We investigate universal actions of the unitary group and universal equivalence relations induced by these actions. We consider the isometric equivalence relation for Hilbertian Polish metric spaces and prove that it is Borel bireducible with the orbit equivalence relation induced by a universal action of the unitary group.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space and let $U(\mathcal{H})$ be the group of all unitary transformations of $\mathcal{H}$. When $U(\mathcal{H})$ is endowed with the strong operator topology it becomes a Polish group, and we denote it by $U_\infty$ (following [7]). A more standard notation for this topological group is $U_\infty(\mathcal{H})$. However, the notation $U_\infty$ stresses our interest in the abstract group and in its arbitrary actions.

In this paper we investigate, collectively, Borel actions of $U_\infty$ on standard Borel spaces and the equivalence relations induced by these actions. The actions can be compared to each other via their mutual embeddability, and the equivalence relations via the notion of Borel reducibility. In either context it is well-known that there are universal elements (c.f. [2]), which are intuitively the most complicated objects. These universal actions and universal equivalence relations will be the focus of this paper.

Previously known examples of $U_\infty$ actions that generate universal equivalence relations are complicated and difficult to work with. One of our objectives in this paper is to identify actions of $U_\infty$ that are fairly simple to define but still induce universal equivalence relations. We show that the natural action of $U_\infty$ on $\mathcal{F}(\mathcal{H})$, the space of all closed subsets of $\mathcal{H}$, is such an action. Moreover, for any Borel action of $U_\infty$ on a standard Borel space $X$, there is a Borel reduction of $X$ into $\mathcal{F}(\mathcal{H})$ which respects the Borelness of invariant sets. In our terminology, $\mathcal{F}(\mathcal{H})$ is faithfully universal. This is not as strong as universal with respect to embeddings.

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of actions, but faithfulness is still a significant concept because of its pertinence to
the Topological Vaught Conjecture. We will elaborate on the details in the text
and give definitions as they become relevant.

As an important application we also identify a natural equivalence relation
that is of the same complexity as the universal equivalence relation given by $U_\infty$
actions. This equivalence relation comes from the classification problem of the
so-called Hilbertian Polish metric spaces. A Hilbertian Polish metric space is a
complete metric space isometrically embeddable into $\mathcal{H}$. The classification problem
is to decide whether two given such spaces are isometric and it is essentially an
equivalence relation. We show that this equivalence relation is the most complicated
among all the orbit equivalence relations induced by unitary group actions.

A careful reader will notice that results in this paper are parallel to some of
those in [6]. Instead of treating the Urysohn space and its full isometry group in
[6], here we are dealing with a more familiar space $\mathcal{H}$ and the unitary group, a
subgroup of the full isometry group.

The paper is organized as follows. In the next section we give the basic defini-
tions and background results. Then in Section 3 we consider the universal actions
and equivalence relations of the unitary group. In section 4 we consider the classifi-
cation problem for Hilbertian Polish metric spaces and prove our main result about
the complexity of the classification problem. Throughout the paper we will always
work with the complex Hilbert space. But all results are valid for the real Hilbert
space and, correspondingly, the orthogonal group.

2. Preliminaries

In this section we review the basic results about Polish group actions and
equivalence relations. We also summarize some results on positive definite functions
and closed subgroups of the unitary group. These will be needed in subsequent
sections.

Recall that a topological space is Polish if it is separable and completely metriz-
able. Likewise, a topological group is called a Polish group if its topology is Polish.
A separable, complete metric space will be called a Polish metric space. For a topo-
logical space $X$, a subset $A \subseteq X$ is called Borel if it is an element of the $\sigma$-algebra
of subsets of $X$ generated by the open sets. The Borel structure of $X$ is just the
$\sigma$-algebra of all Borel subsets of $X$. A space with a $\sigma$-algebra of subsets is called a
standard Borel space if the $\sigma$-algebra is the Borel structure of some Polish topology.
If $X$ and $Y$ are both standard Borel spaces, then a function $f : X \to Y$ is called
Borel (measurable) if for any Borel subset $B$ of $Y$, $f^{-1}(B)$ is a Borel subset of $X$.
A subset $B$ of $Y$ is analytic if there are a standard Borel space $X$, a Borel function
$f : X \to Y$ and a Borel subset $A$ of $X$ such that $f(A) = B$. Basic facts and results
on these concepts can be found in [8].

Let $X$ and $Y$ be standard Borel spaces and $E$ and $F$ be equivalence relations
on $X$ and $Y$ respectively. Then $E$ is Borel reducible to $F$, denoted $E \leq_B F$, if there
is a Borel function $f : X \to Y$ such that, for all $x_1, x_2 \in X$,

$$x_1 E x_2 \iff f(x_1) F f(x_2).$$

In this situation $E$ is intuitively simpler than $F$, or $F$ is more complicated than $E$.
If $E \leq_B F$ and $F \leq_B E$ we say that $E$ and $F$ are Borel bireducible, and intuitively
this means that $E$ and $F$ have the same complexity.
An important class of equivalence relations is that of orbit equivalence relations induced by actions of Polish groups. Let $G$ be a Polish group and $X$ a standard Borel space. We say that $G$ acts on $X$ in a Borel manner, or $X$ is a Borel $G$-space, if there is an action $a : G \times X \to X$ which is a Borel function. Here $G \times X$ is understood to be a standard Borel space since it admits a product Polish topology. For any $x \in X$, the $G$-orbit of $x$ in $X$ is the set

$$[x]_G = \{g \cdot x \mid g \in G\}.$$  

The $G$-orbit equivalence relation $E^X_G$ is defined by

$$x_1 E^X_G x_2 \iff \exists g \in G (g \cdot x_1 = x_2) \iff [x_1]_G = [x_2]_G$$

for all $x_1, x_2 \in X$. Since the action mapping $a : G \times X \to X$ is Borel, $E^X_G$ is an analytic subset of $X \times X$. However, in general, $E^X_G$ does not have to be Borel. The notion of Borel reducibility defined above can be viewed as a weak notion of embeddability of actions and can be used to compare the complexity of all orbit equivalence relations. But often these comparisons can be done via stronger notions of embeddability of actions, which we define below.

Let $X$ and $Y$ be Borel $G$-spaces. A mapping $f : X \to Y$ is a $G$-map if for all $x \in X$ and $g \in G$,

$$f(g \cdot x) = g \cdot f(x).$$

If in addition $f$ is an embedding, then $f$ is called a $G$-embedding. If there is a Borel G-map from $X$ to $Y$ then it follows that $E^X_G \leq_B E^Y_G$.

A Borel $G$-space $X$ (or the action of $G$ on $X$) is called universal if for any Borel $G$-space $Y$ there is a Borel $G$-embedding from $Y$ into $X$. It is known that for any Polish group $G$ there is a universal Borel $G$-space ([2], Theorems 2.6.1). Consequently, if $X$ is a universal Borel $G$-space then for any Borel $G$-space $Y$ we have $E^Y_G \leq_B E^X_G$, i.e., the orbit equivalence relation $E^X_G$ is the most complicated among all $G$-orbit equivalence relations.

In general, if $X$ is a Borel $G$-space, we say that $E^X_G$ is universal if any $G$-orbit equivalence relation is Borel reducible to $E^X_G$, i.e., for any Borel $G$-space $Y$, $E^Y_G \leq_B E^X_G$. It is important to note that the Borel $G$-space $X$ does not need to be universal in order for the orbit equivalence relation to be universal. To emphasize the distinction we offer the following intermediate notion.

For a subset $A$ of $X$, let

$$[A]_G = \{g \cdot x \mid g \in G, a \in A\}.$$  

We call $A$ $G$-invariant if $A = [A]_G$. We say that $E^X_G$ is faithfully universal if for any Borel $G$-space $Y$ there is a Borel reduction $f : Y \to X$ witnessing $E^Y_G \leq_B E^X_G$ such that, for any invariant Borel subset $A$ of $X$, we have that $[f(A)]_G$ is an invariant Borel subset of $Y$. In general, reduction functions satisfying the above property are called faithful Borel reductions. Faithful Borel reductions were first introduced in [4] out of model-theoretic motivations and later studied in [5] (there it was called $F_S$-reducibility) for Borel equivalence relations and $S_\infty$-orbit equivalence relations. The current terminology comes from [7]. The importance of faithful Borel reductions lies in the fact that they carry down the truth of the Topological Vaught Conjecture (c.f. [2], [4], and Theorem 3.7 below). For a Polish group $G$ and a Borel $G$-space $X$, the Topological Vaught Conjecture states that, for every invariant Borel subset $A$ of $X$, either $A$ contains $2^{\aleph_0}$ many $G$-orbits or else $A$ contains only countably many $G$-orbits. If $Y$ and $X$ are both Borel $G$-spaces and there exists a faithful
reduction from $Y$ into $X$, then the Topological Vaught Conjecture holds for $Y$ if it holds for $X$. The Topological Vaught Conjecture for a Polish group $G$ is the statement that the Topological Vaught Conjecture holds for all Borel $G$-spaces. If $X$ is a Borel $G$-space so that $E^X_G$ is faithfully universal, then the Topological Vaught Conjecture for $G$ is equivalent to the instance of it for $X$. The general Topological Vaught Conjecture asserts that the Topological Vaught Conjecture holds for all Polish groups $G$. This conjecture is still open.

