1$ if $x_0 \in B$. In our context this will not quite work — we want our "direct sum" to be an ideal on a countable underlying set, so we cannot directly accommodate uncountably many elements of B without allowing them to interact in some way.

The second difference is that the equivalence relations Adams and Kechris pieced together had a strong kind of mutual ergodicity, such that the restriction of one of the relations to a non-negligible part of the space could not be Borel reducible to any of the other relations. For us we cannot hope to prove, in ZFC alone, the direct analogue of that result, which would be that there is no isomorphic embedding from one of our underlying quotient Boolean algebras to any other. This is because it follows from CH that, for any two ideals $\mathcal{I}$ and $\mathcal{J}$, where $\mathcal{J}$ has the property of Baire, there is an isomorphic embedding from $\mathcal{P}(\omega)/\mathcal{I}$ into $\mathcal{P}(\omega)/\mathcal{J}$.

There will be a collection of underlying ideals with a strong mutual incompatibility property (nonexistence of a Rudin–Keisler isomorphism between restrictions to positive sets), rather reminiscent of the mutual ergodicity of the equivalence relations used by Adams and Kechris. But for the above reason, this property cannot allow us to conclude the nonexistence of isomorphic embeddings between quotient Boolean algebras. Therefore the argument must make strong use of the fact that isomorphisms (as opposed to isomorphic embeddings) are invertible.

1.2. Basic definitions.

Definition 1.1. *If X and Y are Polish spaces and E and F are respectively equivalence relations on X and Y, we say E* is Borel reducible to *F, in symbols $E \leq_B F$, if there is a Borel map $\Theta : X \to Y$ such that, given any $x_0, x_1 \in X$,*

$$x_0 \, E \, x_1 \iff \Theta(x_0) \, F \Theta(x_1)$$

Definition 1.2. *An* ideal *$\mathcal{I}$ on ω is a subset of $\mathcal{P}(\omega)$ such that*

i) *For $A \in \mathcal{I}$, if $B \subseteq A$, then $B \in \mathcal{I}$.*
ii) *For $A, B \in \mathcal{I}$, $A \cup B \in \mathcal{I}$.*
iii) *For every finite $A \subseteq \omega$, $A \in \mathcal{I}$.*

*Elements of $\mathcal{I}$ are said to be $\mathcal{I}$-*null.

Definition 1.3. *For $\mathcal{I}$ an ideal on ω, a set $A \subseteq \omega$ is $\mathcal{I}$-*positive *if $A \notin \mathcal{I}$. The collection of all $\mathcal{I}$-positive subsets of ω is written $\mathcal{I}^+$.*

Definition 1.4. *An ideal $\mathcal{I}$ on ω is* dense *if, for every infinite subset $A \subseteq \omega$, there is an infinite $B \subseteq A$ such that $B \in \mathcal{I}$.*

Definition 1.5. *For $\mathcal{I}$, $\mathcal{J}$ ideals on ω, their* product ideal *$\mathcal{I} \times \mathcal{J}$, on $\omega \times \omega$ is defined as follows: for $A \subseteq \omega \times \omega$,*

$$A \in \mathcal{I} \times \mathcal{J} \iff \{n | \{m | \langle n, m\rangle \in A\} \in \mathcal{J}^+\} \in \mathcal{I}$$

We also write $\mathcal{I} \times \emptyset$ and $\emptyset \times \mathcal{J}$ for the products with the ideal $\{\emptyset\}$ (even though $\{\emptyset\}$ is not officially an ideal by our definition).

Definition 1.6. *Two ideals $\mathcal{I}$ and $\mathcal{J}$ on ω are* Rudin–Keisler isomorphic, *in symbols $\mathcal{I} \cong_{RK} \mathcal{J}$, if there are sets $A, B \subseteq \omega$, with $\omega \setminus A \in \mathcal{I}$ and $\omega \setminus B \in \mathcal{J}$,* *and a bijection* $f : A \to B$*, such that for every* $C \subseteq A$*,*

$$C \in \mathcal{I} \iff f``C \in \mathcal{J}$$

Definition 1.7. *If* $\mathcal{I}$ *is an ideal on* ω *and* $A \subseteq \omega$*, then the* restriction of $\mathcal{I}$ to A*, in symbols* $\mathcal{I} \restriction A$*, is the ideal defined by, for* $C \subseteq \omega$*,*

$$C \in \mathcal{I} \restriction A \iff C \cap A \in \mathcal{I}$$

Definition 1.8. *For* $\mathcal{I}$ *an ideal on* ω *and* $A \subseteq \omega$*, an* ω-partition of A with respect to $\mathcal{I}$ *is an* ω*-sequence* $A_0, A_1, \ldots$ *such that:*

i) *For* $i \neq j$*,* $A_i \cap A_j = \emptyset$*.*
ii) *For each* $i \in \omega$*,* $A_i \in \mathcal{I}^+$
iii) A *is the least upper bound of the* A_i *with respect to* $\mathcal{I}$ *(that is, for each* $i \in \omega$*,* $A_i \setminus A \in \mathcal{I}$*, and if* B *is such that for each* $i \in \omega$*,* $A_i \setminus B \in \mathcal{I}$*, then* $A \setminus B \in \mathcal{I}$*).*

Definition 1.9. *Given an ideal* $\mathcal{I}$*, the* deep ideal of $\mathcal{I}$*, denoted* $\mathrm{DI}(\mathcal{I})$*, is the set of all* $A \subseteq \omega$ *such that* A *does not have an* ω*-partition with respect to* $\mathcal{I}$*. A set is said to be* deep with respect to $\mathcal{I}$ *if it is an element of* $\mathrm{DI}(\mathcal{I})$*.*

Definition 1.10. *An ideal* $\mathcal{I}$ *is* shallow *if there are no positive deep sets with respect to* $\mathcal{I}$*; that is, every* $\mathcal{I}$*-positive set has an* ω*-partition with respect to* $\mathcal{I}$*.*

Definition 1.11. *Given an ideal* $\mathcal{I}$*, the* shallowizing ideal of $\mathcal{I}$*, or* shallowizer of $\mathcal{I}$*, denoted* $\mathrm{SI}\,(\mathcal{I})$*, is the set of all* A *such that* $\mathcal{I} \restriction A$ *is shallow. That is, for* $X \subseteq \omega$*,*

$$X \in \mathrm{SI}\,(\mathcal{I}) \iff \forall Y \subseteq X\,(Y \in \mathrm{DI}(\mathcal{I}) \iff Y \in \mathcal{I})$$

A set is said to be shallow with respect to $\mathcal{I}$ *if it is an element of* $\mathrm{SI}\,(\mathcal{I})$*.*

