Logic and Its Applications

Andreas Blass
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Editors

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Preface

The history of mathematical logic abounds with connections to other areas of mathematics as well as other fields like philosophy. Set theory began when Cantor needed transfinite processes for his analysis of trigonometric series. The basic notions of computability theory are needed in the study of topics ranging from Hilbert's tenth problem (in number theory) to the word problem for groups (in algebra but motivated by topology). Model theory is the foundation of modern infinitesimal analysis. Proof theory elucidates fundamental questions of epistemology.

Like most fields of mathematics, logic has, in addition to its external connections, a rich internal development, motivated by questions arising from within the field itself. Examples include forcing, large cardinals, and inner models in set theory; stability in model theory; functional interpretations and ordinal analysis in proof theory; and the priority method in computability theory. It seems fair to say that, during the last half of the twentieth century, research in mathematical logic was concerned more with such internal issues than with outreach to the rest of mathematics.

Recently, however, outreach has become stronger. The proof theorists' functional interpretations have developed to the point where they can give explicit estimates in analysis, sometimes better than what was obtained by analytic methods. Descriptive set theory, especially the branch that deals with the complexity of equivalence relations, has established connections with fields ranging from abelian group theory to ergodic theory. Technical constructions in model theory have been shown to have fascinating connections with Schanuel's conjecture in transcendental number theory. Computably enumerable sets have infiltrated Riemannian geometry.

The present volume arose from two back-to-back conferences held at the University of Michigan in April, 2003. The first, on "Logic and Its Applications in Algebra and Geometry," sought to bring together logicians (particularly model theorists and set theorists) working in these areas. The second, a workshop on "Combinatorial Set Theory, Excellent Classes, and Schanuel Conjecture," put more emphasis on pure logic, though, as suggested by the presence of Schanuel's conjecture in the title, connections with other areas were present as well.

Yi Zhang, who works on set-theoretic questions in group theory, spent the academic year 2002–2003 in my department at the University of Michigan. Shortly after he arrived, he asked me whether he could organize a conference here. I pointed out that we had no money for that. My comment turned out to be irrelevant; he volunteered to organize a conference anyway. I'm not sure how one gets prominent logicians to travel to a conference when no financial support is available, but Yi
did it. Actually, he managed to get financial support, from existing NSF grants in number theory (thanks to Hugh Montgomery and Trevor Wooley) and set theory, for one invited speaker and one graduate student, but all the other participants came at their own expense.

This seems to be the proper place to record gratitude to the NSF for the money just mentioned; to the Mathematics Department of the University of Michigan for inviting (through colloquium chairman Lizhen Ji) several of the conference speakers to give colloquium talks, for providing meeting rooms, and for supplying food for a reception; to Peter Hinman for providing excellent wine from his renowned cellar for the reception; to John Baldwin and Rami Grossberg for their help in organizing the workshop; to Bart Kastermans for setting up the conference web site and for other technical help; to the participants for making the meeting a great success; to the authors of the present papers for writing up their work for us; and to the American Mathematical Society for agreeing to publish this volume.

Andreas Blass
Ehrenfeucht-Mostowski models in Abstract Elementary Classes

John T. Baldwin

ABSTRACT. Let \( K \) be an abstract elementary class. We give a unified and often simplified exposition of a number of results of Shelah. In particular, we prove the presentation theorem. If \( K \) has the amalgamation property, the joint embedding property and arbitrarily large models, then an analog of Morley’s omitting types theorem holds for Galois types. Further if \( K \) is \( \lambda \)-categorical for a regular cardinal \( \lambda \), \( K \) is stable in all cardinalities less than \( \lambda \). We conclude by explaining the relevance of the notion of tameness.

We work in the context of an abstract elementary class (AEC) with the amalgamation and joint embedding properties and arbitrarily large models. We prove two results using Ehrenfeucht-Mostowski models: 1) Morley’s omitting types theorem – for Galois types. 2) If an AEC (with amalgamation) is categorical in some uncountable power \( \mu \) it is stable in (every) \( \lambda < \mu \).

These results are lemmas towards Shelah’s consideration [14] of the downward transfer of categoricity, which we discuss in Section 6. This paper expounds some of the main ideas of [14], filling in vague allusions to earlier work and trying to separate those results which depend only on the Ehrenfeucht-Mostowski method from those which require more sophisticated stability theoretic tools.

In [17], Shelah proclaims the aim of reconstructing model theory, ‘with no use of even traces compactness’. We analyze here one aspect of this program. Keisler organizes [10] around four kinds of constructions: the Henkin method, Ehrenfeucht-Mostowski models, unions of chains, and ultraproducts. The later history of model theory reveals a plethora of tools arising in stability theory. Fundamental is a notion of dependence which arises from Morley’s study of rank, and passes through various avatars of splitting, strong splitting, and dividing before being fully actualized in the first order setting as forking. We eschew this technique altogether in this paper— to isolate its role. For more general accounts of Abstract Elementary Classes, see [1, 8].

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The axioms of an AEC \((K, \preceq_K)\), were first set down in [19]. We repeat for convenience.

**Definition 0.1.** A class of \(L\)-structures, \((K, \preceq_K)\), is said to be an abstract elementary class: AEC if both \(K\) and the binary relation \(\preceq_K\) are closed under isomorphism and satisfy the following conditions.

- **A1.** If \(M \preceq_K N\) then \(M \subseteq N\).
- **A2.** \(\preceq_K\) is a partial order on \(K\).
- **A3.** If \(\langle A_i : i < \delta \rangle\) is \(\preceq_K\)-increasing chain:
  1. \(\bigcup_{i<\delta} A_i \in K\);
  2. for each \(j < \delta\), \(A_j \preceq_K \bigcup_{i<\delta} A_i\);
  3. if each \(A_i \preceq_K M \in K\) then \(\bigcup_{i<\delta} A_i \preceq_K M\).
- **A4.** (Coherence Axiom) If \(A, B, C \in K\), \(A \preceq_K C, B \preceq_K C\) and \(A \subseteq B\) then \(A \preceq_K B\).
- **A5.** There is a Löwenheim-Skolem number \(\kappa(K)\) such that if \(A \subseteq B \in K\) there is a \(A' \in K\) with \(A \subseteq A' \preceq_K B\) and \(|A'| < |A| + \kappa(K)\).

In English, we often write \(B\) is a strong extension of \(A\) for \(A \preceq_K B\).

Sections 1, 2, 3 define most of the terminology and lay out the basic results. In Sections 4, we show categoricity implies stability and establish the existence of saturated models. Section 5 lifts Morley’s omitting types theorem to the AEC setting. Finally in Section 6, we survey the additional steps needed to prove Shelah’s downward categoricity theorem. I thank Greg Cherlin for some trenchant observations, Tapani Hyttinen for pointing out an error in an earlier draft, and Alex Usvyatsov for a careful reading.

### 1. Assumptions

We work with classes of structures in a fixed vocabulary, \(\tau\). When results are uniform functions of such invariants as the cardinality of \(\tau\) or \(LS(K)\) we may write them in terms of these numbers. We use variants on \(\tau\) to denote vocabularies. In addition to this usage, Shelah uses \(\tau\) as an operator: \(\tau(\Phi)\) denotes the vocabulary of the set of sentences \(\Phi\). We may write \(\tau\)-structure or \(L\)-structure.

**Assumption 1.1.**

1. \(K\) has arbitrarily large models.
2. \(K\) satisfies the amalgamation property and the joint embedding property.

We say \(K\) has the amalgamation property if \(M, N_1, N_2 \in K\) and there are strong embeddings of \(M\) into \(N_1\) and \(N_2\) then there is an \(N_3\) and strong embeddings of \(N_1\) and \(N_2\) into \(N_3\) so that the composition maps agree on \(M\). Joint embedding means that for any two members of \(K\) there is a third into which both can be strongly embedded. Crucially, we amalgamate only over members of \(K\); this distinguishes this context from the context of homogeneous structures. Amalgamation does not imply the existence of arbitrarily large models; the class of initial segments of \(\aleph_1\) with end extension as strong extension is an AEC. An AEC with disjoint amalgamation (the images of \(N_1\) and \(N_2\) in \(N_3\) intersect in the image of \(M\)) and at least two models can easily be seen to have arbitrarily large models.
We stress that amalgamation is a very strong assumption and we make full use of it. However, many of the results can be achieved under some weaker conditions with somewhat more effort; we allude to some of these. Much of the Shelah work involves two kinds of argument of a more local nature: failure of amalgamation in \( \kappa \) implies many models in, say, \( \kappa^+ \) (with various variants), and arguments which assume only amalgamation below (or in) a certain cardinality.

**Notation 1.2.**

1. Let \( H(\lambda, \kappa) \) be the least cardinal \( \mu \) such that if a first order theory \( T \) with \( |T| = \lambda \) has models of every cardinal less than \( \mu \) which omit each of a set \( \Gamma \) of types, with \( |\Gamma| = \kappa \), then there are arbitrarily large models of \( T \) which omit \( \Gamma \).

2. Write \( H(\kappa) \) for \( H(\kappa, \kappa) \).

3. For a similarity type \( \tau \), \( H(\tau) \) means \( H(|\tau|) \).

Note that an old theorem of Morley [13], VII.5.4, VII.5.5, [3] says \( H(\kappa, 2^\kappa) \leq \beth_{(2^\kappa)^+} \). So for practical purposes, we don't have to worry about the number of types we are omitting; we are aiming for eventual results in any case. We have renamed the function \( \mu \) from [13] by the more evocative \( H \). For simplicity, we assume the Löwenheim number is at least \( |\tau| \); this is crucial in the statement of Corollary 2.4.

When \( \text{LS}(K) = |\tau(K)| = \kappa \), \( H(\kappa) \) is sometimes called the Hanf number of \( K \). This is somewhat misleading because a single class cannot have a Hanf number – a Hanf number is a maximum for all similarity types of a given cardinality. It is in fact not the Hanf number of \( K \) but the Hanf number for all AEC with the same Löwenheim number. But as we'll see there is a still wider basis for this name; we will consider other classes of models (which are not AEC) and it is crucial that all of them have the property: for any model \( M \) with \( |M| \geq H(\tau) \), there are models in the class of all cardinalities that omit all types omitted in \( M \).

There is some vestige of compactness here. Both the existence of arbitrarily large models and amalgamation are proved in first order logic using compactness. But they have completely semantic statements and you have to start somewhere.

### 2. The presentation theorem and E-M models

We call the next result: the presentation theorem. It allows us to replace the entirely semantic description of an abstract elementary class by a syntactic one. I find it extraordinary that the notion of an AEC that was designed to give a version of the Fraïssé construction and thus saturated models, also turns out to allow the use of the second great model theoretic technique of the 50's: Ehrenfeucht-Mostowski models.

This result first appears in [19] and again in [17]; our argument is somewhat different.

**Theorem 2.1.** If \( K \) is an AEC with Löwenheim number \( \text{LS}(K) \) (in a vocabulary \( \tau \) with \( |\tau| \leq \text{LS}(K) \)), there is a vocabulary \( \tau' \) with cardinality \( |\text{LS}(K)| \), a first order \( \tau' \)-theory \( T' \) and a set of \( 2^{\text{LS}(K)} \) types \( \Gamma \) such that:
\[ K = \{ M' \upharpoonright \tau : M' \models T' \text{ and } M' \text{ omits } \Gamma \}. \]

Moreover, if \( M' \) is a \( \tau' \)-substructure of \( N' \) where \( M', N' \) satisfy \( T' \) and omit \( \Gamma \) then \( M' \upharpoonright \tau \preceq K N' \upharpoonright \tau \).

Proof. Let \( \tau' \) contain \( n \)-ary function symbols \( F^m_i \) for \( n < \omega \) and \( i < \text{LS}(K) \). We require that for \( i < n \), \( F^m_i(a) = a_i \). We take as \( T' \) the theory which asserts only that its models are nonempty. For any \( \tau' \)-structure \( M' \) and any \( a \in M', \) let \( M'_a \) denote the subset of \( M' \) enumerated by \( \{ F^m_i(a) : i < \text{LS}(K) \} \) where \( n = \lg(a) \). The isomorphism type of \( M'_a \) is determined by the quantifier free \( \tau' \)-type of \( a \). Note that \( M'_a \) may not be a \( \tau' \) or even a \( \tau \)-structure. Let \( \Gamma \) be the set of quantifier free \( \tau' \)-types of finite tuples \( a \) such that \( M'_a \models \tau \notin K \) or for some \( b \subseteq a \), \( M'_b \models \tau \notin K M'_a \models \tau \).

We claim \( T' \) and \( \Gamma \) suffice. That is, if \( K' = \{ M' \upharpoonright \tau : M' \models T' \text{ and } M' \text{ omits } \Gamma \} \) then \( K = K' \). Let the \( \tau' \)-structure \( M' \) omit \( \Gamma \); in particular, each \( M'_a \) is a \( \tau \)-structure. Write \( M' \) as a direct limit of the finitely generated \( \tau \)-structures \( M'_a \). (These may not be closed under the operations of \( \tau' \).) By the choice of \( \Gamma \), each \( M'_a \upharpoonright \tau \in K \) and if \( a \subseteq a' \), \( M'_a \upharpoonright \tau \preceq K M'_{a'} \upharpoonright \tau \), and so by the unions of chains axioms \( M'_a \upharpoonright \tau \in K \). Conversely, if \( M \in K \) we define by induction on \( |a| \), structures \( M_a \) for each finite subset \( a \) of \( M \). Let \( M \) be any \( \preceq K \)-substructure of \( M \) with cardinality \( \text{LS}(K) \) and let the \( \{ F^0_i : i < \text{LS}(K) \} \) be constants enumerating the universe of \( M_0 \). Given a sequence \( b \) of length \( n + 1 \), choose \( M_b \preceq K M \) with cardinality \( \text{LS}(K) \) containing all the \( M_a \) for \( a \subseteq b \) of smaller cardinality. Let \( \{ F^{n+1}_i(b) : i < \text{LS}(K) \} \) enumerate the universe of \( M_b \) (and give the function the same value on any ordering of the range of \( b \)). Now each \( M_a \models \tau \in K \) and if \( b \subseteq c \), \( M_b \preceq K M_c \) so \( M' \) omits \( \Gamma \) as required.

The moreover holds for the partial \( \tau' \)-structures \( M'_a \) directly by the choice of \( \Gamma \) and extends to arbitrary structures by the union of chain axioms on an AEC. In more detail, we have \( M' \) is a direct limit of finite structures \( M'_a \) and \( N' \) is a \( \preceq K \)-direct limit of \( N'_a \) where \( M'_a = N'_a \) for \( a \in M \) because \( M' \upharpoonright \tau \) is a \( \tau \)-substructure of \( N' \upharpoonright \tau \). Each \( M'_a \upharpoonright \tau \preceq K N'_a \upharpoonright \tau \) so the direct limit \( M' \upharpoonright \tau \) is a strong submodel of \( N' \upharpoonright \tau \). \( \square \)

We have represented \( K \) as a \( \text{PCT} \) class in the following sense.

**Definition 2.2.** A \( \text{PC}(T, \Gamma) \) class is the class of reducts to \( \tau \subset \tau' \) of models of a first order theory \( \tau' \)-theory which omit all types from the specified collection \( \Gamma \) of types in finitely many variables over the empty set.

We write \( \text{PC} \) to denote such a class without specifying either \( T \) or \( \Gamma \). And we write \( K \) is \( \text{PC}(\lambda, \mu) \) if \( K \) can be presented as \( \text{PC}(T, \Gamma) \) with \( |T| \leq \lambda \) and \( |\Gamma| \leq \mu \). In the simplest case, we say \( K \) is \( \lambda \)-presented if \( K \) is \( \text{PC}(\lambda, \lambda) \).

In this language we have shown any AEC \( K \) is \( 2^{\text{LS}(K)} \)-presented.

**Remark 2.3.**

(1) There is no use of amalgamation in this theorem.

(2) The only penalty for increasing the size of the language or the Löwenheim number is that the size of \( \tau' \) and the number of types omitted increases as
well. This will mean that for the use of EM models below, the $\theta$ must be chosen larger.

(3) We can observe with Shelah [19] that the class of pairs $(M, N)$ with $M \preceq_K N$ also forms a $PC(\text{LS}(K), 2^{\text{LS}(K)})$. This observation is important for some applications but will not be used here; see Theorem 2.7 and its applications. The moreover clause also appears in Grossberg's account [5] and in Makowsky's [11].

We immediately conclude the required computation of Hanf numbers for abstract elementary classes; we will use in a significant way the fact that $H(\tau)$ is, in fact, the Hanf number for $PCT$ classes where $|\Gamma| \leq 2^{|	au|}$.

**Corollary 2.4.** Let $K$ be an AEC with similarity type $\tau$ with $\text{LS}(K) = |	au|$. If $K$ has a model with cardinality at least $H(\tau)$ then $K$ has arbitrarily large models.

**Notation 2.5.**

1. For any linearly ordered set $X \subseteq M$ where $M$ is a $\tau$-structure we write $D_\tau(X)$ (diagram) for the set of $\tau$-types of finite sequences (in the given order) from $X$. We will omit $\tau$ if it is clear from context.

2. Such a diagram of an order indiscernible set, $D_\tau(X) = \Phi$, is called 'proper for linear orders'.

3. If $X$ is a sequence of $\tau$-indiscernibles with diagram $\Phi = D_\tau(X)$ and any $\tau$ model of $\Phi$ has built in Skolem functions, then for any linear ordering $I$, $EM(I, \Phi)$ denotes the $\tau$-hull of a sequence of order indiscernibles realizing $\Phi$.

4. If $\tau_0 \subset \tau$, the reduct of $EM(I, \Phi)$ to $\tau_0$ is denoted $EM_{\tau_0}(I, \Phi)$.

'Morley's method' (Section 7.2 of [4]) is a fundamental technique in first order model theory. It is essential for the foundations of simplicity theory and for the construction of indiscernibles in infinitary logic. We quote the first order version here; in Lemma 5.1, we prove the analog for abstract elementary classes.

**Lemma 2.6.** If $(X, \prec)$ is a sufficiently long ($|X| > H(|\tau'|)$) linearly ordered subset of a $\tau$-structure $M$, for any $\tau'$ extending $\tau$ (the length needed for $X$ depends on $|\tau'|$) with $\prec$ in $\tau'$ there is a countable sequence $Y$ of $\tau'$-order-indiscernibles (and hence one of arbitrary order type) such that $D_\tau(Y) \subseteq D_\tau(X)$. This implies that the only (first order) $\tau$-types realized in $EM(X, D_\tau(Y))$ were realized in $M$.

Further, we find Skolem models over indiscernibles in an AEC.

**Theorem 2.7.** If $K$ is an abstract elementary class in the vocabulary $\tau$, which is presented as a $PCT$ class witnessed by $\tau', T', \Gamma$ that has arbitrarily large models, there is a $\tau'$-diagram $\Phi$ such that for every linear order $(I, \prec)$ there is a $\tau'$-structure $M = EM(I, \Phi)$ such that:

1. $M \models T'$.
2. The $\tau'$-structure $M = EM(I, \Phi)$ is the Skolem hull of $I$.
3. $I$ is a set of $\tau'$-indiscernibles in $M$. 


(4) $M \models \tau$ is in $K$.
(5) If $I' \subseteq I$ then $EM_\tau(I', \Phi) \preceq K EM_\tau(I, \Phi)$.

Proof. The first four clauses are a direct application of Lemma 2.6, Morley's theorem on omitting types. See also problem 7.2.5 of Chang-Keisler [4] or [3]. It is automatic that $EM(I', \Phi)$ is an $L'$ substructure of $EM(I, \Phi)$. The moreover clause of Theorem 2.1 allows us to extend this to $EM_\tau(I', \Phi) \preceq K EM_\tau(I, \Phi)$. \(\square_{2.7}

Note that we have simplified our presentation of many members of $K$. Inside the class $K$, which is the set of reducts of models which omit $\Gamma$, sits a class $K'$, which is the class of reducts of Skolem hulls of order indiscernibles. In general, $K'$ is a proper subclass of $K$. It may not be an AEC because we don’t know closure under unions of chains. In [16], under strong hypotheses this closure is proved.

Remark 2.8. Silver (Chapter 18 of [10]) gives a simple example of a pseudoelementary class where the categoricity spectrum and its complement are both cofinal in the class of cardinals. The example is the class of models $(M, X)$ where $2^{|X|} \geq |M|$. This class is not an AEC because it is not closed under unions of chains.

The arguments below depend on the classes involved being both AEC and PCT.

3. Galois types and saturation

In this section we take advantage of joint embedding and amalgamation to find a monster model. We then define types in terms of orbits of stabilizers of submodels. This allows an identification of ‘model-homogeneous’ with ‘saturated’. That is, we give an abstract account of Morley-Vaught [12].

Definition 3.1. $M$ is $\mu$-model homogenous if for every $N \preceq K M$ and every $N' \in K$ with $|N'| < \mu$ and $N \preceq K N'$ there is a $K$-embedding of $N'$ into $M$ over $N$.

To emphasize, this differs from the homogenous context because the $N$ must be in $K$. It is easy to show:

Lemma 3.2. If $M_1$ and $M_2$ are $\mu$-model homogenous of cardinality $\mu > LS(K)$ then $M_1 \cong M_2$.

Proof. If $M_1$ and $M_2$ have a common submodel $N$ of cardinality $< \mu$, this is an easy back and forth. Now suppose $N_1, (N_2)$ is a small model of $M_1, (M_2)$ respectively. By the joint embedding property there is a small common extension $N$ of $N_1, N_2$ and by model homogeneity $N$ is embedded in both $M_1$ and $M_2$. \(\square_{3.2}

Note that in the absence of joint embedding, to get uniqueness we would (as in [19]) have to add to the definition of ‘$M$ is model homogeneous’ that all models of cardinality $< \mu$ are embedded in $M$.

Theorem 3.3. If $K$ has the amalgamation property and $\mu^*<\mu^* = \mu^*$ and $\mu^* \geq 2^{LS(K)}$ then there is a model $M$ of cardinality $\mu^*$ which is $\mu^*$-model homogeneous.
We call the model constructed in Theorem 3.3, the monster model. From now on, all structures considered are substructures of $\mathbb{M}$ with cardinality $< \mu^*$. The standard arguments for the use of a monster model in first order model theory ([9, 2]) apply here.

**Definition 3.4.** Let $M \in \mathbb{K}$, $M \preceq \mathbb{K} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of $a$ over $M$ is the orbit of $a$ under the automorphisms of $\mathbb{M}$ which fix $M$.

We freely use the phrase, ‘Galois type of $a$ over $M$’. Note that *a priori* this notion depends on the embedding of $Ma$ into an $N \in \mathbb{K}$ and the embedding of $N$ into $\mathbb{M}$. Since we have assumed amalgamation, our usage is justified as long as the base is an $M \in \mathbb{K}$. In more general situations, the Galois type is an equivalence class of an equivalence relation on triples $(M, a, N)$. This is an equivalence relation on the class of $M$ that are amalgamation bases for extensions in the same cardinality. (See [20, 21].) Since we have amalgamation and have fixed $\mathbb{M}$, we don’t need the extra notation. The following definition and exercise show the connection of the situation as described here with the more complicated description elsewhere. They are needed only to link with the literature.

**Definition 3.5.** For $M \preceq \mathbb{K} N_1 \in \mathbb{K}$, $M \preceq \mathbb{K} N_1 \in \mathbb{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, write $(M, a, N_1) \sim (M, b, N_2)$ if there exist strong embeddings $f_1, f_2$ of $N_1, N_2$ into some $N^*$ which agree on $M$ and with $f_1(a) = f_2(b)$.

**Exercise 3.6.** If $\mathbb{K}$ has amalgamation, $\sim$ is an equivalence relation.

**Exercise 3.7.** Suppose $\mathbb{K}$ has amalgamation and joint embedding. Show $(M, a, N_1) \sim (M, b, N_2)$ if and only if there are embeddings $g_1$ and $g_2$ of $N_1, N_2$ into $\mathbb{M}$ that agree on $M$ and such that $g_1(a)$ and $g_2(b)$ have the same Galois type over $g_1(M)$.

**Definition 3.8.** The set of Galois types over $M$ is denoted $S(M)$.

We distinguish this from $S(M)$ for the usual first order Stone space of $M$. We say a Galois type $p$ over $M$ is realized in $N$ with $M \preceq \mathbb{K} N \preceq \mathbb{K} \mathbb{M}$ if $p \cap N \neq \emptyset$.

**Definition 3.9.** The model $M$ is $\mu$-Galois saturated if for every $N \preceq \mathbb{K} M$ with $|N| < \mu$ and every Galois type $p$ over $N$, $p$ is realized in $M$.

Again, *a priori* this notion depend on the embedding of $M$ into $\mathbb{M}$; but with amalgamation it is well-defined.

**Theorem 3.10.** For $\lambda > \text{LS}(\mathbb{K})$, the model $M$ is $\lambda$-Galois saturated if and only if it is $\lambda$-model homogeneous.

Proof. It is obvious that $\lambda$-model homogeneous implies $\lambda$-Galois saturated. Let $M \preceq \mathbb{K} \mathbb{M}$ be $\lambda$-saturated. We want to show $M$ is $\lambda$-model homogeneous. So fix $M_0 \preceq \mathbb{K} M$ and $N$ with $M_0 \preceq \mathbb{K} N \preceq \mathbb{K} \mathbb{M}$. Say, $|N| = \mu < \lambda$. We must construct an embedding of $N$ into $M$. Enumerate $N - M$ as $(a_i : i < \mu)$. We will define $f_i$ for $i < \mu$ an increasing continuous sequence of maps with domain $N_i$ and range $M_i$ so that $M_0 \preceq \mathbb{K} N_i \preceq \mathbb{K} \mathbb{M}$, $M_0 \preceq \mathbb{K} M_i \preceq \mathbb{K} M$ and $a_i \in N_{i+1}$. The restriction of $\bigcup_{i<\mu} f_i$ to $N$ is required embedding. Let $N_0 = M_0$ and $f_0$ the identity. Suppose $f_i$ has been defined. Choose the least $j$ such that $a_j \in N - N_i$. 

By the model homogeneity of $\mathcal{M}$, $f_i$ extends to an automorphism $\hat{f}_i$ of $\mathcal{M}$. Using the saturation, let $b_j \in M$ realize the Galois type of $\hat{f}_i(a_j)$ over $M_i$. So there is an $\alpha \in \text{aut}\mathcal{M}$ which fixes $M_i$ and takes $b_j$ to $\hat{f}_i(a_j)$. Choose $M_{i+1} \preceq \mathcal{K} M$ with cardinality $\mu$ and containing $M_i b_j$. Now $\hat{f}_i^{-1} \circ \alpha$ maps $M_i$ to $N_i$ and $b_j$ to $a_j$. Let $N_{i+1} = \hat{f}_i^{-1} \circ \alpha(M_{i+1})$ and define $f_{i+1}$ as the restriction of $\alpha^{-1} \circ \hat{f}_i$ to $N_{i+1}$. Then $f_{i+1}$ is as required. \(\square_{3.10}\)

The last argument makes full use of the amalgamation property. We discuss some generalizations in the last paragraph of this article. In the remainder of this section we discuss some important ways in which Galois types behave differently from 'syntactic types'.

Note that if $M \preceq \mathcal{K} N \preceq \mathcal{K} \mathcal{M}$, then $p \in \mathcal{S}(N)$ extends $p' \in \mathcal{S}(N)$ if for some (any) $a$ realizing $p$ and some (any) $b$ realizing $p'$ there is an automorphism $\alpha$ fixing $M$ and taking $a$ to $b$.

We say that a class $\mathcal{K}$ which satisfies the conclusion of the following lemma is $\omega$-compact; $\kappa$-compact for arbitrary $\kappa$ is defined analogously.

**Lemma 3.11.** If $M = \bigcup_{i < \omega} M_i$ is an increasing chain of members of $\mathcal{K}$ and $\{p_i : i < \omega\}$ satisfies $p_{i+1} \upharpoonright M_i = p_i$, there is a $p_\omega \in \mathcal{S}(M)$ with $p_\omega \upharpoonright M_i = p_i$ for each $i$.

Proof. Let $a_i$ realize $p_i$. By hypothesis, for each $i < \omega$, there exists $f_i$ which fixes $M_{i-1}$ and maps $a_i$ to $a_{i-1}$. Let $g_i$ be the composition $g_0 \circ f_1 \circ \ldots f_i$. Then $g_i$ maps $a_i$ to $a_0$, fixes $M_0$ and $g_i \upharpoonright M_{i-1} = g_{i-1} \upharpoonright M_{i-1}$. Let $M'_i$ denote $g_i(M_i)$ and $M'$ their union. Then $\bigcup_{i < \omega} g_i$ is an isomorphism between $M$ and $M'$. So by model-homogeneity there exists an automorphism $h$ of $\mathcal{M}$ with $h \upharpoonright M_i = g_i \upharpoonright M_i$ for each $i$. Let $a_\omega = h^{-1}(a_0)$. Now $g_i^{-1} \circ h$ fixes $M_i$ and maps $a_\omega$ to $a_i$ for each $i$. This completes the proof. \(\square_{3.11}\)

Now suppose we wanted to prove Lemma 3.11 for chains of length $\delta > \omega$. The difficulty can be seen at stage $\omega$. In addition to the assumptions of Lemma 3.11, we are given $\{a_i : i \leq \omega\}$ and $f_{\omega,i}$ which fixes $M_i$ and maps $a_\omega$ to $a_i$. We can construct $g_i$ as in the original proof. The difficulty is to find $g_\omega$ which extends all the $g_i$ and maps $a_\omega$ to $a_0$. In the argument for Lemma 3.11, we found a map $h$ and an element (which we will now call $a'_\omega$ such that $h$ takes $a'_\omega$ to $a_0$ while $h$ extends all the $g_i$. We would be done if $a_\omega$ and $a'_\omega$ realized the same Galois type over $M = M_\omega$. In fact, $a_\omega$ and $a'_\omega$ realized the same Galois type over each $M_i$. So the following locality condition (for chains of length $\omega$) would suffice for this special case. Moreover, by a further induction locality would give Lemma 3.11 for chains of arbitrary length. Locality does not hold for all AEC with amalgamation. Locality is defined in Definition 24 of [17] by a slightly more abstract equivalent to:

**Definition 3.12.** $\mathcal{K}$ has $\kappa$-local Galois types if for every $M = \bigcup_{i < \kappa} M_i$ in a continuous increasing chain of members of $\mathcal{K}$ and for any $p, q \in \mathcal{S}(M)$: if $p \upharpoonright M_i = q \upharpoonright M_i$ for every $i$ then $p = q$. Further, $\mathcal{K}$ has local Galois types if it is $\kappa$-local for all $\kappa$.

Shelah has used the word local in distinct ways in several treatments of this subject [14], [17]. Refining Grossberg and VanDieren, we delineate further variants on locality and the related notion of tameness in Section 6.
We have sketched the proof of:

**Lemma 3.13.** Suppose $K$ has local Galois types. If $M = \bigcup_{i < \kappa} M_i$ is an increasing chain of members of $K$ and $\{p_i : i < \kappa\}$ satisfies $p_{i+1} \upharpoonright M_i = p_i$, there is a $p_\kappa \in S(M)$ with $p_\kappa \upharpoonright M_i = p_i$ for each $i$.

Although, we used uniqueness of chains of arbitrary length to prove the realizability of chains of arbitrary length, the uniqueness is not essential to get isolated results. For example, the argument for Lemma 3.11 actually proves that the union of any increasing chain of Galois-types of cofinality $\omega$ is a Galois type.

Locality and tameness provide key distinctions between the general AEC case and homogenous structures. In homogeneous structures, types are syntactic objects and locality and tameness are both trivial. Thus, as pointed out by Shelah, Hyttinen, and Buechler-Lessmann, Lemma 3.13 applies in the homogeneous context.

### 4. Getting stability

In this section we show that a countable $\lambda$-categorical AEC is $\mu$-stable for $\mu$ above the Löwenheim number and below $\lambda$. The key idea is that for a linear order $I$ and model $EM(I, \Phi)$, automorphisms of $I$ induce automorphisms of $EM(I, \Phi)$. And, automorphisms of $EM(I, \Phi)$ preserve types in any reasonable logic; in particular, automorphisms of $EM(I, \Phi)$ preserve Galois types. Note that a model $N$ is (defined to be) stable if few types are realized in $N$. So if $N$ is a brimful model (Definition 4.2) then the model $N$ is $\sigma$-stable for every $\sigma < |N|$.