If $X$ and $Y$ are Borel $G$-spaces and $f$ is a $G$-embedding from $Y$ into $X$, then the image of any invariant Borel subset of $Y$ under $f$ is an invariant Borel subset of $X$. Thus in particular, if $X$ is a universal Borel $G$-space then $E^X_G$ is a faithfully universal $G$-orbit equivalence relation. To summarize, the following diagram shows the implication relations among the three universality notions.

\[
\begin{array}{c}
\text{universal $G$-space} \\
\downarrow \\
\text{faithfully universal $G$-orbit equivalence relation} \\
\downarrow \\
\text{universal $G$-orbit equivalence relation}
\end{array}
\]

Counterexamples for the inverse implications can be found in [5]. Our results in the next section also provide such examples in the context of $G$ being the unitary group.

Most of the rest of this paper deals with the unitary group $U_\infty$. When there is no ambiguity about the group in question we shall omit the prefix $G$- for the terminology defined above.

We now turn to the unitary group $U_\infty$. We first recall some notation and basic facts about the Hilbert space $\mathcal{H}$ and its unitary group $U(\mathcal{H})$. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is unitary if $\langle Tu, Tv \rangle = \langle u, v \rangle$, for all $u, v \in \mathcal{H}$.

The group operation of $U(\mathcal{H})$ is the composition of unitary operators. The above equation implies that every unitary operator is in fact an isometry of $\mathcal{H}$ considered as a metric space. Moreover, the strong operator topology on $U(\mathcal{H})$ is exactly the pointwise convergence topology when it is considered as a group of isometries. It is well-known that the weak operator topology on $U(\mathcal{H})$ coincides with the strong operator topology, and in the sequel we will use this fact implicitly. More basic facts about Hilbert spaces and unitary operators can be found in [3]. For more advanced background on the unitary group see [9].

In the rest of this section we shall try to give a more or less self-contained account of the results on closed subgroups of $U_\infty$. Most of the following results belong to the folklore and are hard to attribute. Moreover, some of the results, even if well-known to experts, can not be found in the literature in the form we need them.

One of the most useful criteria for deciding whether a Polish group can be topologically embedded into $U_\infty$ goes through the concept of positive definite functions.

Recall that a complex-valued function $f$ on a group $G$ is positive definite if for any $n \geq 1$ and arbitrary $g_1, \ldots, g_n \in G$, $c_1, \ldots, c_n \in \mathbb{C}$,

$$
\sum_{i,j=1}^{n} f(g_j^{-1} g_i) c_i \overline{c_j} \geq 0.
$$
If $G$ is a group, $1_G$ is its identity element and $f$ is a positive definite function on $G$, then among the simplest properties of $f$ are:

(a) $f(1_G) \geq 0$,

(b) $f(g^{-1}) = f(g)$, and

(c) $|f(g)| \leq f(1_G)$,

for all $g \in G$. A less obvious property is a result of Schur that the set of all positive definite functions is closed under multiplication. Proofs of these facts can be found in, e.g., [1], §3.1.

The following characterization theorem is mainly a folklore. An account close to what we need here is given in [9], §30. For the convenience of the reader we give a sketch of the proof afterwards.

**Theorem 2.1.** Let $G$ be a Polish group and $1_G$ its identity element. Then the following are equivalent:

1. $G$ is isomorphic to a (closed) subgroup of $U_\infty$;
2. Continuous positive definite functions on $G$ generate the topology of $G$;
3. Continuous positive definite functions on $G$ generate a neighborhood basis of $1_G$;
4. Continuous positive definite functions on $G$ separate $1_G$ and closed sets not containing $1_G$;
5. There is a continuous positive definite function on $G$ separating $1_G$ and closed sets not containing $1_G$.

A few words about the statements before the proof. Let $\mathcal{F}$ be the collection of all continuous positive definite functions on $G$. A rewording of (2) gives

$(2')$ The topology of $G$ is the weakest topology that makes functions in $\mathcal{F}$ continuous.

Should there be any doubt with the terminology used in (3) and (4), here are the expanded versions with definitions incorporated:

$(3')$ For any open set $V \subseteq G$ with $1_G \in V$, there are functions $f_1, \ldots, f_n \in \mathcal{F}$ and open subsets $O_1, \ldots, O_n$ in $C$ such that

$$1_G \in \bigcap_{i=1}^{n} f_i^{-1}(O_i) \subseteq V.$$  

$(4')$ For any closed set $F \subseteq G$ with $1_G \notin F$, there is an $f \in \mathcal{F}$ such that

$$\sup_{g \in F} |f(g)| < f(1_G).$$

It is also worth noting that, since $G$ is separable, it suffices to require in $(2')$-$$(4')$ that there is a countable family $\mathcal{F}_0 \subseteq \mathcal{F}$ satisfying the corresponding properties.

**Proof of Theorem 2.1.** It is evident that $(2) \Rightarrow (3)$ and $(5) \Rightarrow (4) \Rightarrow (3)$. We show that $(1) \Rightarrow (2)$, $(3) \Rightarrow (1)$ and $(3) \Rightarrow (4) \Rightarrow (5)$.

$(1) \Rightarrow (2)$: Recall that the strong operator topology on $U_\infty$ coincides with the weak operator topology, thus it is the weakest topology to make all the functionals $f_{x,y}(g) = \langle g(x), y \rangle$, $x, y \in \mathcal{H}$, continuous. For any $v \in \mathcal{H}$, let

$$f_v(g) = \langle g(v), v \rangle,$$

for all $g \in U_\infty$. 


Then \( f_n \) is a continuous positive definite function on \( U_\infty \). The continuity is immediate from the definition of the weak operator topology on \( U_\infty \), and the positive definiteness is established by the standard computation:

\[
\sum_{i,j=1}^{n} f_v(g_j^{-1} g_i) c_i c_j = \sum_{i,j=1}^{n} \langle g_j^{-1} g_i(v), v \rangle c_i c_j = \sum_{i,j=1}^{n} \langle g_i(v), g_j(v) \rangle c_i c_j
\]

\[
= \sum_{i,j=1}^{n} \langle c_i g_i(v), c_j g_j(v) \rangle = \left( \sum_{i=1}^{n} c_i g_i(v), \sum_{j=1}^{n} c_j g_j(v) \right) = \left\| \sum_{i=1}^{n} c_i g_i(v) \right\|^2 \geq 0.
\]

Now by the polar identity

\[
4 \langle g(x), y \rangle = \langle g(x + y), x + y \rangle - \langle g(x - y), x - y \rangle
\]

\[
+ i \langle g(x + iy), x + iy \rangle - i \langle g(x - iy), x - iy \rangle,
\]

any topology of \( U_\infty \) that makes the demonstrated positive definite functions continuous must make the functionals \( f_{x,y}(g) = \langle g(x), y \rangle \) continuous as well. Thus (2) is proved.

(3) \( \Rightarrow \) (1): Fix a countable family \( \{ f_n \}_{n \in \mathbb{N}} \) of continuous positive definite functions on \( G \) so that they generate a neighborhood basis of \( 1_G \). Without loss of generality we can assume \( f_n(1_G) \leq 2^{-n} \) for all \( n \in \mathbb{N} \).

Consider the space \( X \) of all complex-valued functions on \( G \) with finite support, i.e., functions \( x : G \to \mathbb{C} \) such that for all but finitely many \( g \in G \), \( x(g) = 0 \). \( X \) is a linear space under addition and scalar multiplication. For \( x, y \in X \), let

\[
\langle x, y \rangle = \sum_{g,h \in G} \sum_{n \in \mathbb{N}} f_n(h^{-1} g)x(g) \overline{y(h)}.
\]

This sesquilinear form is well-defined since for any \( g, h \in G \),

\[
\left| \sum_{n} f_n(h^{-1} g) \right| \leq \sum_{n} |f_n(h^{-1} g)| \leq \sum_{n} f_n(1_G) < \infty.
\]

Let \( N = \{ x \in X \mid \langle x, x \rangle = 0 \} \). Then \( N \) is a linear subspace of \( X \) and it is easy to check that \( \langle \cdot, \cdot \rangle \) is well-defined on the quotient \( X/N \), making it a pre-Hilbert space. Let \( H \) be the completion of \( X/N \) under the induced \( \langle \cdot, \cdot \rangle \). Then \( H \) is a separable complex Hilbert space.