Definition 1.12. *Given an ideal* $\mathcal{I}$*, the* antishallowizing ideal of $\mathcal{I}$*, or* antishallowizer of $\mathcal{I}$*, denoted* $\mathrm{ASI}\,(\mathcal{I})$*, is defined as follows:*

$$X \in \mathrm{ASI}\,(\mathcal{I}) \iff \forall Y \subseteq X\,(Y \in \mathrm{SI}\,(\mathcal{I}) \iff Y \in \mathcal{I})$$

Definition 1.13. *Given an ideal* $\mathcal{I}$*,* $\mathrm{SASI}\,(\mathcal{I})$ *is the ideal generated by the union of* $\mathrm{SI}\,(\mathcal{I})$ *and* $\mathrm{ASI}\,(\mathcal{I})$*:*

$$X \in \mathrm{SASI}\,(\mathcal{I}) \iff (\exists X_0 \in \mathrm{SI}\,(\mathcal{I}))\,(\exists X_1 \in \mathrm{ASI}\,(\mathcal{I}))\,X = X_0 \cup X_1$$

This last definition may be made more concise by adopting the convenient notation according to which, given ideals $\mathcal{I}$ and $\mathcal{J}$, we write $\mathcal{I} \uplus \mathcal{J}$ for the ideal of all sets $X \cup Y$ where $X \in \mathcal{I}$ and $Y \in \mathcal{J}$. Then by definition

$$\mathrm{SASI}\,(\mathcal{I}) = \mathrm{SI}\,(\mathcal{I}) \uplus \mathrm{ASI}\,(\mathcal{J})$$

Note that

$$\mathrm{DI}(\mathcal{I}) \cap \mathrm{SI}\,(\mathcal{I}) = \mathcal{I} = \mathrm{SI}\,(\mathcal{I}) \cap \mathrm{ASI}\,(\mathcal{I})$$

Also note the table of inclusions below:

$$\begin{array}{ccccc} & & \mathrm{SI}\,(\mathcal{I}) & & \\ & \subsetneq & & \supsetneq & \\ \mathcal{I} & & & & \mathrm{SASI}\,(\mathcal{I}) \\ & \subsetneq & & \subsetneq & \\ & \mathrm{DI}\,(\mathcal{I}) & \subseteq & \mathrm{ASI}\,(\mathcal{I}) & \end{array}$$

1.3. Abstractions from previous proof. In [Oli04], it was shown that there are continuum-many nonisomorphic Boolean algebras that are quotients by Borel ideals. Changing notation slightly, and eliding some detail not relevant to our current purposes, the outline of that proof is as follows:

i) In [Oli04, Section 4.2], one defines a family of $\underset{\sim}{\mathbf{\Pi}}^0_3$ ideals $\overline{\mathcal{J}}_x$, x a real, such that each $\overline{\mathcal{J}}_x$ is shallow and dense, and such that, for $x \neq y$, there do not exist $\bar{X} \in \overline{\mathcal{J}}_x^+$, $\bar{Y} \in \overline{\mathcal{J}}_y^+$ such that
$$\overline{\mathcal{J}}_x \restriction \bar{X} \cong_{RK} \overline{\mathcal{J}}_y \restriction \bar{Y}$$
ii) Then, letting $\mathcal{J}_x = \big(\overline{\mathcal{J}}_x \times \emptyset\big) \cap (\emptyset \times \mathrm{Fin})$, and assuming
$$\mathcal{P}(\omega \times \omega)/\mathcal{J}_x \cong \mathcal{P}(\omega \times \omega)/\mathcal{J}_y$$
it is shown that sets $\bar{X}$ and $\bar{Y}$, as above, *do* exist.
iii) One key to showing the existence of RK-isomorphic nontrivial restrictions of $\overline{\mathcal{J}}_x$ and $\overline{\mathcal{J}}_y$ is to "pare down" the underlying sets of the Boolean algebras $\mathcal{P}(\omega\times\omega)/\mathcal{J}_x$ and $\mathcal{P}(\omega\times\omega)/\mathcal{J}_y$ so that the RK-isomorphism between $\mathcal{J}_x \restriction \bar{X}$ and $\mathcal{J}_y \restriction \bar{Y}$ can be read off from the isomorphism between the pared-down versions of the Boolean algebras. However it must be checked that the sets $\bar{X}$ and $\bar{Y}$ thus obtained are still *positive*; otherwise this fact simply trivializes.

    In retrospect, the key to making sure that $\bar{X}$ and $\bar{Y}$ remain positive is to guarantee that the underlying sets pared down from $\omega\times\omega$ (the underlying set of the original Boolean algebras) remain positive with respect to $\mathrm{SASI}\,(\mathcal{J}_x)$ (respectively $\mathrm{SASI}\,(\mathcal{J}_y)$). This is the condition we shall maintain in the new proof.

§2. New construction.

2.1. Idea. As in the Adams-Kechris technique, we want to reduce equality on Borel sets to the relevant isomorphism relation by associating, to each Borel set B, an ideal $\hat{\mathcal{I}}_B$ that somehow sums up all the ideals $\mathcal{J}_x$ for all $x \in B$. Moreover we want the Boolean algebra obtained by modding out by $\hat{\mathcal{I}}_B$ to have, for each $x \in B$, an element $\mathbb{1}_x$ below which the structure of the Boolean algebra is determined by $\mathcal{J}_x$.

Then, if everything works, for B and C distinct Borel sets, there should be no isomorphism between the quotients by $\hat{\mathcal{I}}_B$ and $\hat{\mathcal{I}}_C$, because for an element

$x \in B \setminus C$, the element $\mathbb{1}_x$ should have "nowhere to go", as $\hat{\mathcal{I}}_C$ is composed of ideals that are incompatible with $\mathcal{J}_x$.

But there is a problem: We are summing over (potentially) uncountably many elements of the Borel sets B and C, and the ideals are on countable sets. The idea (due to Hjorth) is to define ideals on the complete binary tree (we will actually use a variant thereof) and consider what happens along branches corresponding to elements of the Borel set.

More precisely: Consider $\omega \times \omega \times 2^{<\omega}$ as a countably infinite collection of complete binary trees; a set $X \subseteq \omega \times \omega \times 2^{<\omega}$ is then a sequence of binary trees, after closing under initial segment. For a real $x \in 2^\omega$, we consider whether the set of all indices n such that the n^{th} tree contains x as a branch, is an element of $\mathcal{J}_x$.

2.2. Formalization and nomenclature. Here we start with the same definitions (for example, of $\mathcal{J}_x$) as in Section 1.3.