Since we deal with reducts and will consider several structures with the same universe; it is crucial to keep the vocabulary of the structure in mind. The AEC under consideration has vocabulary $\tau'$; it is presented as reducts of models of theory $T'$ (which omit certain types) in a vocabulary $\tau'$. In addition, we have the class of linear orderings (LO) in the background.

We really have three AEC’s: $(LO, \subset)$, $K'$ which is $Mod(T')$ with submodel as $\tau'$-closed subset, and $(K, \preceq_K)$. We are describing the properties of the EM-functor between $(LO, \subset)$ and $K'$ or $K$. $K'$ is only a tool that we are singling out to see the steps in the argument. The following definitions hold for any of the three classes and I write $\preceq$ for the notion of substructure. In this section of the paper I am careful to use $\preceq$ when discussing all three cases versus $\preceq_K$ for the AEC.

**Definition 4.1.** $M_2$ is $\sigma$-universal over $M_1$ in $N$ if $M_1 \preceq M_2 \preceq N$ and whenever $M_1 \preceq M'_2 \preceq N$, with $|M_1| \leq |M'_2| \leq \sigma$, there is a $\preceq$-embedding fixing $M_1$ and taking $M'_2$ into $M_2$.

I introduce one term for shorthand. It is related to Shelah’s notion of brimmed in [15] but here the brimful model is bigger than the models it is universal over while brimmed models may have the same cardinality.

**Definition 4.2.** $M$ is brimful if for every $\sigma < |M|$, and every $M_1 \preceq M$ with $|M_1| = \sigma$, there is an $M_2 \preceq M$ with cardinality $\sigma$ that is $\sigma$-universal over $M_1$ in $M$. 
The next notion just makes it easier to write the proof of the following Lemma.

**Notation 4.3.** Let $I \subset J$ be linear orders. We say $a$ and $b$ in $J$ realize the same cut over $I$ and write $a \sim_I b$ if for every $i \in I$, $a < i$ if and only if $b < i$.

**Claim 4.4 (Lemma 3.7 of [18]).** The linear order $I = \lambda^{<\omega}$ is brimful.

Proof. Let $J \subset I$ have cardinality $\theta < \lambda$. Without loss of generality we can assume $J = A^{<\omega}$ for some $A \subset \lambda$. Note that $\sigma \sim_J \tau$ if and only if for the least $n$ such that $\sigma \upharpoonright n \neq \tau \upharpoonright n$, neither is in $J$ and $\sigma(n) \sim_A \tau(n)$. Thus there are only $\theta$ cuts over $J$ realized in $I$. For each cut $C_\alpha$, $\alpha < \theta$, we choose a representative $\sigma_\alpha \in I - J$ of length $n$ such that $\sigma_\alpha \upharpoonright n - 1 \in J$, so $C_\alpha$ is isomorphic to $\{\sigma_\alpha \upharpoonright \tau : \tau \in \lambda^{<\omega}\}$. We can assume any $J^*$ extending $J$ is $J^* = B^{<\omega}$ for some $B \subset \lambda$, say with $\text{otp}(B) = \gamma$. Thus, the intersection of $J^*$ with a cut in $J$ is isomorphic to a subset of $\gamma^{<\omega}$. We finish by noting for any ordinal with $|\gamma| = \theta$, $\gamma^{<\omega}$ can be embedded in $\theta^{<\omega}$. Thus, the required $\theta$-universal set over $J$ is $J \cup \{\sigma_\alpha \upharpoonright \tau : \tau \in \theta^{<\omega}, \alpha < \theta\}$. □

Qing Zhang has provided the following elegant argument for the last claim. First show by induction on $\gamma$ there is a map $g$ embedding $\gamma$ in $\theta^{<\omega}$. (E.g. if $\gamma = \lim_{i < \theta} \gamma_i$, and $g_i$ maps $\gamma_i$ into $\theta^{<\omega}$, let for $\beta < \gamma$, $g(\beta) = \widehat{\gamma_i(\beta)}$ where $\gamma_i \leq \beta < \gamma_{i+}$.) Then let $h$ map $\gamma^{<\omega}$ into $\theta^{<\omega}$ by, for $\sigma \in \gamma^{<\omega}$ of length $n$, setting $h(\sigma) = (g(\sigma(0)), \ldots, g(\sigma(n - 1)))$. □

The argument for Claim 4.4 yields:

**Corollary 4.5.** Suppose $\mu < \lambda$ are cardinals. Then for any $X \subset \mu^{<\omega}$ and any $Y$ with $X \subset Y \subset \lambda^{<\omega}$ and $|X| = |Y| < \mu$, there is an order embedding of $Y$ into $\mu^{<\omega}$ over $X$.

Since every $\tau'$-substructure $N$ of $EM(I, \Phi)$ is contained in a substructure $EM(I_0, \Phi)$ for some subset $I_0$ of $I$ with $|I_0| = |N|$, we have immediately:

**Claim 4.6.** If $I$ is brimful as a linear order, $EM(I, \Phi)$ is brimful as an $\tau'$-structure.

Now using amalgamation and categoricity, we move to the AEC $K$. There are some subtle uses here of the ‘coherence axiom’: $M \subset N \preceq_K N_1$ and $M \preceq_K N_1$ implies $M \preceq_K N$.

**Claim 4.7.** If $I$ is brimful as a linear order, $EM_\tau(I, \Phi)$ is brimful as a member of $K$.

Proof. Let $M = EM(I, \Phi)$; we must show $M \upharpoonright \tau$ is brimful as a member of $K$. Suppose $M_1 \preceq_K M \upharpoonright \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM(I', \Phi)$ with $|I'| = \sigma$ and $M_1 \subset N_1 \subset M$. By Lemma 2.7.5, $N_1 \upharpoonright \tau \preceq_K M \upharpoonright \tau$. By the coherence axiom, $M_1 \preceq_K N_1 \upharpoonright \tau$. Let $M_2$ have cardinality $\sigma$ and $M_1 \preceq_K M_2 \preceq_K M \upharpoonright \tau$. Choose a $\tau'$-substructure $N_2$ of $M$ with cardinality $\sigma$ containing $N_1$ and $M_2$. Now, $N_2$ can be embedded by a map $f$ into the $\sigma$-universal $\tau'$-structure $N_3$ containing $N_1$ which is guaranteed by Claim 4.6. But $f(N_2) \upharpoonright \tau \preceq_K N_3 \upharpoonright \tau$ by the coherence axiom so $N_3 \upharpoonright \tau$ is the required $\sigma$-universal extension of $M_1$. □

**Definition 4.8.** 1) Let $N \subset M$. $N$ is $\lambda$-Galois-stable if for every $M \subset N$ with cardinality $\lambda$, only $\lambda$ Galois types over $M$ are realized in $N$. 
(2) $K$ is $\lambda$-Galois-stable if $M$ is. That is $\text{aut}_M(M)$ has only $\lambda$ orbits for every $M \subset M$ with cardinality $\lambda$.

Since we are usually working in an AEC, we will frequently abuse notation and write stable rather than Galois-stable.

Since for brimful $I$, $M = EM(I, \Phi)$ is brimful, and for $M_0 \preceq_K M_1 \preceq_K M$, each Galois type over $M_0$ realized in $M$ is represented by an $M_1$ with $|M_1| = |M_0|$, Claim 4.7 implies immediately:

**Claim 4.9.** If $K$ is $\lambda$-categorical, the model $M$ with $|M| = \lambda$ is $\sigma$-Galois stable for every $\sigma < \lambda$.

**Theorem 4.10.** If $K$ is categorical in $\lambda$, then $K$ is $\sigma$-Galois-stable for every $\sigma < \lambda$.

**Proof.** Suppose $K$ is not $\sigma$-stable for some $\sigma < \lambda$. Then by Löwenheim-Skolem, there is a model $N$ of cardinality $\sigma^+$ which is not $\sigma$-stable. Let $M$ be the $\sigma$-stable model with cardinality $\lambda$ constructed in Claim 4.9. Categoricity and joint embedding imply $N$ can be embedded in $M$. The resulting contradiction proves the result. □$_{4.10}$

**Corollary 4.11.** Suppose $K$ is categorical in $\lambda$ and $\lambda$ is regular. The model of power $\lambda$ is saturated and so model homogeneous.

**Proof.** Choose in $M_i \preceq_K M$ using $< \lambda$-stability and Löwenheim-Skolem, for $i < \lambda$ so that each $M_i$ has cardinality $< \lambda$ and $M_{i+1}$ realizes all types over $M_i$. By regularity, it is easy to check that $M_\lambda$ is saturated. □$_{4.11}$

The same argument gives saturated models in smaller regular cardinals; more strongly we can demand that the saturated model be an Ehrenfeucht-Mostowski model.

**Corollary 4.12.** Suppose $K$ is an AEC with vocabulary $\tau$ that is categorical in $\lambda$ and $\lambda$ is regular. Then for every regular $\mu$, $\text{LS}(K) < \mu < \lambda$ there is a model $M_\mu = EM_\tau(I_\mu, \Phi)$ which is saturated. In particular, it is $\mu$-model homogeneous.

**Proof.** For any ordered set $J$ of cardinality $\lambda$, let $M = EM_\tau(J, \phi)$ be the model of cardinality $\lambda$. We construct an alternating chain of $K$-submodels of length $\mu$. $M_0 \preceq_K M$ is arbitrary with cardinality $\mu$. $M_{2\alpha+1}$ has cardinality $\mu$ and realizes all types over $M_{2\alpha}$ (possible by Corollary 4.10). $M_{2\alpha+2}$ has cardinality $\mu$, $M_{2\alpha+1} \preceq_K M_{2\alpha+2}$ and $M_{2\alpha+2}$ is $EM_\tau(I_{\alpha+1}, \Phi)$ where $I_{\alpha} \subset I_{\alpha+1} \subset J$ and all $I_{\alpha}$ have cardinality $\mu$. Then $EM_\tau(I_\mu, \Phi) = \bigcup_{\alpha<\mu} EM_\tau(I_{\alpha}, \Phi)$ is saturated by regularity. □$_{4.12}$

Now using stability we can get a still stronger result, eliminating the hypothesis that $\mu$ is regular. We show the proofs of both Corollary 4.12 and Corollary 4.13 since in the first case we constructed a saturated model directly and in the second a model homogeneous structure.

**Corollary 4.13.** Suppose $K$ is an AEC with vocabulary $\tau$ that is categorical in $\lambda$ and $\lambda$ is regular. Then for every $\mu$, $\text{LS}(K) < \mu < \lambda$ there is a model $M_\mu = EM_\tau(\mu^{<\omega}, \Phi)$ which is $\mu$-model homogeneous.
Proof. Represent the categoricity model as $M^* = EM^*(\lambda^{<\omega}, \Phi)$. We show $M_\mu = EM^*(\mu^{<\omega}, \Phi)$ is model homogenous. Suppose $M_1 \not\preceq^K M_\mu$ with $|M_1| = \sigma < |M_\mu|$. Then there is $N_1 = EM^*(I_1, \Phi)$ with $|I_1| = \sigma$, $M_1 \subseteq N_1$ and $I_1 \subseteq \mu^{<\omega}$. Let $M_2$ have cardinality $\sigma$ and $M_1 \not\preceq^K M_2$. By amalgamation, choose $N_2 \in K$ which is an amalgam of $N_1$ and $M_2$ over $M_1$. By the $\lambda$-model homogeneity of $M^*$, there is an embedding of $N_2$ into $M^*$ over $N_1$ say with image $N'_2$. Then $N'_2 \subseteq EM^*(J, \Phi)$ for some $J$ with $I_1 \subseteq J \subseteq \lambda^{<\omega}$ and $|J| = \sigma$. Now by Corollary 4.5 and an argument like that in Claim 4.7, there is an embedding of $EM^*(J, \Phi)$ into $M = EM^*(\mu^{<\omega}, \Phi)$ over $N_1$, and a fortiori over $M_1$ and we finish. $\square_{4.13}$

Remark 4.14. (1) Note that for each $\sigma$ less than the categoricity cardinal $\lambda$, the $\sigma$-universal model that is constructed has the form $EM^*(I', \Phi)$ for some $I'$.

(2) Compare Claim 4.13 to I.3.1 in [20], which has the same conclusion but weakening the amalgamation property to: there are no maximal models. There are two uses of the amalgamation property in the argument for Claim 4.13. The first requires only that $M_1$ be an amalgamation base for models in $K$ of size $\mu$ and so extends easily to prove the analogous result where $K$ has amalgamation is replaced by $K$ has no maximal models. The second is that $M$ is $< \lambda$ model homogenous. This step is done in quite a different way in the proof of I.3.1 in [20]; stability is not used but GCH is.

5. Morley’s method for Galois Types

Now we prove ‘Morley’s method’ for Galois types.

Lemma 5.1. [II.1.5 of Sh394] If $M_0 \leq M$ and $|M| > H(|M_0|)$, we can find an $EM$-set $\Phi$ such that the following hold.

1. The $\tau$-reduct of the Skolem closure of the empty set is $M_0$.
2. For every $I$, $M_0 \leq EM(I, \Phi)$.
3. If $I$ is finite, $EM^*(I, \Phi)$ can be embedded in $M$.
4. $EM^*(I, \Phi)$ omits every galois type over $M_0$ which is omitted in $M$.

Proof. Let $\tau_1$ be the Skolem language given by the presentation theorem and consider $M$ as the reduct of $\tau_1$ structure $M^1$. Add constants for $M_0$ to form $\tau'_1$. Now apply Lemma 2.6 to find an $EM$-diagram $\Phi$ (in $\tau'_1$) with all $\tau$-types of finite subsets of the indiscernible sequence realized in $M$. Now 1) and 2) are immediate. 3) is easy (using clause 5 of Theorem 2.7) since we chose $\Phi$ so all finite subsets of the indiscernible set (and so their Skolem closures) are realized in $M$.

The omission of Galois types is more tricky. Consider both $M$ and $N = EM^*(I, \Phi)$ embedded in $M$. Let $N^1$ denote the $\tau'_1$-structure $EM^*(I, \Phi)$. We need to show that if $a \in N$, $p = ga - tp(a/M_0)$ is realized in $M$. For some $e \in I$, $a$ is in the $\tau_1$-Skolem hull $N_e$ of $e$. (Recall the notation from the presentation theorem.) By 3) there is an embedding $\alpha$ of $N_e$ into $M^1$ over $M_0$. $\alpha$ is also an isomorphism of $N_e \restriction \tau$ into $M$. Now, by the model homogeneity, $\alpha$ extends to an automorphism of $M$, fixing $M_0$ and $\alpha(a) \in M$ realizes $p$. $\square_{5.1}$
This has immediate applications in the direction of transferring categoricity.

**Theorem 5.2.** Suppose $M \in K$ omits a Galois type $p$ over a submodel $M_0$ with $|M| \geq H(|M_0|)$. Then there is no regular cardinal $\lambda \geq |M|$ in which $K$ is categorical.

Proof. By Lemma 5.1, there is a model $N \in K$ with cardinality $\lambda$ which omits $p$. But by Lemma 4.11, the unique model of power $\lambda$ is saturated. $\square_{5.2}$

### 6. Tameness and Downwards Categoricity

In [14] Shelah asserts the following result:

**Theorem 6.1.** If $K$ is categorical in a regular cardinal $\lambda$ and $\lambda > H(H(|\tau|))$ then $K$ is categorical in every $\theta$ with $H(H(|\tau|)) \leq \theta \leq \lambda$.

Here is a sketch of the argument. We have shown that there are saturated models of power $\theta$ for every $\theta < \lambda$. The obstacle to deducing downward categoricity is that Theorem 5.1 only allows us to transfer the omission of types when the model omitting the type is much bigger than the domain of the type. The first step in remedying this problem is to show that all types are determined by ‘relatively small’ subtypes. More precisely, we need the notion that Grossberg and VanDieren [6] have called $\chi$-tame and Shelah [14] refers to has ‘having $\chi$-character’. The following notion is that which is actually claimed in [14].

**Definition 6.2.** We say $K$ is $(\chi, \mu)$-weakly tame if for any saturated $N \in K$ with $|N| = \mu < \lambda$ if $p, q, \in S(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Our notion differs from [6] in two ways: we use parameters for each of the cardinality of the ambient model and the determining small set. This is crucial because the application in [14] involves induction on the cardinality of the ambient model. Secondly, the proof seems to use essentially that the ambient model is saturated. So we have adapted weakly tame for the restriction to saturated models and left the notion of tame to apply for an arbitrary ambient model.

Shelah asserts the following in Sections II.1 and II.2.3 of the published version of [14]. The published proof is inaccurate and incomplete. Shelah provided further clarification and Hyttinen gave a clear and short proof which appears in [1].

**Theorem 6.3.** Suppose $K$ is $\lambda$-categorical for $\lambda \geq H(\tau)$ and $\lambda$ is regular. Then $K$ is $(\chi, \chi_1)$-tame for some $\chi$ and any $\chi_1$ with $\chi < H(\tau) \leq \chi_1 < \lambda$.

The naive argument would give $\chi = H(\tau)$ since one is omitting types. But omitting in every cardinal below $H(\tau)$ is as good as in $H(\tau)$ so the conclusion becomes for some $\chi$ with $\chi < H(\tau)$.

The remainder of the argument for Theorem 6.1 uses such technologies as splitting and minimal types that are beyond the scope of this paper; a fuller account appears in [1]. Under the assumption of $(\chi, \infty)$-tameness for small $\chi$, Grossberg and VanDieren [7] prove upwards categoricity results; see also [1].
Since we were expounding [14] we assumed, as there, that $K$ has arbitrarily large models and the amalgamation and joint embedding properties. We used amalgamation heavily to get monster models and thus get the group theoretic definition of Galois-type. By using the more complicated definition of a Galois type as an equivalence relation on triples, many of these notions can be extended to classes without amalgamation. And one can even prove [17, 1], saturation equals model homogeneity with no amalgamation hypothesis whatsoever. However, I don’t know any way to prove the existence of either saturated model homogeneous models in general AEC without at least some amalgamation hypothesis.

References

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Unsplit Families, Dominating Families, and Ultrafilters

Andreas Blass

ABSTRACT. We study some weakenings of the finite intersection property for families of subsets of the natural numbers. The weakenings involve (1) requiring intersections for only a fixed number of sets from the family and (2) requiring the sets to have elements near each other rather than actually intersecting. These weakenings fit into a chain of implications, none of which are reversible under CH, but almost all of which are consistently reversible. We also connect these properties with weakened domination properties for families for functions on the natural numbers. For unsplit families of sets, the chain of implications collapses from infinitely many properties to just four.

1. Introduction and Background Information

We use the customary notations \( \omega \) for the set of natural numbers, \( \mathcal{P}(\omega) \) for its power set, \( [\omega]^\omega \) for the family of all infinite subsets of \( \omega \), \( \omega^\omega \) for the family of all functions from \( \omega \) to \( \omega \), \( A \subseteq^* B \) for almost inclusion (i.e., \( A - B \) is finite), and \( f \leq^* g \) for eventual majorization (i.e., \( f(n) \leq g(n) \) for all but finitely many \( n \)).

DEFINITION 1.1. A family \( \mathcal{X} \subseteq [\omega]^\omega \) is unsplit if

\[
(\forall A \in \mathcal{P}(\omega))(\exists X \in \mathcal{X}) \ (X \subseteq^* A \ \text{or} \ X \subseteq^* \omega - A).
\]

That is, no single \( A \subseteq \omega \) splits every \( X \in \mathcal{X} \) into two infinite pieces.

DEFINITION 1.2. Following the standard notation for cardinal characteristics of the continuum ([5, 8, 3]), we write

- \( \mathfrak{c} \) for the cardinality of the continuum,
- \( \mathfrak{r} \) for the smallest cardinality of any unsplit family (the unsplitting or refining or reaping number),
- \( \mathfrak{u} \) for the minimum cardinality of any base for a nonprincipal ultrafilter on \( \omega \),
- \( \mathfrak{d} \) for the minimum cardinality of a dominating family, i.e., a family \( D \subseteq \omega^\omega \) such that every \( f \in \omega^\omega \) is \( \leq^* g \) for some \( g \in D \) (the dominating number), and
- \( \mathfrak{g} \) for the groupwise density number, for whose definition we refer to [4] or [3] since we shall not need the details here.

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Notice that a base for a nonprincipal ultrafilter on $\omega$ is the same thing as an unsplit filter base. In particular, $\tau \leq u$. Goldstern and Shelah [6] showed that $\tau < u$ is consistent, but Aubrey [1] showed that $\tau \geq \min\{u, d\}$.

Part of our purpose in the present paper is to make Aubrey’s result more explicit by showing how to obtain, quite directly from an unsplit family, either an ultrafilter base or a dominating family of the same cardinality.

The essential difference between ultrafilters and general unsplit families is that the former (and generating families for ultrafilters) have the strong finite intersection property (the intersection of any finitely many sets from the family is infinite). We shall study the role of the strong finite intersection property by looking at some weakenings of it, both in connection with unsplit families and in more general situations.

The remainder of this introductory section is devoted to fixing some notation and terminology that will be needed later.

**Definition 1.3.** If $\mathcal{X} \subseteq [\omega]^\omega$ and $f : \omega \to \omega$ then

$$f(\mathcal{X}) := \{f(X) : X \in \mathcal{X}\}.$$ 

**Remark 1.4.** The notation $f(\mathcal{X})$ is often used for $\{Y \subseteq \omega : f^{-1}(Y) \in \mathcal{X}\}$. That usage seems to be preferable when dealing with families closed under supersets, but our definition above works better in the situations we shall consider, where no upward closure is required. If $\mathcal{X}$ is closed upward, then the upward closure of our $f(\mathcal{X})$ is the other $f(\mathcal{X})$.

When we use the notation $f(\mathcal{X})$, the function $f$ will usually be finite-to-one and monotone. That is, there is a partition $\Pi$ of $\omega$ into finite intervals such that $f$ is constant on these intervals and strictly increases from one interval to the next. In this situation, $f(\mathcal{X})$ can be visualized as obtained from $\mathcal{X}$ by collapsing each interval in $\Pi$ to a point.

**Lemma 1.5.** If $f$ is finite-to-one and $\mathcal{X}$ is unsplit, then $f(\mathcal{X})$ is also unsplit.

**Proof.** The hypothesis that $f$ is finite-to-one is used only to ensure that $f(\mathcal{X}) \subseteq [\omega]^\omega$. The unsplit property is immediate, for if $A$ were to split every set in $f(\mathcal{X})$ then $f^{-1}(A)$ would split every set in $\mathcal{X}$. \hfill $\square$

**Definition 1.6.** For $X \in [\omega]^\omega$ and $n \in \omega$, let $\text{next}(X, n)$ be the smallest element of $X$ that is $\geq n$.

We recall from [2] the following weakened form of domination.

**Definition 1.7.** For a positive integer $k$, a family $\mathcal{F} \subseteq ^\omega \omega$ is $k$-dominating if every $g \in ^\omega \omega$ is eventually dominated by the (pointwise) maximum of some $k$ functions from $\mathcal{F}$.

2. Meeting, Nearly Meeting, and Just Missing

In this section, we introduce and study some properties that refine and approximate the strong finite intersection property. Throughout this section, $\mathcal{X}$ is an arbitrary subfamily of $[\omega]^\omega$.

The first definition is a natural quantitative version of the strong finite intersection property, replacing “finite” by a specific number.

**Definition 2.1.** Let $k$ be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is $k$-meeting if, for all $X_1, \ldots, X_k \in \mathcal{X}$, $X_1 \cap \cdots \cap X_k$ is infinite.
Thus, $\mathcal{X}$ has the strong finite intersection property if and only if it is $k$-meeting for all $k$.

The next definition is a weakening of $k$-meeting, working modulo a possible collapse of finite intervals to points. We note that the analogous “modulo collapse of finite intervals” weakening of the strong finite intersection property has played a role, for example, in results of Laflamme [7].

**Definition 2.2.** Let $k$ be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is nearly $k$-meeting if there is a finite-to-one, monotone $f:\omega \to \omega$ such that $f(\mathcal{X})$ is $k$-meeting.

The next definition admittedly looks unnatural, because of its dependence on the immediate successor relation on $\omega$. However, it will be used primarily in its “nearly” form, which turns out to be considerably more natural.

**Definition 2.3.** Let $k$ be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is $k$-close if, for all $X_1, \ldots, X_k \in \mathcal{X}$, there are infinitely many $n$ such that, for each $i = 1, \ldots, k$ either $n \in X_i$ or $n+1 \in X_i$.

In other words, instead of requiring the $k$ sets $X_i$ to meet infinitely often, we allow an “error” of 1; when they all hit a two-element interval, that’s as good as actually all meeting.

**Definition 2.4.** Let $k$ be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is nearly $k$-close if there is a finite-to-one, monotone $f:\omega \to \omega$ such that $f(\mathcal{X})$ is $k$-close.

It is often useful to reformulate the “nearly” definitions in terms of the interval partitions corresponding to the finite-to-one monotone functions.

**Lemma 2.5.** $\mathcal{X}$ is nearly $k$-close if and only if there is a partition $\Pi$ of $\omega$ into intervals $[\pi_n, \pi_{n+1})$, where $0 = \pi_0 < \pi_1 < \pi_2 < \ldots$, such that, for every $k$ sets $X_i \in \mathcal{X}$, there are infinitely many $n$ such that all $k$ of the $X_i$ meet the double interval $[\pi_n, \pi_{n+2})$. Nearly $k$-meeting is the same except that the double interval is replaced with the single interval $[\pi_n, \pi_{n+1})$.

In the situation of the lemma, we shall say that $\Pi$ witnesses that $\mathcal{X}$ is nearly $k$-close or nearly $k$-meeting.

It is somewhat surprising that it really makes a difference whether we use the double intervals $[\pi_n, \pi_{n+2})$ or the single intervals $[\pi_n, \pi_{n+1})$, considering that the partition $\Pi$ can be adjusted.

Notice that as $k$ increases, the (nearly) meeting and closeness properties become stronger, in contrast to $k$-dominating, which becomes weaker. In fact, the following proposition exhibits a negative correlation between closeness and domination.

**Proposition 2.6.** $\mathcal{X}$ is nearly $k$-close if and only if $\{\text{next}(X, -) : X \in \mathcal{X}\}$ is not $k$-dominating.

**Proof** First assume that $\mathcal{X}$ is nearly $k$-close and let the interval partition $\Pi$ witness this. Define $g: \omega \to \omega$ by letting $g(n)$ be the left endpoint of the third interval of $\Pi$ after the one containing $n$. So there are two consecutive $\Pi$-intervals included in $[n, g(n))$. We claim that $g$ witnesses that $\{\text{next}(X, -) : X \in \mathcal{X}\}$ is not $k$-dominating. To prove this, suppose toward a contradiction that $X_1, \ldots, X_k$ are $k$ sets in $\mathcal{X}$ such that the maximum of the associated functions $\text{next}(X_i, -)$ eventually majorizes $g$. So, for all but finitely many $n$, we have that at least one $X_i$ has its next element after $n$ greater than the right endpoint of the second $\Pi$-interval after $n$. This means that $X_i$ is disjoint from the two $\Pi$-intervals immediately following the one that contains $n$. Since $n$ is arbitrary (provided it is large enough), we
have that, with finitely many exceptions, no union of two consecutive II-intervals intersects all the $X_i$. This contradicts the assumption that $\mathcal{I}$ witnesses that $\mathcal{X}$ is nearly $k$-close, and so half of the proposition is proved.

For the converse, suppose some function $g : \omega \to \omega$ is not eventually majorized by the maximum of any $k$ of the functions $\text{next}(X, -)$ for $X \in \mathcal{X}$. Let $\Pi$ be a partition of $\omega$ into intervals such that, for each $n$, $g(n)$ is in the next interval after $n$ or earlier. Such a partition is easily constructed by inductively choosing the successive intervals. Given any $k$ sets $X_1, \ldots, X_k \in \mathcal{X}$, the fact that the maximum of the associated functions $\text{next}(X_i, -)$ fails to eventually majorize $g$ means that there are infinitely many $n$ such that all $k$ of the $X_i$ meet the interval $[n, g(n)]$. But this interval is included in the union of two consecutive II-intervals. Thus, there are infinitely many unions of two consecutive II-intervals, each of which meets all $k$ of the sets $X_i$. Thus $\mathcal{X}$ is nearly $k$-close.

REMARK 2.7. This proposition suggests that the notion of "nearly $k$-close" is more natural than it looks at first sight. The proof of the proposition also provides a certain robustness of the notion. For example, where the definition of "$k$-close" requires each of the $k$ sets to contain $n$ or $n + 1$, let us put instead the weaker requirement that each of these sets contains one of $n, n + 1, \ldots, n + p$ for a fixed $p$. Then the first half of the proof of the proposition still works if we just redefine $g(n)$ to be the left endpoint of the $(p + 2)^{\text{nd}}$ (rather than the third) II-interval after $n$. Thus, although we have weakened the notion of "$k$-close", the notion of "nearly $k$-close" is unchanged. The same applies if we let $p$ vary with $n$, provided we appropriately adjust the $g$ in the proof.

REMARK 2.8. To avoid confusion, we point out a potential ambiguity in the phrase "nearly $k$-close for all $k$". Its intended meaning is that, for each $k$ there is a finite-to-one monotone $f_k$ (or equivalently an interval partition $\Pi_k$) witnessing $k$-closeness. But one can easily imagine the same phrase as obtained by prefixing "nearly" to the compound adjective "$k$-close for all $k$". That would mean that a single $f$ (or $\Pi$) works for all $k$ simultaneously, an apparently stronger statement than the previous one. Fortunately, the following lemma shows that the two possible meanings of this phrase are equivalent.

LEMMA 2.9. For any $\mathcal{X} \subseteq [\omega]^\omega$, the following three statements are equivalent.

1. For every $k \in \omega$, $\mathcal{X}$ is nearly $k$-close.
2. There is a finite-to-one $f : \omega \to \omega$ such that, for every $k \in \omega$, $f(\mathcal{X})$ is $k$-close.
3. There is a function $g : \omega \to \omega$ that is not eventually majorized by the maximum of any finitely many of the functions $\text{next}(X, -)$ for $X \in \mathcal{X}$.

Proof Since assertion (1) says that for each $k$ there is a finite-to-one monotone $f_k$ such that $f_k(\mathcal{X})$ is $k$-close and assertion (2) says that a single $f$ works for all $k$ simultaneously, it is clear that (2) implies (1).

Next, we assume (1) and prove (3). By Proposition 2.6, there is for each $k \in \omega$ some $g_k : \omega \to \omega$ that is not eventually majorized by the maximum of any $k$ of the functions $\text{next}(X, -)$ for $X \in \mathcal{X}$. Let $g : \omega \to \omega$ eventually majorize all the $g_k$; for example we could take $g(n) = \max\{g_k(n) : k \leq n\}$. Then $g$ is as required in (3).

Finally, we assume that $g$ is as in (3) and we prove (2). This proof is exactly like the second half of the proof of Proposition 2.6, with the fixed $k$ replaced by an arbitrary finite number, so we do not repeat the argument here. □
Remark 2.10. It is easy to see that any 2k-close family is nearly k-meeting, witnessed by one of the two functions \([n/2]\) and \([n/2]\). Indeed, if \(X_1, \ldots, X_k\) are \(k\) sets in \(\mathcal{X}\) such that only finitely many of the two-point intervals \([2n, 2n+1]\) meet all the \(X_i\), and if \(Y_1, \ldots, Y_k\) are \(k\) sets in \(\mathcal{X}\) such that only finitely many \([2n+1, 2n+2]\) meet all the \(Y_i\), then there are only finitely many \(m\) such that each of the \(2k\) sets \(X_i\) and \(Y_i\) contains \(m\) or \(m+1\). That is, \(\mathcal{X}\) is not 2k-close.

Of course, it follows immediately that any nearly 2k-close family is nearly \(k\)-meeting. Indeed, if \(\Pi\) witnesses “nearly 2k-close”, then we can witness “nearly \(k\) meeting” with a partition obtained by merging pairs of consecutive intervals from \(\Pi\).

In this context with “nearly”, the following theorem lets us improve 2k to \(k+1\) by merging (perhaps) considerably longer blocks of intervals.

**Theorem 2.11.** Every nearly \((k+1)\)-close family is nearly \(k\)-meeting.

**Proof.** It suffices to prove that every \((k+1)\)-close family \(\mathcal{X}\) is nearly \(k\)-meeting, for if \(\mathcal{X}\) is only nearly \((k+1)\)-close then we can apply the argument to a \((k+1)\)-close family of the form \(f(\mathcal{X})\).