We consider the representation of \( G \) in \( U(H) \) given by the definition that, for each \( g \in G \) and \( x \in X \),

\[
T_g x(h) = x(g^{-1} h).
\]

For any \( g_0 \in G \) and \( x, y \in X \), we have that

\[
\langle T_{g_0} x, T_{g_0} y \rangle = \sum_{g,h,n} f_n(h^{-1} g)x(g_0^{-1} g) \overline{y(g_0^{-1} h)}
\]

\[
= \sum_{g,h,n} f_n(h^{-1} g)x(g) \overline{y(h)} = \langle x, y \rangle.
\]

The map \( g \mapsto T_g \) is obviously a group homomorphism. For any \( g_0 \in G \) and \( x \in X \), we have that

\[
\langle T_{g_0} x, x \rangle = \sum_{g,h,n} f_n(h^{-1} g)x(g_0^{-1} g) \overline{x(g_0^{-1} h)} = \sum_{g,h,n} f_n(h^{-1} g_0 x(g) \overline{y(h)}).
\]

It follows immediately from this and the continuity of \( f_n \) that \( g \mapsto T_g \) is continuous.
We next check that \( g \mapsto T_g \) is injective. For this it suffices to show that for \( g_0 \neq 1_G, T_{g_0} \neq I \). Suppose \( g_0 \neq 1_G \). There exists some \( n \in \mathbb{N} \) such that \( f_n(g_0) \neq f_n(1_G) \). Consider \( x_0 \in \mathcal{X} \) defined by

\[
x_0(g) = \begin{cases} 1, & \text{if } g = g_0, \\ 0, & \text{otherwise}. \end{cases}
\]

Then the \( n \)-th term of \( \langle T_{g_0}x_0 - x_0, T_{g_0}x_0 - x_0 \rangle \) is

\[
\sum_{g,h} f_n(h^{-1}g)(x_0(g_0^{-1}g) - x_0(g))(x_0(g_0^{-1}h) - x_0(h)) = 2(f_n(1_G) - \text{Re} f_n(g_0)) > 0.
\]

It follows that \( \langle T_{g_0}x_0 - x_0, T_{g_0}x_0 - x_0 \rangle > 0 \), and therefore \( T_{g_0} \neq I \).

Now to establish that \( g \mapsto T_g \) is a topological group isomorphic embedding, the only thing that remains to be checked is that the inverse of \( g \mapsto T_g \) is continuous. For this suppose \( T_{g_m} \to T_{g_\infty} \), as \( m \to \infty \), for group elements \( g_m, g_\infty \in G \). We are to show that \( g_m \to g_\infty \), as \( m \to \infty \). Consider

\[
x_0(g) = \begin{cases} 1, & \text{if } g = 1_G, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and } y_0(h) = \begin{cases} 1, & \text{if } g = g_\infty, \\ 0, & \text{otherwise}. \end{cases}
\]

Then \( \langle T_{g_m}x_0, y_0 \rangle \to \langle T_{g_\infty}x_0, y_0 \rangle \) as \( m \to \infty \). But a straightforward computation shows that

\[
\langle T_{g_m}x_0, y_0 \rangle = \sum_n f_n(g_\infty^{-1}g_m),
\]

and

\[
\langle T_{g_\infty}x_0, y_0 \rangle = \sum_n f_n(1_G).
\]

These imply that for all \( n \in \mathbb{N} \), \( f_n(g_\infty^{-1}g_m) \to f_n(1_G) \) as \( m \to \infty \). From our assumption on \( \{ f_n \} \) it follows that \( g_\infty^{-1}g_m \to 1_G \), or \( g_m \to g_\infty \), as \( m \to \infty \). Thus \((3) \Rightarrow (1)\) is proved.

\((3) \Rightarrow (4):\) This is an application of the lemma of Schur (a product of positive definite functions is positive definite) and other properties of positive definite functions. Suppose \( F \subseteq G \) is closed and \( 1_G \notin F \). Then there are continuous positive definite functions \( f_1, \ldots, f_n \) and open sets \( O_1, \ldots, O_n \) in \( \mathbb{C} \) such that

\[
1_G \in \bigcap_{i=1}^n f_i^{-1}(O_i) \subseteq G \setminus F.
\]

Without loss of generality assume that \( f_i(1_G) = 1 \) for \( 1 \leq i \leq n \). Then there are \( \epsilon_1, \ldots, \epsilon_n > 0 \) such that, for any \( x \in F \), there is \( 1 \leq i \leq n \) with

\[
|f_i(x)| \leq 1 - \epsilon_i.
\]

Now the function \( f = f_1 \cdots f_n \) is continuous and positive definite on \( G \). Moreover, \( f(1_G) = 1 \) and, letting \( \epsilon = \min \{ \epsilon_1, \ldots, \epsilon_n \} \), we have that for any \( x \in F \),

\[
|f(x)| = |f_1(x)| \cdots |f_n(x)| \leq 1 - \epsilon < 1.
\]

\((4) \Rightarrow (5):\) Given a countable family \( \{ f_n \}_{n \in \mathbb{N}} \) of continuous positive definite functions so that they separate \( 1_G \) and closed sets not containing \( 1_G \), we can first scale them down so that \( f_n(1_G) \leq 2^{-n} \), for all \( n \in \mathbb{N} \). Then the function \( f \) defined by

\[
f(x) = \sum_{n \in \mathbb{N}} f_n(x)
\]
is continuous, positive definite on $G$ and separates $1_G$ and closed sets not containing $1_G$.

Statement (5) can be strengthened further to

(5') There is a real-valued continuous positive definite function on $G$ separating $1_G$ and closed sets not containing $1_G$.

In fact, if $f$ is a complex-valued such function, then $f' = \text{Re} f$ is as required. Similarly, functions in statements (3) and (4) can be taken to be real-valued as well.

An identical proof gives similar characterizations for closed subgroups of the orthogonal group of the infinite-dimensional separable real Hilbert space. Statements (3)-(5) are unchanged in such a theorem. Thus the closed subgroups of the unitary group and those of the orthogonal group are the same class of topological groups. In particular it implies that statement (2) in Theorem 2.1 can be replaced by one for real-valued continuous positive definite functions as well.

Also clear from the proof is that Theorem 2.1 can be generalized to second countable topological groups. The metrizability is implied by the existence of enough many continuous positive definite functions. The completeness requirement is unnecessary.

As an application of Theorem 2.1, we consider the Banach spaces $L_p([0, 1])$ and $l_p$ for $1 \leq p \leq 2$. The additive groups of these spaces are Polish groups. Concerning positive definite functions on these groups, the following result of Schoenberg is well-known.

**Theorem 2.2 (Schoenberg [11]).** For $1 \leq p \leq 2$, the function $e^{-|x|^p}$ is positive definite on $\mathbb{R}$, $l_p$ or $L_p([0, 1])$.

Thus we have

**Corollary 2.3.** For $1 \leq p \leq 2$, the additive groups of $l_p$ and $L_p([0, 1])$ are isomorphic to some closed subgroups of $U_\infty$.

**Proof.** The function $e^{-|x|^p}$ is continuous and obviously it separates the origin (which is the identity of the additive group) and closed sets not containing it, as the function is norm dependent.

The formulation of Theorem 2.1 was based on communications with Pestov and Megrelishvili. Schoenberg's result and the corollary were brought to our attention by Megrelishvili.

Now let $\text{Iso}(\mathcal{H})$ be the group of all isometries of $\mathcal{H}$ (onto itself) endowed with the topology of pointwise convergence. Then it is easy to see that $\text{Iso}(\mathcal{H})$ is a Polish group and that $U_\infty$ is a closed subgroup of $\text{Iso}(\mathcal{H})$. In fact it is well-known that $\text{Iso}(\mathcal{H})$ is a semi-direct product of $U_\infty$ with the abelian group of translations in $\mathcal{H}$ (i.e., the additive group $\mathcal{H}$). In Section 4 we will need the following theorem of Uspenskiy.