Given a set $B \subseteq 2^\omega$, we define an ideal $\mathcal{I}_B$ on $\omega \times \omega \times 2^{<\omega}$ as follows: For $x \in 2^\omega$, $X \subseteq \omega \times \omega \times 2^{<\omega}$, put

$$\begin{aligned} \mathbb{1}_x \quad &= \quad \{\langle n, m, x \restriction k\rangle \mid n, m, k \in \omega\} \\ X \in \mathcal{I}_{\{x\}} &\iff \{\langle n, m\rangle \mid (\exists^\infty k)\langle n, m, x \restriction k\rangle \in X\} \in \mathcal{J}_x \\ X \in \mathcal{I}_B &\iff (\forall x \in B)\ X \in \mathcal{I}_{\{x\}} \end{aligned}$$

Note that the last definition could as well have been written

$$X \in \mathcal{I}_B \iff (\forall x \in B)\ X \cap \mathbb{1}_x \in \mathcal{I}_{\{x\}}$$

and that, for $x \in B$,

$$\mathcal{I}_B \restriction \mathbb{1}_x = \mathcal{I}_{\{x\}}.$$

2.3. Shallowizers, antishallowizers, etc. It is easy to see that the ideal $\mathcal{I}_{\{x\}}$ defined above is RK-isomorphic to the ideal $\big(\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}\big) \cap (\emptyset \times \mathrm{Fin} \times \mathrm{Fin})$. (The RK-isomorphism is literal equality for elements of $\mathbb{1}_x$, which is an $\mathcal{I}_{\{x\}}$-conull set).

We defer until Section 4 the proof of the following identities for the derived ideals defined in the previous section (here we identify $\mathcal{I}_{\{x\}}$ with $\big(\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}\big) \cap (\emptyset \times \mathrm{Fin} \times \mathrm{Fin})$).

$$\begin{aligned} \mathrm{DI}\left(\mathcal{I}_{\{x\}}\right) &= (\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}) \cap (\mathrm{Fin} \times \mathrm{Fin} \times \mathrm{Fin}) \\ \mathrm{SI}\left(\mathcal{I}_{\{x\}}\right) &= \emptyset \times \mathrm{Fin} \times \mathrm{Fin} \\ \mathrm{ASI}\left(\mathcal{I}_{\{x\}}\right) &= \overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin} \\ \mathrm{SASI}\left(\mathcal{I}_{\{x\}}\right) &= \overline{\mathcal{J}}_x \times \mathrm{Fin} \times \mathrm{Fin} \end{aligned}$$

2.4. Finite-threaded ideal. The ideal $\mathcal{I}_B$ defined above is in general $\utilde{\mathbf{\Pi}}^1_1$ since its definition contains a universal real quantifier that is not in any obvious way eliminable; it is quite plausible that $\mathcal{I}_B$ may be $\utilde{\mathbf{\Pi}}^1_1$-complete even if B itself is Borel. To obtain Borel ideals, we make the following modification.

DEFINITION 2.1. *A subset $X \subseteq \omega \times \omega \times 2^{<\omega}$ is* finite-threaded *if there is a finite set $x_0, x_1, \ldots, x_{k-1} \in 2^\omega$ such that*

$$X \subseteq \bigcup_{i<k} \mathbb{1}_{x_i}$$

We write FT for the collection of all finite-threaded sets.

LEMMA 2.1. FT *is a* $\underset{\sim}{\mathbf{\Sigma}}^0_2$ *ideal.*

PROOF. FT is obviously an ideal. It is $\underset{\sim}{\mathbf{\Sigma}}^0_2$ since $X \in \mathrm{FT}$ just in case

$$(\exists k)\,(\forall \langle n_0, m_0, s_0\rangle, \langle n_1, m_1, s_1\rangle, \ldots, \langle n_k, m_k, s_k\rangle \in X)\,(\exists i \neq j \leq k)(s_i \not\perp s_j).$$

⊣

Now, for $B \subseteq 2^\omega$, we define

$$\hat{\mathcal{I}}_B = \mathcal{I}_B \cap \mathrm{FT}$$

LEMMA 2.2. *If $B \subseteq 2^\omega$ is Borel, then $\hat{\mathcal{I}}_B$ is a Borel ideal.*

PROOF. Both $\mathcal{I}_B$ and FT are $\underset{\sim}{\mathbf{\Pi}}^1_1$. It follows that $\hat{\mathcal{I}}_B$ is $\underset{\sim}{\mathbf{\Pi}}^1_1$. But $\hat{\mathcal{I}}_B$ is also $\underset{\sim}{\mathbf{\Sigma}}^1_1$ since $X \in \hat{\mathcal{I}}_B$ just in case

$$\exists x_0, x_1, \ldots, x_{k-1} \in 2^\omega \left[X \subseteq \bigcup_{i<k} \mathbb{1}_{\{x_i\}} \;\&\; \forall i < k \;\left(x_i \in B \Rightarrow X \cap \mathbb{1}_{x_i} \in \mathcal{I}_{\{x_i\}}\right)\right]$$

(where easily $\mathcal{I}_{\{x_i\}}$ is Borel, tracing the definitions back to the $\underset{\sim}{\mathbf{\Pi}}^0_3$ ideal $\overline{\mathcal{J}}_{x_i}$ from Section 1.3). ⊣

§3. Main theorem.

We are now ready to state the main result of this paper. We shall state the theorem in a rather technical form; the "quotable" result will actually be a corollary.

THEOREM 3.1. *There is a Borel function $f : \omega^\omega \to \omega^\omega$ such that:*

i) *If x is a Borel code for a Borel subset of 2^ω, then $f(x)$ is a Borel code for a Borel ideal on $\omega \times \omega \times 2^{<\omega}$. In this case we write $\mathcal{K}_x$ for the ideal with Borel code $f(x)$.*
ii) *If x and y are Borel codes for the* same *Borel set, then $\mathcal{K}_x = \mathcal{K}_y$.*
iii) *If x and y are Borel codes for* distinct *Borel sets, then $\mathcal{P}(\omega \times \omega \times 2^{<\omega})/\mathcal{K}_x \ncong \mathcal{P}(\omega \times \omega \times 2^{<\omega})/\mathcal{K}_y$.*

The proof of Theorem 3.1 will comprise Sections 3.1, 3.2, 3.3, 3.4, 3.5, and 3.6.