Assume therefore that \(\mathcal{X}\) is \((k+1)\)-close. Consider the partition of \(\omega\) into the intervals of length two, \([2n, 2n+1]\). If every \(X \in \mathcal{X}\) meets all but finitely many of these intervals, then this partition witnesses that \(\mathcal{X}\) is nearly \(k\)-meeting (and in fact nearly \(k\)-meeting for all \(l \in \omega\)).

So we may assume, without loss of generality, that there is some \(Z \in \mathcal{X}\) that misses infinitely many of the intervals \([2n, 2n+1]\). Fix such a \(Z\), and enumerate as \(n_1 < n_2 < \ldots\) the integers \(n\) such that \(Z \cap [2n, 2n+1] = \emptyset\). Let \(\Pi\) be the partition of \(\omega\) into the intervals \([2n_i+1, 2n_{i+1}]\), along with the initial interval \([0, 2n_1]\). In other words, the partition \(\Pi\) breaks \(\omega\) in the middle of each of the intervals \([2n, 2n+1]\) that \(Z\) misses. We intend to show that \(\Pi\) witnesses that \(\mathcal{X}\) is nearly \(k\)-meeting, i.e., that given any \(k\) elements of \(\mathcal{X}\), we can find an interval \([2n_i+1, 2n_{i+1}]\) \(\in \Pi\) that meets all \(k\) of them.

So let any \(k\) sets \(X_1, \ldots, X_k \in \mathcal{X}\) be given. Apply the assumption that \(\mathcal{X}\) is \((k+1)\)-close to the \(k+1\) sets \(Z, X_1, \ldots, X_k\). We obtain infinitely many \(m\) such that the interval \([m, m+1]\) meets \(Z\) and all \(k\) of the \(X_i\). Such an \(m\) cannot be of the form \(2n_i\), for then the interval \([m, m+1] = [2n_i, 2n_i + 1]\) would miss \(Z\) (by definition of \(n_i\)). Therefore, \([m, m+1]\) lies entirely within one of the intervals of \(\Pi\), and so that interval of \(\Pi\) meets all the \(X_i\). Since this happens for infinitely many \(m\), the proof is complete.

In view of the theorem, we have the following chain of implications between the “nearly” meeting and close properties.

\[
\begin{align*}
(1) \quad \ldots & \quad \implies \text{nearly (k+1)-close} \implies \\
& \quad \implies \text{nearly k-meeting} \implies \text{nearly k-close} \implies \ldots \\
& \quad \implies \text{nearly 4-close} \implies \text{nearly 3-meeting} \implies \text{nearly 3-close} \implies \\
& \quad \implies \text{nearly 2-meeting} \implies \text{nearly 2-close} \implies \text{nearly 1-meeting}.
\end{align*}
\]

Of course, the last item in the chain is always true, since we have \(\mathcal{X} \subseteq [\omega]^\omega\). At the other end of the chain, we can consider the conjunction of all the properties listed.

That conjunction is item (1) from Lemma 2.9. We can now extend that lemma by including properties involving meeting rather than closeness.

**Corollary 2.12.** For any \(\mathcal{X} \subseteq [\omega]^\omega\), the following are equivalent.
(1) The three statements from Lemma 2.9.
(2) For every \( k \in \omega \), \( \mathcal{X} \) is nearly \( k \)-meeting.
(3) There is a finite-to-one \( f : \omega \rightarrow \omega \) such that, for every \( k \in \omega \), \( f(\mathcal{X}) \) is \( k \)-meeting.

Proof The chain of implications above immediately shows that item (2) here and item (1) of Lemma 2.9 are equivalent, and it is trivial that in the present corollary (3) implies (2). To complete the proof, it suffices to deduce the present (3) from item (2) of Lemma 2.9.

For this purpose, notice first that, in the proof of Theorem 2.11, the partition witnessing that \( \mathcal{X} \) is nearly \( k \)-meeting was obtained without reference to the value of \( k \). Thus, if \( \mathcal{X} \) is \((k+1)\)-close for all \( k \), then there is a function \( g \), independent of \( k \), such that \( g(\mathcal{X}) \) is \( k \)-meeting for all \( k \). Now if (2) of Lemma 2.9 holds, then we have an \( f \) such that \( f(\mathcal{X}) \) is \((k+1)\)-close for all \( k \), and so we get \( g \) such that \( g f(\mathcal{X}) \) is \( k \)-meeting for all \( k \). That is, we get (3) of the present corollary, as required. ☐

3. Consistency Results

In this section, we consider the reversibility of the implications in the chain (1). The last implication in the chain is not reversible, for \([\omega]^\omega \) is 1-meeting but not nearly 2-close. All the other reversals, however, are independent of ZFC, as the following theorems show. In connection with the first of these theorems, we recall the result from [4] that the inequality \( u < g \) is consistent relative to ZFC.

**Theorem 3.1.** Assume \( u < g \). Then all the implications except the last in the chain (1) are reversible. Thus, if \( \mathcal{X} \subseteq [\omega]^\omega \) is nearly 2-close then there is a finite-to-one, monotone \( f : \omega \rightarrow \omega \) such that \( f(\mathcal{X}) \) has the strong finite intersection property.

**Proof** Under the assumption \( u < g \), it is shown in [2, Theorem 6.5] that every \( k \)-dominating family of monotone functions from \( \omega \) to \( \omega \), for any \( k < \omega \), is 2-dominating. By Proposition 2.6, it follows that every nearly 2-close family \( \mathcal{X} \subseteq [\omega]^\omega \) is nearly \( k \)-close for all \( k \in \omega \). This immediately gives the first conclusion of the theorem. The second follows by Corollary 2.12. ☐

The next two theorems show that the continuum hypothesis (or Martin’s axiom, or even weaker hypotheses) gives a picture diametrically opposed to the picture in Theorem 3.1 under \( u < g \).

**Theorem 3.2.** Assume \( \tau \geq \delta \). Then there is, for any \( k \geq 1 \), a \( k \)-meeting family that is not nearly \( k + 1 \)-close.

Before beginning the proof, we recall some information from [3] and relate it to the present situation. Consider two interval partitions (i.e., partitions of \( \omega \) into finite intervals), \( \Pi \) and \( \Pi' \). We say that \( \Pi \) dominates \( \Pi' \) if each interval in \( \Pi \), with finitely many exceptions, includes an interval in \( \Pi' \). It is easy to see (and explicitly shown in [3]) that there is a family of \( \delta \) interval partitions such that every interval partition is dominated by one from this family.

**Lemma 3.3.** Suppose \( \Pi \) dominates \( \Pi' \). Then, with finitely many exceptions, every union of two consecutive intervals from \( \Pi' \) is included in the union of two consecutive intervals from \( \Pi \). Thus, if \( \Pi' \) witnesses that a family \( \mathcal{X} \) is nearly \( k \)-close for a certain \( k \), then so does \( \Pi \).
Proof. If the first conclusion failed, then, infinitely often, the union of some two consecutive intervals from \( \Pi' \) would meet three or more consecutive intervals from \( \Pi \). Then any of these three or more \( \Pi \)-intervals except the first and last would include no entire interval from \( \Pi' \). This contradicts the assumption that \( \Pi \) dominates \( \Pi' \), so the first conclusion is proved. The second conclusion follows by the definition of what it means to witness near closeness. \( \square \)

Lemma 3.4. Given any natural number \( n \geq 2 \) and any family \( A \) of fewer than \( r \) infinite subsets of \( \omega \), we can partition \( \omega \) into \( n \) pieces each of which has infinite intersection with each set in \( A \).

Proof. For \( n = 2 \), this is just the definition of \( r \). If the result is true for \( n \), then to get it for \( n + 1 \), first partition \( \omega \) into \( n \) pieces of the desired sort, then choose one of the pieces \( P \), and apply the definition of \( r \) within \( P \), splitting \( P \) into two pieces each meeting infinitely all the sets \( P \cap A \) for \( A \in A \). \( \square \)

Proof of Theorem 3.2. Fix \( k \geq 1 \), and fix a family \( \{ \Pi_\alpha : \alpha < \varnothing \} \) of \( \varnothing \) interval partitions dominating all interval partitions. We shall produce a \( k \)-meeting family \( \mathcal{X} \) such that no \( \Pi_\alpha \) witnesses that \( \mathcal{X} \) is nearly \((k+1)\)-close. According to Lemma 3.3, no other partition can witness that \( \mathcal{X} \) is nearly \((k+1)\)-close, and so the proof will be complete.

The construction of \( \mathcal{X} \) is an induction of length \( \varnothing \). At stage \( \alpha \), we shall put into \( \mathcal{X} \) some \( k + 1 \) sets \( A_0^\alpha, \ldots, A_k^\alpha \) such that no two consecutive intervals of \( \Pi_\alpha \) meet all \( k + 1 \) of them. Thus, stage \( \alpha \) will ensure that \( \Pi_\alpha \) does not witness that \( \mathcal{X} \) is nearly \((k+1)\)-close. During the construction, we shall take care that every \( k \) sets that we put into \( \mathcal{X} \) have infinitely many points in common, so the final \( \mathcal{X} \) will be \( k \)-meeting.

We now describe stage \( \alpha \). Since \( \alpha \) will be fixed during this description, we omit it from the notation, writing simply \( \Pi \) for \( \Pi_\alpha \), and writing \( A_i \) for the \( k + 1 \) sets to be added. We also write \( I_n \) for the \( n \)th interval of \( \Pi \). Let \( \mathcal{Y} \) be the family of sets already put into \( \mathcal{X} \) at earlier stages. We assume, as an induction hypothesis, that \( \mathcal{Y} \) is \( k \)-meeting. Since only finitely many sets are added to \( \mathcal{X} \) at any stage, and since our induction has length \( \varnothing \), we have \( |\mathcal{Y}| < \varnothing \).

To each \((\leq k)\)-element subfamily \( \mathcal{K} \subseteq \mathcal{Y} \), associate the set \( B_\mathcal{K} \) of those \( n \in \omega \) such that the intersection of the sets in \( \mathcal{K} \) meets the \( n \)th interval \( I_n \) of \( \Pi \). Since \( \mathcal{Y} \) is \( k \)-meeting, each \( B_\mathcal{K} \) is infinite. Furthermore, the number of such sets \( B_\mathcal{K} \) is, like \( |\mathcal{Y}| \), smaller than \( \varnothing \) and therefore, by the hypothesis of the theorem, smaller than \( r \).

We shall need a co-infinite set \( Z \subseteq \omega \) that meets each \( B_\mathcal{K} \) infinitely often. Since the number of \( B_\mathcal{K} \)'s is \( < r \), we can simply split \( \omega \) into two pieces that each meet each \( B_\mathcal{K} \) infinitely, and then let \( Z \) be one of the pieces. (In fact, \( r \) isn't really relevant here; for any family of \( < r \) infinite subsets of \( \omega \) there is a co-infinite set meeting them all infinitely.) Fix such a \( Z \), and view it as the union of some intervals \( J_n \) separated by members of \( \omega - Z \). Since \( Z \) is infinite and co-infinite, the intervals \( J_n \) are finite and there are infinitely many of them. Let \( C_\mathcal{K} \) be the set of those \( n \in \omega \) such that \( B_\mathcal{K} \) meets \( J_n \). Our choice of \( Z \) ensures that each \( C_\mathcal{K} \) is infinite. If we write \( \tilde{J}_n \) for \( \bigcup_{m \in J_n} I_m \) then \( C_\mathcal{K} \) is the set of \( n \) such that \( \tilde{J}_n \) meets all of the sets in \( \mathcal{K} \).

Since the \( C_\mathcal{K} \)'s form a family of \( < r \) infinite subsets of \( \omega \), Lemma 3.4 lets us partition \( \omega \) into \( k + 1 \) pieces \( Z_0, \ldots, Z_k \), each meeting every \( C_\mathcal{K} \) infinitely often.
Define, for $0 \leq i \leq k$,

$$A_i = \bigcup_{n \in Z - Z_i} \bar{J}_n.$$  

(Note that the union is not over $n$ in $Z_i$ but rather over $n$ in the union of the other $k$ pieces $Z_j$.) These $A_i$'s are the sets that we add to $\mathcal{X}$ at the current stage of the construction. It remains to verify that they have the required properties.

Consider any two consecutive intervals of $\Pi$, say $I_m$ and $I_{m+1}$; we must check that the $A_i$ do not all meet $I_m \cup I_{m+1}$. Our construction of $Z$ and the $J_n$ ensures that, if $m$ and $m+1$ are both in $Z$ then they are in the same $J_n$. So we can fix an $n$ such that each of $m$ and $m+1$ is either in $J_n$ or outside $Z$. Let $i$ be the (unique) index such that $n \in Z_i$, and observe that $A_i$ does not meet $\bar{J}_n$, nor does $A_i$ meet any $I_p$ whose index $p$ is outside $Z$. It follows that $I_m \cup I_{m+1}$ cannot meet $A_i$, which is what we wanted to check.

Finally, we must check that the $k$-meeting property of $\mathcal{Y}$ is preserved when we adjoin $A_0, \ldots, A_k$. Any family of $k$ sets from $\mathcal{Y} \cup \{A_0, \ldots, A_k\}$ is the union of some ($\leq k$)-element subfamily $\mathcal{K}$ of $\mathcal{Y}$ and a subfamily $\mathcal{A}$ of $\{A_0, \ldots, A_k\}$ that omits at least one $A_i$. Fix such $\mathcal{K}$ and $i$. Our choice of $Z_i$ ensures that it contains infinitely many elements of $C \mathcal{K}$. For each of these elements $n$, $\bar{J}_n$ meets the intersection of the sets in $\mathcal{K}$ and is a subset of all the sets in $\mathcal{A}$. Thus, all the sets in $\mathcal{K} \cup \mathcal{A}$ have a common element in $\bar{J}_n$. Since this happens for infinitely many $n$, the proof is complete.

The preceding theorem says that under the hypothesis $\tau \geq \mathfrak{d}$, and thus in particular under $\text{CH}$, half of the implications in the chain (1) cannot be reversed. The next theorem does the same for the other half, but it uses a stronger hypothesis, still weaker than $\text{CH}$.

**Theorem 3.5.** Assume $\tau = \mathfrak{c}$. Then there is, for each $k \geq 2$, a $k$-close family that is not nearly $k$-meeting.

**Proof** Fix $k$ and list all interval partitions of $\omega$ as $\Pi_\alpha$, indexed by $\alpha < \mathfrak{c}$. We shall construct, in an induction of length $\mathfrak{c}$, a $k$-close family $\mathcal{X} \subseteq [\omega]^\omega$ that is not nearly $k$-meeting. At each stage $\alpha$, we shall ensure that $\Pi_\alpha$ does not witness that $\mathcal{X}$ is nearly $k$-meeting by adding to $\mathcal{X}$ some $k$ sets $A_1, \ldots, A_k$ that do not all meet any interval of $\Pi_\alpha$.

We now describe stage $\alpha$. Since $\alpha$ will be fixed during this description, we omit it from the notation, writing simply $\Pi$ for $\Pi_\alpha$, and writing $A_i$ for the $k$ sets to be added. We also write $I_n$ for the $n^{\text{th}}$ interval of $\Pi$. Let $\mathcal{Y}$ be the family of sets already put into $\mathcal{X}$ at earlier stages. We assume, as an induction hypothesis, that $\mathcal{Y}$ is $k$-close. Since only finitely many sets are added to $\mathcal{X}$ at any stage, and since our induction has length $\mathfrak{c}$, we have $|\mathcal{Y}| < \mathfrak{c}$.

To each ($\leq k$)-element subfamily $\mathcal{K} \subseteq \mathcal{Y}$, associate the set $B_\mathcal{K}$ of those $n \in \omega$ such that some two-point interval $[p, p + 1]$ with $p \in I_n$ meets all of the sets in $\mathcal{K}$. Since $\mathcal{Y}$ is $k$-close, each $B_\mathcal{K}$ is infinite. Furthermore, the number of such sets $B_\mathcal{K}$ is, like $|\mathcal{Y}|$, smaller than $\mathfrak{c}$ and therefore, by the hypothesis of the theorem, smaller than $\tau$.

By Lemma 3.4, partition $\omega$ into $k$ sets $Z_1, \ldots Z_k$ each meeting each $B_\mathcal{K}$ infinitely often. Define

$$A_i = \bigcup_{n \notin Z_i} I_n.$$
These are the $k$ sets to be added to $\mathcal{X}$ at the current stage. We must check that they have the required properties.

It is clear that no $I_n$ meets all the $A_i$; indeed, if $n \in Z_i$ then $I_n$ is disjoint from $A_i$.

So it remains to prove that $\mathcal{Y} \cup \{A_1, \ldots, A_k\}$ is $k$-close. Consider therefore any $k$ members of this family. If they are all in $\mathcal{Y}$, then the induction hypothesis gives what we want. If they are the $k$ newly added $A_i$’s then we find infinitely many two-point intervals meeting them all as follows. Since the $Z_i$’s partition $\omega$ and are all infinite, there are infinitely many $n$ such that $n$ is in one $Z_i$ and $n + 1$ is in a different $Z_j$. For such an $n$, the last point $p$ in the interval $I_n$ belongs (like that whole interval) to all the A’s except $A_i$, while $p + 1$, being in $I_{n+1}$, belongs to all the A’s except $A_j$. Thus, each of the A’s meets $[p, p + 1]$, as required.

Finally, we consider the case of $k$ sets, some of which are from $\mathcal{Y}$ and some from the newly added A’s. So these $k$ sets form the union of some ($\leq k$)-element subfamily $\mathcal{K}$ of $\mathcal{Y}$ and some subfamily of $\{A_1, \ldots, A_k\}$ that omits at least one $A_i$. Fix such $\mathcal{K}$ and $i$. By our construction, there are infinitely many elements $n$ in $Z_i \cap B_\mathcal{K}$. For each such $n$, there is, by definition of $B_\mathcal{K}$, some $p \in I_n$ such that $[p, p + 1]$ meets all the sets in $\mathcal{K}$. Since $I_n$ is also included in all the A’s except $A_i$, it follows that $[p, p + 1]$ meets all the $k$ sets we began with, and so the proof is complete.

The contrast between the reversibility of all but the last of the implications in (1) under $u < g$ (Theorem 3.1) and their irreversibility under $r = c$ (Theorems 3.2 and 3.5) suggests questions about partial reversibility. For example, the referee asked about the consistency of “there is a 7-meeting family that is not 8-close but every 5-meeting family is 6-close.” I do not know the answers to such questions. I also do not know whether the hypothesis $r = c$ in Theorem 3.5 can be replaced by the weaker hypothesis $r \geq d$. For example, in the Sacks model, where $r = d = \aleph_1 < c = \aleph_2$, is every $k$-close family nearly $k$-meeting?

4. Unsplit Families

In this section, we investigate what happens to the previous results under the additional hypothesis that the family $\mathcal{X}$ is unsplit. The first result shows that most of the chain (1) collapses.

**Theorem 4.1.** Any 3-meeting unsplit family is an ultrafilter base, modulo finite sets.

**Proof** Let $\mathcal{X}$ be a 3-meeting unsplit family. It suffices to show that it is a filter base modulo finite sets, i.e., that the intersection of any two members of $\mathcal{X}$ almost includes another member of $\mathcal{X}$. The “ultra” part of the conclusion then follows immediately from the “unsplit” assumption.

So let $A, B \in \mathcal{X}$. Since $\mathcal{X}$ is unsplit, it contains some $C$ that is either almost included in $A \cap B$ or almost disjoint from $A \cap B$. The former alternative is what we want. The latter is absurd as $A, B, C$ would be a counterexample to the “3-meeting” assumption.

**Corollary 4.2.** For unsplit families, 3-meeting implies $k$-meeting, and therefore nearly 3-meeting implies nearly $k$-meeting, for all $k$.

With a little more work, we can collapse one more implication in the chain (1).

**Theorem 4.3.** For unsplit families, nearly 3-close implies nearly 4-close.
Proof. It suffices to show that every 3-close unsplit family is nearly 4-close, for if we are given an unsplit family $\mathcal{X}$ that is only nearly 3-close then we can work instead with a 3-close family of the form $f(\mathcal{X})$.

So assume $\mathcal{X}$ is unsplit and 3-close. We shall show that the partition $\Pi$ of $\omega$ into two-point intervals $[2n, 2n+1]$ witnesses that $\mathcal{X}$ is nearly 4-close. Assume, toward a contradiction, that we have four sets $A, B, C, D \in \mathcal{X}$ such that only finitely many unions $[2n, 2n+3]$ of two consecutive intervals of $\Pi$ meet all four of $A, B, C, D$.

Let $P$ be the union of those two-point intervals $[p, p+1]$ that meet both $A$ and $B$, and similarly let $Q$ be the union of those two-point intervals $[q, q+1]$ that meet both $C$ and $D$. Then $P$ and $Q$ are disjoint, for if $n$ were in both then the 3-point interval $[n-1, n+1]$ would meet all four of $A, B, C, D$, and that is impossible as every 3-point interval is included in a 4-point interval starting at an even number, i.e., in one of the intervals $[2n, 2n+3]$ that we assumed do not meet all of $A, B, C, D$.

Since $\mathcal{X}$ is unsplit, it must contain some set $E$ almost disjoint from one of $P$ and $Q$. But if $E \cap P$ is finite, then there are only finitely many 2-point intervals meeting all three of $A, B, E$, and similarly if $E \cap Q$ is finite, then there are only finitely many 2-point intervals meeting all three of $C, D, E$. In either case, we have contradicted the assumption that $\mathcal{X}$ is 3-close, and so the proof is complete. \qed

The following corollary combines the preceding two theorems.

**Corollary 4.4.** For unsplit families, “nearly 3-close” is equivalent to all the earlier properties in the chain (1) of implications and to “ultrafilter base modulo finite”.

Thus, for unsplit families, the implication chain (1) collapses to

\[(2) \quad \text{ultrafilter base mod finite } \iff \text{nearly 3-close } \implies \text{nearly 2-meeting } \implies \text{nearly 2-close } \implies \text{nearly 1-meeting}\]

The next corollary is our amplification of the result from [1] that $r \geq \min\{u, d\}$.

**Corollary 4.5.** If $\mathcal{X}$ is unsplit, then either there is a finite-to-one monotone $f : \omega \to \omega$ such that $f(\mathcal{X})$ is an ultrafilter base modulo finite sets, or the family $\{\text{next}(X, -) : X \in \mathcal{X}\}$ is 3-dominating.

Proof. If $\mathcal{X}$ is nearly 3-close, then we have the first alternative in the conclusion, by Corollary 4.4. If $\mathcal{X}$ is not nearly 3-close, then we have the second alternative by Proposition 2.6. \qed

5. Consistency Results for Unsplit Families

In this section we consider the question of reversing the arrows in the shorter chain (2) for unsplit families. Our old counterexample, $[\omega]^{\omega}$, to the reversal of the last arrow still works in the present situation, as $[\omega]^{\omega}$ is obviously unsplit. Under the assumption $u < g$, the two implications other than the last are reversible by Theorem 3.1 even without the “unsplit” assumption. The question that remains is whether these two implications are reversible in ZFC. We shall show that they are not; in particular the reversals fail under CH.

**Proposition 5.1.** Any 2-meeting family can be enlarged to a 2-meeting unsplit family.

Proof. Given a 2-meeting family, use Zorn’s lemma to enlarge it to a maximal 2-meeting family $\mathcal{X}$. We show that $\mathcal{X}$ is unsplit. So let any potential splitting set
$S \subseteq \omega$ be given. If $S \in \mathcal{X}$ then $S$ doesn’t split $\mathcal{X}$ because it doesn’t split itself. So we may assume that $S \notin \mathcal{X}$ and therefore, by maximality, $\mathcal{X} \cup \{S\}$ is not 2-meeting. This means, since $\mathcal{X}$ is 2-meeting, that some $X \in \mathcal{X}$ is almost disjoint from $S$. But then $S$ again fails to split $\mathcal{X}$.

**Corollary 5.2.** If $\tau \geq \theta$ then there is an unsplit, 2-meeting family that is not nearly 3-close.

**Proof.** By Theorem 3.2, there is a 2-meeting family that is not nearly 3-close. By Proposition 5.1, we can enlarge this family to become unsplit, while keeping it 2-meeting. Enlargement obviously preserves the property of not being nearly 3-close, so the proof is complete.

**Corollary 5.3.** Whether “3-dominating” in Corollary 4.5 can be improved to “2-dominating” is independent of ZFC. The improvement is correct if $\mu < \mathfrak{g}$ but incorrect if $\tau \geq \mathfrak{d}$.

**Theorem 5.4.** If $\tau = \mathfrak{c}$ then there is an unsplit, 2-close family that is not nearly 2-meeting.

**Proof.** We use the proof of Theorem 3.5, for $k = 2$, with some extra work inserted into the inductive construction of the family $\mathcal{X}$ to ensure that it is unsplit. Let $\Pi_\alpha$ be as in that earlier proof. List all subsets of $\omega$ as $S_\alpha$, indexed by $\alpha < \mathfrak{c}$. As in the proof of Theorem 3.5, we construct $\mathcal{X}$ in an inductive process of length $\mathfrak{c}$, ensuring at stage $\alpha$ that (as before) $\Pi_\alpha$ does not witness that $\mathcal{X}$ is nearly 2-meeting, and also (the new work) that $S_\alpha$ does not split $\mathcal{X}$. As before, we shall add only finitely many sets to $\mathcal{X}$ at each stage, and we shall take care that $\mathcal{X}$ remains 2-close.

We now describe stage $\alpha$, omitting the subscripts $\alpha$ as before. Letting $\mathcal{Y}'$ be the part of $\mathcal{X}$ already constructed, known to be 2-close by induction hypothesis, begin by adding $A_0$ and $A_1$ exactly as before to ensure that $\Pi$ will not witness that $\mathcal{X}$ is 2-meeting. Let $\mathcal{Y}'$ be the resulting family of sets, still 2-close by construction. Next, we ensure that $S$ will not split $\mathcal{X}$ by putting either $S$ or its complement $\omega - S$ into $\mathcal{X}$. The choice of which set to put into $\mathcal{X}$ is dictated by the need to keep $\mathcal{X}$ 2-close (as in the earlier proof). So we must ensure that the newly added set, $S$ or $\omega - S$, and any set $Y$ previously added both meet infinitely many 2-point intervals. To complete the proof, we suppose that neither choice for the newly added set works, and we deduce a contradiction as follows.

Since adding $S$ doesn’t work, there is $Y_1 \in \mathcal{Y}'$ such that only finitely many 2-point intervals meet both $S$ and $Y_1$. Similarly, there is $Y_2 \in \mathcal{Y}'$ such that only finitely many 2-point intervals meet both $\omega - S$ and $Y_2$. Since $\mathcal{Y}'$ is 2-close, find infinitely many 2-point intervals $[n, n + 1]$ meeting both $Y_1$ and $Y_2$. If infinitely many of these $n$’s are in $S$, then those 2-point intervals $[n, n + 1]$ contradict our choice of $Y_1$. If not, then infinitely many of these $n$’s are in $\omega - S$ and we similarly contradict our choice of $Y_2$.

**References**


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Several proofs of PA-unprovability

Andrey Bovykin

ABSTRACT. In this article we give unprovability proofs of several combinatorial statements using the method of indiscernibles. Our first two statements are strong versions of KM (the Kanamori-McAloon principle) and KM(2), (the Kanamori-McAloon principle for pairs) that both allow unexpectedly simple proofs. We proceed to discuss results by A. Weiermann and G. Lee on $I\Sigma_k$-provability/unprovability of the principles $PH_{\log(n)}^{k+1}$ and $KM_{\log(n)}^{k+1}$ for different $n \in \mathbb{N}$. We contrast a result by G. Lee on provability of $KM_{\log(n)}^{k+1}$ for $n \geq k$ with the following result. If the condition “for all $x$, $y$ in $H$, $x < y$ implies $2^x < y$” is imposed on the min-homogeneous set $H$ then the modified statement $KM_{\log(n)}^{k+1}$ becomes $I\Sigma_k$-unprovable for all $n, k \in \mathbb{N}$. For $k \geq 2$, the restriction $|H| > 2^c$, where $c$ is the second element of $H$ also makes the modified $KM_{\log(n)}^{k+1}$ unprovable in $I\Sigma_k$ for all $n \in \mathbb{N}$.

The article is directed at a broad audience and is intended to be suitable for expository purposes.

Here, we present several model-theoretic proofs of unprovability (in Peano Arithmetic, PA, and in its fragments $I\Sigma_k$, the theories of induction for formulas containing not more than $k$ quantifiers) using the method of indiscernibles. We give full proofs and intend this article to be suitable for expository purposes and accessible to mathematicians interested in “how is it possible to prove that a concrete statement about natural numbers is unprovable?”.

1. Introduction

Peano Arithmetic (PA) is the first-order theory in the language

$$L = \{+, \times, <, 0, 1\}$$

consisting of the following axioms: associativity and commutativity of $+$ and $\times$, the neutral elements are 0 and 1 respectively, distributivity, discrete linear order.

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axioms for $<$, 1 is the successor of 0, $x < y \rightarrow x + z < y + z$, and the induction scheme: for every $\mathcal{L}$-formula $\varphi(x, \bar{y})$, we have an axiom

\[ \forall \bar{y}[ \varphi(0, \bar{y}) \land \forall x(\varphi(x, \bar{y}) \rightarrow \varphi(x + 1, \bar{y})) \rightarrow \forall x\varphi(x, \bar{y})]. \]

For $n \in \mathbb{N}$, define a $\Sigma_n$-formula as an $\mathcal{L}$-formula of the form

\[ \exists x_1 \forall x_2 \exists x_3 \ldots \varphi(x, x_1, x_2, \ldots, x_n, \bar{y}), \]

where $\varphi$ does not contain unbounded quantifiers, i.e., a $\Sigma_n$-formula is a formula (possibly with free variables) that starts with an existential quantifier and contains not more than $n$ quantifiers altogether.

If we restrict the induction scheme to $\Sigma_n$-formulas then the theory obtained is denoted by $I\Sigma_n$, “induction for $\Sigma_n$-formulas”. Clearly, for every $n \in \mathbb{N}$, $I\Sigma_n \subseteq I\Sigma_{n+1}$ and $\text{PA} = \bigcup_{n=1}^{\infty} I\Sigma_n$. Also, it is known and easily provable nowadays that for every $n \in \mathbb{N}$, $I\Sigma_n \neq I\Sigma_{n+1}$.

An alternative axiomatization of $I\Sigma_n$ uses the instance of the least-number principle

\[ \forall \bar{y}[ \exists x\varphi(x, \bar{y}) \rightarrow \exists z(\varphi(z, \bar{y}) \land \forall w < z \neg \varphi(w, \bar{y})] \]

for every $\Sigma_n$-formula $\varphi(x, \bar{y})$.

It is common to identify theorems of PA with “finite mathematics”, that is the world of mathematical theorems which can be formulated in $\mathcal{L}$ and whose proof does not require the use of any notion of “infinite set” in an essential way. As typical arbitrary old examples of such “theorems of finite mathematics” I would quote:

1. if $\pi(x)$ is the number of primes less than $x$ then $\pi(x) \sim \frac{x}{\ln x}$, $x \to \infty$
   (in order to define the notion of an integral in the language of arithmetic we should only use Darboux partitions with rational ends; the notion of a limit is arithmetized easily, also, all necessary instances of theorems of complex analysis can be conducted in Peano Arithmetic);

2. for every concrete continuous function $f : [a, b) \to \mathbb{R}$ definable in the language of arithmetic (this includes all functions working mathematicians encounter): \[ \frac{d}{dx} \left( \int_a^x f(t)dt \right) = f(x); \]

3. for every concrete $n \times n$ matrix $A(t)$ continuous on $[a, b)$, if $\Phi(t)$ is a matrix whose columns are solutions of $\frac{d}{dt} = A(t)x$ then $W(t) = \text{det } \Phi(t)$ can be calculated as $W(t) = W(a) \cdot \exp(\int_a^t \text{Tr}A(s)ds)$.

It is then an easy excercise to check that the usual proofs of the above theorems can be conducted in Peano Arithmetic.

However, it is widely believed that all $\mathcal{L}$-theorems of existing mathematics (apart from logicians’ discoveries we shall be talking about below) can be proved even in $I\Sigma_2$. It would be a big surprise and an interesting result if someone managed to find an existing theorem in mathematics that can be formulated in $\mathcal{L}$ but does not have a proof formalizable in $I\Sigma_2$.