**Theorem 2.4 (Uspenskiy).** $\text{Iso}(\mathcal{H})$ is isomorphic to a closed subgroup of $U_\infty$.

**Proof.** For each $v \in \mathcal{H}$ define a positive definite function $p_v$ by

$$p_v(g) = e^{-\|g(v) - v\|^2}$$

for $g \in \text{Iso}(\mathcal{H})$. Each $p_v$ is continuous since $\text{Iso}(\mathcal{H})$ has the pointwise convergence topology. To see that each $p_v$ is positive definite, note that for any $g_1, \ldots, g_n \in \mathcal{H}$,

$$p_v(g_1 \cdots g_n) = e^{-\|g_v(g_1 \cdots g_n) - v\|^2} = e^{-\|g_v(g_1) \cdots g_v(g_n) - v\|^2} = p_v(g_1) \cdots p_v(g_n).$$
Iso(ℋ) and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} p_v (g_j^{-1} g_i) c_i c_j = \sum_{i,j=1}^{n} e^{-\|g_j^{-1} g_i (v) - v\|^2} c_i c_j$$

$$= \sum_{i,j=1}^{n} e^{-\|g_j^{-1} g_i (v) - g_j^{-1} g_i (v)\|^2} c_i c_j$$

$$= \sum_{i,j=1}^{n} e^{-\|g_i (v) - g_j (v)\|^2} c_i c_j.$$ 

This last expression is non-negative by Schoenberg’s Theorem 2.2 ($e^{-|x|^2}$ on $\mathbb{R}$), Schur’s lemma on products of positive definite functions, and the fact that pointwise limits of positive definite functions are positive definite.

Finally, it is straightforward to see that \{\text{eqn}\} generates the topology of Iso(ℋ), and thus by Theorem 2.1 (2) Iso(ℋ) is a closed subgroup of $U_\infty$. □

3. Universal actions and universal equivalence relations

For any Polish space $X$, let $\mathcal{F}(X)$ be the space of all closed subsets of $X$ endowed with the Effros Borel structure (c.f., e.g., 12.B of [8]). Specifically, the Borel structure on $\mathcal{F}(X)$ is generated by sets of the form

\( \{ F \in \mathcal{F}(X) \mid F \cap U \neq \emptyset \} \)

where $U$ is an open subset of $X$. Then $\mathcal{F}(X)$ becomes a standard Borel space. If moreover a Polish group $G$ acts on $X$ in a Borel manner then the induced action of $G$ on $\mathcal{F}(X)$ is also Borel.

When general results of Becker and Kechris on the existence of universal Borel $G$-spaces are applied to the special case of the unitary group $U_\infty$, we obtain the following realizations of the universal objects ([2], Theorems 2.6.1 and 3.5.3).

**Theorem 3.1 (Becker-Kechris).** The left translation of $U_\infty$ on $\mathcal{F}(U_\infty)^N$ is a universal Borel $U_\infty$-action. The left translation of $U_\infty$ on $\mathcal{F}(U_\infty)$ induces a universal $U_\infty$-orbit equivalence relation.

Because of the complicated structure of $\mathcal{F}(U_\infty)$ it is desirable to find simpler realizations of the universal Borel $U_\infty$-space. One approach is to consider the application actions of $U_\infty$ induced by that on $\mathcal{H}$. Kechris and the author ([6] defined a general action for an arbitrary isometry group that resembles the logic action for $S_\infty$. Since $U_\infty$ is naturally a group of isometries, one can deduce the following result (Corollary 9.3 of [6]).

**Theorem 3.2 (Gao-Kechris).** For each natural number $n > 0$, let $U_\infty$ act on $\mathcal{F}(\mathcal{H}^n)$ by coordinatewise and pointwise application. Then the product space

$$\prod_{n > 0} \mathcal{F}(\mathcal{H}^n)$$

is a universal Borel $U_\infty$-space.

The following theorem is the main theorem of this section. It shows that, if we are willing to loosen up the notion of universality, then some significantly simpler realization of the universal space does exist.
Theorem 3.3. Let $U_\infty$ act on $\mathcal{F}(\mathcal{H})$ by pointwise application and let $E$ be the orbit equivalence relation. Then $E$ is a faithfully universal $U_\infty$-orbit equivalence relation.

The rest of the section is devoted to the proof of Theorem 3.3. This will be done in two steps. Denote $X = \prod_{n>0} \mathcal{F}(\mathcal{H}^n)$ and $Y = \mathcal{F}(\mathcal{H})^\mathbb{N}$.

Our first step is to show that there is a faithful Borel reduction from $E_X$ to $E_Y$. By Theorem 3.2, this implies that $E_Y$ is faithfully universal among $U_\infty$-orbit equivalence relations. Then in the second step, we define a faithful Borel reduction from $Y$ into $\mathcal{F}(\mathcal{H})$, and thus complete the proof.

For each natural number $n > 0$, we endow $\mathcal{H}^n$ with the inner product $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_n = \frac{1}{n} \sum_{i=1}^n \langle x_i, y_i \rangle$, for $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathcal{H}^n$. Thus a norm on $\mathcal{H}^n$ is given by $\| (x_1, \ldots, x_n) \|_n = \sqrt{\frac{1}{n} \sum_{i=1}^n \| x_i \|^2}$, and $\mathcal{H}^n$ becomes a Hilbert space unitarily isomorphic to $\mathcal{H}$.

Now for each $k \in \mathbb{N}$ consider the linear isomorphic embedding $T_k: \mathcal{H}^{2k} \rightarrow \mathcal{H}^{2k+1}$

$T_k$ is unitary, i.e., it preserves inner products. Hence each $\mathcal{H}^{2k}$ can be regarded as a closed subspace of $\mathcal{H}^{2k+1}$ in view of the canonical $T_k$. We have

$\mathcal{H} \subset \mathcal{H}^2 \subset \mathcal{H}^4 \subset \cdots \subset \mathcal{H}^{2k} \subset \mathcal{H}^{2k+1} \subset \cdots$

Now take the direct limit

$\mathcal{H}^\infty = \bigcup_{k \in \mathbb{N}} \mathcal{H}^{2k}$,

and let $Z$ be the completion of $\mathcal{H}^\infty$. Then $Z$ is unitarily isomorphic to $\mathcal{H}$ since $\mathcal{H}^\infty$ is a pre-Hilbert space with the inner product

$\langle \bar{x}, \bar{y} \rangle_\infty = \langle \bar{x}, \bar{y} \rangle_{2k}$,

where the number $k$ is large enough such that $\bar{x}, \bar{y} \in \mathcal{H}^{2k}$.

Moreover, for $\varphi \in U(\mathcal{H})$ and $n > 0$, let $\varphi^{(n)} \in U(\mathcal{H}^n)$ be given by

$\varphi^{(n)}(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$.

Then we have

$\varphi = \varphi^{(1)} \subset \varphi^{(2)} \subset \varphi^{(4)} \subset \cdots \subset \varphi^{(2k)} \subset \varphi^{(2k+1)} \subset \cdots$,

i.e., each $\varphi^{(2k+1)}$ coincides with $\varphi^{(2k)}$ on $\mathcal{H}^{2k} \subset \mathcal{H}^{2k+1}$. Therefore we can let

$\varphi^{(\infty)} = \bigcup_{k \in \mathbb{N}} \varphi^{(2k)}$. 

which is a unitary transformation on $\mathcal{H}\infty$. Let $\varphi^Z$ be the unique extension of $\varphi^{(\infty)}$ to $Z$. Then $\varphi^Z \in U(Z)$. Denote

$$U^Z_\infty = \{ \varphi^Z \in U(Z) \mid \varphi \in U(\mathcal{H}) \}.$$

Then $U^Z_\infty$ is a closed subgroup of $U(Z)$. The following lemma is in essence a characterization of $U^Z_\infty$ in $U(Z)$.

**Lemma 3.4.** There is a sequence $\{D_m\}_{m \in \mathbb{N}}$ of closed subsets of $Z$ such that

$$U^Z_\infty = \{ \Phi \in U(Z) \mid \forall m \ (\Phi(D_m) = D_m) \}.$$

**Proof.** It is enough to show that, for each $n > 0$, there is a sequence $\{K_{n,l}\}_{l \in \mathbb{N}}$ of closed subsets of $\mathcal{H}^n$ such that

$$\{ \varphi^{(n)} \in U(\mathcal{H}^n) \mid \varphi \in U(\mathcal{H}) \} = \{ \Phi \in U(\mathcal{H}^n) \mid \forall l \ (\Phi(K_{n,l}) = K_{n,l}) \}.$$

Then the sequence $\{D_m\}_{m \in \mathbb{N}}$ can be taken to be any enumeration of the set $\{K_{n,l} \mid n, l \in \mathbb{N}\}$.