COROLLARY 3.1. *For any Polish space X and any Borel equivalence relation E on X, E is Borel reducible to the equivalence relation of isomorphism between quotient Boolean algebras by Borel ideals.*

PROOF OF COROLLARY 3.1. To any element x of X, assign, as a complete E-invariant, the ideal $\mathcal{K}_{\theta(x)}$, where $\theta : X \to \omega^\omega$ is a Borel function picking out a Borel code for the Borel set $[x]_E$, the E-equivalence class of x. ⊣

3.1. Nonisomorphism between threads. We need to show that if A and B are Borel sets, $x \in A$, $y \in B$, $x \neq y$, then

$$\mathcal{P}(\omega \times \omega \times 2^{<\omega}) / \left(\hat{\mathcal{I}}_A \restriction \mathbb{1}_x\right) \ncong \mathcal{P}(\omega \times \omega \times 2^{<\omega}) / \left(\hat{\mathcal{I}}_B \restriction \mathbb{1}_y\right)$$

Indeed, we shall go further and show that if $X \subseteq \mathbb{1}_x$ is SASI $\left(\hat{\mathcal{I}}_A\right)$-positive, then there is no $Y \subseteq \mathbb{1}_y$ such that

$$\mathcal{P}(\omega \times \omega \times 2^{<\omega}) / \left(\hat{\mathcal{I}}_A \restriction X\right) \cong \mathcal{P}(\omega \times \omega \times 2^{<\omega}) / \left(\hat{\mathcal{I}}_B \restriction Y\right)$$

Lemma 3.1. *Let $X \subseteq \mathbb{1}_x$ and $Y \subseteq \mathbb{1}_y$, with $X \notin \mathrm{SASI}\left(\hat{\mathcal{I}}_A\right)$, and suppose*

$$\mathcal{P}(X)/\hat{\mathcal{I}}_A \cong \mathcal{P}(Y)/\hat{\mathcal{I}}_B$$

Then there are sets $\bar{X}, \bar{Y} \subseteq \omega$, with $\bar{X} \notin \overline{\mathcal{J}}_x$ and $\bar{Y} \notin \overline{\mathcal{J}}_y$, such that

$$\overline{\mathcal{J}}_x \restriction \bar{X} \cong_{RK} \overline{\mathcal{J}}_y \restriction \bar{Y}$$

In particular, by Section 1.3, it follows that $x = y$.

The proof of this lemma is spread out over Sections 3.1, 3.2, and 3.3.

Suppose ϕ is an isomorphism from $\mathcal{P}(X)/\hat{\mathcal{I}}_A$ to $\mathcal{P}(Y)/\hat{\mathcal{I}}_B$. We fix a lift of ϕ to a map $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$; that is, when it is convenient, we shall write $\phi(X')$, where $X' \subseteq X$, to mean some arbitrary representative of $\phi\left([X']_{\hat{\mathcal{I}}_A}\right)$, chosen in advance. We do likewise for the inverse isomorphism ϕ^{-1}.

For convenience, we fix a notation for the "slices" of the underlying space, with first coordinate fixed:

$$A_n = X \cap (\{n\} \times \omega \times \omega)$$
$$B_i = Y \cap (\{i\} \times \omega \times \omega)$$

Now we define the S relation associated to ϕ by

$$nSi \iff \phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right) \wedge [B_i]_{\mathcal{I}_{\{y\}}} > 0$$

Claim 3.1. *For each n, $\{i | nSi\}$ is finite, and for each i, $\{n | nSi\}$ is finite.*

Proof. First note that a subset W of A_n is $\mathcal{I}_{\{x\}}$-positive if and only if $\{\langle k, \ell\rangle | \langle n, k, \ell\rangle \in W\} \notin \mathrm{Fin} \times \mathrm{Fin}$. Therefore there is no ω-partition of A_n (or any subset thereof) with respect to $\mathcal{I}_{\{x\}}$ (see Claim 4.1 below). But suppose $\{i | nSi\}$ were infinite. Then we could choose an infinite, yet $\overline{\mathcal{J}}_y$-null, subset I of $\{i | nSi\}$, and note that for each $i \in I$ and any subset V of $\phi(A_n) \cap B_i$, V is $\mathcal{I}_{\{y\}}$-positive if and only if $\{\langle k, \ell\rangle | \langle i, k, \ell\rangle \in V\} \notin \mathrm{Fin} \times \mathrm{Fin}$. It follows that $\mathcal{I}_{\{y\}} \restriction \phi(A_n) \cap \bigcup_{i \in I} B_i \cong_{RK} \emptyset \times \mathrm{Fin} \times \mathrm{Fin}$. But this is impossible, because there are no ω-partitions of any positive subsets of $\phi(A_n)$.

For the second assertion, we apply the symmetrical reasoning to ϕ^{-1}. ⊣

Claim 3.2. *For each n,*

$$\phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right) = \bigvee_{i | nSi} \left(\phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right) \wedge [B_i]_{\mathcal{I}_{\{y\}}}\right)$$

and for each i,

$$\phi^{-1}\left([B_i]_{\mathcal{I}_{\{y\}}}\right) = \bigvee_{n|nSi} \left(\phi^{-1}([B_i]_{\mathcal{I}_{\{y\}}}) \wedge [A_n]_{\mathcal{I}_{\{x\}}}\right)$$

Proof. We prove the first assertion; the second will follow symmetrically. It is immediate that

$$\bigvee_{i|nSi} \left(\phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right) \wedge [B_i]_{\mathcal{I}_{\{y\}}}\right) \leq \phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right)$$

Thus we need only refute the possibility

$$\bigvee_{i|nSi} \left(\phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right) \wedge [B_i]_{\mathcal{I}_{\{y\}}}\right) < \phi\left([A_n]_{\mathcal{I}_{\{x\}}}\right)$$

Suppose the above were true. Then

$$\bigcup_{i|\neg nSi} \left(\phi\left(A_n\right) \cap B_i\right) \notin \left(\overline{\mathcal{J}}_y \times \emptyset \times \mathrm{Fin}\right) \cap \left(\emptyset \times \mathrm{Fin} \times \mathrm{Fin}\right)$$

and thus

$$\bigcup_{i|\neg nSi} \left(\phi\left(A_n\right) \cap B_i\right) \notin \overline{\mathcal{J}}_y \times \emptyset \times \mathrm{Fin}$$

by the definition of the S relation. But this is impossible, because A_n is deep with respect to $\mathcal{I}_{\{x\}}$, so $\cup_{i|\neg nSi} \left(\phi\left(A_n\right) \cap B_i\right)$ (being a subset of $\phi\left(A_n\right)$) is deep with respect to $\mathcal{I}_{\{y\}}$, but $\overline{\mathcal{J}}_y \times \emptyset \times \mathrm{Fin}$ is shallow. ⊣

3.2. Paring. For each n such that $A_n \notin \mathcal{I}_{\{x\}}$, let i_n be least such that nSi_n.

$$\begin{aligned} B'_{i_n} &= B_{i_n} \cap \phi\left(A_n\right) \\ A'_n &= \phi^{-1}(B'_{i_n}) \cap A_n \\ X' &= \bigcup_{n\in\omega} A'_n \end{aligned}$$

This "pares" the union of the A_n—that is, X—down to a subset X'; then ϕ restricts to an isomorphism $\phi : \mathcal{P}(X')/\mathcal{I}_{\{x\}} \to \mathcal{P}\left(\phi\left(X'\right)\right)/\mathcal{I}_{\{y\}}$.