For a very long time it was believed that PA comprises an axiomatization of the set of all ‘truths’ about natural numbers and finite sets until in 1931 K. Gõdel proved that for every consistent recursive theory $T$ containing PA, there is an $\mathcal{L}$-formula $\varphi$ such that neither $\varphi$ nor $\neg \varphi$ can be derived in $T$. An example of a sentence that
is neither provable nor refutable in $T$ is $\text{Con}_T$, the arithmetical formula expressing consistency of $T$ ("for every natural number $n$, $n$ is not a code of a sequence of formulas ending with $\exists x \ x \neq x$ and such that every formula in the sequence is either an axiom of $T$ or is obtained from the previous ones by a rule of inference of the predicate calculus").

The first PA-unprovable statements of ‘mathematical’ character (i.e., not referring to arithmetization of syntax and provability) appeared in 1976 in the work of J.Paris (building upon joint work with L.Kirby [19]) and led to formulation of the Paris-Harrington Principle [33]:

$$\text{PH} \iff \forall m n c \exists N \left[ \text{for every } f : [N]^m \to c, \text{ there is an } f\text{-homogeneous } H \subseteq N, |H| \geq n \text{ such that } |H| > \min H \right].$$

PH is not provable in Peano Arithmetic. Moreover, for every $k \in \mathbb{N}$, the statement $\text{PH}^{(k+1)}$ defined as

$$\text{PH}^{(k+1)} \iff \forall n c \exists N \left[ \text{for every } f : [N]^{k+1} \to c, \text{ there is an } f\text{-homogeneous } H \subseteq N, |H| \geq n \text{ such that } |H| > \min H \right]$$

is $I\Sigma_k$-unprovable [35] and is equivalent to $\text{RFN}_{\Sigma_1} (I\Sigma_k)$, the 1-consistency of $I\Sigma_k$:

$$\forall \varphi \in \Sigma_1 \left( \text{Pr}_{I\Sigma_k} (\varphi) \to \varphi \right).$$

It says "for all $\Sigma_1$-statements $\varphi$, if $I\Sigma_k$ proves $\varphi$ then $\varphi$ holds". Unprovability of $\text{RFN}_{\Sigma_1} (I\Sigma_k)$ in $I\Sigma_k$ easily follows from Gödel’s Theorem: put $\varphi$ to be $\exists x \ x \neq x$ to observe that $\text{RFN}_{\Sigma_1} (I\Sigma_k)$ implies $\text{Con}_{I\Sigma_k}$. Many statements equivalent to PH have been studied since: the Hercules-Hydra battle and termination of Goodstein sequences by L.Kirby and J.Paris [36], the flipping principle of L.Kirby [20], the arboreal statement by G.Mills [31], Pudlák’s Principle [12], the kiralic and regal principles by P.Clote and K.McAloon [6].

An important PA-unprovable statement was introduced in [16] by A.Kanamori and K.McAloon. A function $f$ in $m$ arguments is called regressive if

$$f(x_0, x_1, \ldots, x_{m-1}) < x_0 \text{ for all } x_0 < x_1 < \cdots < x_{m-1}.$$

For regressive functions of $m$ arguments, we cannot guarantee existence of a homogeneous set of size $(m+1)$, e.g., for $f(x_0, x_1, \ldots, x_{m-1}) = x_0 - 1$, every set of size $(m+1)$ is not homogeneous. However, we can talk about min-homogeneous sets: a set $H$ is called min-homogeneous if for all $c_0 < c_1 < \cdots < c_{m-1}$ and $c_0 < d_1 < \cdots < d_{m-1}$ in $H$, $f(c_0, c_1, \ldots, c_{m-1}) = f(c_0, d_1, \ldots, d_{m-1})$. Now, KM is defined as:

$$\text{KM} \iff \forall m a n b \exists H \left[ \text{for every regressive function } f \text{ defined on } [a, b]^m, \text{ there is a min-homogeneous set } H \subseteq [a, b] \text{ of size at least } n \right].$$

The statement KM is unprovable in PA and is equivalent to PH. Also, KM$^{(k)}$, the version of KM restricted to $k$-tuples, is equivalent to PH$^{(k)}$.

The historical prototypes of the Paris-Harrington Principle and the earlier PA-unprovable statements [34] are large cardinal axioms (for an early discussion of this connection, see [18]). In the case of arithmetic, the closedness properties postulated by large cardinal axioms correspond to closedness properties of initial segments of models of arithmetic. So far, the idea to look closely at arithmetical versions of different large cardinal axioms (and to go beyond J.Paris’ semi-regularity, regularity,
strength, extendibility and Ramseyness of initial segments \([19], [35]\)) has not been really explored. We believe that it will eventually be very fruitful.

Very often an unprovable statement can be viewed as a ‘miniaturization’ of an infinitary theorem. A spectacular example of miniaturization is H.Friedman’s Theorem \([42]\) on unprovability of a finite version of the following theorem by J.Kruskal. Define a tree as a partially ordered set with the least element and such that the set of all predecessors of every point is linearly ordered. Then if \(\{T_i\}_{i \in \mathbb{N}}\) is a countable sequence of finite trees then there are \(i < j \in \mathbb{N}\) such that \(T_i \preceq T_j\), i.e., there is an inf-preserving embedding from \(T_i\) into \(T_j\). Friedman’s Theorem says that

\[
\forall k \exists N \left[ \begin{array}{l}
\text{if } \{T_i\}_{i=1}^N \text{ is a sequence of finite trees such that for all } i \leq N \text{ we have } |T_i| \leq k + i \text{ then } \\
\text{there are } i, j \leq N \text{ such that } i < j \text{ and } T_i \preceq T_j 
\end{array} \right]
\]

is not provable in \(\text{ATR}_0\), a theory stronger than Peano Arithmetic. It was later shown by M.Loebl and J.Matoušek \([26]\) that if the condition \(|T_i| \leq k + i\) is replaced by \(|T_i| \leq k + \frac{1}{2} \log i\) then the statement becomes \(I\Sigma_1\)-provable but for the condition \(|T_i| \leq k + 4 \log i\) the statement is \(\text{PA}\)-unprovable. What happens between \(\frac{1}{2}\) and 4 was recently resolved by A.Weiermann \([46]\). Let \(\alpha\) be the Otter’s constant (the radius of convergence of \(\sum_{i=0}^{\infty} t_i z^i\), where \(t_i\) is the number of finite trees of size \(i\)), \(\alpha \approx 2.955765\ldots\). Then for any primitive recursive real number \(r\),

1. if \(r \leq \frac{1}{\log \alpha}\) then the statement with the condition \(|T_i| \leq k + r \log i\) is \(I\Sigma_1\)-provable;
2. if \(r > \frac{1}{\log \alpha}\) then it is \(\text{PA}\)-unprovable.

Another example is a theorem by H.Friedman, N.Robertson and P.Seymour on unprovability of the Graph Minor Theorem \([10]\) (for graphs \(G\) and \(H\), we say that \(H\) is a minor of \(G\) if \(H\) is obtained from \(G\) by a succession of three elementary operations: edge removal, edge contraction and removal of an isolated vertex):

\[
\forall k \exists N \left[ \begin{array}{l}
\text{if } \{G_i\}_{i=1}^N \text{ is a sequence of finite graphs such that for all } i \leq N \text{ we have } |G_i| \leq k + i \\
\text{then for some } i < j \leq N, G_i \text{ is a minor of } G_j 
\end{array} \right]
\]

is not provable in \(\Pi^1_1\)-\(\text{CA}_0\), a very strong subsystem of the second-order arithmetic.

Many other examples of unprovable statements can be found in the Contemporary Mathematics volume 65 \([43]\) devoted entirely to arithmetical unprovability results. Among the developments that escaped this volume I would like to mention K.McAloon’s \([29], [30]\) and Z.Ratajczyk’s \([38],[39]\) theories of iterations of the Paris-Harrington Principle, Cichon’s treatment of Goodstein sequences \([5]\), the treatment of \(\text{PH}\) by S.Kripke and S.Kochen \([22]\) using ultraproducts, the Friedman-McAloon-Simpson early fundamental article \([8]\) (a version of Galvin-Prikry partition theorem unprovable in \(\text{ATR}_0\)) and the subsequent article by S.Shelah \([44]\), the Buchholz Hydra \([3]\) and Friedman’s early results on combinatorial statements unprovable in \(\text{ZFC+ large cardinals}\) \([9], [11]\) (see also the article \([40]\) by J.-P. Ressayre). Also, there is a whole range of recent results achieved by H.Friedman.

But of course this list is far from complete. The purpose of this introduction was to show some landmarks and to give an adequate impression of what a good theorem in this subject should look like.

Each independence proof of an arithmetical statement so far falls into one of the two categories:
(1) model-theoretic constructions showing how, assuming the statement, a model of a given theory can be built directly, "by hands";

(2) combinatorial proofs springing from the Ketonen-Solovay article [17] (showing combinatorially that the function arising from our statement eventually dominates every function of the Grzegorczyk-Wainer hierarchy (since all PA-provably recursive functions occur in this hierarchy, the result follows) or from the study of well-quasi-ordered sets [42]. Most of the proofs in [43] are of this category, as well as the articles [27] and [28] by M.Loebl and J. Nešetřil.

Apart from the original articles we mentioned above, other good sources reporting on proofs of category (1) would be the book [13] by P.Hájek and P.Pudlák and the papers [1], [2] by J.Avigad and R.Sommers (proof-theoretic aspects). A recent manuscript [23] by H.Kotlarski is an exposition of both approaches as well as of many different proofs of Gödel’s Theorem.

Apparently, there is also a third category of unprovability proofs, proofs that interpret directly independence statements as reduction strategies for proof systems. We have little to say about it and refer the reader to a recent article by L.Carlucci [4] and the articles [14] and [15] by M.Hamano and M.Okada.

The proofs in this article are typical of category (1). We aimed to formulate simple statements whose proofs would be very short and rid of unnecessary combinatorial manipulations. The combinatorial statements discussed in this article can be shown to be provable in the second-order arithmetic, by an easy application of the Infinite Ramsey Theorem and we omit these proofs.

Also, as it often happens in this subject, all our statements imply 1-consistency of the theory we prove them to be independent of. We conjecture but do not prove that they are equivalent to its 1-consistency. So, apart from our model-theoretic proofs, direct combinatorial arguments are expected to be eventually found, showing our statements independent of $I\Sigma_k$ to be equivalent to $PH^{(k+1)}$ or $KM^{(k+1)}$ and the statements independent of PA to be equivalent to PH or KM.

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2. Monochromatic paths in oriented graphs

Let $G_{ab}$ be the following oriented graph\footnote{The words ‘directed graph’ (‘digraph’) and ‘edges’ are often used instead of ‘oriented graph’ and ‘arrows’.}: the vertices of $G_{ab}$ are the natural numbers $a, a+1, \ldots, b$; there is an arrow from $x$ to $y$ if and only if $x < y$. The set of all arrows originating in $x$ will be denoted by $E_x$. In $G_{ab}$, $|E_x| = b - x$.

Let $G$ be an oriented graph and suppose a colouring $\chi$ of arrows of $G$ is fixed. A path through $G$ (a way along arrows) is called monochromatic with respect to $\chi$ if any two arrows originating in the same vertex of the path and finishing somewhere on the path (i.e., any two chords with the same origin) have the same colour.

**Theorem 1.**

Let $A(a, b, n)$ be the following formula: “for any colouring $\chi$ of arrows of $G_{ab}$ such that $|\chi(E_x)| \leq 2^{nx}$, there is a monochromatic path of length $n$”. Define

$$MPG \iff \forall a \exists b A(a, b, n).$$
Then $I\Sigma_1 \not\vdash \text{MPG}$. Moreover, for every $a \in \mathbb{N}$, the function $F(a,n) = F_a(n) = \min c \ A(a,a+c,n)$ eventually dominates every primitive recursive function.

In the proof we shall use the fact that primitive recursive functions are provably recursive in $I\Sigma_1$. The converse (every $I\Sigma_1$-provably recursive function is primitive recursive) is also true and is a nontrivial theorem proved independently by G. Mints [32] and C. Parsons [37] (also G. Takeuti [45]). By $I\Delta_0$ we denote the theory $I\Sigma_0$ (as defined above), the theory of induction for formulas without unbounded quantifiers. The expression $x = \langle x_1, \ldots, x_n \rangle$ means that $x$ is the code of the sequence $x_1, \ldots, x_n$, using some standard coding function. We shall build a model of $I\Sigma_1 + \neg \text{MPG}$, thus showing that MPG cannot be a theorem of $I\Sigma_1$.

**Proof.**

Let $M \models I\Sigma_1$ be nonstandard, $a, b, n \in M$, $\mathbb{N} < n$. Assume that $M \models A(a,b,n) \land \forall y < b \neg A(a,y,n)$. Let $\psi_1(u,v), \ldots, \psi_n(u,v)$ be the first $n$ $\Delta_0$-formulas in the two free variables shown. Introduce the coloring: for the arrow $xy$, set

$$\chi(xy) = \begin{cases} \{p < x \mid \psi_1(y,p)\}, & \\
\vdots & \\
\{p < x \mid \psi_n(y,p)\}. & 
\end{cases}$$

Since for every $x \in [a, b]$, $|\chi(E_x)| \leq 2^{n^2}$, we choose in $G_{ab}$ a monochromatic path $C = \{c_i\}_{i=1}^n$. Notice that for any $c_{i_1} < c_{i_2} < c_{i_3}$ in $C$, the fact that $c_{i_1}c_{i_2}$ and $c_{i_1}c_{i_3}$ have the same colour implies that for any $\Delta_0$-formula $\psi(u,v)$ and any $p < c_{i_1}$,

$$M \models \psi(c_{i_2},p) \leftrightarrow \psi(c_{i_3},p). \quad (\ast)$$

Define the initial segment $I = \sup_{i \in \mathbb{N}} c_i$. It can easily be checked from (\ast) that $I$ is closed under addition and multiplication: for any $c_{i_1} < c_{i_2}$ in $C$, if for some $p < c_{i_1}$ we have $\left[\frac{c_{i_2}}{2}\right] = p$ then, by (\ast), $\left[\frac{c_{i_2}}{p}\right] = p$ for all $i > i_1$. Hence all $c_i$ for $i > i_1$ would coincide or differ by 1 which is not the case. Hence $2p < c_{i_2}$ and $I$ is closed under addition. For multiplication, the proof is similar: if there was $p < c_{i_1}$ such that for all $i > i_1$, $\left[\frac{c_{i_2}}{p}\right] = p$ then there would be at most $2p + 1$ such elements of $C$, namely $p^2, p^2 + 1, \ldots, p^2 + 2p$. But this set can not accommodate infinitely-many elements of $C$ since we already proved that the distance between them is greater than $p$.

In order to show that $I \models I\Sigma_1$, we check the least-element principle for an arbitrary $\Sigma_1$-formula. Consider a $\Sigma_1$-formula $\exists y \varphi(x,y,p)$ such that $I \models \exists y \varphi(e,y,p)$ for $(e,p) < c_{i_1} \in I$. Now, considering the formula $\psi(u,v)$ defined as $\psi(u, (x,p)) \leftrightarrow \exists y < u \varphi(x,y,p)$, and applying (\ast) to it, we observe that

$$I \models \exists y \varphi(e,y,p) \leftrightarrow M \models \exists y < c_{i_1+1} \varphi(e,y,p). \quad (***)$$

Now, since $M \models I\Delta_0$, there is $x^* < c_{i_1}$ such that

$$M \models \exists y < c_{i_1+1} \varphi(x^*,y,p) \land \forall x < x^* \forall y < c_{i_1+1} \neg \varphi(x,y,p).$$

Hence, by (**), $I \models \exists y \varphi(x^*,y,p) \land \forall x < x^* \forall y \neg \varphi(x,y,p)$ which means that $I \models I\Sigma_1$. Now, if it was the case that $I\Sigma_1 \models \forall a \exists y A(a,y,n)$ then there would exist $b' \in I$ such that $M \models A(a,b',n)$ contradicting the choice of $b$ as the minimal one. Hence $I\Sigma_1 \not\vdash \forall a \exists y A(a,y,n)$.

Let $a \in \mathbb{N}$. To show that $F_a(n)$ eventually dominates every primitive recursive function, consider an arbitrary $\Delta_0$-formula $\varphi(x,y)$ such that $I\Sigma_1 \models \forall x \exists y \varphi(x,y)$. Let $B(n) = \exists m > n \exists z \ (A(a,z,m) \land \forall y < z \neg \varphi(m,y))$. We have just shown that for every nonstandard $n \in M$, $M \models \neg B(n)$. Hence there is $n^* \in \mathbb{N}$ such
that $M \vDash \neg B(n^*)$ (otherwise it would contradict the induction instance for the $\Sigma_1$-formula $B(n)$) which means that for all natural numbers $m > n^*$, $\mathbb{N} \vDash F_\varphi(m) > \min y$$\varphi(m, y)$.

A nice feature of all proofs of this kind is that they do not use Gödel’s Theorem: we constructed a model $I \vDash I\Sigma_1 + \neg \text{MPG}$ by hands.

Note that in order to establish $I\Sigma_1$-unprovability of our statement we only needed the ground model $M$ to satisfy $I\Delta_0 + \exp$ (i.e., $I\Delta_0$ plus the axiom that exponentiation is a total function). However, in order to show that the function is fast-growing we needed $M$ to satisfy the full theory $I\Sigma_1$.

Notice also that MPG easily implies $\text{KM}^{(2)}$ (as well as $\text{PH}^{(2)}$ and 1-consistency of $I\Sigma_1$).

A usual non-model-theoretic argument showing that a certain function $F$ dominates all primitive recursive functions would be as follows. For every $n, k \in \mathbb{N}$, let $A_1(n) = n + 1$, $A_{k+1}(n) = A_k^{(n)}(n)$. Consider the Ackermann function $A(n) = A_n(n)$. It is well-known that $A(n)$ dominates all primitive recursive functions. If a lower bound is found for $F$ in terms of compositions of the Ackermann function with unbounded primitive recursive functions then $F$ dominates all primitive recursive functions. Such is the elementary combinatorial proof by M.Kojman and S.Shelah [21] of the fact that the function associated with $\text{KM}^{(2)}$ dominates all primitive recursive functions (and, hence, $\text{KM}^{(2)}$ is $I\Sigma_1$-unprovable).

3. Functions enumerating finite families of sets

Here, we present a statement independent of Peano Arithmetic such that the model-theoretic proof of its PA-unprovability is very simple and appeals directly to the Ramsey reason of unprovability (truth values of formulas as colours).

Let $P(k)$ denote the set of all subsets of $\{0, 1, \ldots, k - 1\}$. Let $X \Rightarrow (n)^{k+1}$ be a shorthand for the formula expressing: if for every $x_0 < x_1 < \cdots < x_k$ in $X$ we fix a function $f_{x_0x_1\ldots x_k} : n \to P(x_0)$ then there is $H \subset X$ of size $n$ such that for all $x_0 < x_1 < \cdots < x_k$ and $x_0 < y_1 < \cdots < y_k$ in $H$, $f_{x_0x_1\ldots x_k} \equiv f_{x_0y_1\ldots y_k}$.

**Theorem 2.** For every $k \in \mathbb{N}$, $I\Sigma_k \not\vDash \forall n\exists b \ [a, b] \Rightarrow (n)^{k+1}$.

In particular, PA does not prove $\forall m\exists b \ [a, b] \Rightarrow (n)^{m}$.

**Proof.**

Let $M \vDash I\Sigma_1$, $n, a, b \in M \setminus \mathbb{N}$, $n < a < b$ and $b = \min y[a, y] \Rightarrow (n)^{k+1}$. Let $\psi_1(x_0, x_1, \ldots, x_k), \ldots, \psi_n(x_0, x_1, \ldots, x_k)$ be the first $n$ $\Delta_0$-formulas in $(k + 1)$ free variables. For every $x_0 < x_1 < \cdots < x_k$ in $[a, b]$, let $f_{x_0x_1\ldots x_k} : n \to P(x_0)$ be defined as $f_{x_0x_1\ldots x_k}(i) = \{p < x_0 \mid M \vDash \psi_i(p, x_1, \ldots, x_k)\}$. Now, since $[a, b] \Rightarrow (n)^{k+1}$, there is $H = \{c_0, c_1, \ldots, c_{n-1}\}$ such that for all $i_0 < i_1 < \cdots < i_k$ and $i_0 < j_1 < \cdots < j_k$, we have $f_{ci_0c_1\ldots c_k} \equiv f_{ci_0j_1\ldots j_k}$, i.e., for all $i < n$ and all $p < c_{i_0}$,

$M \vDash \psi_i(p, c_{i_1}, \ldots, c_{i_k}) \iff \psi_i(p, c_{j_1}, \ldots, c_{j_k})$.

Define $I = \sup\{c_i \mid i \in \mathbb{N}\}$. We can show that $I$ is closed under addition and multiplication exactly as we did in the previous section. Let us show that $I \vDash I\Sigma_k$. Consider a formula $\exists x_1 \forall x_2 \ldots \psi(x, p, x_1, x_2, \ldots, x_k)$ such that $I \vDash \exists x_1 \forall x_2 \ldots \psi(e, p, x_1, x_2, \ldots, x_k)$ for $(e, p) < c_i \in I$. Again, it is easy to see that the following holds:

$I \vDash \exists x_1 \forall x_2 \ldots \varphi(e, p, x_1, \ldots, x_k)$ if and only if
\[ M \models \exists x_1 < c_{i_1+1} \forall x_2 < c_{i_1+2} \ldots \varphi(e, p, x_1, \ldots, x_k). \quad (**) \]

Let \( x^* \in M \) be such that \( M \models \left( x^* \text{ is the minimal } x < c_{i_1} \text{ such that } \exists x_1 < c_{i_1+1} \forall x_2 < c_{i_1+2} \ldots \varphi(x, p, x_1, \ldots, x_k) \right) \).

Then, by (**) \( I \models x^* \) is the minimal \( x \) such that \( \exists x_1 \forall x_2 \ldots \varphi(x, p, x_1, \ldots, x_k) \).

Hence \( I \models I \Sigma_k \). However, as \( M \models [b = \min y[a, y] \Rightarrow (n)^{k+1}] \) and the relation \( x \Rightarrow (z)^w \text{ is } \Delta_0 \),

\[ I \models I \Sigma_k + \forall y[a, y] \not\Rightarrow (n)^{k+1}. \]

Thus \( I \Sigma_k \not\models \forall \exists b \, b \models [a, b] \Rightarrow (n)^{k+1} \).

It is easily provable that \( \forall \exists b \, [a, b] \Rightarrow (n)^k \) implies \( \text{KM}^{(k)} \). For that, notice that every regressive function satisfies the condition \( f(x_0, x_1, \ldots, x_{k-1}) < 2^{k \cdot x_0} \). Of course, a simple argument from [33] also goes through to show straight away that \( \forall \exists b \, [a, b] \Rightarrow (n)^{k+1} \) implies \( \text{RFN}_{\Sigma_1} (I \Sigma_k) \). We conjecture that \( \forall \exists b \, [a, b] \Rightarrow (n)^{k+1} \) is equivalent to \( \text{RFN}_{\Sigma_1} (I \Sigma_k) \).

We have to mention that the 2
\[ n \] -trick used in this and the previous section appeared in the article [16] as well as (implicitly) in the original article [33] and in other writings on the subject. We incorporate it into our combinatorial principles and suggest that this becomes a short easy way to teach model-theoretic unprovability proofs.

### 4. The story of \( \text{PH}^{(k)}_{\text{log}(n)} \) and \( \text{KM}^{(k)}_{\text{log}(n)} \)

Let \( \log^{(n)}(x) = \log(\log \ldots \log(x)) \), Let \( 2_n(x) \) be defined as \( 2_0(x) = x \), \( 2_{n+1}(x) = 2^{2^\ldots^2(x)} \) \( n \) times. Also, define \( \log^*(m) \) as the minimal \( n \) such that \( 2_n(2) \geq m \). In the notation for the Ramsey number \( R(m, k, c) \), \( m \) is the size of a homogeneous set, \( k \) is the length of tuples and \( c \) is the number of colours.

For every function \( F(x) \), define

\[ \text{PH}^{(k)}_F \leftrightarrow \forall n \exists N \left[ \begin{array}{c} \text{for every } f : [N]^k \to c, \text{there is } \\ \text{a homogeneous } H \subseteq N, |H| \geq n, F(\min H) < |H| \end{array} \right]. \]

We say that \( f \) is \( F \)-regressive if for all \( x_0 < x_1 < \ldots < x_{k-1}, \) we have

\[ f(x_0, x_1, \ldots, x_{k-1}) < F(x_0). \]

Now define

\[ \text{KM}^{(k)}_F \leftrightarrow \forall n \exists N \left[ \begin{array}{c} \text{for every } F \text{-regressive } f \text{ defined on } [N]^k \text{ there is } \\ \text{a min-homogeneous subset of } N \text{ of size at least } n \end{array} \right]. \]

Also, define \( \text{PH}_F \leftrightarrow \forall k \text{ } \text{PH}^{(k)}_F \) and \( \text{KM}_F \leftrightarrow \forall k \text{ } \text{KM}^{(k)}_F \). It is easy to see that for every strictly increasing \( F \), \( \text{PH}_F \rightarrow \text{PH} \) and \( \text{KM}_F \rightarrow \text{KM} \) thus making these statements PA-unprovable. A. Weiermann proved [47] that for every \( n \in \mathbb{N}, \text{PH}^{(n)}_{\text{log}(n)} \) is PA-unprovable but \( \text{PH}^{(n)}_{\text{log}^*} \) is PA-provable.

The following interesting result has been proved recently by Gyesik Lee [24]:

1. if \( n < k - 1 \) then \( \text{KM}^{(k+1)}_{\text{log}(n)} \) is \( I \Sigma_k \)-unprovable;
2. if \( n > k - 1 \) then \( I \Sigma_1 \) proves \( \text{KM}^{(k+1)}_{\text{log}(n)} \).

The case \( n = k - 1 \) is at the moment an open problem. Similar theorems hold for the family \( \text{PH}^{(k)}_{\text{log}(n)} \):
(1) if $n < k$ then $I\Sigma_k$ does not prove $\text{PH}_{\log(n)}^{(k+1)}$ (A. Weiermann [47]);

(2) if $n > k$ then $I\Sigma_1$ proves $\text{PH}_{\log(n)}^{(k+1)}$ (G. Lee [24]).

The case $n = k$ is currently an open problem. However, rather complete solutions in the case $n = k = 1$ have been recently obtained by A. Weiermann and G. Lee [25]: if $A^{-1}$ is the inverse of the Ackermann function and $\{F_m\}_{m \in \mathbb{N}}$ is the Grzegorczyk hierarchy of primitive recursive functions then:

1. $I\Sigma_1 \not\vdash \text{PH}_{A^{-1}}^{(2)}$;
2. for every $m \in \mathbb{N}$, $I\Sigma_1 \vdash \text{PH}_{F^{-1}_m}^{(2)}$;
3. $I\Sigma_1 \not\vdash \text{KM}_{f(x) = x^{A^{-1}(x)}}^{(2)}$;
4. for every $m \in \mathbb{N}$, $I\Sigma_1 \vdash \text{KM}_{f_m}^{(2)}$, where $f_m(x) = x^{F^{-1}_m(x)}$.

In particular, $\text{KM}_{\log(n)}^{(2)}$ is provable but $\text{PH}_{\log(n)}^{(2)}$ is unprovable.

The reason for $I\Sigma_1$-provability of $\text{PH}_{\log(n)}^{(k+1)}$ and $\text{KM}_{\log(n)}^{(k+1)}$ for large $n$ comes from the Erdős-Rado theorem [7], which implies that an upper bound for the Ramsey number $R(m, k + 1, m)$ is $2_n(m)$ for some large enough $n$ depending only on $k$. Given $m$ and $c$, let $\ell = \max\{m, c\}$. Consider any coloring $f : [0, 2_n(\ell)]^{k+1} \rightarrow c$. By Ramsey Theorem, there is $H \subseteq [0, 2_n(\ell)]$ of size at least $\ell$ which is $f$-homogeneous. Also,

$$\log(n)(\min H) < \log(n)(2_n(\ell)) \leq |H|.$$ 

Thus $I\Sigma_1 \vdash \text{PH}_{\log(n)}^{(k+1)}$.

A similar argument for $I\Sigma_1$-provability of $\text{KM}_{\log(n)}^{(k+1)}$ for large $n$ goes as follows: consider $n$ such that $R(m, k + 1, m) < 2_n(m)$ for all $m$. Let $f$ be a $\log(n)$-regressive function defined on $[0, 2_n(m)]$. Then the image of $f$ is contained in $[0, m]$. Hence there is a homogeneous (thus also min-homogeneous) subset $H \subseteq [0, 2_n(m)]$ of size at least $m$.

To determine the smallest $n$ which makes these principles $I\Sigma_k$-provable is a nontrivial open problem.

The unprovability results of A. Weiermann and G. Lee are obtained entirely by combinatorial means refering to the fast-growing hierarchy of recursive functions. At the moment no obvious connection can be seen between the direct combinatorial approach and the model-theoretic approach to unprovability proofs.

5. Spreading the indiscernibles

In order to return $\text{KM}_{\log(n)}^{(k)}$ to the realm of unprovability, we add a new condition “for all $x, y \in H$, $x < y \rightarrow 2^x < y$" on the min-homogeneous set and make the modified $\text{KM}_{\log(n)}^{(k+1)}$, $I\Sigma_k$-unprovable for all $n \in \mathbb{N}$. Of course, an easy application of the Infinite Ramsey Theorem shows our statements to be provable in second-order arithmetic. For a function $F$, we define

$$A_F^k \leftrightarrow \forall an \exists b \left[\begin{array}{l}
\text{for every } F\text{-regressive } f \text{ defined on } [a, b]^k \\
\text{there is a min-homogeneous subset } H \\
of [a, b] \text{ of size at least } n \text{ such that} \\
\text{for all } x, y \in H \ (x < y \rightarrow 2^x < y)
\end{array}\right].$$
THEOREM 3. For every \( k, n \in \mathbb{N} \), \( I\Sigma_k \nvdash A_{\log(n)}^{k+1} \).

In the proof, we shall use that for all \( x > 2^e \), \( \log x < \frac{x}{e} \).

PROOF.
For the rest of the proof we fix \( k, n \in \mathbb{N} \). Let \( M \models I\Sigma_1, d > e > \mathbb{N}, a \geq 2^{n+2}(e) \) and let \( \psi_1(x_0, x_1, \ldots, x_k), \ldots, \psi_e(x_0, x_1, \ldots, x_k) \) be the first \( e \Delta_0 \)-formulas in \((k+1)\) free variables. Let \( b > a \) be minimal such that for any \( f: [a, b]^{k+1} \rightarrow [a, b] \) such that \( f(x_0, x_1, \ldots, x_k) < \log(n)x_0 \), there is a min-homogeneous set \( H \) such that \(|H| \geq d\) and for all \( x, y \in H \), \( x < y \) implies \( 2^e < y \). Define

\[
(f(x_0, x_1, \ldots, x_k) = \left\{ \begin{array}{ll}
p < \log^{(n+2)}x_0 & | \psi_1(p, x_1, \ldots, x_k) \\
\vdots & \\
p < \log^{(n+2)}x_0 & | \psi_e(p, x_1, \ldots, x_k) \end{array} \right. \right. .
\]

Notice that, since \( \log^{(n+1)}x_0 > 2^e \), we have \( f(x_0, x_1, \ldots, x_k) < 2^{e: \log^{(n+2)}x_0} < 2^{e: \frac{1}{e} \cdot \log^{(n+1)}x_0} = \log(n)x_0 \). By the choice of \( b \), we obtain a set \( H \subseteq [a, b], H = \{c_i\}_{i=1}^{\lfloor H \rfloor} \), satisfying the following condition \((*)_{n+2}: |H| \geq d\), and for any \( c_0 < c_1 < \cdots < c_k, c_0 < d_1 < \cdots < d_k \) in \( H \), any \( p < \log^{(n+2)}c_0 \) and any \( \Delta_0 \)-formula \( \psi \) in \((k+1)\) free variables, we have

\[
2^{c_0} < c_1 \text{ and } M \models \psi(p, c_1, \ldots, c_k) \iff \psi(p, d_1, \ldots, d_k).
\]

Let us show that for \( I = \sup\{c_i \mid i \in \mathbb{N}, c_i \in H\} \), we have \( I \models I\Sigma_k \).

First, notice that for every \( m \in \mathbb{N} \), if \( x \in I \) then \( 2_m(x) \in I \): if \( x < c_i \) then \( 2_m(x) < c_{i+m} \).