Fix $n > 0$ and denote the group on the left side by $U^{(n)}_\infty$. For each tuple $\overline{p} = (p_1, \ldots, p_n)$ of positive rational numbers, let

$$K_{\overline{p}} = \{ (x_1, \ldots, x_n) \in \mathcal{H}^n \mid \|x_i\| \leq p_i, i = 1, \ldots, n \}.$$

Each $K_{\overline{p}}$ is closed in $\mathcal{H}^n$. Similarly, for each tuple $\overline{q} = (q_{ij})_{1 \leq i, j \leq n}$ of positive rational numbers, let

$$L_{\overline{q}} = \{ (x_1, \ldots, x_n) \in \mathcal{H}^n \mid \forall 1 \leq i, j \leq n \ (\|x_i - x_j\| \leq q_{ij}) \}.$$

Then each $L_{\overline{q}}$ is also closed in $\mathcal{H}^n$. Denote

$$G = \{ \Phi \in U(\mathcal{H}^n) \mid \forall \overline{p} (\Phi(K_{\overline{p}}) = K_{\overline{p}}) \ \text{and} \ \forall \overline{q} (\Phi(L_{\overline{q}}) = L_{\overline{q}}) \}.$$

We verify that $G = U^{(n)}_\infty = \{ \varphi^{(n)} \mid \varphi \in U^{(n)}_\infty \}$.

It is clear that $G$ consists exactly of $\Phi \in U(\mathcal{H}^n)$ such that

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathcal{H}^n$, if $\Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$,
then for all $1 \leq i, j \leq n$, $\|x_i\| = \|y_i\|$ and $\|x_i - x_j\| = \|y_i - y_j\|$. 

Thus immediately $U^{(n)}_\infty \subseteq G$. For the opposite inclusion, assume $\Phi \in G$. Note that $\Phi(\mathcal{H}) = \mathcal{H}$. We put $\varphi = \Phi \restriction \mathcal{H}$. By considering $\Phi \circ (\varphi^{(n)})^{-1}$ in place of $\Phi$, we may assume without loss of generality that $\varphi$ is the identity transformation, toward proving that $\Phi$ is also the identity.

Suppose $\Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ and $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$. Then for some $1 \leq i \leq n$, $x_i \neq y_i$. Let $i$ be the least such index and let $u = x_i$. Our assumption gives that

$$\Phi(u, \ldots, u) = (u, \ldots, u).$$

Therefore by linearity of $\Phi$, we have

$$\Phi(x_1 - u, \ldots, x_n - u) = (y_1 - u, \ldots, y_n - u).$$

Since $\Phi \in G$, we must have $\|x_i - u\| = 0 = \|y_i - u\|$, so $y_i = u = x_i$, a contradiction.

To complete the proof of the lemma, we let $\{K_{n,l}\}_{l \in \mathbb{N}}$ be any enumeration of the countable set of all $K_{\overline{p}}$ and $L_{\overline{q}}$. □
Now let \( \{D_m\}_{m \in \mathbb{N}} \) be a fixed sequence of closed subsets of \( Z \) given by Lemma 3.4. We are ready to define a Borel reduction \( f: X \to Y \). In doing so we tacitly identify \( Z \) with \( \mathcal{H} \) since they are unitarily isomorphic. For a sequence
\[
\bar{R} = (R_n)_{n > 0} \in \prod_{n > 0} \mathcal{F}(\mathcal{H}^n) = X,
\]
i.e., each \( R_n \) being a closed subset of \( \mathcal{H}^n \), we let
\[
f(\bar{R}) = (D_0, R_1, D_1, D_2, R_2, \ldots, R_n, D_n, \ldots) \in \mathcal{F}(\mathcal{Z})^\mathbb{N}.
\]
It is clear that \( f \) is a Borel function. We claim that \( f \) is a faithful reduction from \( E^{X}_{U,\infty} \) to \( E^{Y}_{U,\infty} \).

To see that it is a reduction, suppose \( \bar{R}, \bar{S} \in X \). Note that \( \bar{R}E^{Y}_{U,\infty}\bar{S} \) iff there is \( \varphi \in U(\mathcal{H}) \) such that \( \varphi(R_n) = S_n \) for all \( n > 0 \). When all \( R_n \) and \( S_n \) are regarded as subsets of \( Z \), \( \varphi(R_n) = S_n \) iff \( \varphi^2(R_n) = S_n \). Thus by Lemma 3.4, we have
\[
\bar{R}E^{Y}_{U,\infty}\bar{S} \iff \exists \Phi \in U^{\mathcal{Z}}_{U,\infty} \forall n > 0 (\Phi(R_n) = S_n) \iff \exists \Phi \in U(\mathcal{Z}) (\Phi(R_n) = S_n \text{ for all } n > 0 \text{ and } \Phi(D_m) = D_m \text{ for all } m \in \mathbb{N}) \iff f(\bar{R})E^{Y}_{U,\infty}f(\bar{S}).
\]

To see that \( f \) is faithful, let \( A \) be an invariant Borel subset of \( X \). We use the abbreviation \( f(\bar{R}) = (\bar{R}, \bar{D}) \), and have that
\[
f(A) = \{ (\bar{R}, \bar{D}) \mid \bar{R} \in A \}.
\]
Let \( \bar{0} \) be the element of \( X \) whose each coordinate is the set \( \{0\} \subset \mathcal{H} \). By a theorem of Miller (c.f., [2], Corollary 2.3.3), the orbit \( [f(\bar{0})]_{U,\infty} \) in \( Y \) is Borel. Now for arbitrary \( (\bar{S}, \bar{K}) \in Y \), from
\[
(\bar{S}, \bar{K}) \in [f(A)]_{U,\infty} \iff (\bar{0}, \bar{K}) \in [f(\bar{0})]_{U,\infty} \text{ and } \bar{S} \in A,
\]
we know that \( [f(A)]_{U,\infty} \) is Borel. This finishes our first step.

The second step of our proof is fulfilled by the following lemma which is actually stronger than what we need.

**Lemma 3.5.** There is a Borel \( U_{\infty} \)-embedding from \( Y = \mathcal{F}(\mathcal{H})^{\mathbb{N}} \) into \( \mathcal{F}(\mathcal{H}) \).

**Proof.** For \( n \in \mathbb{N} \) we first define \( S_n: \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}) \) by
\[
S_n(F) = \begin{cases} 
2nx + \frac{x}{\|x\|} + 1 + \frac{\|x\|}{\|y\|} \mid x \in F, \|y\| = n+1, & \text{if } 0 \notin F, \\
\{ y \in \mathcal{H} \mid \|y\| = 2n+1 \}, & \text{if } 0 \in F.
\end{cases}
\]
For any \( F \in \mathcal{F}(\mathcal{H}) \) and \( y \in S_n(F) \), we have \( 2n \leq \|y\| \leq 2n+1 \). Now for \( \tilde{F} = (F_0, F_1, \ldots) \in \mathcal{F}(\mathcal{H})^{\mathbb{N}} \), we let
\[
S(\tilde{F}) = \bigcup_{n \in \mathbb{N}} S_n(F_n).
\]
We claim that \( S(\tilde{F}) \) is closed. To see this, let \( \{y_k\}_{k \in \mathbb{N}} \) be a sequence such that for each \( k \in \mathbb{N}, y_k \in S_{n_k}(F_{n_k}) \) for some \( n_k \), and \( y_k \to y_\infty \) as \( k \to \infty \) for some \( y_\infty \in \mathcal{H} \).
Since for \( n \neq m \) the distance between \( S_n(F_n) \) and \( S_m(F_m) \) is at least 1, we may assume that for some fixed \( n_0 \), \( y_k \in S_{n_0}(F_{n_0}) \) for all \( k \in \mathbb{N} \). If for infinitely many \( k \in \mathbb{N} \), \( \|y_k\| = 2n_0 + 1 \), then we have \( \|y_\infty\| = 2n_0 + 1 \) and so \( y_\infty \in S_{n_0}(F_{n_0}) \). If for infinitely many \( k \in \mathbb{N} \), \( \|y_k\| = 2n_0 \), then we have that \( 0 \in F_{n_0} \) and \( \|y_\infty\| = 2n_0 \), hence also \( y_\infty \in S_{n_0}(F_{n_0}) \). Otherwise, we may assume that none of the \( y_k \) has norm \( 2n_0 \) or \( 2n_0 + 1 \). Therefore there are \( x_k \in F_{n_0} \) such that

\[
y_k = \frac{2n_0x_k}{\|x_k\|} + \frac{x_k}{1 + \|x_k\|}.
\]