Our goal here is to establish that, without loss of generality, we may assume that the S relation is a partial function (that is, we want to replace X by X'). For that we need to see that the S relation restricts correctly; that is, that for any n such that A_n is positive, and any $i \in \omega$,

$$\phi\left(\left[X' \cap A_n\right]_{\mathcal{I}_{\{x\}}}\right) \wedge \left[\phi\left(X'\right) \cap B_i\right]_{\mathcal{I}_{\{y\}}} > 0 \iff i = i_n$$

and that $X' \notin \mathrm{SASI}\left(\mathcal{I}_{\{x\}}\right)$ (so therefore $\phi\left(X'\right) \notin \mathrm{SASI}\left(\mathcal{I}_{\{y\}}\right)$).

The first point is by construction. For the second point, note that since $X \notin \mathrm{SASI}\left(\mathcal{I}_{\{x\}}\right)$, it follows that $\left\{n|A_n \notin \mathcal{I}_{\{x\}}\right\} \notin \overline{\mathcal{J}}_x$. This uses the fact that

$$A_n \notin \mathcal{I}_{\{x\}} \iff \{\langle k,\ell\rangle|\langle n,k,\ell\rangle \in A_n\} \notin \mathrm{Fin} \times \mathrm{Fin}$$

and the characterization above of $\mathrm{SASI}\left(\mathcal{I}_{\{x\}}\right)$. Thus, for a $\overline{\mathcal{J}}_x$-positive collection of n, $\phi\left(A_n\right)$ is positive, so B'_{i_n} is positive, so A'_n is positive. But since each A'_n

is a subset of $\{n\} \times \omega \times \omega$, this says exactly that their union, namely X', is SASI $\left(\mathcal{I}_{\{x\}}\right)$-positive.

3.3. RK isomorphism. Section 3.2 establishes that we may assume that the S relation is a function, by paring X down to a smaller set and recovering an isomorphism between smaller Boolean algebras. By applying the same technique to the inverse isomorphism ϕ^{-1}, we may in fact assume that S is a bijection between a $\overline{\mathcal{J}}_x$-positive subset $\bar{X}$ of ω and a $\overline{\mathcal{J}}_y$-positive subset $\bar{Y}$ of ω, such that ϕ is an isomorphism between $\mathcal{P}(X)/\mathcal{I}_{\{x\}}$ and $\mathcal{P}(Y)/\mathcal{I}_{\{y\}}$, $\bar{X}$ is the set of all indices n such that $X \cap A_n \notin \mathcal{I}_{\{x\}}$, and $\bar{Y}$ is the set of all indices i such that $Y \cap B_i \notin \mathcal{I}_{\{y\}}$.

We now show that, in this situation, S in fact induces an RK-isomorphism between $\overline{\mathcal{J}}_x \restriction \bar{X}$ and $\overline{\mathcal{J}}_y \restriction \bar{Y}$. We need to see that, for any subset C of $\bar{X}$,

$$C \in \overline{\mathcal{J}}_x \iff S``C \in \overline{\mathcal{J}}_y$$

By symmetry it is enough to show one direction, so assume $C \notin \overline{\mathcal{J}}_x$. Then $\cup_{n\in C} A_n \notin \overline{\mathcal{J}}_x \times \mathrm{Fin} \times \mathrm{Fin}$, so $\cup_{n\in C} A_n \notin \mathrm{SASI}\left(\mathcal{I}_{\{x\}}\right)$ by Section 2.3. Therefore $\phi\left(\cup_{n\in C} A_n\right) \notin \mathrm{SASI}\left(\mathcal{I}_{\{y\}}\right)$. Letting D denote $\{i | \phi\left(\cup_{n\in C} A_n\right) \cap B_i \notin \mathcal{I}_{\{y\}}\}$, we have $D \notin \overline{\mathcal{J}}_y$. But for any $i \in D$, we have that $\phi^{-1}\left(\phi\left(\cup_{n\in C} A_n\right) \cap B_i\right) \notin \mathcal{I}_{\{x\}}$, so $[\cup_{n\in C} A_n]_{\mathcal{I}_{\{x\}}} \wedge \phi^{-1}([B_i]_{\mathcal{I}_{\{x\}}})$ is positive. But we also know that $\phi^{-1}([B_i]_{\mathcal{I}_{\{y\}}}) \leq [A_{S^{-1}(i)}]_{\mathcal{I}_{\{x\}}}$ (see Claim 3.2 and recall that S is a bijection). Therefore $S^{-1}(i) \in C$. Thus it follows that $S^{-1}``D \subseteq C$, so $D \subseteq S``C$. Since $D \notin \overline{\mathcal{J}}_y$, we have that $S``C \notin \overline{\mathcal{J}}_y$, as required. This finishes the proof of Lemma 3.1. $\dashv$

3.4. Nonisomorphism between whole algebras. Suppose there were an isomorphism

$$\phi : \mathcal{P}(\omega \times \omega \times 2^{<\omega})/\hat{\mathcal{I}}_A \cong \mathcal{P}(\omega \times \omega \times 2^{<\omega})/\hat{\mathcal{I}}_B$$

where $A \neq B$. Without loss of generality, suppose $A \setminus B$ is nonempty, and let $x \in A \setminus B$. Then ϕ restricts to an isomorphism between $\mathcal{P}(\omega \times \omega \times 2^{<\omega})/\hat{\mathcal{I}}_A \restriction [\mathbb{1}_x]_{\hat{\mathcal{I}}_A}$ and $\mathcal{P}(\omega \times \omega \times 2^{<\omega})/\hat{\mathcal{I}}_B \restriction \phi\left([\mathbb{1}_x]_{\hat{\mathcal{I}}_A}\right)$. We know that $\mathbb{1}_x \in \mathrm{SASI}\left(\hat{\mathcal{I}}_A\right)^+$, so $\phi\left(\mathbb{1}_x\right) \in \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)^+$.

If we could identify a $y \in B$ such that $\phi\left(\mathbb{1}_x\right) \cap \mathbb{1}_y \in \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)^+$, we would be done, because we could then apply ϕ^{-1} to $\phi\left(\mathbb{1}_x\right) \cap \mathbb{1}_y$ and find a SASI $\left(\hat{\mathcal{I}}_A\right)$-positive subset of $\mathbb{1}_x$ and a SASI $\left(\hat{\mathcal{I}}_B\right)$-positive subset of $\mathbb{1}_y$ such that the restrictions of the corresponding Boolean algebras were isomorphic, contradicting Lemma 3.1.