Now, let \( \varphi(x, x_0, x_1, \ldots, x_k) \) be an arbitrary \( \Delta_0 \)-formula and for \( p \in I \) we have \( I \models \exists x_0 \exists x_1 \forall x_2 \cdots \varphi(x, p, x_1, \ldots, x_k) \), i.e., for some \( e \in I \),

\[
I \models \exists x_1 \forall x_2 \cdots \varphi(e, p, x_1, \ldots, x_k).
\]

Since for every \( m \), \( I \) is closed under \( 2_m(x) \), there is \( i \in \mathbb{N} \) such that \( \langle e, p \rangle < \log^{(n+2)}c_i \). Now, by \((*)_{n+2} \), for every \( x < \log^{(n+2)}c_i \), the following condition \((**) \) holds: \( I \models \exists x_1 \forall x_2 \cdots \varphi(x, p, x_1, \ldots, x_k) \) if and only if \( M \models \exists x_1 < c_{i+1} \forall x_2 < c_{i+2} \cdots \varphi(x, p, x_1, \ldots, x_k) \). Since \( M \models I\Delta_0 \), there is \( x^* < c_i \) such that

\[
M \models x^* \text{ is minimal such that } \exists x_1 < c_{i+1} \forall x_2 < c_{i+2} \cdots \varphi(x^*, p, x_1, \ldots, x_k).
\]

Then, by \((***) \),

\[
I \models x^* \text{ is minimal such that } \exists x_1 \forall x_2 \cdots \varphi(x^*, p, x_1, \ldots, x_k).
\]

This completes the proof. \( \square \)

6. Applying the pigeonhole principle and number theory to indiscernibles

Here, we introduce another condition that would make the modified \( K M^{(k+1)}_{\log(n)} I\Sigma_k \)-unprovable for all \( n \in \mathbb{N}, k \geq 2 \).
THEOREM 4. For every \( n \in \mathbb{N} \), every \( k \geq 2 \), the statement
\[
\forall a \forall m \exists b \left[ \text{for every } \log^{(n)} \text{-regressive function } f \text{ defined on } [a, b]^{k+1}, \text{ there is an } f\text{-min-homogeneous subset } H \subset [a, b] \text{ of size at least } m \text{ and such that } 2^{(\text{second element of } H)} < |H| \right]
\]
is unprovable in \( I\Sigma_k \).

PROOF.
Let \( M \models I\Sigma_1 \) be nonstandard, \( a, d \in M \setminus \mathbb{N} \) and \( b \in M \) be minimal such that for every \( \log^{(n)} \)-regressive function \( f \) defined on \([a, b]^{k+1}\), there is a \( f\)-min-homogeneous \( H \subset [a, b] \) of size at least \( d \) and such that \( 2^{(\text{second element of } H)} < |H| \).

Let \( \mathbb{N} < e < \log^{(n+2)} a \) and \( \psi_1(x_0, x_1, \ldots, x_k), \ldots, \psi_e(x_0, x_1, \ldots, x_k) \) be the first \( e \Delta_0 \)-formulas in \((k + 1)\) free variables. Define, as in the previous section,
\[
f(x_0, x_1, \ldots, x_k) = \left\{ \begin{array}{ll}
\{ p < \log^{(n+2)} x_0 \mid \psi_1(p, x_1, \ldots, x_k) \} \\
\vdots \\
\{ p < \log^{(n+2)} x_0 \mid \psi_e(p, x_1, \ldots, x_k) \}
\end{array} \right\}.
\]

Notice that again, since \( \log^{(n+1)} x_0 > 2^c \), we have \( f(x_0, x_1, \ldots, x_k) < 2^{e \log^{(n+2)} x_0} < 2^{e \cdot 1 \cdot \log^{(n+1)} x_0} = \log^{(n)} x_0 \).

Let \( H \subset [a, b] \) be an \( f\)-min-homogeneous subset such that
\[
2^{(\text{second element of } H)} < |H|
\]
and \( |H| \geq d \). Let us again write down the indiscernibility condition we have for \( H \): for any \( c_0 < c_1 < \cdots < c_k \), \( c_0 < d_1 < \cdots < d_k \) in \( H \), any \( p < \log^{(n+2)} c_0 \) and any \( \Delta_0 \)-formula \( \psi \) in \((k + 1)\) free variables, we have
\[
M \models \psi(p, c_1, \ldots, c_k) \leftrightarrow \psi(p, d_1, \ldots, d_k).
\]
In particular, for any \( c_1 < \cdots < c_k \) and \( d_1 < \cdots < d_k \) in \( H \setminus \{\min H\} \) and any \( \Delta_0 \)-formula \( \varphi \) in \( k \) free variables,
\[
M \models \varphi(c_1, \ldots, c_k) \leftrightarrow \varphi(d_1, \ldots, d_k).
\]

Let us show that for \( I = \sup\{c_i \mid i \in \mathbb{N}, c_i \in H\} \), we have \( I \models I\Sigma_k \). Once we show that \( I \) is closed under exponentiation, we can demonstrate that \( I \models I\Sigma_k \) by the same argument as in the previous section, so we omit this argument here.

Now, let \( c < d \) be the first two elements of \( H \setminus \{\min H\} \). We are going to demonstrate that \( 2^c < d \). Then, since \( k \geq 2 \), by indiscernibility, for any pair \( x < y \) in \( H \setminus \{\min H\} \), we would have \( 2^x < y \).

Suppose there is \( p < \log^{(n+2)} c \) such that
\[
2_{n+3}(p) < d \leq 2_{n+3}(p + 1).
\]
Then, by indiscernibility, all elements of \( H \) which are greater than \( c \) belong to
\[
(2_{n+3}(p), 2_{n+3}(p + 1)).
\]
However, there are fewer than \( 2_{n+3}(p + 1) - 2 \) elements in this interval, while
\[
2_{n+3}(p + 1) - 2 < 2_{n+3}(\log^{(n+2)} c) - 2 \leq 2^c - 2 < |H \setminus \{\min H, c\}|
\]
and, by the pigeonhole principle, there is no space for all elements of \( H \setminus \{ \min H, c \} \)
there. Hence \( 2^x < y \) for all \( x < y \) in \( H \setminus \{ \min H \} \), hence \( I \) is closed under exponentiation. This completes the proof. \( \square \)

Note that theorems similar to Theorems 3 and 4 can be proved for

\[ PH_{\log(n)}^{(n+1)} \]

with similar proofs.

It can easily be seen from the proof that we can replace the condition \( 2^c < |H| \)
by \( q^c < |H| \) for any rational number \( q > 1 \).

We can further improve Theorem 4 by fusing the pigeonhole argument with the
following straightforward number-theoretic observation. We suggest that a version
of this observation may be very relevant in other circumstances when we have to
estimate the cardinality of a set of indiscernibles.

Let \( \ell = \log(n+2) \) and \( p_i \) be the \( i \)th prime. Then for every \( i \leq \ell \) and any
\( c_1, c_2 \in H \setminus \{ \min H, c \} \), \( c_1 \equiv c_2 (\text{mod } p_i) \). Hence, the number of elements of \( H \)
among any \( K \) consecutive elements of \([a, b] \) does not exceed

\[
A = K \cdot \prod_{i \leq \ell} \frac{K + p_i - 1}{Kp_i}.
\]

By opening up the brackets in \( A \), we obtain

\[
A \leq K \cdot \prod_{i \leq \ell} \frac{1}{p_i} + 1 \leq \frac{K}{\ell!}.
\]

Hence the condition we impose on the min-homogeneous set can be weakened to

\[
\frac{q^c}{(\log(n+2) \cdot c)!} < |H|,
\]

where \( c \) is the second element of \( H \). Of course, for large \( n \), it is only a slight
improvement. If we use the inequality \( m \cdot \ln m < p_m \) from [41] instead of \( m < p_m \)
above then we can show that the condition can be weakened further to

\[
q^c \cdot \left( \frac{1}{\ln \ell} \right)^\ell \cdot \frac{e^{Li(\ell)}}{\ell!} < |H|,
\]

where \( \ell = \log(n+2) \) and \( c \) is the second element of \( H \). For that notice that for
all \( k \in \mathbb{N} \), all \( i \leq \ell + k \) and any \( c_1, c_2 \in H \setminus \{ \min H, c \} \), \( c_1 \equiv c_2 (\text{mod } p_i) \). Now,

\[
\prod_{i \leq \ell + k} \frac{1}{\ln i} = \exp(-\sum_{i \leq \ell + k} \ln \ln i)
\]

and

\[
\sum_{i \leq \ell + k} \ln \ln i \geq \int_2^{\ell+k'} \ln \ln x dx \quad \text{for some } k' \leq k.
\]

But we also have

\[
\int_2^{\ell+k'} \ln \ln x dx = (\ell + k') \cdot \ln(\ell + k') - 2 \ln 2 - \int_2^{\ell+k'} \frac{dx}{\ln x} \geq \ell \cdot \ln \ln \ell - Li(\ell).
\]

The last inequality follows from the growth rate of \( \int_2^{x} \frac{dt}{\ln t} \) and the fact that \( \ell \) is
nonstandard (for any \( C \in \mathbb{N} \), the increment of \( x \ln \ln x \) on \([\ell, \ell + k']\) is greater than
the increment of \( \int_2^{x} \frac{dt}{\ln t} \) by more than \( C \)).
Other slight improvements can be obtained by applying refined versions of the number-theoretic observation above. We conjecture that the condition $\min H < |H|$ should be enough to guarantee $I\Sigma_k$-unprovability of our principle for all $n, k \in \mathbb{N}$.

7. Future

Time has come to be able to do the following two things which are within reach.

(1) Interbreed the model theory and combinatorics that lead to the independence phenomenon with the hardcore number theory. This should result in formulating many consistent number-theoretic statements implying PH or Con$_{\text{PA}}$ or, hopefully, even stronger statements. This is not as easy as it sounds, since, as we mentioned in the introduction, all of the existing number theory is (to our knowledge) formalizable in Peano Arithmetic.

(2) Of utmost importance for foundations of mathematics would be to find first-order statements in the language of arithmetic independent of strong theories (higher-order arithmetics, type theory, set theories). For that, we shall have to learn to build models of these theories. Here, we would like to mention the pioneering successful attempts by H. Friedman [9] [11].

References


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Ultrafilter semirings and nonstandard submodels of the Stone-Čech Compactification of the Natural Numbers

Mauro Di Nasso and Marco Forti

Abstract. We investigate the possibility of natural extensions of addition and multiplication of natural numbers to subsets of their Stone-Čech Compactification $\beta \mathbb{N}$. We give a necessary and sufficient condition for a subset of $\beta \mathbb{N}$ to be a supersemiring of $\mathbb{N}$. In particular we characterize the ultrafilters contained in such semirings.

Introduction

A problem that has been repeatedly considered in the literature is the possibility of extending the sum and product operations of natural numbers to the space $\beta \mathbb{N}$ of ultrafilters over $\mathbb{N}$. (Natural numbers being identified with principal ultrafilters.)

There is in fact a natural way of defining a sum $k \oplus U$ and a product $k \odot U$ for all $k \in \mathbb{N}$ and $U \in \beta \mathbb{N}$:

\[
A \in k \oplus U \iff \{ n \in \mathbb{N} \mid n + k \in A \} \in U
\]
\[
A \in k \odot U \iff \{ n \in \mathbb{N} \mid n \cdot k \in A \} \in U.
\]

Consistently, one can also define a sum $U \oplus U = 2 \cdot U$ and a product $U \odot U = U^2$ of any ultrafilter $U$ by itself:

\[
A \in 2 \cdot U \iff \{ n \in \mathbb{N} \mid 2n \in A \} \in U
\]
\[
A \in U^2 \iff \{ n \in \mathbb{N} \mid n^2 \in A \} \in U.
\]

However, it seems that there is no "natural" algebraic way of defining commutative sums $U \oplus V$ and products $U \odot V$ for nonprincipal $U \neq V$. More important, it has been shown that no extensions of sum and product can be continuous, when $\beta \mathbb{N}$ is topologized as the Stone-Čech compactification of the discrete space $\mathbb{N}$ (see e.g. [8]).

In the early days of nonstandard analysis, the question was raised as to whether $\beta \mathbb{N}$ can be given a structure of nonstandard model of the natural numbers. Again, the answer was in the negative (see the discussion in A. Robinson’s paper [10]).

In the seventies, the compact space $\beta \mathbb{N}$ began to be studied as an algebraic object. Inspired by the proof of the Finite Sums Theorem given by F. Galvin

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and S. Glazer, reported in [8], Neil Hindman started a deep investigation of the structures \((\beta\mathbb{N}, \oplus)\) and \((\beta\mathbb{N}, \odot)\), where the operations given above are generalized in the following manner:

\[
\begin{align*}
A \in \mathcal{U} \oplus \mathcal{V} &\iff \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid n + m \in A\} \in \mathcal{V}\} \in \mathcal{U} \\
A \in \mathcal{U} \odot \mathcal{V} &\iff \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid n \cdot m \in A\} \in \mathcal{V}\} \in \mathcal{U}.
\end{align*}
\]

Remark that these structures are “highly non-commutative” since their centers reduce to \(\mathbb{N}\) in both cases. Nevertheless they revealed a powerful tool in combinatorial number theory and topological dynamics, thanks to the interplay with the Stone-Čech compact topology. The reader interested in applications is advised to consult the recent book [8], where H. Hindman and D. Strauss prove several remarkable results in Ramsey Theory appealing to algebraic properties of \(\beta\mathbb{N}\).

In this paper we show that extending the entire commutative semiring structure of \(\mathbb{N}\) is in fact possible, provided one reduces to convenient subsets of \(\beta\mathbb{N}\). To this end, we introduce the notion of \textit{ultrafilter semiring} as a suitable subset of \(\beta\mathbb{N}\) where the sum and product operations of \(\mathbb{N}\) can be extended in a natural way that preserves the property of being “the positive half” of an ordered ring. The resulting structures are then shown to come with a much richer structure: namely, they are nonstandard models of the natural numbers, i.e. complete elementary extensions.

The existence of proper ultrafilter semirings is an open problem. In fact it is equivalent to the existence of a special class of ultrafilters, named \textit{Hausdorff} in [5]. The existence of plenty of Hausdorff ultrafilters follows e.g. from the Continuum Hypothesis. However it has been recently announced by T. Bartoszynski and S. Shelah that it is independent of Zermelo-Fraenkel set theory.

The paper is organized as follows. In Section 1, we introduce ultrafilter semirings and we characterize them, as subspaces of \(\beta\mathbb{N}\), in terms of the Rudin-Keisler preorder of ultrafilters. In Section 2 we prove that such subspaces of \(\beta\mathbb{N}\) are nonstandard models. Section 3 contains a few remarks concerning the relationship of ultrafilter semirings with two more general ways of introducing nonstandard extensions. We conclude with a sketchy account of the set theoretic strength of Hausdorff ultrafilters and with some open questions.

In general, we refer to [3] and [8] for definitions and facts concerning ultrapowers, ultrafilters, nonstandard models, and the space \(\beta\mathbb{N}\).

1. **Ultrafilter Semirings**

The notion of \textit{positive semiring} is intended as the “nonnegative part” of an ordered ring, like \(\mathbb{N}\) with respect to \(\mathbb{Z}\). Having this in mind, we adopt the following

**Definition 1.1.** \((A, \oplus, \odot)\) is a positive semiring if

1. \(\oplus\) is associative and commutative;
2. \(\odot\) is associative and distributive with respect to \(\oplus\);
3. for all \(a, b \in A\) there exists a unique \(c\) such that \(a \odot c = b \) or \(a = b \oplus c\);
4. if \(a \odot c = b\) and \(b \odot c = a\), then \(a = b\).

Clearly every positive semiring \(A\) has a unique element 0 s.t. \(x \oplus 0 = x\) for all \(x \in A\). Moreover \(A\) has a natural ordering \(<\) defined by putting \(x < y\) if there exists \(z \neq 0\) s.t. \(x \oplus z = y\). It is easily seen that adding to \(A\) a new element \(-x\) for each \(x \neq 0\), and extending operations and ordering in the natural way, one obtains an ordered ring.
A possible construction of the Stone-Čech compactification $\beta \mathbb{N}$ of the discrete space $\mathbb{N}$ is by means of ultrafilters. Precisely, one can define:

- $\beta \mathbb{N}$ is the collection of all ultrafilters $\mathcal{U}$ on $\mathbb{N}$, where each natural number is identified with the corresponding principal ultrafilter;
- The family of sets of the form $X^* = \{ U \in \beta \mathbb{N} \mid X \subseteq U \}$ with $X \subseteq \mathbb{N}$, is a basis of (cl)open subsets;
- For every $f : \mathbb{N} \to \mathbb{N}$, the corresponding (unique) continuous extension $f^* : \beta \mathbb{N} \to \beta \mathbb{N}$ is the function where $f^*(U) = \{ X \subseteq \mathbb{N} \mid f^{-1}(X) \in U \}$.

We use the notation $f \equiv_\mathcal{U} g$ to mean that the functions $f$ and $g$ are equal $\mathcal{U}$-almost everywhere, i.e. $\{ n \in \mathbb{N} \mid f(n) = g(n) \} \in \mathcal{U}$.

We say that a subset $A \subseteq \beta \mathbb{N}$ is invariant if $f^*(U) \in A$ for all $f : \mathbb{N} \to \mathbb{N}$ and all $U \in A$.

We are now ready to state our main definition.

**Definition 1.2.** Let $A \subseteq \beta \mathbb{N}$ be invariant. A positive semiring $(A, \oplus, \odot)$ is an ultrafilter semiring if the following conditions are satisfied for all $f, g : \mathbb{N} \to \mathbb{N}$ and all $U \in A$:

1. $f^*(U) \oplus g^*(U) = (f + g)^*(U)$;
2. $f^*(U) \odot g^*(U) = (f \cdot g)^*(U)$.

The ultrafilters belonging to an ultrafilter semiring share a noteworthy algebraic property, namely

**Lemma 1.3.** Let $A$ be an ultrafilter semiring. Then every ultrafilter $\mathcal{U} \in A$ satisfies the following condition:

(H) For all $f, g : \mathbb{N} \to \mathbb{N}$, $f^*(\mathcal{U}) = g^*(\mathcal{U}) \iff f \equiv_\mathcal{U} g$.

**Proof.** If $f \equiv_\mathcal{U} g$, then $f^{-1}(X) \in \mathcal{U}$ if and only if $g^{-1}(X) \in \mathcal{U}$. So the left-pointing arrow holds for all ultrafilters. On the other hand, assume by contradiction that $f^*(U) = g^*(U)$ and $f \not\equiv_\mathcal{U} g$. Then either $f$ is greater than $g$ $\mathcal{U}$-almost everywhere, or conversely. Assume w.l.o.g. that there exists a function $h : \mathbb{N} \to \mathbb{N}$ such that $f \equiv_\mathcal{U} g + h$, say. Then $f^*(U) = g^*(U) \oplus h^*(U)$, by (i). Hence $h^*(U) = 0$, and so $h$ is the constant 0 on a set in $\mathcal{U}$, a contradiction.

The above property of ultrafilters has been first considered as “property (C)” in [4], where it is derived from the “3-arrow property” of [1]. It has been renamed (H) in [5], where ultrafilters satisfying it are called Hausdorff, because they are exactly those providing ultrapowers whose natural topology is Hausdorff (see also [9]). Hausdorff ultrafilters are extensively studied in [6].

As we identify natural numbers with the corresponding principal ultrafilters, every ultrafilter semiring is a supersemiring of $(\mathbb{N}, +, \cdot)$. It is in fact an end-extension, namely

**Lemma 1.4.** Every ultrafilter semiring $A$ is discretely ordered. In particular all nonprincipal ultrafilters of $A$ are greater than all principal ones.

---

1 Recalling that the Rudin-Keisler-preordering $\leq_{RK}$ on ultrafilters is defined by $\mathcal{U} \leq_{RK} \mathcal{V}$ if there exists $f : \mathbb{N} \to \mathbb{N}$ s.t. $\mathcal{U} = f^*(\mathcal{V})$, one has that $A \subseteq \beta(\mathbb{N})$ is invariant if and only if it is $RK$-downward closed, i.e. $\mathcal{U} \leq_{RK} \mathcal{V} \in A$ implies $\mathcal{U} \in A$. 
PROOF. For $k \in \mathbb{N}$, define the function $s_k : \mathbb{N} \to \mathbb{N}$ by $s_k(m) = 0$ for $m < k$ and $s_k(m) = m - k$ otherwise. If $\mathcal{U}$ is nonprincipal, then $s_k + k \equiv \text{id}_\mathcal{U}$, where $\text{id}$ is the identity function. So we obtain from (i) that $s_k^*(\mathcal{U}) \oplus k = \text{id}^*(\mathcal{U}) = \mathcal{U}$. \hfill \Box

It follows immediately that every non-zero element $\mathcal{U} \in \mathcal{A}$ has an immediate predecessor $s_1^*(\mathcal{U})$, and also an immediate successor $\mathcal{U} \oplus 1$. The key property of ultrafilter semirings is the following

**Lemma 1.5.** Every ultrafilter semiring $A$ is filtered, i.e. for all $\mathcal{U}, \mathcal{V} \in A$ there exist $f, g : \mathbb{N} \to \mathbb{N}$ and $\mathcal{W} \in A$ such that $f^*(\mathcal{W}) = \mathcal{U}$ and $g^*(\mathcal{W}) = \mathcal{V}$.\footnote{In terms of the Rudin-Keisler preordering, this property says that $A$ is upward directed.}

**Proof.** Define the function $\sqrt{\cdot} : \mathbb{N} \to \mathbb{N}$ that maps $n$ to the largest integer whose square does not exceed $n$. Clearly $(\sqrt{n})^2 \leq n < ((\sqrt{n} + 1)^2).

Put $f(n) = n - (\sqrt{n})^2$, $g(n) = \sqrt{n} - f(n)$, and $h(n) = (\sqrt{n + 1})^2 - n - 1$.

By (ii) we have, for all $\mathcal{W} \in A$, $\mathcal{W} = (\sqrt{\cdot}^*(\mathcal{W}))^2 \oplus f^*(\mathcal{W})$ and $\sqrt{\cdot}^*(\mathcal{W}) + 1 = \mathcal{W} \oplus 1 \oplus h^*(\mathcal{W})$. It follows that $\sqrt{\cdot}^*$ maps any $\mathcal{W} \in A$ to the greatest element of $A$ whose square does not exceed $\mathcal{W}$.

Given $\mathcal{U}, \mathcal{V} \in A$, put $\mathcal{W} = (\mathcal{U} \oplus \mathcal{V})^2 \oplus \mathcal{U}$. Then clearly $\sqrt{\cdot}^*(\mathcal{W}) = \mathcal{U} \oplus \mathcal{V}$, and we have at once $\mathcal{U} = f^*(\mathcal{W})$ and $\mathcal{V} = g^*(\mathcal{W})$. \hfill \Box

It follows from the above lemma that the sum and product operations are determined by the conditions (i) and (ii) of Definition 1.2. Hence any invariant subset of $\beta \mathbb{N}$ admits at most one structure of ultrafilter semiring. More precisely, we have the following characterization

**Theorem 1.6.** Let $A$ be an invariant subset of $\beta \mathbb{N}$. Then $A$ is an ultrafilter semiring if and only if $A$ is filtered and contains only Hausdorff ultrafilters.

For sake of brevity, call $FIH$ a subset of $\beta \mathbb{N}$ satisfying the conditions of Theorem 1.6, i.e. a filtered invariant collection of Hausdorff ultrafilters. The fact that every ultrafilter semiring is $FIH$ is the content of Lemmata 1.3 and 1.5. The converse implication will follow from the stronger fact that every $FIH$ subset of $\beta \mathbb{N}$ comes with a natural structure of nonstandard model of the natural numbers. (Obviously any such nonstandard model $\mathbb{N}^*$ satisfies both conditions (i) and (ii).) We devote the next section to prove this fact.

2. **Nonstandard submodels of $\beta \mathbb{N}$**

Recall that, by definition, a nonstandard model of the natural numbers is a (proper) elementary extension $\mathbb{N}^*$ of the complete first order structure over $\mathbb{N}$, i.e. the structure on $\mathbb{N}$ with respect to the complete language $\mathcal{L}_\mathbb{N}$ containing one symbol for each constant, function and relation on $\mathbb{N}$ (see [3] §4.4). Any $FIH$ subset of $\beta \mathbb{N}$ becomes an $\mathcal{L}_\mathbb{N}$-structure, when symbols are interpreted as follows:

- every constant $k \in \mathbb{N}$ is interpreted by the corresponding principal ultrafilter (that we already identified with $k$);
- every $k$-place function $F : \mathbb{N}^k \to \mathbb{N}$ is interpreted by the function $F^* : \mathcal{A}^k \to \mathcal{A}$ such that, for all $\mathcal{U} \in \mathcal{A}$ and all $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{N}$,

$$F^*(f_1^*(\mathcal{U}), \ldots, f_k^*(\mathcal{U})) = (F \circ (f_1, \ldots, f_k))^*(\mathcal{U}),$$\footnote{If $f_1, \ldots, f_k : \mathbb{N} \to \mathbb{N}$ is the function $n \mapsto F(f_1(n), \ldots, f_k(n))$.}
• every \( k \)-place relation \( R \subseteq \mathbb{N}^k \) is interpreted by the relation \( R^* \subseteq A^k \) such that, for all \( U \in A \) and all \( f_1, \ldots, f_k : \mathbb{N} \to \mathbb{N} \)

\[
(f_1^*(U), \ldots, f_k^*(U)) \in R^* \iff \{ n \in \mathbb{N} \mid (f_1(n), \ldots, f_k(n)) \in R \} \in U.
\]

The defining properties of FIH sets are all what is needed for the above definitions of \( F^* \) and \( R^* \) to be well posed. In fact

**Lemma 2.1.** Let \( A \) be an FIH set and let \( U, V \in A \) and \( f_i, g_i : \mathbb{N} \to \mathbb{N} \) be such that \( g_i^*(V) = f_i^*(U) \) for \( i = 1, \ldots, k \). Then for all \( F : \mathbb{N}^k \to \mathbb{N} \)

\[
(F \circ (f_1, \ldots, f_k))^*(U) = (F \circ (g_1, \ldots, g_k))^*(V)
\]

and for all \( R \subseteq \mathbb{N}^k \)

\[
\{ n \in \mathbb{N} \mid (f_1(n), \ldots, f_k(n)) \in R \} \in U \iff \{ n \in \mathbb{N} \mid (g_1(n), \ldots, g_k(n)) \in R \} \in V
\]

Moreover, for any \( k \)-tuple \( (V_1, \ldots, V_k) \in A^k \) there exist an ultrafilter \( U \in A \) and functions \( f_i : \mathbb{N} \to \mathbb{N} \) such that \( f_i^*(U) = V_i \) for \( i = 1, \ldots, k \).

**Proof.** The last assertion of the lemma is just an iterated application of filtration. By filtration, pick \( W \in A \) and \( p, q : \mathbb{N} \to \mathbb{N} \) s.t. \( p^*(W) = U \) and \( q^*(W) = V \). Compositions commute with \( * \), hence \( (f_i \circ p)^*(W) = f_i^*(U) = g_i^*(V) = (g_i \circ q)^*(W) \). Since \( W \) is Hausdorff, we deduce that \( f_i \circ p \equiv_W g_i \circ q \), whence also \( F \circ (f_1, \ldots, f_k) \circ p \equiv_W F \circ (g_1, \ldots, g_k) \circ q \), and the first equality follows.

Moreover

\[
\{ n \mid (f_1(n), \ldots, f_k(n)) \in R \} \in U \iff P = \{ n \mid (f_1(p(n)), \ldots, f_k(p(n))) \in R \} \in W
\]

and

\[
\{ n \mid (g_1(n), \ldots, g_k(n)) \in R \} \in V \iff Q = \{ n \mid (g_1(q(n)), \ldots, g_k(q(n))) \in R \} \in W.
\]

The equalizer \( E \) of the functions \( (f_1, \ldots, f_k) \circ p \) and \( (g_1, \ldots, g_k) \circ q \) belongs to \( W \). Since \( E \cap P = E \cap Q \) we get \( P \in W \) if and only if \( Q \in W \), and the thesis follows.

\[\square\]

We can now state a suitable version of the Loś theorem, namely

**Theorem 2.2.** Let \( A \) be an FIH subset of \( \beta\mathbb{N} \) and let \( \varphi(x_1, \ldots, x_k) \) be a formula in the complete first-order language \( L_\mathbb{N} \) over \( \mathbb{N} \). Then the following holds for all \( f_1, \ldots, f_k : \mathbb{N} \to \mathbb{N} \) and all \( U \in A \):

\[
A \models \varphi[f_1^*(U), \ldots, f_k^*(U)] \iff \{ n \in \mathbb{N} \mid \varphi(f_1(n), \ldots, f_k(n)) \} \in U
\]

In particular \( A \) is a complete elementary extension of \( \mathbb{N} \).

**Proof.** We proceed by induction on the complexity of the formula \( \varphi \). The case of atomic \( \varphi \) is precisely the content of Lemma 2.1. Conjunction and negation follow from the basic properties of ultrafilters, so we are left with existential quantification. Assume that \( \varphi(x_1, \ldots, x_k) = \exists x_0 \psi(x_0, x_1, \ldots, x_k) \). Then

\[
A \models \varphi[f_1^*(U), \ldots, f_k^*(U)] \iff
\]

\[
\exists W \in A . \ A \models \psi[V, f_1^*(U), \ldots, f_k^*(U)] \iff \exists p, q : \mathbb{N} \to \mathbb{N} . \ q^*(W) = U \&
\]

\[
\exists W \in A . \ A \models \psi[p^*(W), (f_1 \circ q)^*(W), \ldots, (f_k \circ q)^*(W)] \iff \exists W \in A . \ A \models \psi[p(n), f_1(q(n)), \ldots, f_k(q(n))] \in W \iff \]

\[
\exists W \in A . \ A \models \psi[p(n), f_1(q(n)), \ldots, f_k(q(n))] \in W \iff
\]

\[
\exists W \in A . \ A \models \psi[p(n), f_1(q(n)), \ldots, f_k(q(n))] \in W \iff
\]

\[
\exists W \in A . \ A \models \psi[p(n), f_1(q(n)), \ldots, f_k(q(n))] \in W \iff
\]
\begin{align*}
\iff & \exists p : \mathbb{N} \rightarrow \mathbb{N} . \{ n \in \mathbb{N} | \psi(p(n), f_1(n), \ldots, f_k(n)) \} \in \mathcal{U} \\
\iff & \{ n \in \mathbb{N} | \exists m. \psi(m, f_1(n), \ldots, f_k(n)) \} \in \mathcal{U} \\
\iff & \{ n \in \mathbb{N} | \varphi(f_1(n), \ldots, f_k(n)) \} \in \mathcal{U}
\end{align*}

We can now combine the previous results and conclude this section by stating our main result:

**Corollary 2.3.** Every ultrafilter semiring is a nonstandard model of the natural numbers.

3. **Final remarks**

Having proved that an ultrafilter semiring is necessarily a filtered invariant subset of $\beta\mathbb{N}$ made up of Hausdorff ultrafilters, one could obtain the results of Section 2 by referring to the topological extensions introduced in [5]. In fact, it is proved there in full generality that if a set $X$ is a discrete dense subspace of the Hausdorff space $X^*$ and every function $f : X \rightarrow X$ has a continuous extension $f^* : X^* \rightarrow X^*$, then $X^*$ can be given a natural structure of nonstandard extension if and only if $X^*$ is homeomorphic to an $FIH$ subset of $\beta X$. In order to make this paper self-contained, we gave here a direct “logical” proof of Theorem 2.2, avoiding any use of topological tools.

For the same reasons, we did not exploit another alternative way of proving Theorem 2.2, namely by appealing to the “functional” characterization of complete elementary extensions given in [7]. The latter characterization makes use of three properties: the first one (called accessibility) is the counterpart of our filtration property; the remaining two properties (preservation of composition and diagonal) could be proved by appropriately resorting to uniqueness of the continuous extensions of functions to the Stone-Cech Compactification.