Note that we have from the above equality

\[
\|y_k\| = 2n_0 + \frac{\|x_k\|}{1 + \|x_k\|},
\]

and hence

\[
\|x_k\| = \frac{\|y_k\| - 2n_0}{2n_0 + 1 - \|y_k\|}.
\]

Moreover,

\[
x_k = \left( \frac{2n_0(2n_0 + 1 - \|y_k\|)}{\|y_k\| - 2n_0} + \left( 1 + \frac{\|y_k\| - 2n_0}{2n_0 + 1 - \|y_k\|} \right)^{-1} \right)^{-1} y_k.
\]

Now there are three cases. Case (1): \( \|y_\infty\| = 2n_0 + 1 \). In this case \( y_\infty \in S_{n_0}(F_{n_0}) \) and there is nothing to prove. Case (2): \( \|y_\infty\| = 2n_0 \). In this case we deduce that

\[
\|y_k\| - \|y_\infty\| = \frac{\|x_k\|}{1 + \|x_k\|} \to 0 \text{ as } k \to \infty.
\]

Hence \( \|x_k\| \to 0 \) and \( x_k \to 0 \). This shows that \( 0 \in F_{n_0} \) since \( F_{n_0} \) is closed. Therefore \( y_\infty \in S_{n_0}(F_{n_0}) \). Case (3): \( \|y_\infty\| \neq 2n_0 \) or \( 2n_0 + 1 \). Then by letting \( k \to \infty \) in the last of the displayed formulae in the preceding paragraph we obtain some \( x_\infty \) such that \( x_k \to x_\infty \) as \( k \to \infty \). Thus \( x_\infty \in F_{n_0} \) and \( x_\infty \neq 0 \). Moreover, we have that

\[
y_\infty = \frac{2n_0x_\infty}{\|x_\infty\|} + \frac{x_\infty}{1 + \|x_\infty\|},
\]

so \( y_\infty \in S_{n_0}(F_{n_0}) \) as needed.

We verify that \( S \) is a Borel \( U_\infty \)-embedding. It is easy to see that \( S \) is Borel and is an embedding. To see that \( S \) respects the group action, it suffices to show that, for each \( n \in \mathbb{N} \), \( T \in U(H) \) and \( F \in F(H) \),

\[
T(S_n(F)) = S_n(T(F)).
\]

If \( 0 \not\in F \), then \( 0 \not\in T(F) \) and we only need to note that

\[
T(S_n(F)) = T\left( \left\{ \frac{2nx}{\|x\|} + \frac{x}{1 + \|x\|} \mid x \in F \right\} \cup \{ y \in H \mid \|y\| = 2n + 1 \} \right)
\]

\[
= \left\{ \frac{2nT(x)}{\|T(x)\|} + \frac{T(x)}{1 + \|T(x)\|} \mid x \in F \right\} \cup \{ y \in H \mid \|y\| = 2n + 1 \}
\]

\[
= S_n(T(F)).
\]
If $0 \in F$, the same identity holds except that $0 \in T(F)$ and the part \( \{ y \in H \mid \|y\| = 2n \} \) must be simultaneously added to all terms. This finishes the proof of the lemma.

We have thus finished the proof of Theorem 3.3. The following corollary is immediate from the proof of Lemma 3.5.

**Corollary 3.6.** Let $r_2 > r_1 \geq 0$ be real numbers. The application action of $U_{\infty}$ on $F(\{ x \in H \mid r_1 \leq \|x\| \leq r_2 \})$ induces a faithfully universal $U_{\infty}$-orbit equivalence relation. In particular, if $B_1(H)$ is the unit ball of $H$, then the application action of $U_{\infty}$ on $F(B_1(H))$ induces a faithfully universal $U_{\infty}$-orbit equivalence relation.

We would like to add one more remark about the proof. The reader can compare the first step of our proof to the argument in [6] for the main theorem there and readily see that the two proofs have the same structure. In [6] we dealt with the Urysohn space $U$ instead of $H$ and the isometry group $\text{Iso}(U)$ instead of $U(H)$. Our argument here for the faithful universality of $E^H_{U_{\infty}}$ can be repeated for the objects in [6] to obtain the following improvement.

**Theorem 3.7.** Let $\text{Iso}(U)$ act on $\mathcal{F}(U)^{\mathbb{N}}$ by coordinatewise and pointwise application. Then the equivalence relation induced by this action is a faithfully universal orbit equivalence relation for all Borel actions of Polish groups. Consequently, the Topological Vaught Conjecture for this space is equivalent to the general Topological Vaught Conjecture.

4. Hilbertian Polish metric spaces

In this section we consider the isometric classification of Hilbertian Polish metric spaces. A *Polish metric space* is a separable complete metric space. We call a metric space *Hilbertian* if it can be isometrically embedded into the Hilbert space $H$.

Each Hilbertian Polish metric space has an isometric copy as a closed subset of $H$. Conversely, each closed subset of $H$, with its induced metric from $H$, is Hilbertian. Hence the space $F(H)$ can be naturally identified as the space of all Hilbertian Polish metric spaces.

Note that in the above definition it is equivalent to take the Hilbert space to be real. In fact, this will simplify some of the computations later in this section, so we make the convention here that all Hilbert spaces mentioned in this section will be real. In [11] Schoenberg gave some interesting characterizations of Hilbertian metric spaces in terms of real-valued positive definite functions and negative definite functions. To state these characterizations we need to recall the notion of positive or negative definite functions on a general space $X$.

A function $f : X^2 \to \mathbb{R}$ is called *positive definite on $X$* if for any $x, y \in X$, $f(x, x) = 0$ and $f(x, y) = f(y, x)$, and for any natural number $n > 1$, arbitrary real numbers $r_1, \ldots, r_n \in \mathbb{R}$ and elements $x_1, \ldots, x_n \in X$,

$$\sum_{1 \leq i,j \leq n} f(x_i, x_j) r_i r_j \geq 0.$$  

On the other hand, $f$ is called *negative definite on $X$* if for any $x, y \in X$, $f(x, x) = 0$ and $f(x, y) = f(y, x)$, and for any natural number $n > 1$, arbitrary real numbers
In case \( X \) is a metric space and \( d \) is the metric on \( X \), a function \( g : \mathbb{R} \to \mathbb{R} \) is called positive (negative) definite on \( X \) if \( g(d(x, y)) \) is positive (negative) definite on \( X \).

Schoenberg [11] showed that \( e^{-\lambda x^2} \) are positive definite on \( \mathcal{H} \) for all \( \lambda > 0 \). It turns out that these functions characterize Hilbertian spaces, in the following sense.

**Theorem 4.1** (c.f. [10] and [11]). The following are equivalent for a metric space \((X, d)\):

1. \( X \) is Hilbertian;
2. \( e^{-\lambda x^2} \) are positive definite on \( X \) for all \( \lambda > 0 \);
3. \( e^{-\lambda x^2} \) are positive definite on \( X \) for a decreasing sequence of positive values \( \lambda \) converging to 0;
4. \( d^2 \) is negative definite on \( X \);
5. (Menger) \( X \) is separable and for every natural number \( n > 1 \), every set of \( n+1 \) distinct points of \( X \) can be isometrically embedded into the space \( \mathbb{R}^n \).

The theorem provides another descriptively simple way of coding Hilbertian Polish spaces. In general if we code a Polish metric space by specifying the metric on a countable dense subset, then by Theorem 4.1 (iv), in the space of all codes for Polish metric spaces the subset of codes for Hilbertian Polish metric spaces is a standard Borel space.

We define the isometry relation \( \cong \) for \( K, L \in \mathcal{F}(\mathcal{H}) \) by

\[ K \cong L \iff \text{there is an isometry from } K \text{ onto } L. \]

Note that \( \cong \) is different from the orbit equivalence relation induced by the \( U_\infty \) action on \( \mathcal{F}(\mathcal{H}) \) considered in the previous section. Rather, it can be identified with the \( \text{Iso}(\mathcal{H}) \)-orbit equivalence relation on \( \mathcal{F}(\mathcal{H}) \), which we prove below.

**Theorem 4.2.** The isometry relation \( \cong \) for Hilbertian Polish metric spaces is Borel bireducible with the orbit equivalence relation of the application action of \( \text{Iso}(\mathcal{H}) \) on \( \mathcal{F}(\mathcal{H}) \).