3.5. Key structural property. A key structural property of $\hat{\mathcal{I}}_A \restriction \mathbb{1}_x$ is that, while $\mathbb{1}_x$ has no ω-partition into deep sets, [1] it has an ω-partition into deep sets modulo shallow sets. That is, there are $X_0, X_1, \ldots \subseteq \mathbb{1}_x$ such that:

[1]In the two-dimensional analogue, suppose $X_0, X_1, \ldots$ is an ω-partition, into deep sets, of $\omega \times \omega$ with respect to $\mathcal{J}_x = \left(\overline{\mathcal{J}}_x \times \emptyset\right) \cap (\emptyset \times \mathrm{Fin})$. Note that no vertical slice $\{n\} \times \omega$ can meet infinitely many X_i (otherwise we could easily find an upper bound strictly below $\omega \times \omega$) so each vertical slice has infinite intersection with at least one X_i. But since each X_i is deep, it's in $\mathrm{Fin} \times \mathrm{Fin}$. So from each X_i choose one element of each infinite intersection with a vertical

i) $(\forall i) X_i \subseteq \mathbb{1}_x$
ii) $(\forall i \neq j) X_i \cap X_j = \emptyset$
iii) $(\forall i) X_i \in \mathrm{DI}\left(\hat{\mathcal{I}}_A\right)$
iv) $(\forall Y \subseteq \mathbb{1}_x) \left[(\forall i) Y \cap X_i \in \hat{\mathcal{I}}_A\right] \Longrightarrow Y \in \mathrm{SI}\left(\hat{\mathcal{I}}_A\right)$

Specifically, take X_i to equal $\{i\} \times \omega \times \{x \restriction k \,|k \in \omega\}$.

Therefore $\phi\left(\mathbb{1}_x\right)$ — recall that by this notation we mean some arbitrary representative of $\phi\left([\mathbb{1}_x]_{\hat{\mathcal{I}}_A}\right)$ — has the same property; specifically, there are $Y_0, Y_1, \ldots \subseteq \phi\left(\mathbb{1}_x\right)$ such that:

i) $(\forall i) Y_i \subseteq \phi\left(\mathbb{1}_x\right)$
ii) $(\forall i \neq j) Y_i \cap Y_j = \emptyset$
iii) $(\forall i) Y_i \in \mathrm{DI}\left(\hat{\mathcal{I}}_B\right)$
iv) $(\forall Y \subseteq \phi\left(\mathbb{1}_x\right)) \left[(\forall i) Y \cap Y_i \in \hat{\mathcal{I}}_B\right] \Longrightarrow Y \in \mathrm{SI}\left(\hat{\mathcal{I}}_B\right)$

For property (ii), what literally follows from the isomorphism is that $Y_i \cap Y_j$ is $\hat{\mathcal{I}}_B$-null. However it is easy to disjointify the Y_i and verify that the remaining properties still hold.

3.6. Use of FT ideal. By the argument of Section 3.4, we would *also* have a contradiction if there were a finite sequence $y_0, y_1, \ldots, y_{k-1} \in B$ such that

$$\phi\left(\mathbb{1}_x\right) \subseteq \bigcup_{i<k} \mathbb{1}_{y_i}$$

because then, for at least one $i < k$, we would have

$$\phi\left(\mathbb{1}_x\right) \cap \mathbb{1}_{y_i} \in \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)^+$$

and could then apply the same argument. Therefore $\phi\left(\mathbb{1}_x\right) \notin \mathrm{FT}$.

The plan is to find an infinite set $Z \subseteq \mathcal{P}(\omega \times \omega \times 2^{<\omega})$ with the following properties:

i) Z is a subset of $\phi\left(\mathbb{1}_x\right)$
ii) the projection of Z to the "tree coordinate" is an antichain in $2^{<\omega}$
iii) each Y_i contains at most one element of Z.

The contradiction will then be that $Z \in SI\left(\hat{\mathcal{I}}_B\right)$ by (iii) above and property (iv) of Section 3.5, but $\hat{\mathcal{I}}_B \restriction Z \cong_{RK} \mathrm{Fin}$, because a subset of Z is positive if infinite (because not finite-threaded), and obviously null if finite.

The plan for constructing Z is to start with $\phi\left(\mathbb{1}_x\right)$, and consider each Y_i in turn, trying to add an element of Z that belongs to both Y_i and the "current set" (that is, $\phi\left(\mathbb{1}_x\right)$ less whatever we've cut away from it). To make sure that the tree coordinates of the elements of Z form an antichain, we try to find two elements of Y_i, in the current set, with incompatible tree coordinates, and take one of them in an attempt to build the antichain, removing from the current set everything whose tree coordinate is compatible with that of the chosen element,

slice, and put them all together – the horizontal projection is all of ω, so the set is positive, but its intersection with each X_i is finite, so null.

so that later choices will continue to build up an antichain. We do so in a way that preserves the SASI $\left(\hat{\mathcal{I}}_B\right)$-positivity of the current set.

Sometimes this may work ("Case III" below), but there are two things that can go wrong: Y_i may have trivial intersection with the current set ("Case I"), or it may not be possible to find elements with incompatible tree coordinate ("Case II"). In Case II, there are infinitely many elements of the intersection of Y_i and the current set whose tree coordinates lie along a single branch; we keep track of that branch for possible use in constructing Z.

By recursion we define, for each $i<\omega$, sets $C_i, D_i \subseteq i$ and $W_i \subseteq \mathcal{P}(\omega\times\omega\times 2^{<\omega})$, and $y_i \in 2^\omega$ whenever $i \in C_{i+1}$, and $\langle n_i, m_i, s_i\rangle \in Y_i \cap W_i$ whenever $i \in D_{i+1}$, such that:

i) $C_i \subseteq C_{i+1}$, $D_i \subseteq D_{i+1}$
ii) $C_i \cap D_i = \emptyset$
iii) $W_i \notin \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$
iv) $W_i \supseteq W_{i+1}$
v) $[W_0]_{\hat{\mathcal{I}}_B} \leq \phi\left([\mathbb{1}_x]_{\hat{\mathcal{I}}_B}\right)$
vi) $(\forall j<i)\,[j \in C_i \Longrightarrow (\forall \langle n,m,t\rangle \in W_i) t \not\subseteq y_j]$
vii) $(\forall j<i)\,[j \in D_i \Longrightarrow (\forall \langle n,m,t\rangle \in W_i) t \perp s_j]$
viii) $i \in C_{i+1} \Longrightarrow Y_i \cap W_i \cap \mathbb{1}_{y_i} \notin \hat{\mathcal{I}}_B$

Start with C_0 and D_0 empty, and W_0 equal to $\phi(\mathbb{1}_x)$; easily all the above conditions are met. Now assume we have met all the conditions for all $j<i$, and we will attempt to pick an element of the antichain from $Y_i \cap W_i$. There are three possible cases:

Case I: $Y_i \cap W_i \in \hat{\mathcal{I}}_B$. Then we "do nothing": that is, we take $C_{i+1} = C_i$, $D_{i+1} = D_i$, $W_{i+1} = W_i$, and do not assign values to y_i or $\langle n_i, m_i, s_i\rangle$.