By Lemma 1.3 the existence of nontrivial ultrafilter rings yields the existence of nonprincipal Hausdorff ultrafilters over $\mathbb{N}$. The converse implication also holds. Namely, given a nonprincipal Hausdorff ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the subspace $\{ f^*(\mathcal{U}) | f : \mathbb{N} \rightarrow \mathbb{N} \}$ of $\beta\mathbb{N}$ is $FIH$, hence an ultrafilter semiring. (Actually, it is isomorphic to the ultrapower $\mathbb{N}^\mathcal{U}$. The set theoretic strength of this hypothesis is not yet completely known. Assuming the Continuum Hypothesis, or even Martin’s Axiom, one obtains a lot of nonisomorphic Ramsey (= selective) ultrafilters, which are well known to be Hausdorff. More precisely, as a consequence of results in [6], one obtains ultrafilter semirings isomorphic to iterated ultrapowers as well as ultrafilter semirings which are not ultrapowers. On the other hand, in ZFC alone, the existence of Hausdorff ultrafilters cannot be proved. In fact, it has been recently announced to the authors that T. Bartoszynski and S. Shelah are building a forcing model of ZFC without Hausdorff ultrafilters.\footnote{Added in proof. The manuscript: A. Bartoszynski, S. Shelah, \textit{There may be no Hausdorff ultrafilters}, is now available at \url{http://front.math.ucdavis.edu/math.LO/0311064}.}

The definition of \textit{positive semiring} has been chosen in order to obtain by algebraic means a uniform upper bound for any two given ultrafilters. The question arises as to whether this goal can be reached by means of weaker assumptions. E.g.
1. Can one prove Lemma 1.5 assuming only the properties 1-2 of Definition 1.1 (together with (i) and (ii) of Definition 1.2)?

2. Can one prove Lemma 1.5 for ultrafilter semigroups, i.e. structures with only one operation $\oplus$ satisfying the properties 1,3,4 of Definition 1.1 and (i) of Definition 1.2?

One can also try a similar algebraic approach to hyperreal fields included in the Stone-Cech Compactification $\beta\mathbb{R}$ of $\mathbb{R}$ (considered as a discrete space). Call ultrafilter field an invariant subset of $\beta\mathbb{R}$, carrying a structure of ordered (real) field and satisfying the analogues of (i) and (ii) of Definition 1.2:

3. Can one obtain for ultrafilter fields the analogues of the results of Section 1?

References


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Unitary Group Actions and Hilbertian Polish Metric Spaces

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Abstract. We investigate universal actions of the unitary group and universal equivalence relations induced by these actions. We consider the isometric equivalence relation for Hilbertian Polish metric spaces and prove that it is Borel bireducible with the orbit equivalence relation induced by a universal action of the unitary group.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space and let $U(\mathcal{H})$ be the group of all unitary transformations of $\mathcal{H}$. When $U(\mathcal{H})$ is endowed with the strong operator topology it becomes a Polish group, and we denote it by $U_\infty$ (following [7]). A more standard notation for this topological group is $U(\mathcal{H})_s$. However, the notation $U_\infty$ stresses our interest in the abstract group and in its arbitrary actions.

In this paper we investigate, collectively, Borel actions of $U_\infty$ on standard Borel spaces and the equivalence relations induced by these actions. The actions can be compared to each other via their mutual embeddability, and the equivalence relations via the notion of Borel reducibility. In either context it is well-known that there are universal elements (c.f. [2]), which are intuitively the most complicated objects. These universal actions and universal equivalence relations will be the focus of this paper.

Previously known examples of $U_\infty$ actions that generate universal equivalence relations are complicated and difficult to work with. One of our objectives in this paper is to identify actions of $U_\infty$ that are fairly simple to define but still induce universal equivalence relations. We show that the natural action of $U_\infty$ on $\mathcal{F}(\mathcal{H})$, the space of all closed subsets of $\mathcal{H}$, is such an action. Moreover, for any Borel action of $U_\infty$ on a standard Borel space $X$, there is a Borel reduction of $X$ into $\mathcal{F}(\mathcal{H})$ which respects the Borelness of invariant sets. In our terminology, $\mathcal{F}(\mathcal{H})$ is faithfully universal. This is not as strong as universal with respect to embeddings.
of actions, but faithfulness is still a significant concept because of its pertinence to the Topological Vaught Conjecture. We will elaborate on the details in the text and give definitions as they become relevant.

As an important application we also identify a natural equivalence relation that is of the same complexity as the universal equivalence relation given by $U_\infty$ actions. This equivalence relation comes from the classification problem of the so-called Hilbertian Polish metric spaces. A Hilbertian Polish metric space is a complete metric space isometrically embeddable into $\mathcal{H}$. The classification problem is to decide whether two given such spaces are isometric and it is essentially an equivalence relation. We show that this equivalence relation is the most complicated among all the orbit equivalence relations induced by unitary group actions.

A careful reader will notice that results in this paper are parallel to some of those in [6]. Instead of treating the Urysohn space and its full isometry group in [6], here we are dealing with a more familiar space $\mathcal{H}$ and the unitary group, a subgroup of the full isometry group.

The paper is organized as follows. In the next section we give the basic definitions and background results. Then in Section 3 we consider the universal actions and equivalence relations of the unitary group. In section 4 we consider the classification problem for Hilbertian Polish metric spaces and prove our main result about the complexity of the classification problem. Throughout the paper we will always work with the complex Hilbert space. But all results are valid for the real Hilbert space and, correspondingly, the orthogonal group.

2. Preliminaries

In this section we review the basic results about Polish group actions and equivalence relations. We also summarize some results on positive definite functions and closed subgroups of the unitary group. These will be needed in subsequent sections.

Recall that a topological space is Polish if it is separable and completely metrizable. Likewise, a topological group is called a Polish group if its topology is Polish. A separable, complete metric space will be called a Polish metric space. For a topological space $X$, a subset $A \subseteq X$ is called Borel if it is an element of the $\sigma$-algebra of subsets of $X$ generated by the open sets. The Borel structure of $X$ is just the $\sigma$-algebra of all Borel subsets of $X$. A space with a $\sigma$-algebra of subsets is called a standard Borel space if the $\sigma$-algebra is the Borel structure of some Polish topology. If $X$ and $Y$ are both standard Borel spaces, then a function $f : X \to Y$ is called Borel (measurable) if for any Borel subset $B$ of $Y$, $f^{-1}(B)$ is a Borel subset of $X$. A subset $B$ of $Y$ is analytic if there are a standard Borel space $X$, a Borel function $f : X \to Y$ and a Borel subset $A$ of $X$ such that $f(A) = B$. Basic facts and results on these concepts can be found in [8].

Let $X$ and $Y$ be standard Borel spaces and $E$ and $F$ be equivalence relations on $X$ and $Y$ respectively. Then $E$ is Borel reducible to $F$, denoted $E \leq_B F$, if there is a Borel function $f : X \to Y$ such that, for all $x_1, x_2 \in X$,

$$x_1 E x_2 \iff f(x_1) F f(x_2).$$

In this situation $E$ is intuitively simpler than $F$, or $F$ is more complicated than $E$. If $E \leq_B F$ and $F \leq_B E$ we say that $E$ and $F$ are Borel bireducible, and intuitively this means that $E$ and $F$ have the same complexity.
An important class of equivalence relations is that of orbit equivalence relations induced by actions of Polish groups. Let $G$ be a Polish group and $X$ a standard Borel space. We say that $G$ acts on $X$ in a Borel manner, or $X$ is a Borel $G$-space, if there is an action $a : G \times X \to X$ which is a Borel function. Here $G \times X$ is understood to be a standard Borel space since it admits a product Polish topology. For any $x \in X$, the $G$-orbit of $x$ in $X$ is the set 
\[ [x]_G = \{ g \cdot x \mid g \in G \}. \]
The $G$-orbit equivalence relation $E^X_G$ is defined by 
\[ x_1 E^X_G x_2 \iff \exists g \in G (g \cdot x_1 = x_2) \iff [x_1]_G = [x_2]_G \]
for all $x_1, x_2 \in X$. Since the action mapping $a : G \times X \to X$ is Borel, $E^X_G$ is an analytic subset of $X \times X$. However, in general, $E^X_G$ does not have to be Borel. The notion of Borel reducibility defined above can be viewed as a weak notion of embeddability of actions and can be used to compare the complexity of all orbit equivalence relations. But often these comparisons can be done via stronger notions of embeddability of actions, which we define below.

Let $X$ and $Y$ be Borel $G$-spaces. A mapping $f : X \to Y$ is a $G$-map if for all $x \in X$ and $g \in G$, 
\[ f(g \cdot x) = g \cdot f(x). \]
If in addition $f$ is an embedding, then $f$ is called a $G$-embedding. If there is a Borel $G$-map from $X$ to $Y$ then it follows that $E^X_G \leq_B E^Y_G$.

A Borel $G$-space $X$ (or the action of $G$ on $X$) is called universal if for any Borel $G$-space $Y$ there is a Borel $G$-embedding from $Y$ into $X$. It is known that for any Polish group $G$ there is a universal Borel $G$-space ([2], Theorems 2.6.1). Consequently, if $X$ is a universal Borel $G$-space then for any Borel $G$-space $Y$ we have $E^Y_G \leq_B E^X_G$, i.e., the orbit equivalence relation $E^X_G$ is the most complicated among all $G$-orbit equivalence relations.

In general, if $X$ is a Borel $G$-space, we say that $E^X_G$ is universal if any $G$-orbit equivalence relation is Borel reducible to $E^X_G$, i.e., for any Borel $G$-space $Y$, $E^Y_G \leq_B E^X_G$. It is important to note that the Borel $G$-space $X$ does not need to be universal in order for the orbit equivalence relation to be universal. To emphasize the distinction we offer the following intermediate notion.

For a subset $A$ of $X$, let 
\[ [A]_G = \{ g \cdot x \mid g \in G, a \in A \}. \]
We call $A$ $G$-invariant if $A = [A]_G$. We say that $E^X_G$ is faithfully universal if for any Borel $G$-space $Y$ there is a Borel reduction $f : Y \to X$ witnessing $E^Y_G \leq_B E^X_G$ such that, for any invariant Borel subset $A$ of $X$, we have that $[f(A)]_G$ is an invariant Borel subset of $Y$. In general, reduction functions satisfying the above property are called faithful Borel reductions. Faithful Borel reductions were first introduced in [4] out of model-theoretic motivations and later studied in [5] (there it was called FS-reducibility) for Borel equivalence relations and $S_\infty$-orbit equivalence relations. The current terminology comes from [7]. The importance of faithful Borel reductions lies in the fact that they carry down the truth of the Topological Vaught Conjecture (c.f. [2], [4], and Theorem 3.7 below). For a Polish group $G$ and a Borel $G$-space $X$, the Topological Vaught Conjecture states that, for every invariant Borel subset $A$ of $X$, either $A$ contains $2^{\aleph_0}$ many $G$-orbits or else $A$ contains only countably many $G$-orbits. If $Y$ and $X$ are both Borel $G$-spaces and there exists a faithful
reduction from \( Y \) into \( X \), then the Topological Vaught Conjecture holds for \( Y \) if it holds for \( X \). The Topological Vaught Conjecture for a Polish group \( G \) is the statement that the Topological Vaught Conjecture holds for all Borel \( G \)-spaces. If \( X \) is a Borel \( G \)-space so that \( E^X_G \) is faithfully universal, then the Topological Vaught Conjecture for \( G \) is equivalent to the instance of it for \( X \). The general Topological Vaught Conjecture asserts that the Topological Vaught Conjecture holds for all Polish groups \( G \). This conjecture is still open.

If \( X \) and \( Y \) are Borel \( G \)-spaces and \( f \) is a \( G \)-embedding from \( Y \) into \( X \), then the image of any invariant Borel subset of \( Y \) under \( f \) is an invariant Borel subset of \( X \). Thus in particular, if \( X \) is a universal Borel \( G \)-space then \( E^X_G \) is a faithfully universal \( G \)-orbit equivalence relation. To summarize, the following diagram shows the implication relations among the three universality notions.

universal \( G \)-space
\[ \Downarrow \]
faithfully universal \( G \)-orbit equivalence relation
\[ \Downarrow \]
universal \( G \)-orbit equivalence relation

Counterexamples for the inverse implications can be found in [5]. Our results in the next section also provide such examples in the context of \( G \) being the unitary group.

Most of the rest of this paper deals with the unitary group \( U_\infty \). When there is no ambiguity about the group in question we shall omit the prefix \( G \)- for the terminology defined above.

We now turn to the unitary group \( U_\infty \). We first recall some notation and basic facts about the Hilbert space \( \mathcal{H} \) and its unitary group \( U(\mathcal{H}) \). A linear operator \( T: \mathcal{H} \to \mathcal{H} \) is unitary if

\[ \langle Tu, Tv \rangle = \langle u, v \rangle, \text{ for all } u, v \in \mathcal{H}. \]

The group operation of \( U(\mathcal{H}) \) is the composition of unitary operators. The above equation implies that every unitary operator is in fact an isometry of \( \mathcal{H} \) considered as a metric space. Moreover, the strong operator topology on \( U(\mathcal{H}) \) is exactly the pointwise convergence topology when it is considered as a group of isometries. It is well-known that the weak operator topology on \( U(\mathcal{H}) \) coincides with the strong operator topology, and in the sequel we will use this fact implicitly. More basic facts about Hilbert spaces and unitary operators can be found in [3]. For more advanced background on the unitary group see [9].

In the rest of this section we shall try to give a more or less self-contained account of the results on closed subgroups of \( U_\infty \). Most of the following results belong to the folklore and are hard to attribute. Moreover, some of the results, even if well-known to experts, can not be found in the literature in the form we need them.

One of the most useful criteria for deciding whether a Polish group can be topologically embedded into \( U_\infty \) goes through the concept of positive definite functions.

Recall that a complex-valued function \( f \) on a group \( G \) is positive definite if for any \( n \geq 1 \) and arbitrary \( g_1, \ldots, g_n \in G \), \( c_1, \ldots, c_n \in \mathbb{C} \),

\[ \sum_{i,j=1}^{n} f(g_j^{-1}g_i)c_i\overline{c_j} \geq 0. \]
If $G$ is a group, $1_G$ is its identity element and $f$ is a positive definite function on $G$, then among the simplest properties of $f$ are:

(a) $f(1_G) \geq 0$,

(b) $f(g^{-1}) = f(g)$, and

(c) $|f(g)| \leq f(1_G)$,

for all $g \in G$. A less obvious property is a result of Schur that the set of all positive definite functions is closed under multiplication. Proofs of these facts can be found in, e.g., [1], §3.1.

The following characterization theorem is mainly a folklore. An account close to what we need here is given in [9], §30. For the convenience of the reader we give a sketch of the proof afterwards.

**Theorem 2.1.** Let $G$ be a Polish group and $1_G$ its identity element. Then the following are equivalent:

1. $G$ is isomorphic to a (closed) subgroup of $U_\infty$;
2. Continuous positive definite functions on $G$ generate the topology of $G$;
3. Continuous positive definite functions on $G$ generate a neighborhood basis of $1_G$;
4. Continuous positive definite functions on $G$ separate $1_G$ and closed sets not containing $1_G$;
5. There is a continuous positive definite function on $G$ separating $1_G$ and closed sets not containing $1_G$.

A few words about the statements before the proof. Let $\mathcal{F}$ be the collection of all continuous positive definite functions on $G$. A rewording of (2) gives

(2') The topology of $G$ is the weakest topology that makes functions in $\mathcal{F}$ continuous.

Should there be any doubt with the terminology used in (3) and (4), here are the expanded versions with definitions incorporated:

(3') For any open set $V \subseteq G$ with $1_G \in V$, there are functions $f_1, \ldots, f_n \in \mathcal{F}$ and open subsets $O_1, \ldots, O_n$ in $\mathbb{C}$ such that

$$1_G \in \bigcap_{i=1}^n f_i^{-1}(O_i) \subseteq V.$$ 

(4') For any closed set $F \subseteq G$ with $1_G \notin F$, there is an $f \in \mathcal{F}$ such that

$$\sup_{g \in F} |f(g)| < f(1_G).$$ 

It is also worth noting that, since $G$ is separable, it suffices to require in (2')-(4') that there is a countable family $\mathcal{F}_0 \subseteq \mathcal{F}$ satisfying the corresponding properties.

**Proof of Theorem 2.1.** It is evident that (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) $\Rightarrow$ (3). We show that (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2): Recall that the strong operator topology on $U_\infty$ coincides with the weak operator topology, thus it is the weakest topology to make all the functionals $f_{x,y}(g) = \langle g(x), y \rangle$, $x, y \in \mathcal{H}$, continuous. For any $v \in \mathcal{H}$, let

$$f_v(g) = \langle g(v), v \rangle$$

for all $g \in U_\infty$. 

Then \( f_v \) is a continuous positive definite function on \( U_\infty \). The continuity is immediate from the definition of the weak operator topology on \( U_\infty \), and the positive definiteness is established by the standard computation:

\[
\sum_{i,j=1}^{n} f_v(g_j^{-1}g_i)c_ic_j = \sum_{i,j=1}^{n} \langle g_j^{-1}g_i(v), v \rangle c_ic_j = \sum_{i,j=1}^{n} \langle g_i(v), g_j(v) \rangle c_ic_j
\]

\[
= \sum_{i,j=1}^{n} \langle c_i g_i(v), c_j g_j(v) \rangle = \left\langle \sum_{i=1}^{n} c_i g_i(v), \sum_{j=1}^{n} c_j g_j(v) \right\rangle = \left\| \sum_{i=1}^{n} c_i g_i(v) \right\|^2 \geq 0.
\]

Now by the polar identity

\[
4\langle g(x), y \rangle = \langle g(x + y), x + y \rangle - \langle g(x - y), x - y \rangle + i\langle g(x + iy), x + iy \rangle - i\langle g(x - iy), x - iy \rangle,
\]

any topology of \( U_\infty \) that makes the demonstrated positive definite functions continuous must make the functionals \( f_{x,y}(g) = \langle g(x), y \rangle \) continuous as well. Thus (2) is proved.

(3) \( \Rightarrow \) (1): Fix a countable family \( \{f_n\}_{n \in \mathbb{N}} \) of continuous positive definite functions on \( G \) so that they generate a neighborhood basis of \( 1_G \). Without loss of generality we can assume \( f_n(1_G) \leq 2^{-n} \) for all \( n \in \mathbb{N} \).

Consider the space \( X \) of all complex-valued functions on \( G \) with finite support, i.e., functions \( x : G \to \mathbb{C} \) such that for all but finitely many \( g \in G \), \( x(g) = 0 \). \( X \) is a linear space under addition and scalar multiplication. For \( x, y \in X \), let

\[
\langle x, y \rangle = \sum_{g,h \in G} \sum_{n \in \mathbb{N}} f_n(h^{-1}g)x(g)y(h).
\]

This sesquilinear form is well-defined since for any \( g, h \in G \),

\[
\left| \sum_{n} f_n(h^{-1}g) \right| \leq \sum_{n} |f_n(h^{-1}g)| \leq \sum_{n} f_n(1_G) < \infty.
\]

Let \( N = \{ x \in X | \langle x, x \rangle = 0 \} \). Then \( N \) is a linear subspace of \( X \) and it is easy to check that \( \langle \cdot, \cdot \rangle \) is well-defined on the quotient \( X/N \), making it a pre-Hilbert space. Let \( H \) be the completion of \( X/N \) under the induced \( \langle \cdot, \cdot \rangle \). Then \( H \) is a separable complex Hilbert space.

We consider the representation of \( G \) in \( U(H) \) given by the definition that, for each \( g \in G \) and \( x \in X \),

\[
T_gx(h) = x(g^{-1}h).
\]

For any \( g_0 \in G \) and \( x, y \in X \), we have that

\[
\langle T_{g_0}x, T_{g_0}y \rangle = \sum_{g,h,n} f_n(h^{-1}g)x(g_0^{-1}g)y(g_0^{-1}h) = \sum_{g,h,n} f_n(h^{-1}g)x(g)y(h) = \langle x, y \rangle.
\]

The map \( g \mapsto T_g \) is obviously a group homomorphism. For any \( g_0 \in G \) and \( x \in X \), we have that

\[
\langle T_{g_0}x, x \rangle = \sum_{g,h,n} f_n(h^{-1}g)x(g_0^{-1}g)y(h) = \sum_{g,h,n} f_n(h^{-1}g_0g)x(g)y(h).
\]

It follows immediately from this and the continuity of \( f_n \) that \( g \mapsto T_g \) is continuous.
We next check that \( g \mapsto T_g \) is injective. For this it suffices to show that for \( g_0 \neq 1_G, T_{g_0} \neq I \). Suppose \( g_0 \neq 1_G \). There exists some \( n \in \mathbb{N} \) such that \( f_n(g_0) \neq f_n(1_G) \). Consider \( x_0 \in X \) defined by

\[
x_0(g) = \begin{cases} 
1, & \text{if } g = g_0, \\
0, & \text{otherwise}.
\end{cases}
\]

Then the \( n \)-th term of \( \langle T_{g_0} x_0 - x_0, T_{g_0} x_0 - x_0 \rangle \) is

\[
\sum_{g,h} f_n(h^{-1}g)(x_0(g_0^{-1}g) - x_0(g))(x_0(g_0^{-1}h) - x_0(h))
\]

\[
= 2(f_n(1_G) - \text{Re} f_n(g_0)) > 0.
\]

It follows that \( \langle T_{g_0} x_0 - x_0, T_{g_0} x_0 - x_0 \rangle > 0 \), and therefore \( T_{g_0} \neq I \).

Now to establish that \( g \mapsto T_g \) is a topological group isomorphic embedding, the only thing that remains to be checked is that the inverse of \( g \mapsto T_g \) is continuous. For this suppose \( T_{g_m} \to T_{g_\infty} \), as \( m \to \infty \), for group elements \( g_m, g_\infty \in G \). We are to show that \( g_m \to g_\infty \), as \( m \to \infty \). Consider

\[
x_0(g) = \begin{cases} 
1, & \text{if } g = 1_G, \\
0, & \text{otherwise},
\end{cases}
\]

and \( y_0(h) = \begin{cases} 
1, & \text{if } g = g_\infty, \\
0, & \text{otherwise}.
\end{cases} \)

Then \( \langle T_{g_m} x_0, y_0 \rangle \to \langle T_{g_\infty} x_0, y_0 \rangle \) as \( m \to \infty \). But a straightforward computation shows that

\[
\langle T_{g_m} x_0, y_0 \rangle = \sum_n f_n(g_\infty^{-1} g_m),
\]

and

\[
\langle T_{g_\infty} x_0, y_0 \rangle = \sum_n f_n(1_G).
\]

These imply that for all \( n \in \mathbb{N} \), \( f_n(g_\infty^{-1} g_m) \to f_n(1_G) \) as \( m \to \infty \). From our assumption on \( \{ f_n \} \) it follows that \( g_\infty^{-1} g_m \to 1_G \), or \( g_m \to g_\infty \), as \( m \to \infty \). Thus (3)\( \Rightarrow \)(1) is proved.

(3)\( \Rightarrow \)(4): This is an application of the lemma of Schur (a product of positive definite functions is positive definite) and other properties of positive definite functions. Suppose \( F \subseteq G \) is closed and \( 1_G \notin F \). Then there are continuous positive definite functions \( f_1, \ldots, f_n \) and open sets \( O_1, \ldots, O_n \) in \( \mathbb{C} \) such that

\[
1_G \in \bigcap_{i=1}^n f_i^{-1}(O_i) \subseteq G \setminus F.
\]

Without loss of generality assume that \( f_i(1_G) = 1 \) for \( 1 \leq i \leq n \). Then there are \( \epsilon_1, \ldots, \epsilon_n > 0 \) such that, for any \( x \in F \), there is \( 1 \leq i \leq n \) with

\[
|f_i(x)| \leq 1 - \epsilon_i.
\]

Now the function \( f = f_1 \cdots f_n \) is continuous and positive definite on \( G \). Moreover, \( f(1_G) = 1 \) and, letting \( \epsilon = \min \{ \epsilon_1, \ldots, \epsilon_n \} \), we have that for any \( x \in F \),

\[
|f(x)| = |f_1(x)| \cdots |f_n(x)| \leq 1 - \epsilon < 1.
\]

(4)\( \Rightarrow \)(5): Given a countable family \( \{ f_n \}_{n \in \mathbb{N}} \) of continuous positive definite functions so that they separate \( 1_G \) and closed sets not containing \( 1_G \), we can first scale them down so that \( f_n(1_G) \leq 2^{-n} \), for all \( n \in \mathbb{N} \). Then the function \( f \) defined by

\[
f(x) = \sum_{n \in \mathbb{N}} f_n(x)
\]
is continuous, positive definite on $G$ and separates $1_G$ and closed sets not containing $1_G$. □

Statement (5) can be strengthened further to

(5') There is a real-valued continuous positive definite function on $G$ separating $1_G$ and closed sets not containing $1_G$.

In fact, if $f$ is a complex-valued such function, then $f' = \text{Re}f$ is as required. Similarly, functions in statements (3) and (4) can be taken to be real-valued as well.

An identical proof gives similar characterizations for closed subgroups of the orthogonal group of the infinite-dimensional separable real Hilbert space. Statements (3)-(5) are unchanged in such a theorem. Thus the closed subgroups of the unitary group and those of the orthogonal group are the same class of topological groups. In particular it implies that statement (2) in Theorem 2.1 can be replaced by one for real-valued continuous positive definite functions as well.

Also clear from the proof is that Theorem 2.1 can be generalized to second countable topological groups. The metrizability is implied by the existence of enough many continuous positive definite functions. The completeness requirement is unnecessary.

As an application of Theorem 2.1, we consider the Banach spaces $L_p([0,1])$ and $l_p$ for $1 \leq p \leq 2$. The additive groups of these spaces are Polish groups. Concerning positive definite functions on these groups, the following result of Schoenberg is well-known.

**Theorem 2.2** (Schoenberg [11]). For $1 \leq p \leq 2$, the function $e^{-|x|^p}$ is positive definite on $\mathbb{R}$, $l_p$ or $L_p([0,1])$.

Thus we have

**Corollary 2.3.** For $1 \leq p \leq 2$, the additive groups of $l_p$ and $L_p([0,1])$ are isomorphic to some closed subgroups of $U_\infty$.

**Proof.** The function $e^{-|x|^p}$ is continuous and obviously it separates the origin (which is the identity of the additive group) and closed sets not containing it, as the function is norm dependent. □

The formulation of Theorem 2.1 was based on communications with Pestov and Megrelishvili. Schoenberg's result and the corollary were brought to our attention by Megrelishvili.

Now let $\text{Iso}(\mathcal{H})$ be the group of all isometries of $\mathcal{H}$ (onto itself) endowed with the topology of pointwise convergence. Then it is easy to see that $\text{Iso}(\mathcal{H})$ is a Polish group and that $U_\infty$ is a closed subgroup of $\text{Iso}(\mathcal{H})$. In fact it is well-known that $\text{Iso}(\mathcal{H})$ is a semi-direct product of $U_\infty$ with the abelian group of translations in $\mathcal{H}$ (i.e., the additive group $\mathcal{H}$). In Section 4 we will need the following theorem of Uspenskiy.

**Theorem 2.4** (Uspenskiy). $\text{Iso}(\mathcal{H})$ is isomorphic to a closed subgroup of $U_\infty$.

**Proof.** For each $v \in \mathcal{H}$ define a positive definite function $p_v$ by

$$p_v(g) = e^{-\|g(v) - v\|^2}$$

for $g \in \text{Iso}(\mathcal{H})$. Each $p_v$ is continuous since $\text{Iso}(\mathcal{H})$ has the pointwise convergence topology. To see that each $p_v$ is positive definite, note that for any $g_1, \ldots, g_n \in$
\[ \sum_{i,j=1}^{n} p_v(g_j^{-1} g_i) c_i c_j = \sum_{i,j=1}^{n} e^{-\|g_{j}^{-1} g_i(v) - v\|^{2}} c_i \overline{c_j} \]
\[ = \sum_{i,j=1}^{n} e^{-\|g_{j}^{-1} g_i(v) - g_j^{-1} g_j(v)\|^{2}} c_i \overline{c_j} \]
\[ = \sum_{i,j=1}^{n} e^{-\|g_i(v) - g_j(v)\|^{2}} c_i \overline{c_j}. \]

This last expression is non-negative by Schoenberg’s Theorem 2.2 (\(e^{-|x|^{2}}\) on \(\mathbb{R}\)), Schur’s lemma on products of positive definite functions, and the fact that pointwise limits of positive definite functions are positive definite.

Finally, it is straightforward to see that \(\{p_v \mid v \in \mathcal{H}\}\) generates the topology of \(\text{Iso}(\mathcal{H})\), and thus by Theorem 2.1 (2) \(\text{Iso}(\mathcal{H})\) is a closed subgroup of \(U_\infty\). \(\square\)

### 3. Universal actions and universal equivalence relations

For any Polish space \(X\), let \(\mathcal{F}(X)\) be the space of all closed subsets of \(X\) endowed with the Effros Borel structure (c.f., e.g., 12.B of [8]). Specifically, the Borel structure on \(\mathcal{F}(X)\) is generated by sets of the form

\[ \{F \in \mathcal{F}(X) \mid F \cap U \neq \emptyset\} \]

where \(U\) is an open subset of \(X\). Then \(\mathcal{F}(X)\) becomes a standard Borel space. If moreover a Polish group \(G\) acts on \(X\) in a Borel manner then the induced action of \(G\) on \(\mathcal{F}(X)\) is also Borel.

When general results of Becker and Kechris on the existence of universal Borel \(G\)-spaces are applied to the special case of the unitary group \(U_\infty\), we obtain the following realizations of the universal objects ([2], Theorems 2.6.1 and 3.5.3).

**Theorem 3.1** (Becker-Kechris). The left translation of \(U_\infty\) on \(\mathcal{F}(U_\infty)^N\) is a universal Borel \(U_\infty\)-action. The left translation of \(U_\infty\) on \(\mathcal{F}(U_\infty)\) induces a universal \(U_\infty\)-orbit equivalence relation.

Because of the complicated structure of \(\mathcal{F}(U_\infty)\) it is desirable to find simpler realizations of the universal Borel \(U_\infty\)-space. One approach is to consider the application actions of \(U_\infty\) induced by that on \(\mathcal{H}\). Kechris and the author [6] defined a general action for an arbitrary isometry group that resembles the logic action for \(S_\infty\). Since \(U_\infty\) is naturally a group of isometries, one can deduce the following result (Corollay 9.3 of [6]).

**Theorem 3.2** (Gao-Kechris). For each natural number \(n > 0\), let \(U_\infty\) act on \(\mathcal{F}(\mathcal{H}^n)\) by coordinatewise and pointwise application. Then the product space

\[ \prod_{n>0} \mathcal{F}(\mathcal{H}^n) \]

is a universal Borel \(U_\infty\)-space.

The following theorem is the main theorem of this section. It shows that, if we are willing to loosen up the notion of universality, then some significantly simpler realization of the universal space does exist.
Theorem 3.3. Let $U_\infty$ act on $\mathcal{F}(\mathcal{H})$ by pointwise application and let $E$ be the orbit equivalence relation. Then $E$ is a faithfully universal $U_\infty$-orbit equivalence relation.

The rest of the section is devoted to the proof of Theorem 3.3. This will be done in two steps. Denote

$$X = \prod_{n>0} \mathcal{F}(\mathcal{H}^n) \quad \text{and} \quad Y = \mathcal{F}(\mathcal{H})^N.$$

Our first step is to show that there is a faithful Borel reduction from $E_{U_\infty}^X$ to $E_{U_\infty}^Y$. By Theorem 3.2, this implies that $E_{U_\infty}^Y$ is faithfully universal among $U_\infty$-orbit equivalence relations. Then in the second step, we define a faithful Borel reduction from $Y$ into $\mathcal{F}(\mathcal{H})$, and thus complete the proof.

For each natural number $n > 0$, we endow $\mathcal{H}^n$ with the inner product

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_n = \frac{1}{n} \sum_{i=1}^{n} \langle x_i, y_i \rangle,$$

for $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathcal{H}^n$. Thus a norm on $\mathcal{H}^n$ is given by

$$\| (x_1, \ldots, x_n) \|_n = \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} \| x_i \|_2^2 \right)^{1/2},$$

and $\mathcal{H}^n$ becomes a Hilbert space unitarily isomorphic to $\mathcal{H}$.