**Proof.** Let \( E \) denote the orbit equivalence relation \( E_{\text{Iso}(\mathcal{H})}^{\mathcal{F}(\mathcal{H})} \). We need to show that \( (\cong_1) \leq_B E \) and \( E \leq_B (\cong_1) \). Before defining the reductions let us recall a basic fact of the metric geometry in Hilbert and Euclidean spaces:

If \( A \) and \( B \) are subsets of \( \mathbb{R}^n \), \( n > 0 \) or \( \mathcal{H} \), and \( \varphi : A \to B \) is an isometry from \( A \) onto \( B \), then \( \varphi \) can be extended to an isometry from \( \langle A \rangle \) onto \( \langle B \rangle \), where \( \langle \cdot \rangle \) denote the affine subspace generated by the set.

We now define the reduction witnessing \( (\cong_1) \leq_B E \). Fix an isometric embedding \( f \) of \( \mathcal{H} \) into \( \mathcal{H} \) so that \( f(\mathcal{H})^\perp \) has infinite dimension. Consider the Borel map \( K \mapsto f(K) \) from \( \mathcal{F}(\mathcal{H}) \) into \( \mathcal{F}(\mathcal{H}) \). If \( K, L \in \mathcal{F}(\mathcal{H}) \) are not isometric to each other, then \( f(K) \) and \( f(L) \) are in no way \( E \)-equivalent. On the other hand, if \( K \cong L \), then \( f(K) \cong f(L) \). Let \( \varphi \) be an isometry from \( f(K) \) onto \( f(L) \). Then by the above fact, \( \varphi \) can be extended to an isometry \( \varphi^* \) from \( f(K) \) onto \( f(L) \). By
our construction, both \((f(K))^\perp\) and \((f(L))^\perp\) have infinite dimension, and therefore there is an isometry \(\psi\) between them. Finally the isometry \(\varphi^* \oplus \psi\) is an isometry of the whole space sending \(f(K)\) to \(f(L)\), and therefore \(f(K)Ef(L)\).

In order to establish the converse reduction we first show that \(E \leq_B (\equiv_i) \times \text{id}_{\mathbb{N}}\), where the equivalence relation on the right-hand side is defined on \(\mathcal{F}(\mathcal{H}) \times \mathbb{N}\) by

\[
((K,n),(L,m)) \in (\equiv_i) \times \text{id}_{\mathbb{N}} \iff K \cong_i L \text{ and } m = n.
\]

For this we simply associate with any given \(K \in \mathcal{F}(\mathcal{H})\) the pair \((K,n)\), where \(n = 0\) if \(\dim((K)^\perp)\) is infinite and \(n = \dim((K)^\perp) + 1\) otherwise. Note that this is a Borel map since it is Borel to check if the dimension of \((K)^\perp\) is finite or not. It is easy to see that this map is a reduction from \(E\) to \((\equiv_i) \times \text{id}_{\mathbb{N}}\), using again the aforementioned fact and by a similar argument as in the preceding paragraph.

It remains to see that \((\equiv_i) \times \text{id}_{\mathbb{N}} \leq_B (\equiv_i).\) This requires us to associate with any pair \((K,n)\), where \(K\) is a Hilbertian Polish metric space and \(n\) a natural number, another Hilbertian Polish metric space \(K_n\), so that

\[
K \cong_i L \text{ and } n = m \iff K_n \cong_i L_m.
\]

For this suppose we are given \(K\) and \(n\), and \(d_K\) is the metric on \(K\). Let \(K'\) be a metric space with underlying set \(K\) and a new metric \(d_K'\) defined by

\[
d_K'(x,y) = \sqrt{1 - e^{-d_K(x,y)}}.
\]

To see that \(d_K'\) is a metric, note that the function \(1 - e^{-t}\) is monotone increasing for \(t > 0\), which implies that \(1 - e^{-d_K(x,y)}\) is a metric, and that the square root of any metric is a metric. Also note that the transformation from \(d_K\) to \(d_K'\) is a homeomorphism, which implies that \(d_K'\) is a complete metric iff \(d_K\) is. Obviously \(d_K' < 1\), and thus the diameter of \(K'\) is \(\leq 1\).

We claim that \(K'\) is a Hilbertian Polish metric space. By Theorem 4.1 it suffices to check that \((d_K')^2\) is negative definite. Let \(x_1,\ldots,x_n \in X\) and \(r_1,\ldots,r_n \in \mathbb{R}\) with \(\sum_{i=1}^n r_i = 0\). Then

\[
\sum_{i,j=1}^n d_K'(x_i,x_j)^2 r_i r_j = \sum_{i,j=1}^n (1 - e^{-d_K(x_i,x_j)}) r_i r_j = -\sum_{i,j=1}^n e^{-d_K(x_i,x_j)} r_i r_j.
\]

By a theorem of Scheonberg (Corollary 1 of [11]) the space \((K',\sqrt{d_K'})\) is Hilbertian. Thus by Theorem 4.1(ii) the function \(e^{-d_K(x,y)}\) is positive definite on \(K\). Thus the above displayed formula must be non-positive. This shows that \(K'\) is Hilbertian. We then identify \(K'\) with a closed subset of \(\mathcal{H}\) so that \((K')^\perp\) has infinite dimension.

We are now ready to define \(K_n\). Let \(K_n\) be the union of \(K'\) with the set of all points \(u \in \mathcal{H}\) such that the distance between \(u\) and \((K')^\perp\) is exactly \(n + 1\). It is easy to see that \(K_n\) is a closed subset of \(\mathcal{H}\). Moreover, the part \(K_n - K'\) is a connected clopen subset of \(K_n\) with infinite diameter. We check that this construction works. Let \((K,n)\) and \((L,m)\) be given, and \(K_n\) and \(L_m\) are constructed. If \(K \cong_i L\) then \(K' \cong_i L'\) and therefore if in addition \(n = m\) then \(K_n \cong_i L_m\). Conversely, suppose \(K_m \cong_i L_n\) and \(\varphi\) is an isometry witnessing it. Then \(\varphi(K_n - K')\) must be a connected clopen subset of \(L_m\) with infinite diameter, hence it must be \(L_n - L'\). It follows that \(\varphi(K') = L'\) and that \(n = m\). The same \(\varphi\) witnesses that \(K \cong_i L\) by a parallel change of metric on both spaces.
This finishes our proof that \((\cong_i) \times \text{id}_N \leq_B (\cong_i)\), and therefore of the theorem.

\[\square\]

We can now establish the main theorem of this paper, that is the universality of these equivalence relations among all \(U_\infty\)-orbit equivalence relations.

**Theorem 4.3.** The isometry equivalence relation for Hilbertian Polish metric spaces is Borel bireducible to a universal \(U_\infty\)-orbit equivalence relation.

**Proof.** By a theorem of Mackey (c.f. Theorem 2.3.5 of [8]) and Theorem 2.4, any \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation is Borel reducible to some \(U_\infty\)-orbit equivalence relation. Thus it follows from Theorems 4.2 and 3.3 that \((\cong_i) \leq_B E_{U_\infty}^{\mathcal{F}(\mathcal{H})}\).

For the converse, we consider the set \(X = \{x \in \mathcal{H} \mid 1 \leq \|x\| \leq 2\}\) and the application action of \(U_\infty\) on \(\mathcal{F}(X)\). By Corollary 3.6, the equivalence relation \(E_{U_\infty}^{\mathcal{F}(X)}\) is a universal \(U_\infty\)-orbit equivalence relation. We claim that \(E_{U_\infty}^{\mathcal{F}(X)} \leq_B (\cong_i)\).

To define the reduction, let \(S_3 = \{x \in \mathcal{H} \mid \|x\| = 3\}\) be the sphere of radius 3 in \(\mathcal{H}\). Given any \(K \in \mathcal{F}(X)\), let \(K' = K \cup \{0\} \cup S_3\). We check that the map \(K \mapsto K'\) is a required reduction. If \(K, L \in \mathcal{F}(X)\) and there is a unitary transformation \(U \in U(\mathcal{H})\) mapping \(K\) onto \(L\), then certainly \(U(0) = 0\) and \(U\) maps \(S_3\) onto \(S_3\), therefore in particular \(K' \cong L'\). Conversely, suppose \(K' \cong L'\) and this is witnessed by an isometry \(\varphi\). Then \(\varphi\) can be extended to an isometry of \(\mathcal{H}\) onto \(\mathcal{H}\), by the fact we noted in the proof of Theorem 4.2. Since \(S_3\) is a connected clopen subset of \(K\) with diameter 6, its image under \(\varphi\) or its extension must be \(S_3\) as a subset of \(L\). From this it follows that \(\varphi(0) = 0\) since the origin can be metrically characterized as the only point in \(K\) with distance 3 to every point of \(S_3\). It then follows that the extension of \(\varphi\) is an isometry of the whole space sending 0 to 0. Thus this extension must be a unitary transformation and moreover, it sends \(K\) onto \(L\).

\[\square\]

Since \(U_\infty\) is naturally a closed subgroup of \(\text{Iso}(\mathcal{H})\), we have the following immediate corollary.