Case II: Case I fails, but there do not exist incompatible nodes s and s' in $2^{<\omega}$ and natural numbers n, m, n', m' such that $\langle n,m,s\rangle$ and $\langle n',m',s'\rangle$ are both in $Y_i \cap W_i$. In that case, all tree coordinates of elements of $Y_i \cap W_i$ lie along a single branch of the complete binary tree. We call that branch y_i and throw i into C_{i+1}; that is, $C_{i+1} = C_i \cup \{i\}$, $D_{i+1} = D_i$. Now we form W_{i+1} by removing from W_i everything whose tree coordinate is compatible with y_i; that is, $W_{i+1} = W_i \setminus \mathbb{1}_{y_i}$. We leave $\langle n_i, m_i, s_i\rangle$ undefined.

Note here that $y_i \neq x$, because $\mathbb{1}_x \in \hat{\mathcal{I}}_B$. Therefore we must have $W_i \cap \mathbb{1}_{y_i} \in \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$; otherwise we could look at the isomorphism ϕ restricted to $\phi^{-1}(W_i \cap \mathbb{1}_{y_i})$ and recover a contradiction to Lemma 3.1. Thus $W_{i+1} = W_i \setminus \mathbb{1}_{y_i} \notin \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$.

Case III: There do exist $\langle n,m,s\rangle, \langle n',m',s'\rangle \in Y_i \cap W_i$ with $s \perp s'$. Then let $E = \{\langle k,\ell,t\rangle \in \omega\times\omega\times 2^{<\omega} | t \perp s\}$, $E' = \{\langle k,\ell,t\rangle \in \omega\times\omega\times 2^{<\omega} | t \perp s'\}$, and note $\omega\times\omega\times 2^{<\omega} \setminus (E \cup E')$ consists of triples whose tree coordinates have bounded length, so it is $\hat{\mathcal{I}}_B$-null, and thus certainly SASI $\left(\hat{\mathcal{I}}_B\right)$-null. Thus either $E \cap W_i \notin \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$, or $E' \cap W_i \notin \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$. In the former case, take $\langle n_i, m_i, s_i\rangle = \langle n,m,s\rangle$ and $W_{i+1} = E \cap W_i$; in the latter case, do the symmetrical thing. In either case we add i to D_{i+1}; that is, $C_{i+1} = C_i$, $D_{i+1} = D_i \cup \{i\}$, and y_i is not defined.

Most of the properties are clearly inductively preserved by construction; however property (iii) deserves comment. In Case I there is nothing to do; in case II, we noted that $W_{i+1} = W_i \setminus \mathbb{1}_{y_i} \notin \mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$. In Case III we specifically choose E or E' so that intersecting with W_i will preserve $\mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$-positivity.

Now Case I cannot happen cofinitely often, because if it did, then let i be large enough that Case I happens at step i and all later steps. Then notice that, for all $j > i$, $W_j = W_i$, and therefore $W_i \cap Y_j$ is $\hat{\mathcal{I}}_B$-null for all $j \geq i$. But there are only finitely many Y_j for j *smaller* than i, and they are all elements of $\mathrm{DI}\left(\hat{\mathcal{I}}_B\right)$ and therefore of $\mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$, so we can subtract them all from W_i, and be left with a $\mathrm{SASI}\left(\hat{\mathcal{I}}_B\right)$-positive set whose intersection with every Y_j is $\hat{\mathcal{I}}_B$-null. But that contradicts property (iv) of Section 3.5.

At the end, let $C = \bigcup_{i<\omega} C_i$, $D = \bigcup_{i<\omega} D_i$. By the previous paragraph, at least one of C, D must be infinite. If D is infinite, then the $\langle n_i, m_i, s_i \rangle$ for $i \in D$ are the desired antichain Z. If C is infinite, then for each $i \in C$, it follows by property (viii) that we have a new branch $y_i \in 2^\omega$ such that there are infinitely many triples $\langle n, m, y_i \restriction k \rangle \in W_i \cap Y_i$, so we can easily choose our desired antichain Z from those.

This finishes the proof of Theorem 3.1. We simply define $\mathcal{K}_x$ to be $\hat{\mathcal{I}}_A$, where A is the Borel set coded by the Borel code x. To see that there is a function f returning a Borel code $f(x)$ for $\mathcal{K}_x$ involves noticing that there are uniform-in-x $\underset{\sim}{\mathbf{\Pi}}^1_1$ codes for $\mathcal{K}_x$ and its complement (by the proof of Lemma 2.2), and applying the Suslin–Kleene theorem (for which see [Mosch80, 7B]). ⊣

§4. Calculation of derived ideals. Here we sketch the proofs of the identities stated in Section 2.3:

Claim 4.1 (Folklore). *$\omega \times \omega$ has no ω-partition with respect to* Fin × Fin *(so* $\mathrm{DI}(\mathrm{Fin} \times \mathrm{Fin}) = \mathcal{P}(\omega \times \omega)$*).*

Proof. Suppose $C_0 \cup C_1 \cup \ldots = \omega \times \omega$, where the C_n are pairwise disjoint and all Fin × Fin-positive. Then let n_0 be least such that $C_0 \cap (\{n_0\} \times \omega)$ is infinite, and let $C_0' = C_0 \setminus (\{n_0\} \times \omega)$. Let n_1 be least *greater than* n_0 such that $C_1 \cap (\{n_1\} \times \omega)$ is infinite, and let $C_1' = C_1 \setminus (\{n_1\} \times \omega)$, and so on.

Now for each k, $C_k \setminus C_k'$ is in Fin × Fin, but letting $C' = \bigcup_{k<\omega} C_k'$, we have that $(\omega \times \omega) \setminus C'$ is Fin × Fin-positive.

⊣

Lemma 4.1. $\mathrm{DI}\left(\mathcal{I}_{\{x\}}\right) = (\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}) \cap (\mathrm{Fin} \times \mathrm{Fin} \times \mathrm{Fin})$

Proof.

- $\supseteq$: Suppose X is an element of the right-hand side. If $X \in \mathcal{I}_{\{x\}}$ there is nothing to do, so assume X is $\mathcal{I}_{\{x\}}$-positive. Since by hypothesis $X \in \overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}$, this can only be because $X \notin \emptyset \times \mathrm{Fin} \times \mathrm{Fin}$, but again by hypothesis, $X \in \mathrm{Fin} \times \mathrm{Fin} \times \mathrm{Fin}$.

Thus there is a finite but nonempty collection of first coordinates such that the slice of X corresponding to that coordinate is $\mathrm{Fin} \times \mathrm{Fin}$-positive. The restriction of $\mathcal{I}_{\{x\}}$ to each such slice is RK-isomorphic to $\mathrm{Fin} \times \mathrm{Fin}$, so $\mathcal{I}_{\{x\}}$ restricted to X is the direct sum of finitely many copies of $\mathrm{Fin} \times \mathrm{Fin}$, which is easily RK-isomorphic to $\mathrm{Fin} \times \mathrm{Fin}$ itself. Then by Claim 4.1, X is deep with respect to $\mathcal{I}_{\{x\}}$.