Now for each $k \in \mathbb{N}$ consider the linear isomorphic embedding

$$T_k : \mathcal{H}^{2^k} \hookrightarrow \mathcal{H}^{2^{k+1}}$$

$$\tilde{x} \mapsto (\tilde{x}, \tilde{x})$$

$T_k$ is unitary, i.e., it preserves inner products. Hence each $\mathcal{H}^{2^k}$ can be regarded as a closed subspace of $\mathcal{H}^{2^{k+1}}$ in view of the canonical $T_k$. We have

$$\mathcal{H} \subset \mathcal{H}^2 \subset \mathcal{H}^4 \subset \cdots \subset \mathcal{H}^{2^k} \subset \mathcal{H}^{2^{k+1}} \subset \cdots .$$

Now take the direct limit

$$\mathcal{H}^\infty = \bigcup_{k \in \mathbb{N}} \mathcal{H}^{2^k},$$

and let $Z$ be the completion of $\mathcal{H}^\infty$. Then $Z$ is unitarily isomorphic to $\mathcal{H}$ since $\mathcal{H}^\infty$ is a pre-Hilbert space with the inner product

$$\langle \tilde{x}, \tilde{y} \rangle_\infty = \langle \tilde{x}, \tilde{y} \rangle_{2^k},$$

where the number $k$ is large enough such that $\tilde{x}, \tilde{y} \in \mathcal{H}^{2^k}$.

Moreover, for $\varphi \in U(\mathcal{H})$ and $n > 0$, let $\varphi^{(n)} \in U(\mathcal{H}^n)$ be given by

$$\varphi^{(n)}(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n)).$$

Then we have

$$\varphi = \varphi^{(1)} \subset \varphi^{(2)} \subset \varphi^{(4)} \subset \cdots \subset \varphi^{(2^k)} \subset \varphi^{(2^{k+1})} \subset \cdots ,$$

i.e., each $\varphi^{(2^{k+1})}$ coincides with $\varphi^{(2^k)}$ on $\mathcal{H}^{2^k} \subset \mathcal{H}^{2^{k+1}}$. Therefore we can let

$$\varphi^{(\infty)} = \bigcup_{k \in \mathbb{N}} \varphi^{(2^k)}.$$
which is a unitary transformation on $\mathcal{H}^\infty$. Let $\varphi^Z$ be the unique extension of $\varphi^{(\infty)}$ to $Z$. Then $\varphi^Z \in U(Z)$. Denote

$$U^Z_\infty = \{ \varphi^Z \in U(Z) \mid \varphi \in U(\mathcal{H}) \}.$$ 

Then $U^Z_\infty$ is a closed subgroup of $U(Z)$. The following lemma is in essence a characterization of $U^Z_\infty$ in $U(Z)$.

**Lemma 3.4.** There is a sequence $\{D_m\}_{m \in \mathbb{N}}$ of closed subsets of $Z$ such that

$$U^Z_\infty = \{ \Phi \in U(Z) \mid \forall m (\Phi(D_m) = D_m) \}.$$ 

**Proof.** It is easy to show that, for each $n > 0$, there is a sequence $\{K_{n,l}\}_{l \in \mathbb{N}}$ of closed subsets of $\mathcal{H}^n$ such that

$$\{ \varphi^{(n)} \in U(\mathcal{H}^n) \mid \varphi \in U(\mathcal{H}) \} = \{ \Phi \in U(\mathcal{H}^n) \mid \forall l (\Phi(K_{n,l}) = K_{n,l}) \}.$$ 

Then the sequence $\{D_m\}_{m \in \mathbb{N}}$ can be taken to be any enumeration of the set $\{K_{n,l} \mid n, l \in \mathbb{N}\}$.

Fix $n > 0$ and denote the group on the left side by $U^{(n)}_\infty$. For each tuple $\vec{p} = (p_1, \ldots, p_n)$ of positive rational numbers, let

$$K_{\vec{p}} = \{(x_1, \ldots, x_n) \in \mathcal{H}^n \mid \|x_i\| \leq p_i, i = 1, \ldots, n \}.$$ 

Each $K_{\vec{p}}$ is closed in $\mathcal{H}^n$. Similarly, for each tuple $\vec{q} = (q_{ij})_{1 \leq i, j \leq n}$ of positive rational numbers, let

$$L_{\vec{q}} = \{(x_1, \ldots, x_n) \in \mathcal{H}^n \mid \forall 1 \leq i, j \leq n (\|x_i - x_j\| \leq q_{ij}) \}.$$ 

Then each $L_{\vec{q}}$ is also closed in $\mathcal{H}^n$. Denote

$$G = \{ \Phi \in U(\mathcal{H}^n) \mid \forall \vec{p} (\Phi(K_{\vec{p}}) = K_{\vec{p}}) \text{ and } \forall \vec{q} (\Phi(L_{\vec{q}}) = L_{\vec{q}}) \}.$$ 

We verify that $G = U^{(n)}_\infty = \{\varphi^{(n)} \mid \varphi \in U_\infty\}$.

It is clear that $G$ consists exactly of $\Phi \in U(\mathcal{H}^n)$ such that

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathcal{H}^n$, if $\Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$,

then for all $1 \leq i, j \leq n$, $\|x_i\| = \|y_i\|$ and $\|x_i - x_j\| = \|y_i - y_j\|$.

Thus immediately $U^{(n)}_\infty \subseteq G$. For the opposite inclusion, assume $\Phi \in G$. Note that $\Phi(\mathcal{H}) = \mathcal{H}$. We put $\varphi = \Phi \upharpoonright \mathcal{H}$. By considering $\Phi \circ (\varphi^{(n)})^{-1}$ in place of $\Phi$, we may assume without loss of generality that $\varphi$ is the identity transformation, toward proving that $\Phi$ is also the identity.

Suppose $\Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ and $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$. Then for some $1 \leq i \leq n, x_i \neq y_i$. Let $i$ be the least such index and let $u = x_i$. Our assumption gives that

$$\Phi(u, \ldots, u) = (u, \ldots, u).$$ 

Therefore by linearity of $\Phi$, we have

$$\Phi(x_1 - u, \ldots, x_n - u) = (y_1 - u, \ldots, y_n - u).$$ 

Since $\Phi \in G$, we must have $\|x_i - u\| = 0 = \|y_i - u\|$, so $y_i = u = x_i$, a contradiction.

To complete the proof of the lemma, we let $\{K_{n,l}\}_{l \in \mathbb{N}}$ be any enumeration of the countable set of all $K_{\vec{p}}$ and $L_{\vec{q}}$. □
Now let \( \{D_m\}_{m \in \mathbb{N}} \) be a fixed sequence of closed subsets of \( Z \) given by Lemma 3.4. We are ready to define a Borel reduction \( f : X \to Y \). In doing so we tacitly identify \( Z \) with \( \mathcal{H} \) since they are unitarily isomorphic. For a sequence 
\[
R = (R_n)_{n > 0} \in \prod_{n > 0} \mathcal{F}(\mathcal{H}^n) = X,
\]
i.e., each \( R_n \) being a closed subset of \( \mathcal{H}^n \), we let 
\[
f(R) = (D_0, R_1, D_1, R_2, D_2, \ldots, R_n, D_n, \ldots) \in \mathcal{F}(Z)^\mathbb{N}.
\]
It is clear that \( f \) is a Borel function. We claim that \( f \) is a faithful reduction from \( E^X_{U_\infty} \) to \( E^Y_{U_\infty} \).

To see that it is a reduction, suppose \( \bar{R}, \bar{S} \in X \). Note that \( \bar{R} E^Y_{U_\infty} \bar{S} \) iff there is \( \varphi \in U(\mathcal{H}) \) such that \( \varphi(R_n) = S_n \) for all \( n > 0 \). When all \( R_n \) and \( S_n \) are regarded as subsets of \( Z \), \( \varphi(R_n) = S_n \) iff \( \varphi^Z(R_n) = S_n \). Thus by Lemma 3.4, we have
\[
\bar{R} E^Y_{U_\infty} \bar{S} \iff \exists \Phi \in U_Z \forall n > 0 (\Phi(R_n) = S_n) \iff \exists \Phi \in U(Z) (\Phi(R_n) = S_n \text{ for all } n > 0 \text{ and } \Phi(D_m) = D_m \text{ for all } m \in \mathbb{N}) \iff f(\bar{R}) E^Y_{U_\infty} f(\bar{S}).
\]

To see that \( f \) is faithful, let \( A \) be an invariant Borel subset of \( X \). We use the abbreviation \( f(\bar{R}) = (\bar{R}, \bar{D}) \), and have that
\[
f(A) = \{ (\bar{R}, \bar{D}) \mid \bar{R} \in A \}.
\]
Let \( \bar{0} \) be the element of \( X \) whose each coordinate is the set \( \{0\} \subset \mathcal{H} \). By a theorem of Miller (c.f., [2], Corollary 2.3.3), the orbit \( [f(\bar{0})]_{U_\infty} \) in \( Y \) is Borel. Now for arbitrary \( (\bar{S}, \bar{K}) \in Y \), from
\[
(\bar{S}, \bar{K}) \in [f(A)]_{U_\infty} \iff (\bar{0}, \bar{K}) \in [f(\bar{0})]_{U_\infty} \text{ and } \bar{S} \in A,
\]
we know that \( [f(A)]_{U_\infty} \) is Borel. This finishes our first step.

The second step of our proof is fulfilled by the following lemma which is actually stronger than we need.

**Lemma 3.5.** There is a Borel \( U_\infty \)-embedding from \( Y = \mathcal{F}(\mathcal{H})^\mathbb{N} \) into \( \mathcal{F}(\mathcal{H}) \).

**Proof.** For \( n \in \mathbb{N} \) we first define \( S_n : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}) \) by
\[
S_n(F) = \begin{cases}
\left\{ \frac{2nx}{\|x\|} + \frac{x}{1 + \|x\|} \mid x \in F \right\} \cup \\
\left\{ y \in \mathcal{H} \mid y = 2n + 1 \right\}, & \text{if } 0 \notin F,
\end{cases}
\]
\[
\begin{cases}
\left\{ \frac{2nx}{\|x\|} + \frac{x}{1 + \|x\|} \mid x \in F, x \neq 0 \right\} \cup \\
\left\{ y \in \mathcal{H} \mid y = 2n \text{ or } 2n + 1 \right\}, & \text{if } 0 \in F.
\end{cases}
\]
For any \( F \in \mathcal{F}(\mathcal{H}) \) and \( y \in S_n(F) \), we have \( 2n \leq \|y\| \leq 2n + 1 \). Now for \( \bar{F} = (F_0, F_1, \ldots) \in \mathcal{F}(\mathcal{H})^\mathbb{N} \), we let 
\[
S(\bar{F}) = \bigcup_{n \in \mathbb{N}} S_n(F_n).
\]
We claim that \( S(\bar{F}) \) is closed. To see this, let \( \{y_k\}_{k \in \mathbb{N}} \) be a sequence such that for each \( k \in \mathbb{N} \), \( y_k \in S_{n_k}(F_{n_k}) \) for some \( n_k \), and \( y_k \to y_\infty \) as \( k \to \infty \) for some \( y_\infty \in \mathcal{H} \).
Since for \( n \neq m \) the distance between \( S_n(F_n) \) and \( S_m(F_m) \) is at least 1, we may assume that for some fixed \( n_0, y_k \in S_{n_0}(F_{n_0}) \) for all \( k \in \mathbb{N} \). If for infinitely many \( k \in \mathbb{N}, \|y_k\| = 2n_0 + 1 \), then we have \( \|y_\infty\| = 2n_0 + 1 \) and so \( y_\infty \in S_{n_0}(F_{n_0}) \). If for infinitely many \( k \in \mathbb{N}, \|y_k\| = 2n_0 \), then we have that \( 0 \in F_{n_0} \) and \( \|y_\infty\| = 2n_0 \), hence also \( y_\infty \in S_{n_0}(F_{n_0}) \). Otherwise, we may assume that none of the \( y_k \) has norm \( 2n_0 \) or \( 2n_0 + 1 \). Therefore there are \( x_k \in F_{n_0} \) such that

\[
y_k = \frac{2n_0x_k}{\|x_k\|} + \frac{x_k}{1 + \|x_k\|}.
\]

Note that we have from the above equality

\[
\|y_k\| = 2n_0 + \frac{\|x_k\|}{1 + \|x_k\|},
\]

and hence

\[
\|x_k\| = \frac{\|y_k\| - 2n_0}{2n_0 + 1 - \|y_k\|}.
\]

Moreover,

\[
x_k = \left( \frac{2n_0(2n_0 + 1 - \|y_k\|)}{\|y_k\| - 2n_0} + \left( 1 + \frac{\|y_k\| - 2n_0}{2n_0 + 1 - \|y_k\|} \right)^{-1} \right)^{-1} y_k.
\]

Now there are three cases. Case (1): \( \|y_\infty\| = 2n_0 + 1 \). In this case \( y_\infty \in S_{n_0}(F_{n_0}) \) and there is nothing to prove. Case (2): \( \|y_\infty\| = 2n_0 \). In this case we deduce that

\[
\|y_k\| - \|y_\infty\| = \frac{\|x_k\|}{1 + \|x_k\|} \to 0 \text{ as } k \to \infty.
\]

Hence \( \|x_k\| \to 0 \) and \( x_k \to 0 \). This shows that \( 0 \in F_{n_0} \) since \( F_{n_0} \) is closed. Therefore \( y_\infty \in S_{n_0}(F_{n_0}) \). Case (3): \( \|y_\infty\| \neq 2n_0 \) or \( 2n_0 + 1 \). Then by letting \( k \to \infty \) in the last of the displayed formulae in the preceding paragraph we obtain some \( x_\infty \) such that \( x_k \to x_\infty \) as \( k \to \infty \). Thus \( x_\infty \in F_{n_0} \) and \( x_\infty \neq 0 \). Moreover, we have that

\[
y_\infty = \frac{2n_0x_\infty}{\|x_\infty\|} + \frac{x_\infty}{1 + \|x_\infty\|},
\]

so \( y_\infty \in S_{n_0}(F_{n_0}) \) as needed.

We verify that \( S \) is a Borel \( U_\infty \)-embedding. It is easy to see that \( S \) is Borel and is an embedding. To see that \( S \) respects the group action, it suffices to show that, for each \( n \in \mathbb{N} \), \( T \in U(\mathcal{H}) \) and \( F \in \mathcal{F}(\mathcal{H}) \),

\[
T(S_n(F)) = S_n(T(F)).
\]

If \( 0 \not\in F \), then \( 0 \not\in T(F) \) and we only need to note that

\[
T(S_n(F))
= T\left\{ \frac{2nx}{\|x\|} + \frac{x}{1 + \|x\|} \mid x \in F \right\} \cup \left\{ y \in \mathcal{H} \mid \|y\| = 2n + 1 \right\}
= \left\{ \frac{2nT(x)}{\|T(x)\|} + \frac{T(x)}{1 + \|T(x)\|} \mid x \in F \right\} \cup \left\{ y \in \mathcal{H} \mid \|y\| = 2n + 1 \right\}
= S_n(T(F)).
\]
If $0 \in F$, the same identity holds except that $0 \in T(F)$ and the part $\{ y \in H \mid \|y\| = 2n \}$ must be simultaneously added to all terms. This finishes the proof of the lemma. \qed

We have thus finished the proof of Theorem 3.3. The following corollary is immediate from the proof of Lemma 3.5.

Corollary 3.6. Let $r_2 > r_1 \geq 0$ be real numbers. The application action of $U_\infty$ on $\mathcal{F} \{ x \in H \mid r_1 \leq \|x\| \leq r_2 \}$ induces a faithfully universal $U_\infty$-orbit equivalence relation. In particular, if $B_1(H)$ is the unit ball of $H$, then the application action of $U_\infty$ on $\mathcal{F}(B_1(H))$ induces a faithfully universal $U_\infty$-orbit equivalence relation.

We would like to add one more remark about the proof. The reader can compare the first step of our proof to the argument in [6] for the main theorem there and readily see that the two proofs have the same structure. In [6] we dealt with the Urysohn space $U$ instead of $H$ and the isometry group $\text{Iso}(U)$ instead of $U(H)$. Our argument here for the faithful universality of $E_{U_\infty}^Y$ can be repeated for the objects in [6] to obtain the following improvement.

Theorem 3.7. Let $\text{Iso}(U)$ act on $\mathcal{F}(U)^\mathbb{N}$ by coordinatewise and pointwise application. Then the equivalence relation induced by this action is a faithfully universal orbit equivalence relation for all Borel actions of Polish groups. Consequently, the Topological Vaught Conjecture for this space is equivalent to the general Topological Vaught Conjecture.

4. Hilbertian Polish metric spaces

In this section we consider the isometric classification of Hilbertian Polish metric spaces. A *Polish metric space* is a separable complete metric space. We call a metric space *Hilbertian* if it can be isometrically embedded into the Hilbert space $H$.

Each Hilbertian Polish metric space has an isometric copy as a closed subset of $H$. Conversely, each closed subset of $H$, with its induced metric from $H$, is Hilbertian. Hence the space $\mathcal{F}(H)$ can be naturally identified as the space of all Hilbertian Polish metric spaces.

Note that in the above definition it is equivalent to take the Hilbert space to be real. In fact, this will simplify some of the computations later in this section, so we make the convention here that all Hilbert spaces mentioned in this section will be real. In [11] Schoenberg gave some interesting characterizations of Hilbertian metric spaces in terms of real-valued positive definite functions and negative definite functions. To state these characterizations we need to recall the notion of positive or negative definite functions on a general space $X$.

A function $f : X^2 \to \mathbb{R}$ is called *positive definite on $X$* if for any $x, y \in X$, $f(x, x) = 0$ and $f(x, y) = f(y, x)$, and for any natural number $n > 1$, arbitrary real numbers $r_1, \ldots, r_n \in \mathbb{R}$ and elements $x_1, \ldots, x_n \in X$,

$$\sum_{1 \leq i, j \leq n} f(x_i, x_j) r_i r_j \geq 0.$$ 

On the other hand, $f$ is called *negative definite on $X$* if for any $x, y \in X$, $f(x, x) = 0$ and $f(x, y) = f(y, x)$, and for any natural number $n > 1$, arbitrary real numbers
Let \( r_1, \ldots, r_n \in \mathbb{R} \) and elements \( x_1, \ldots, x_n \in X \),
\[
\sum_{i=1}^{n} r_i = 0 \implies \sum_{1 \leq i, j \leq n} f(x_i, y_i)r_ir_j \leq 0.
\]

In case \( X \) is a metric space and \( d \) is the metric on \( X \), a function \( g : \mathbb{R} \to \mathbb{R} \) is called positive (negative) definite on \( X \) if \( g(d(x, y)) \) is positive (negative) definite on \( X \).

Schoenberg [11] showed that \( e^{-\lambda x^2} \) are positive definite on \( \mathcal{H} \) for all \( \lambda > 0 \). It turns out that these functions characterize Hilbertian spaces, in the following sense.

**Theorem 4.1** (c.f. [10] and [11]). *The following are equivalent for a metric space \( (X, d) \):*

(i) \( X \) is Hilbertian;

(ii) \( e^{-\lambda x^2} \) are positive definite on \( X \) for all \( \lambda > 0 \);

(iii) \( e^{-\lambda x^2} \) are positive definite on \( X \) for a decreasing sequence of positive values \( \lambda \) converging to 0;

(iv) \( d^2 \) is negative definite on \( X \);

(v) (Menger) \( X \) is separable and for every natural number \( n > 1 \), every set of \( n+1 \) distinct points of \( X \) can be isometrically embedded into the space \( \mathbb{R}^n \).

The theorem provides another descriptively simple way of coding Hilbertian Polish spaces. In general if we code a Polish metric space by specifying the metric on a countable dense subset, then by Theorem 4.1 (iv), in the space of all codes for Polish metric spaces the subset of codes for Hilbertian Polish metric spaces is a closed set. This is another way to see that the space of all Hilbertian Polish metric spaces is a standard Borel space.

We define the isometry relation \( \cong_i \) for \( K, L \in \mathcal{F}(\mathcal{H}) \) by
\[
K \cong_i L \iff \text{there is an isometry from } K \text{ onto } L.
\]

Note that \( \cong_i \) is different from the orbit equivalence relation induced by the \( U_\infty \) action on \( \mathcal{F}(\mathcal{H}) \) considered in the previous section. Rather, it can be identified with the \( \text{Iso}(\mathcal{H}) \)-orbit equivalence relation on \( \mathcal{F}(\mathcal{H}) \), which we prove below.

**Theorem 4.2.** *The isometry relation \( \cong_i \) for Hilbertian Polish metric spaces is Borel bireducible with the orbit equivalence relation of the application action of \( \text{Iso}(\mathcal{H}) \) on \( \mathcal{F}(\mathcal{H}) \).*

**Proof.** Let \( E \) denote the orbit equivalence relation \( E^\mathcal{F}(\mathcal{H}) \). We need to show that \( (\cong_i) \leq_B E \) and \( E \leq_B (\cong_i) \). Before defining the reductions let us recall a basic fact of the metric geometry in Hilbert and Euclidean spaces:

If \( A \) and \( B \) are subsets of \( \mathbb{R}^n \), \( n > 0 \) or \( \mathcal{H} \), and \( \varphi : A \to B \) is an isometry from \( A \) onto \( B \), then \( \varphi \) can be extended to an isometry from \( \langle A \rangle \) onto \( \langle B \rangle \), where \( \langle \cdot \rangle \) denote the affine subspace generated by the set.

We now define the reduction witnessing \( (\cong_i) \leq_B E \). Fix an isometric embedding \( f \) of \( \mathcal{H} \) into \( \mathcal{H} \) so that \( f(\mathcal{H}) \) has infinite dimension. Consider the Borel map \( K \mapsto f(K) \) from \( \mathcal{F}(\mathcal{H}) \) into \( \mathcal{F}(\mathcal{H}) \). If \( K, L \in \mathcal{F}(\mathcal{H}) \) are not isometric to each other, then \( f(K) \) and \( f(L) \) are in no way \( E \)-equivalent. On the other hand, if \( K \cong_i L \), then \( f(K) \cong_i f(L) \). Let \( \varphi \) be an isometry from \( f(K) \) onto \( f(L) \). Then by the above fact, \( \varphi \) can be extended to an isometry \( \varphi^* \) from \( \langle f(K) \rangle \) onto \( \langle f(L) \rangle \). By
our construction, both \( \langle f(K) \rangle^\perp \) and \( \langle f(L) \rangle^\perp \) have infinite dimension, and therefore there is an isometry \( \psi \) between them. Finally the isometry \( \varphi^* \oplus \psi \) is an isometry of the whole space sending \( f(K) \) to \( f(L) \), and therefore \( f(K)E f(L) \).

In order to establish the converse reduction we first show that \( E \leq_B (\cong_i) \times \text{id}_\mathbb{N} \), where the equivalence relation on the right-hand side is defined on \( \mathcal{F}(\mathcal{H}) \times \mathbb{N} \) by

\[
(\langle K, n \rangle, \langle L, m \rangle) \in (\cong_i) \times \text{id}_\mathbb{N} \iff K \cong_i L \text{ and } m = n.
\]

For this we simply associate with any given \( K \in \mathcal{F}(\mathcal{H}) \) the pair \( (K, n) \), where \( n = 0 \) if \( \dim(\langle K \rangle^\perp) \) is infinite and \( n = \dim(\langle K \rangle^\perp) + 1 \) otherwise. Note that this is a Borel map since it is Borel to check if the dimension of \( \langle K \rangle^\perp \) is finite or not. It is easy to see that this map is a reduction from \( E \) to \( (\cong_i) \times \text{id}_\mathbb{N} \), using again the aforementioned fact and by a similar argument as in the preceding paragraph.

It remains to see that \( (\cong_i) \times \text{id}_\mathbb{N} \leq_B (\cong_i) \). This requires us to associate with any pair \( (K, n) \), where \( K \) is a Hilbertian Polish metric space and \( n \) a natural number, another Hilbertian Polish metric space \( K_n \), so that

\[
K \cong_i L \text{ and } n = m \iff K_n \cong_i L_m.
\]

For this suppose we are given \( K \) and \( n \), and \( d_K \) is the metric on \( K \). Let \( K' \) be a metric space with underlying set \( K \) and a new metric \( d_K' \) defined by

\[
d_K'(x, y) = \sqrt{1 - e^{-d_K(x, y)}}.
\]

To see that \( d_K' \) is a metric, note that the function \( 1 - e^{-t} \) is monotone increasing for \( t > 0 \), which implies that \( 1 - e^{-d_K(x, y)} \) is a metric, and that the square root of any metric is a metric. Also note that the transformation from \( d_K \) to \( d_K' \) is a homeomorphism, which implies that \( d_K' \) is a complete metric iff \( d_K \) is. Obviously \( d_K' < 1 \), and thus the diameter of \( K' \) is \( \leq 1 \).

We claim that \( K' \) is a Hilbertian Polish metric space. By Theorem 4.1 it suffices to check that \( (d_K')^2 \) is negative definite. Let \( x_1, \ldots, x_n \in X \) and \( r_1, \ldots, r_n \in \mathbb{R} \) with \( \sum_{i=1}^n r_i = 0 \). Then

\[
\sum_{i, j=1}^n d_K'(x_i, x_j)^2 r_i r_j = \sum_{i, j=1}^n (1 - e^{-d_K(x_i, x_j)}) r_i r_j = - \sum_{i, j=1}^n e^{-d_K(x_i, x_j)} r_i r_j.
\]

By a theorem of Scheonberg (Corollary 1 of [11]) the space \( (K, \sqrt{d_K}) \) is Hilbertian. Thus by Theorem 4.1(ii) the function \( e^{-d_K(x, y)} \) is positive definite on \( K \). Thus the above displayed formula must be non-positive. This shows that \( K' \) is Hilbertian. We then identify \( K' \) with a closed subset of \( \mathcal{H} \) so that \( \langle K' \rangle^\perp \) has infinite dimension.

We are now ready to define \( K_n \). Let \( K_n \) be the union of \( K' \) with the set of all points \( u \in \mathcal{H} \) such that the distance between \( u \) and \( \langle K' \rangle \) is exactly \( n + 1 \). It is easy to see that \( K_n \) is a closed subset of \( \mathcal{H} \). Moreover, the part \( K_n - K' \) is a connected clopen subset of \( K_n \) with infinite diameter. We check that this construction works. Let \( (K, n) \) and \( (L, m) \) be given, and \( K_n \) and \( L_m \) are constructed. If \( K \cong_i L \) then \( K' \cong_i L' \) and therefore if in addition \( n = m \) then \( K_n \cong_i L_m \). Conversely, suppose \( K_m \cong_i L_n \) and \( \varphi \) is an isometry witnessing it. Then \( \varphi(K_n - K') \) must be a connected clopen subset of \( L_m \) with infinite diameter, hence it must be \( L_m - L' \). It follows that \( \varphi(K') = L' \) and that \( n = m \). The same \( \varphi \) witnesses that \( K \cong_i L \) by a parallel change of metric on both spaces.
This finishes our proof that \((\cong_i) \times \text{id}_N \leq_B (\cong_i)\), and therefore of the theorem. 

We can now establish the main theorem of this paper, that is the universality of these equivalence relations among all \(U_\infty\)-orbit equivalence relations.

**Theorem 4.3.** The isometry equivalence relation for Hilbertian Polish metric spaces is Borel bireducible to a universal \(U_\infty\)-orbit equivalence relation.

**Proof.** By a theorem of Mackey (c.f. Theorem 2.3.5 of [8]) and Theorem 2.4, any \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation is Borel reducible to some \(U_\infty\)-orbit equivalence relation. Thus it follows from Theorems 4.2 and 3.3 that \((\cong_i) \leq_B E^{\mathcal{F}(\mathcal{H})}_{U_\infty}\).

For the converse, we consider the set \(X = \{x \in \mathcal{H} \mid 1 \leq \|x\| \leq 2\}\) and the application action of \(U_\infty\) on \(\mathcal{F}(X)\). By Corollary 3.6, the equivalence relation \(E_{U_\infty}^{\mathcal{F}(X)}\) is a universal \(U_\infty\)-orbit equivalence relation. We claim that \(E_{U_\infty}^{\mathcal{F}(X)} \leq_B (\cong_i)\). To define the reduction, let \(S_3 = \{x \in \mathcal{H} \mid \|x\| = 3\}\) be the sphere of radius 3 in \(\mathcal{H}\). Given any \(K \in \mathcal{F}(X)\), let \(K' = K \cup \{0\} \cup S_3\). We check that the map \(K \mapsto K'\) is a required reduction. If \(K, L \in \mathcal{F}(X)\) and there is a unitary transformation \(U \in U(\mathcal{H})\) mapping \(K\) onto \(L\), then certainly \(U(0) = 0\) and \(U\) maps \(S_3\) onto \(S_3\), therefore in particular \(K' \cong_i L'\). Conversely, suppose \(K' \cong_i L'\) and this is witnessed by an isometry \(\varphi\). Then \(\varphi\) can be extended to an isometry of \(\mathcal{H}\) onto \(\mathcal{H}\), by the fact we noted in the proof of Theorem 4.2. Since \(S_3\) is a connected clopen subset of \(K\) with diameter 6, its image under \(\varphi\) or its extension must be \(S_3\) as a subset of \(L\). From this it follows that \(\varphi(0) = 0\) since the origin can be metrically characterized as the only point in \(K\) with distance 3 to every point of \(S_3\). It then follows that the extension of \(\varphi\) is an isometry of the whole space sending 0 to 0. Thus this extension must be a unitary transformation and moreover, it sends \(K\) onto \(L\).

Since \(U_\infty\) is naturally a closed subgroup of \(\text{Iso}(\mathcal{H})\), we have the following immediate corollary.

**Corollary 4.4.** The following equivalence relations are pairwise Borel bireducible:

1. the isometry of Hilbertian Polish metric spaces;
2. the universal \(U_\infty\)-orbit equivalence relation;
3. the universal \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation;
4. the \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation on \(\mathcal{F}(\mathcal{H})\);
5. the \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation on \(\mathcal{F}(\mathcal{H})^N\);
6. the \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation on \(\prod_n \mathcal{F}(\mathcal{H}^n)\).

Before closing the paper we would like to explore a bit further the faithfulness of the universal equivalence relations. A general theorem in [6] (Corollary 9.2 of [6]) implies that the space \(\prod_n \mathcal{F}(\mathcal{H}^n)\) is a universal Borel \(\text{Iso}(\mathcal{H})\)-space. As the space improves to \(\mathcal{F}(\mathcal{H})^N\), we lose this property but retain faithful universality. This is the content of the theorem to follow. We do not know if there is a faithful reduction further down to \(\mathcal{F}(\mathcal{H})\).

**Theorem 4.5.** The application action of \(\text{Iso}(\mathcal{H})\) on \(\mathcal{F}(\mathcal{H})^N\) induces a faithfully universal \(\text{Iso}(\mathcal{H})\)-orbit equivalence relation.
The proof is similar to the first step of that of Theorem 3.3 in the last section. We again consider the spaces

\[ X = \prod_{n>0} \mathcal{F}(H^n) \quad \text{and} \quad Y = \mathcal{F}(\mathcal{H})^\mathbb{N}, \]

and show that there is a faithful Borel reduction from $E^X_{\text{Iso}(\mathcal{H})}$ to $E^Y_{\text{Iso}(\mathcal{H})}$.

One can repeat the setup of the proof of Theorem 3.3 to the point that the spaces $\mathcal{H}^n$, $\mathcal{H}^\infty$ and $Z$, as well as inner products on them, are defined and each $\varphi \in \text{Iso}(\mathcal{H})$ naturally induces $\varphi^{(n)} \in \text{Iso}(\mathcal{H}^n)$, $\varphi^{(\infty)} \in \text{Iso}(\mathcal{H}^\infty)$ and $\varphi^Z \in \text{Iso}(Z)$. Denote

\[ G^Z = \{ \varphi^Z \mid \varphi \in \text{Iso}(\mathcal{H}) \}. \]

Then we have a lemma similar to Lemma 3.4 (but with a different proof) below.

**Lemma 4.6.** There is a sequence $\{D_m\}_{m \in \mathbb{N}}$ of closed subsets of $Z$ such that

\[ G^Z = \{ \Phi \in \text{Iso}(Z) \mid \forall m (\Phi(D_m) = D_m) \}. \]

**Proof.** We again show that for each $n > 1$ there is a sequence $\{K_m\}_{m \in \mathbb{N}}$ of closed subsets $\mathcal{H}^n$ such that

\[ \{ \varphi^{(n)} \in \text{Iso}(\mathcal{H}^n) \mid \varphi \in \text{Iso}(\mathcal{H}) \} = \{ \Phi \in \text{Iso}(\mathcal{H}^n) \mid \forall m (\Phi(K_m) = K_m) \}. \]

Denote the group on the left side by $G^{(n)}$. For each tuple $\bar{q} = (q_{i,j})_{1 \leq i,j \leq n}$ of positive rational numbers, let

\[ L_{\bar{q}} = \{(x_1, \ldots, x_n) \in \mathcal{H}^n \mid \forall 1 \leq i,j \leq n (\|x_i - x_j\| \leq q_{i,j}) \}. \]

Also define

\[ G = \{ \Phi \in \text{Iso}(\mathcal{H}^n) \mid \forall \bar{q} (\Phi(L_{\bar{q}}) = L_{\bar{q}}) \}; \]

in other words, $G$ consists exactly of $\Phi \in \text{Iso}(\mathcal{H}^n)$ such that

- for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathcal{H}^n$, if $\Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$,
  - then for all $1 \leq i,j \leq n$, $\|x_i - x_j\| = \|y_i - y_j\|$.