**Corollary 4.4.** The following equivalence relations are pairwise Borel bireducible:

1. the isometry of Hilbertian Polish metric spaces;
2. the universal \(U_\infty\)-orbit equivalence relation;
3. the universal \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation;
4. the \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation on \(\mathcal{F}(\mathcal{H})\);
5. the \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation on \(\mathcal{F}(\mathcal{H})^N\);
6. the \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation on \(\prod_n \mathcal{F}(\mathcal{H}^n)\).

Before closing the paper we would like to explore a bit further the faithfulness of the universal equivalence relations. A general theorem in [6] (Corollary 9.2 of [6]) implies that the space \(\prod_n \mathcal{F}(\mathcal{H}^n)\) is a universal Borel \(\text{Iso}(\mathcal{H})\)-space. As the space improves to \(\mathcal{F}(\mathcal{H})^N\), we lose this property but retain faithful universality. This is the content of the theorem to follow. We do not know if there is a faithful reduction further down to \(\mathcal{F}(\mathcal{H})\).

**Theorem 4.5.** The application action of \(\text{Iso}(\mathcal{H})\) on \(\mathcal{F}(\mathcal{H})^N\) induces a faithfully universal \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation.
The proof is similar to the first step of that of Theorem 3.3 in the last section. We again consider the spaces
\[ X = \prod_{n>0} F(H^n) \quad \text{and} \quad Y = F(H)^\mathbb{N}, \]
and show that there is a faithful Borel reduction from \( E^X_{\text{ISO}(H)} \) to \( E^Y_{\text{ISO}(H)} \).

One can repeat the setup of the proof of Theorem 3.3 to the point that the spaces \( H^n, H^\infty \) and \( Z \), as well as inner products on them, are defined and each \( \varphi \in \text{Iso}(H) \) naturally induces \( \varphi^{(n)} \in \text{Iso}(H^n) \), \( \varphi^{(\infty)} \in \text{Iso}(H^\infty) \) and \( \varphi^Z \in \text{Iso}(Z) \). Denote
\[ G^Z = \{ \ varphi^Z \mid \ varphi \in \text{Iso}(H) \}. \]

Then we have a lemma similar to Lemma 3.4 (but with a different proof) below.

**Lemma 4.6.** There is a sequence \( \{D_n\}_{n \in \mathbb{N}} \) of closed subsets of \( Z \) such that
\[ G^Z = \{ \Phi \in \text{Iso}(Z) \mid \forall n (\Phi(D_n) = D_n) \}. \]

**Proof.** We again show that for each \( n > 1 \) there is a sequence \( \{K_n\}_{m \in \mathbb{N}} \) of closed subsets \( H^n \) such that
\[ \{ \varphi^{(n)} \in \text{Iso}(H^n) \mid \varphi \in \text{Iso}(H) \} = \{ \Phi \in \text{Iso}(H^n) \mid \forall m (\Phi(K_m) = K_m) \}. \]

Denote the group on the left side by \( G^{(n)} \). For each tuple \( \bar{q} = (q_{ij})_{1 \leq i, j \leq n} \) of positive rational numbers, let
\[ L_{\bar{q}} = \{(x_1, \ldots, x_n) \in H^n \mid \forall 1 \leq i, j \leq n (\|x_i - x_j\| \leq q_{ij})\}. \]

Also define
\[ G = \{ \Phi \in \text{Iso}(H^n) \mid \forall \bar{q} (\Phi(L_{\bar{q}}) = L_{\bar{q}}) \}; \]
in other words, \( G \) consists exactly of \( \Phi \in \text{Iso}(H^n) \) such that
- for all \((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in H^n\), if \( \Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \),
- then for all \( 1 \leq i, j \leq n \), \( \|x_i - x_j\| = \|y_i - y_j\| \).

We verify that \( G^{(n)} = G \).

It is clear that \( G^{(n)} \subseteq G \). For the other inclusion we argue that if \( \Phi \in G \) is the identity map on \( H \subseteq H^n \) then it is identity everywhere on \( H^n \).

Suppose that \( \Phi \in G \) and that for any \( x \in H \),
\[ \Phi(x, \ldots, x) = (x, \ldots, x). \quad (4.1) \]

In particular, \( \Phi(0, \ldots, 0) = (0, \ldots, 0) \) and thus \( \Phi \) is a unitary transformation of \( H^n \).

For any \( 1 \leq i \leq n \) and \( x \in H \), let \( u^x_i \) denote the element of \( H^n \) whose \( i \)-th coordinate is \( x \) and other coordinates are all 0. We also present \( u^x_i \) as \((0, \ldots, 0, x, 0, \ldots, 0)\) if \( i \) is understood properly. Since \( \Phi \) is a linear map, it is enough to show that for any \( 1 \leq i \leq n \) and \( x \in H \), \( \Phi(u^x_i) = u^x_i \).

Fix \( 1 \leq i \leq n \) and \( 0 \neq x \in H \), and assume
\[ \Phi(u^x_i) = \Phi(0, \ldots, 0, x, 0, \ldots, 0) = (y_1, \ldots, y_n). \]

Since \( \Phi \in G \), we have that \( y_1 = \cdots = y_{i-1} = y_{i+1} = y_n \). Let \( y_0 \) denote this common element. We also have
\[ \|y_i - y_0\| = \|x\|. \quad (4.2) \]

By (5.1) and the unitarity of \( \Phi \), we have that for any \( z \in H \),
\[ (x, z) = \sum_{i=1}^{n} \langle y_i, z \rangle. \quad (4.3) \]
Thus any \( u \) orthogonal to \( x \) in \( \mathcal{H} \) is also orthogonal to each \( y_i, 1 \leq i \leq n \). It follows that for every \( 1 \leq i \leq n \), \( y_i \) is in the subspace generated by \( x \), and hence is a scalar multiple of \( x \). Let \( \alpha, \beta \in \mathbb{R} \) such that \( y_0 = \alpha x \) and \( y_i = \beta x \). Then (5.2) yields
\[
|\alpha - \beta| = 1.
\]
Also by the linearity of \( \Phi \), our assumption and (5.1), for any \( z \in \mathcal{H} \),
\[
\Phi(-z, \ldots, -z, x - z, -z, \ldots, -z) = (y_0 - z, \ldots, y_0 - z, y_0 - z, y_0 - z, \ldots, y_0 - z).
\]
Thus by the unitarity of \( \Phi \) again, the norms of the vectors on both sides are the same. Thus
\[
(n - 1)\|z\|^2 + \|x - z\|^2 = (n - 1)\|\alpha x - z\|^2 + \|\beta x - z\|^2.
\]
In (5.5) if we plug in \( z = \lambda x \) for arbitrary \( \lambda \), the equation can be simplified to
\[
(n - 1)\lambda^2 + (1 - \lambda)^2 = (n - 1)(\alpha - \lambda)^2 + (\beta - \lambda)^2
\]
and further to
\[
-2\lambda + 1 = -2(n - 1)\lambda - 2\beta\lambda + (n - 1)\alpha^2 + \beta^2.
\]
Since \( \lambda \) is arbitrary, we must have
\[
1 = (n - 1)\alpha + \beta
\]
and
\[
1 = (n - 1)\alpha^2 + \beta^2.
\]
Solving the equation system formed by (5.4), (5.6) and (5.7) we get two sets of solutions:
\[
\begin{align*}
\left\{ \begin{array}{l}
\alpha = 0 \\
\beta = 1
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
\alpha = 2/n \\
\beta = 2/n - 1.
\end{array} \right.
\end{align*}
\]
Let \( v_i^x \) denote the element of \( \mathcal{H}^n \) given by the second set of solutions, i.e.,
\[
v_i^x = (\frac{2}{n} x, \ldots, \frac{2}{n} x, (\frac{2}{n} - 1)x, \frac{2}{n} x, \ldots, \frac{2}{n} x),
\]
where the term with coefficient \( 2/n - 1 \) appears as the \( i \)-th coordinate. \( \square \)

The rest of the proof is a verbatim repetition of the part following Lemma 3.4.

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References


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