- $\subseteq$: Suppose X is not in the right-hand side. Then either there is a $\overline{\mathcal{J}}_x$-positive collection of first coordinates such that the corresponding slices are $\emptyset \times \mathrm{Fin}$-positive, or there are infinitely many first coordinates such that the corresponding slices are $\mathrm{Fin} \times \mathrm{Fin}$-positive.

  In the first case, cut X down so that for each of the $\overline{\mathcal{J}}_x$-positively many first coordinates mentioned, exactly one second coordinate is represented; call the resulting set X'. Then X' is $\mathcal{I}_{\{x\}}$-positive, and $\mathcal{I}_{\{x\}} \restriction X'$ is RK-isomorphic to $(\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}) \restriction X'$. But this last ideal is shallow (see [Oli04, Lemma 4.1]), so $X \notin \mathrm{DI}\left(\mathcal{I}_{\{x\}}\right)$.

  In the second case, from the infinitely many first coordinates whose slices are $\mathrm{Fin} \times \mathrm{Fin}$-positive, choose an infinite $\overline{\mathcal{J}}_x$-null subset (this is possible because $\overline{\mathcal{J}}_x$ is dense) and cut X down to an X' with first coordinates only in that subset. Then X' is $\mathcal{I}_{\{x\}}$-positive and $\mathcal{I}_{\{x\}} \restriction X'$ is RK-isomorphic to $\emptyset \times \mathrm{Fin} \times \mathrm{Fin}$. But $\omega \times \omega \times \omega$ has an ω- partition with respect to $\emptyset \times \mathrm{Fin} \times \mathrm{Fin}$ (just cut it up into slices corresponding to different first coordinates), so again $X' \notin \mathrm{DI}\left(\mathcal{I}_{\{x\}}\right)$.

$\dashv$

Lemma 4.2. $\mathrm{SI}\left(\mathcal{I}_{\{x\}}\right) = \emptyset \times \mathrm{Fin} \times \mathrm{Fin}$

Proof.

- $\supseteq$: Suppose X is an element of the right-hand side. Then if X is positive, it can only be because X is $\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}$- positive, and $\mathcal{I}_{\{x\}} \restriction X = \overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin} \restriction X$, which is shallow. Therefore $X \in \mathrm{SI}\left(\mathcal{I}_{\{x\}}\right)$.
- $\subseteq$: Suppose X is not an element of the right-hand side. Then there is at least one first coordinate such that the corresponding slice of X is $\mathrm{Fin} \times \mathrm{Fin}$-positive. Call that slice X'. Then $\mathcal{I}_{\{x\}} \restriction X' \cong_{\mathrm{RK}} \mathrm{Fin} \times \mathrm{Fin}$, so $X \notin \mathrm{SI}\left(\mathcal{I}_{\{x\}}\right)$ by Claim 4.1.

$\dashv$

Lemma 4.3. $\mathrm{ASI}\left(\mathcal{I}_{\{x\}}\right) = \overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}$

Proof.

- $\supseteq$: Suppose X is an element of the right-hand side. Then if $X' \subseteq X$ is positive, it can only be because $X' \notin \emptyset \times \mathrm{Fin} \times \mathrm{Fin}$. But now by Lemma 4.2, $X' \notin \mathrm{SI}\left(\mathcal{I}_{\{x\}}\right)$. Since X' was an arbitrary positive subset of X, it follows that $X \in \mathrm{ASI}\left(\mathcal{I}_{\{x\}}\right)$.
- $\subseteq$: Suppose X is not in the right-hand side. Then there is a $\overline{\mathcal{J}}_x$-positive set of first coordinates whose corresponding slices of X are $\emptyset \times \mathrm{Fin}$-positive. Pare X down so that exactly one second coordinate is represented for each of the aforementioned first coordinates; call this new set X'. Then X' is an

$\mathcal{I}_{\{x\}}$-positive element of $\emptyset \times \mathrm{Fin} \times \mathrm{Fin}$, which equals $\mathrm{SI}\left(\mathcal{I}_{\{x\}}\right)$. Therefore $X \notin \mathrm{ASI}\left(\mathcal{I}_{\{x\}}\right)$.

⊣

LEMMA 4.4. $\mathrm{SASI}\left(\mathcal{I}_{\{x\}}\right) = \overline{\mathcal{J}}_x \times \mathrm{Fin} \times \mathrm{Fin}$

PROOF. By Lemmata 4.2 and 4.3, and using the notation mentioned in the remark following Definition 1.13, we want

$$(\emptyset \times \mathrm{Fin} \times \mathrm{Fin}) \,\underline{\cup}\, (\overline{\mathcal{J}}_x \times \emptyset \times \mathrm{Fin}) = \overline{\mathcal{J}}_x \times \mathrm{Fin} \times \mathrm{Fin}$$

For simplicity we show instead the two-dimensional version (the three-dimensional version is not essentially different):

$$(\emptyset \times \mathrm{Fin}) \,\underline{\cup}\, (\overline{\mathcal{J}}_x \times \emptyset) = \overline{\mathcal{J}}_x \times \mathrm{Fin}$$

- $\supseteq$: Let X be an element of the right-hand side. Then letting A be the set of all first coordinates whose slices in X are infinite, we have that $A \in \overline{\mathcal{J}}_x$. Let X' be the set of all elements of X whose first coordinates are in A. Then $X = X' \cup (X \setminus X')$, with $X' \in \overline{\mathcal{J}}_x \times \emptyset$ and $X \setminus X' \in \emptyset \times \mathrm{Fin}$.
- $\subseteq$: Immediate by the inclusions $\emptyset \times \mathrm{Fin} \subseteq \overline{\mathcal{J}}_x \times \mathrm{Fin}$ and $\overline{\mathcal{J}}_x \times \emptyset \subseteq \overline{\mathcal{J}}_x \times \mathrm{Fin}$, and the fact that $\overline{\mathcal{J}}_x \times \mathrm{Fin}$ is an ideal.

⊣

REFERENCES

[AdKech00] SCOT ADAMS and ALEXANDER S. KECHRIS, *Linear algebraic groups and countable Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 13 (2000), no. 4, pp. 909–943.

[Far04] ILIJAS FARAH, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?*, ***Illinois Journal of Mathematics***, vol. 46 (2002), no. 4, pp. 999–1033.

[Mosch80] YIANNIS MOSCHOVAKIS, ***Descriptive set theory***, North–Holland, 1980.

[Oli04] MICHAEL RAY OLIVER, *Continuum-many Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, $\mathcal{I}$ Borel*, this JOURNAL, vol. 9 (2004), pp. 799–816.

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