We verify that $G^{(n)} = G$.

It is clear that $G^{(n)} \subseteq G$. For the other inclusion we argue that if $\Phi \in G$ is the identity map on $\mathcal{H} \subseteq \mathcal{H}^n$ then it is identity everywhere on $\mathcal{H}^n$.

Suppose that $\Phi \in G$ and that for any $x \in \mathcal{H}$,

\[ \Phi(x, \ldots, x) = (x, \ldots, x). \quad (4.1) \]

In particular, $\Phi(0, \ldots, 0) = (0, \ldots, 0)$ and thus $\Phi$ is a unitary transformation of $\mathcal{H}^n$.

For any $1 \leq i \leq n$ and $x \in \mathcal{H}$, let $u_i^x$ denote the element of $\mathcal{H}^n$ whose $i$-th coordinate is $x$ and other coordinates are all $0$. We also present $u_i^x$ as $(0, \ldots, 0, x, 0, \ldots, 0)$ if $i$ is understood properly. Since $\Phi$ is a linear map, it is enough to show that for any $1 \leq i \leq n$ and $x \in \mathcal{H}$, $\Phi(u_i^x) = u_i^x$.

Fix $1 \leq i \leq n$ and $0 \neq x \in \mathcal{H}$, and assume

\[ \Phi(u_i^x) = (y_1, \ldots, 0, x, 0, \ldots, 0) = (y_1, \ldots, y_n). \]

Since $\Phi \in G$, we have that $y_1 = \cdots = y_{i-1} = y_{i+1} = y_n$. Let $y_0$ denote this common element. We also have

\[ \|y_i - y_0\| = \|x\|. \quad (4.2) \]

By (5.1) and the unitarity of $\Phi$, we have that for any $z \in \mathcal{H}$,

\[ \langle x, z \rangle = \sum_{i=1}^n \langle y_i, z \rangle. \quad (4.3) \]
Thus any $u$ orthogonal to $x$ in $\mathcal{H}$ is also orthogonal to each $y_i$, $1 \leq i \leq n$. It follows that for every $1 \leq i \leq n$, $y_i$ is in the subspace generated by $x$, and hence is a scalar multiple of $x$. Let $\alpha, \beta \in \mathbb{R}$ such that $y_0 = \alpha x$ and $y_i = \beta x$. Then (5.2) yields

$$|\alpha - \beta| = 1.$$  \hspace{1cm} (4.4)

Also by the linearity of $\Phi$, our assumption and (5.1), for any $z \in \mathcal{H}$,

$$\Phi(-z, \ldots, -z, x - z, -z, \ldots, -z) = (y_0 - z, \ldots, y_0 - z, y_i - z, y_0 - z, \ldots, y_0 - z).$$

Thus by the unitarity of $\Phi$ again, the norms of the vectors on both sides are the same. Thus

$$(n - 1)\|z\|^2 + \|x - z\|^2 = (n - 1)\|\alpha x - z\|^2 + \|\beta x - z\|^2.$$  \hspace{1cm} (4.5)

In (5.5) if we plug in $z = \lambda x$ for arbitrary $\lambda$, the equation can be simplified to

$$(n - 1)\lambda^2 + (1 - \lambda)^2 = (n - 1)(\alpha - \lambda)^2 + (\beta - \lambda)^2$$

and further to

$$-2\lambda + 1 = -2(n - 1)\alpha \lambda - 2\beta \lambda + (n - 1)\alpha^2 + \beta^2.$$  \hspace{1cm} (4.5)

Since $\lambda$ is arbitrary, we must have

$$1 = (n - 1)\alpha + \beta$$  \hspace{1cm} (4.6)

and

$$1 = (n - 1)\alpha^2 + \beta^2.$$  \hspace{1cm} (4.7)

Solving the equation system formed by (5.4), (5.6) and (5.7) we get two sets of solutions:

$$\left\{ \begin{array}{l}
\alpha = 0 \\
\beta = 1
\end{array} \right. \text{ and } \left\{ \begin{array}{l}
\alpha = 2/n \\
\beta = 2/n - 1.
\end{array} \right.$$

Let $v_i^x$ denote the element of $\mathcal{H}^n$ given by the second set of solutions, i.e.,

$$v_i^x = \left(\frac{2}{n}, \ldots, \frac{2}{n}, \frac{2}{n} - 1, \frac{2}{n}, \frac{2}{n}, \ldots, \frac{2}{n}\right),$$

where the term with coefficient $2/n - 1$ appears as the $i$-th coordinate. \hfill \Box

The rest of the proof is a verbatim repetition of the part following Lemma 3.4.

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References


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ABSTRACT DECOMPOSITION THEOREM AND APPLICATIONS

RAMI GROSSBERG AND OLIVIER LESSMANN

ABSTRACT. In this paper, we prove a decomposition theorem for abstract elementary classes $\mathcal{K}$ with the amalgamation property, under the assumption that certain axioms regarding independence, existence of some prime models, and regular types are satisfied. This context encompasses the following:

1. $\mathcal{K}$ is the class of models of an $\aleph_0$-stable first order theory.
2. $\mathcal{K}$ is the class of $\mathbb{C}_{\aleph_0}$-saturated models of a superstable first order theory.
3. $\mathcal{K}$ is the class of models of an excellent Scott sentence $\psi \in L_{\omega_1,\omega}$.
4. $\mathcal{K}$ the class of locally saturated models of a superstable good diagram $D$.
5. $\mathcal{K}$ is the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental good diagram $D$.

We also prove the nonstructure part necessary to obtain a Main Gap theorem for (5), which appears in the second author’s Ph.D. thesis [Le]. The main gap in the contexts (1) and (2) are theorems of Shelah [Sh a]; (3) is by Grossberg and Hart [GrHa]; (4) by Hyttinen and Shelah [HySh2].

INTRODUCTION

This paper has two purposes. The first is to present an abstract setting lifting the essential features of classifiable first order theories, to settings which are not first order. The second is to present, as an application, a new Main Gap theorem in the context of homogeneous model theory.

In his celebrated paper [Sh 131], Saharon Shelah proved the so-called Main Gap Theorem for the class of $\aleph_\varepsilon$-saturated models of a complete first order theory $T$. The result consists of showing that, if there are fewer than the maximum number of nonisomorphic models of cardinality $\lambda > |T|$, then the theory $T$ is superstable and satisfies NDOP, every $\aleph_\varepsilon$-saturated model has a decomposition in terms of an independent tree of small models, and furthermore, the tree is well-founded. This implies that the number of nonisomorphic models in each cardinal is bounded by a slow growing function. This exponential vs. slow growing dichotomy in the number of nonisomorphic models is what is referred to as the main gap.

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This main gap phenomenon is not limited to the first order case. About six years after [Sh 131], Rami Grossberg and Bradd Hart [GrHa] proved the main gap for the class of models of an excellent Scott sentence (see [Sh87b] or Lessmann’s paper in this volume for a definition of excellence). The crucial property allowing a decomposition is also NDOP.

Another non first order context with main gap phenomena is homogeneous model theory. Homogeneous model theory was introduced by Shelah in [Sh 3]. It is quite general and includes first order logic, e.c. models, Banach space model theory, many generic constructions, classes of models with amalgamation over sets (infinitary, $L^n$), as well as concrete cases like expansions of Hilbert spaces, and free groups. We have good notions of omega-stability/total transcendence [Le1], superstability ([HySh2], [HyLe]), stability ([Sh 54], [Gr1], [Gr2], [GrLe], [HySh1]), and simplicity [BuLe]. For an exposition on homogeneous model theory, see [Le2]. Hyttinen and Shelah proved a Main Gap theorem for locally saturated models of a superstable diagram [HySh2] (4); NDOP is also the dividing line. Our result for the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram (5) also uses NDOP. These results were proved independently and are incomparable approximations to the main gap conjecture for the class of $\aleph_0$-saturated models (it was proved under simplicity by Hyttinen and Lessmann).

The Main Gap proofs have two components: On the one hand we have a structure result (each model is decomposed into an independent tree of small models), and on the other hand we have a non-structure result (undesirable features, for example if the tree is not well-founded, produce the maximal number of models). In this paper, we present a framework in which we prove the structure part. This framework is defined in the context of Shelah’s Abstract Elementary Classes [Sh88] with the Amalgamation Property and the Joint Embedding Property. See [Gr3] for an exposition on this subject. We postulate the existence of an independence relation which is well-behaved over models. We also postulate the existence of a special kind of prime models, called primary, which exist over certain sets and behave well with respect to the independence relation (dominance). Finally, we posit the existence of certain types, called regular, which must be dense, and we ask for good behaviour (essentially capturing basic orthogonality calculus). This framework is general enough to include in the same proof [Sh 131], [GrHa], [HySh2], and the case of $(D, \mu)$-homogeneous models of a totally transcendental diagram $D$ introduced in [Le1].

In addition to providing a useful result allowing one to bypass all the technical aspects of a structure proof, we try to isolate and understand the essential features that ensure a good structural theory. The motivation is similar to Shelah’s good frames (see [Sh 600], [Sh 705]), in his work around categoricity for abstract elementary classes. Here, we think that a decomposition theorem under NDOP is a good indication that a good structure theory is possible. The next step is to see if geometric model theory can be developed in this context. There are also other good indications that this
is possible; a recent result of the second author with Hyttinen and Shelah [HLS] generalises one of the basic results of first order geometric model theory obtained for (1) – (2) to (3) – (5); we believe that generalisations to the context we isolate is possible.

Our result is a modest step towards the following conjecture of Shelah (late 1990s):

**Conjecture** (Shelah). Let \( \mathcal{K} \) be an AEC. Denote by \( \delta \) the ordinal \( (2^{\text{LS}(\mathcal{K})})^+ \). If \( \mathcal{K} \) has at least one model, but fewer than the maximal number of models in some cardinal \( \lambda > \beth_\delta \), then the number of nonisomorphic models of size \( \aleph_\alpha \) is bounded by \( \beth_\delta (\aleph_0^+ + |\alpha|) \) for each ordinal \( \alpha \).

The paper is organised as follows: In Section 1, we introduce an axiomatic framework for abstract elementary classes with AP and JEP. We have axioms postulating the existence of a good independence relation capturing the essential features of the superstable case, in a spirit similar to Baldwin [Ba]. We have axioms on primary models, their existence over certain sets, their uniqueness, and their behavior with respect to the independence relation. We also have axioms regarding the existence of regular types and how they connect with independence and primary models. The axioms are numbered separately and are given names, which are used in the proof. We prove a decomposition theorem in this axiomatic framework under NDOP (Theorem 1.32). The key difference with Shelah’s abstract treatment of his main gap theorems [Sh c] is that his relies on compactness, whereas ours does not. We also describe how (1) – (4) fall within this framework.

In Section 2, we present the necessary orthogonality calculus to show that the class of \( (D, \aleph_0) \)-homogeneous models of a totally transcendental diagram \( D \) satisfies the axioms of Section 1. This implies that under NDOP, every \( (D, \aleph_0) \)-homogeneous model is prime and minimal over an independent tree of small models. We also prove several additional lemmas that will allow us to complete the main gap for this class.

In Section 3, we introduce DOP (the negation of NDOP) for the class of \( (D, \aleph_0) \)-homogeneous models of a totally transcendental diagram \( D \). We show that DOP implies the existence of many nonisomorphic models (Theorem 3.4). For nonstructure results using DOP (the failure of NDOP), the axiomatization needs several levels of saturation (or homogeneity, or fullness). We give a proof of the nonstructure parts of the theorem in the context of Chapter IV (of [Sh c]). This gives the main gap for the class \( \mathcal{K} \) of \( (D, \mu) \)-homogeneous models of a totally transcendental diagram \( D \) (for any infinite \( \mu \)). Note that, since finite diagrams generalise the first order case, it is easy to see that the failure of a finite diagram to be totally transcendental does not imply the existence of many models. All the basic tools in place, we can also show, using the methods of [Sh c] or [Ha] that \( \lambda \mapsto I(\lambda, \mathcal{K}) \) is weakly monotonic (Morley’s Conjecture) for sufficiently large \( \lambda \).

In Section 4, we introduce depth for the class of \( (D, \aleph_0) \)-homogeneous models. We prove that if a class is deep then it has many nonisomorphic
models (Theorem 4.23). Finally, we derive the main gap (Theorem 4.25) for
this class. Using the same methods, we can also derive the main gap for the
class of \((D, \mu)\)-homogeneous models of a totally transcendental diagram \(D\).

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1. The axiomatic framework and decomposition theorem

We fix \((\mathcal{K}, \prec)\) an abstract elementary class [Sh88] (AEC for short), i.e.
we assume that \((\mathcal{K}, \prec)\) satisfies the following axioms:

**Definition 1.1** (Abstract Elementary Class). \(\mathcal{K}\) is a class of models in the
same similarity type \(L\). The relation \(\prec\) is a partial order on \(\mathcal{K}\).

1. \(\mathcal{K}\) is closed under isomorphism;
2. If \(M, N \in \mathcal{K}\) and \(M \prec N\) then \(M \subseteq N\), i.e. \(M\) is a submodel of \(N\);
3. If \(M, N, M^* \in \mathcal{K}\) with \(M \subseteq N\) and \(M, N \prec M^*\) then \(M \prec N\);
4. There is a cardinal \(\text{LS}(\mathcal{K})\) such that for all \(M \in \mathcal{K}\) and \(A \subseteq M\) there
   is \(N \prec M\) containing \(A\) of size at most \(|A| + \text{LS}(\mathcal{K})\).
5. \(\mathcal{K}\) is closed under Tarski-Vaught chains: Let \((M_i : i < \lambda)\) be a \(\prec\)-
increasing and continuous chain of models of \(\mathcal{K}\). Then \(\bigcup_{i < \lambda} M_i \in \mathcal{K}\).
   Also \(M_0 \prec \bigcup_{i < \lambda} M_i\) and further, if \(M_i \prec N \in \mathcal{K}\) for each \(i < \lambda\), then
   \(\bigcup_{i < \lambda} M_i \prec N\).

It is not difficult to see that if \(I\) is a directed partially ordered set and
\((M_s : s \in I)\) is such that \(M_s \in \mathcal{K}\) and \(M_s \prec M_t\) for \(s < t\) in \(I\), then
\(\bigcup_{s \in I} M_s \in \mathcal{K}\).

We now define naturally \(\mathcal{K}\)-embedding to be those embeddings preserving
\(\prec\). We will work under the additional hypothesis that \(\mathcal{K}\) has the Joint
Embedding Property (JEP) and the Amalgamation Property (AP).

**Axiom 1 (Joint Embedding Property).** Let \(M_0, M_1 \in \mathcal{K}\). Then there
is \(M^* \in \mathcal{K}\) and \(\mathcal{K}\)-embeddings \(f_\ell : M_\ell \to M^*\) for \(\ell = 1, 2\).

**Axiom 2 (Amalgamation Property).** For \(M_\ell \in \mathcal{K}\) (\(\ell = 0, 1, 2\)) and \(\mathcal{K}\-
embeddings \(f_\ell : M_0 \to M_\ell\), for \(\ell = 1, 2\), there exist \(M^* \in \mathcal{K}\) and \(\mathcal{K}\-
embeddings \(g_\ell : M_\ell \to M^*\), for \(\ell = 1, 2\), such that \(g_1 \circ f_1 = g_2 \circ f_2\). In
other words, the following diagram commutes:

\[
\begin{array}{ccc}
M_1 & \xrightarrow{g_1} & M^* \\
\downarrow f_1 & & \downarrow g_2 \\
M_0 & \xrightarrow{f_2} & M_2
\end{array}
\]

**Remark 1.2.** Notice that when \(\mathcal{K}\) is an AEC then the amalgamation prop-
erty is equivalent to: For every \(M_\ell \in \mathcal{K}\) (for \(\ell = 0, 1, 2\)) such that \(M_0 \prec M_\ell\)
(for \(\ell = 1, 2\)) there are \(M^* \in \mathcal{K}\) with \(M_2 \prec M^*\) and a \(\mathcal{K}\)-embedding
$g : M_1 \rightarrow M^*$ such that the following diagram commutes:

![Diagram](image)

Recall the next definition:

**Definition 1.3.** $M \in \mathcal{K}$ is $\lambda$-model homogeneous if whenever $N_0 < M$ and $N_1 \in \mathcal{K}$ with $N_0 < N_1$ and $N_1$ has size less than $\lambda$, then exists a $\mathcal{K}$-embedding $f : N_1 \rightarrow M$ which is the identity on $N_0$.

Assuming that $\mathcal{K}$ has AP and JEP, it is possible to construct $\lambda$-model homogeneous models for arbitrarily large $\lambda$. Notice that since we are not assuming the existence of large models in $\mathcal{K}$, a $\lambda$-model homogeneous model $M$ may be small, even though $\lambda$ is big. If $M$ is $\lambda$-model homogeneous, then any $N \in \mathcal{K}$ of size less than $\lambda$ $\mathcal{K}$-embeds into $M$. We can then use model-homogeneous models as monster models (more on this later). We make the following convenient convention.

**Convention 1.4.** We fix a $\kappa$-model homogeneous model $\mathcal{C} \in \mathcal{K}$, for a suitably large cardinal $\kappa$. We will work inside $\mathcal{C}$; all sets and models are assumed to be inside $\mathcal{C}$ of size less than $\kappa$.

We now postulate the existence of an independence relation on subsets of $\mathcal{C}$, i.e. a relation on triples of sets $A, B,$ and $C$,

$$A \perp C, \quad B$$

satisfying the some axioms. These are similar to the standard first order axioms for non-forking in the context of superstability; i.e. Local Character is with respect to a finite set. They are weaker in one respect: we deal mainly with models. This appears in the phrasing of Symmetry, Transitivity, and Local Character. We do not assume Extension; this will be done addressed later when we deal with stationarity.

**Axiom 3 (Independence).** Let $A, B, C$ and $D$ be sets. Let $M$ be a model.

1. (Definition) $A \perp C$ if and only if $A \perp B \cup C$;
2. (Triviality) Let $M$ be a model. Then $A \not\perp A$;
3. (Finite Character) $A \perp C$ if and only if $A' \perp C'$, for all finite $A' \subseteq M$ $A$, $C' \subseteq C$;
4. (Monotonicity) If $A \perp C$ and $B \subseteq B_1 \subseteq M_1$ and $C' \subseteq C$, then $A \perp C'$. $B_1$
(5) **(Local Character)** Let \((M_i : i < \omega)\) is an \(<\)-increasing sequence of models and \(M = \bigcup_{i<\omega} M_i\). Then, for every \(a\) there is \(i < \omega\) such that \(a \downarrow M\).

(6) **(Transitivity)** If \(M_0 \subseteq M_1 \subseteq C\), then \(A \downarrow C\) and \(A \downarrow M_1\) if and only if \(A \downarrow C\).

(7) **(Symmetry over models)** \(A \downarrow C\) if and only if \(C \downarrow A\).

(8) **(Invariance)** Let \(f\) be a \(K\)-embedding with \(A \cup B \cup C \subseteq \text{dom}(f)\). Then \(A \downarrow C\) if and only if \(f(A) \downarrow f(C)\).

We are now concerned with prime models.

**Definition 1.5.** We say that a model \(M \in K\) is **prime over** \(A\), if for every \(N \in K\) containing \(A\), there exists a \(K\)-embedding \(f: M \rightarrow N\), which is the identity on \(A\).

We work with a special kind of prime models, called **primary models**. We isolate the main property of **bona fide** primary models that we are going to use, namely that any two primary models over the same set are isomorphic (but we do not assume that all prime models have this property):

**Axiom 4 (Uniqueness of primary models).** Let \(M \in K\) be primary over \(A\). Then \(M\) is prime over \(A\). Moreover, if \(M' \in K\) is another primary model over \(A\), then \(M\) and \(M'\) are isomorphic over \(A\).

First, we need to define the notion of independent system. We will say that a set of finite sequences \(I\) is a **tree** if it is closed under initial segment. We will use the notation \(\eta < \nu\) to mean that \(\eta\) is an initial segment of \(\nu\).

**Definition 1.6.** Let \(I\) be a tree, we say that \(\langle M_\eta : \eta \in I \rangle \subseteq M_1\) is a **system** if \(M_\eta \in K\) for each \(\eta \in I\) and \(M_\eta \subseteq M_\nu\) when \(\eta < \nu \in I\).

The concept in the next definition is called **system in stable amalgamation** by Shelah (see [Sh87b] and [Sh c]).

**Definition 1.7.** We say that \(\langle M_\eta : \eta \in I \rangle\) is an **independent system** if it is a system satisfying in addition:

\[
M_\eta \downarrow \bigcup_{\eta < \nu} M_\nu, \quad \text{for every } \eta \in I.
\]

Where \(\eta^-\) is the predecessor of \(\eta\), i.e. \(\eta^- := \eta \uparrow (\ell(\eta) - 1)\).

**Axiom 5 (Existence of primary models).**

1. Let \(M \in K\). There is a primary model \(M' < M\) over the empty set;
2. If \(\bar{a} \in N \setminus M\) (where \(\bar{a}\) is finite) then there is a primary model \(M' < N\) over \(M \cup \bar{a}\);
(3) If \( \langle M_\eta \mid \eta \in I \rangle \subseteq N \) is an independent system, then there exists a primary model \( M' \prec N \) over \( \bigcup_{\eta \in I} M_\eta \).

The next axiom is one of the key properties of primary models.

**Axiom 6 (Dominance).** Suppose that \( A \downarrow C \) and \( M(C) \) is primary over \( M \). Then \( A \downarrow M(C) \). This implies the Concatenation property of independence. We prove it in detail, but will be terser in the future.

**Lemma 1.8 (Concatenation).** If \( A \downarrow BC \) and \( C \downarrow B \) then \( C \downarrow BA \).

**Proof.** By Finite Character, we may assume that \( A, B, \) and \( C \) are finite. Let \( M' \prec M'' \) with \( M' \) primary over \( M \cup B \) and \( M'' \) primary over \( M \cup BC \). By Dominance, we have \( A \downarrow M'' \), so \( A \downarrow M'C \) by Monotonicity, \( A \downarrow C \) by Definition and Monotonicity, so \( C \downarrow A \) by Symmetry. By Dominance, we also have \( C \downarrow M' \). Thus \( C \downarrow M'A \) by Transitivity, so \( C \downarrow BA \) by Monotonicity.

**Lemma 1.9.** Let \( M \prec N \in \mathcal{K} \) and assume that \( ab \downarrow N \). Then \( a \downarrow b \) if and only if \( a \downarrow b \).

**Proof.** By Monotonicity, \( a \downarrow N \), so if \( a \downarrow b \), then also \( a \downarrow b \) by Transitivity.

For the converse, choose a primary model \( M' \) over \( M \cup b \). We claim that \( a \downarrow Nb \). To see this, notice that \( N \downarrow ab \), so by Dominance \( N \downarrow M'' \), where \( M'' \) is a primary model over \( M \cup ab \), which implies \( N \downarrow M'a \) and \( N \downarrow a \) by Monotonicity, so the claim follows by Symmetry. Now assume that \( a \downarrow b \). By Dominance, we have \( a \downarrow M' \), so \( a \downarrow b \) by Transitivity using the claim.

**Definition 1.10.** We say that \( \{B_i \mid i < \alpha \} \) is independent over \( M \) if

\[
B_i \downarrow \bigcup_{M} \{B_j \mid j \neq i, j < \alpha \},
\]

for every \( i < \alpha \).

We now prove the usual result on independent families.

**Lemma 1.11.** Let \( \{B_i \mid i < \alpha \} \) be a family of sets and assume that

\[
(*) \quad B_{i+1} \downarrow \bigcup_{M} \{B_j \mid j < i \}, \quad \text{for every } i < \alpha.
\]

Then \( \{B_i \mid i < \alpha \} \) is independent over \( M \).
Proof. By finite character of independence, it is enough to prove this statement for $\alpha$ finite. We do this by induction on the integer $\alpha$.

For $\alpha = 1$, (*) implies that $B_1 \downarrow B_0$, so by symmetry over models we have $B_0 \downarrow B_1$, which shows that $\{B_0, B_1\}$ is independent over $M$.

Assume by induction that the statement is true for $\alpha < \omega$. Let $i \leq \alpha + 1$ be given. We must show that

\[(**)
B_i \downarrow \bigcup_M \{B_j \mid j \leq \alpha + 1, j \neq i\}.
\]

If $i = \alpha + 1$, then this is (*), so we may assume that $i \neq \alpha + 1$ and therefore (***) can be rewritten as

\[
B_i \downarrow \bigcup_M \{B_j \mid j \leq \alpha, j \neq i\} \cup B_{\alpha+1}.
\]

Notice that by induction hypothesis

\[
B_i \downarrow \bigcup_M \{B_j \mid j \leq \alpha, j \neq i\}.
\]

And further by (*) we have

\[
B_{\alpha+1} \downarrow \bigcup_M \{B_j \mid j \leq \alpha, j \neq i\} \cup B_i.
\]

Hence, the result follows by Concatenation.

Under our axioms, independent systems are quite independent. For $J$ a subtree of $I$, denote $M_J = \bigcup\{M_\eta : \eta \in J\}$ (note that $M_J$ is not necessarily a model). The following lemma is a version of Shelah’s generalised Symmetry Lemma. It also appears in a similar form in Makkai [Ma].

**Lemma 1.12.** Let $\langle M_\eta \mid \eta \in I \rangle$ be an independent system. Then, for any $I_1, I_2$ subtrees of $I$, we have:

\[(*)
M_{I_1} \downarrow_{M_{I_1 \cap I_2}} M_{I_2}
\]

*Proof. By the finite character of independence, it is enough to prove (*) for finite trees $I$. We prove this by induction on $|I_1 \cup I_2|$. First, if $I_2 \subseteq I_1$, then it is obvious. Thus, assume that there is $\eta \in I_2 \setminus I_1$, and choose $\eta$ of maximal length. Let $J_2 := I_2 \setminus \{\eta\}$. Notice that by choice of $\eta$, we have $M_{I_1 \cap J_2} = M_{I_1 \cap I_2}$. By induction hypothesis, we have that

\[(*)
M_{I_1} \downarrow_{M_{I_1 \cap I_2}} M_{J_2}.
\]

Since $M_{\eta^-} \subseteq M_{J_2}$, by monotonicity (*) implies that

\[(**)
M_{I_1} \downarrow_{M_{\eta^-}} M_{J_2}.
\]
By definition of independent system and monotonicity we have

\[
M_{\eta} \downarrow M_{I_1 \cup M_{J_2}}.
\]

Therefore, by concatenation applied to (**) and (***) we can conclude that

\[
M_{I_1} \downarrow M_{I_2}.
\]

Now, using (*) and monotonicity we have

\[
M_{I_1} \downarrow M_{I_1 \cap I_2}.
\]

Thus, the transitivity property applied to (†) and (‡), implies that

\[
M_{I_1} \downarrow M_{I_2}.
\]

This finishes the proof.

In the context of an abstract elementary class with amalgamation, Shelah introduced a natural notion of types over models. The material we are about to cover can be found in more details in [Gr3]. Consider the following equivalence relation $\sim$ on triples $(a, M, N)$, where $M, N \in \mathcal{K}$, $M \prec N$ and $a \in N$. We say that

\[
(a_1, M, N_1) \sim (a_2, M, N_2)
\]

if there is $N^* \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f_\ell : N_\ell \to N^*$ which are the identity on $M$ and such that $f_1(a_1) = f_2(a_2)$. The picture is

\[
\begin{array}{ccc}
N_1 & \xrightarrow{f_1} & N^* \\
\downarrow \text{id} & & \uparrow f_2 \\
M & \xrightarrow{\text{id}} & N_2
\end{array}
\]

The amalgamation property ensures that $\sim$ is an equivalence relation. The equivalence class $(a, M, N)/\sim$ is written $\text{ga-tp}(a/M, N)$; it is the galois type of $a$ over $M$ in $N$.

**Definition 1.13.** Let $M \in \mathcal{K}$. Then

\[
\text{ga-S}(M) = \{\text{ga-tp}(a/M, N) : \text{For some } a \in N, M \prec N \in \mathcal{K}\}.
\]

As is common in first order model theory, we use the letters $p, q$, and $r$ for types. We will say that $N'$ realizes $\text{ga-tp}(a/M, N)$ if $M \prec N'$ and there is $b \in N'$ such that $\text{ga-tp}(b/M, N') = \text{ga-tp}(a/M, N)$. We continue to write $b \models p$, if $a$ realizes $p$. Similarly, we can define $p \upharpoonright M'$ for $M' \prec M$, and $p \subseteq q$. The amalgamation property guarantees that the types are well-behaved; for example the union of an $\omega$-chain of of galois-types is a galois-type.

We have the following striking correspondence between $\lambda$-saturation and $\lambda$-model homogeneity. Recall first:
Definition 1.14. Let $M \in \mathcal{K}$. Then $M$ is $\lambda$-galois saturated if $M$ realizes each $p \in \text{ga-S}(N)$ for $N \prec M$ of size less than $\lambda$.

The next fact is due to Shelah [Sh 576] (see also [Gr3] for a proof).

Fact 1.15 (Shelah). Let $\mathcal{K}$ be an abstract elementary class with AP and JEP. For any $\lambda > \text{LS}(\mathcal{K})$ we have that $M$ is $\lambda$-model homogeneous if and only if $M$ is $\lambda$-galois saturated.

This justifies further our use of $\mathcal{C}$ as a monster model: all relevant types are realized in $\mathcal{C}$. From now on, since we may only consider types of the form $\text{ga-tp}(a/M, \mathcal{C})$, we will omit $\mathcal{C}$.

The invariance of the independence relation makes it natural to extend the independence relation to types.

Definition 1.16. Let $M \in \mathcal{K}$.

1. We say that $p \in \text{ga-S}(M)$ is free over $N \prec M$ if for every $\bar{a} \in \mathcal{C}$ realizing $p$, we have $\bar{a} \perp M$.

2. We say that $p \in \text{ga-S}(M)$ is stationary if for every $N \in \mathcal{K}$ containing $M$, there is a unique extension $p_N \in \text{ga-S}(N)$ of $p$ such that $p_N$ is free over $M$.

3. We say that the stationary type $p \in \text{ga-S}(M)$ is based on $N$ if $p$ is free over $N$.

It is clear that ‘for every’ is equivalent to ‘for some’ in (1). Notice that in this context, the existence of a free extension to a stationary type (2), not just its uniqueness, is quite important. Thus, the next axiom tells us the we have Extension for types over models in addition to uniqueness of free extensions.

Axiom 7 (Existence of Stationary types). Let $M \in \mathcal{K}$. Then any $p \in \text{ga-S}(M)$ is stationary.

The next lemma follows from the definition, Local Character, and Transitivity.

Lemma 1.17. Let $p \in \text{ga-S}(M)$ and let $(M_i : i < \lambda)$ be an $\prec$-increasing and continuous chain of models such that $\bigcup_{i<\lambda} M_i = M$. Then there is $i < \lambda$ such that $p$ is based on $M_i$. In particular, there is always $N \prec M$ of size $\text{LS}(\mathcal{K})$ such that $p$ is based on $N$.

We now introduce a strong independence between stationary types: orthogonality.

Definition 1.18. Let $p \in \text{ga-S}(M)$ and $q \in \text{ga-S}(N)$. We say that $p$ is orthogonal to $q$, written $p \perp q$, if for every $M_1 \in \mathcal{K}$ containing $M \cup N$ and for every $a \models p_{M_1}$ and $b \models q_{M_1}$, we have $a \perp b$.

By symmetry of independence, $p \perp q$ if and only if $q \perp p$. Also, if $p \in \text{ga-S}(M)$, $q \in \text{ga-S}(N)$ and with $M \prec N$, then by definition $p \perp q$ if and only if $p_N \perp q$. In fact, more is true:
Lemma 1.19 (Parallelism). Let $M \prec N$ and $p, q \in \text{ga-S}(M)$. Then $p \perp q$ if and only if $p_N \perp q_N$.

Proof. We have already shown the left to right direction, so suppose that $p_N \perp q_N$ and let $M_1 \in \mathcal{K}$ with $M \prec M_1$, and suppose, for a contradiction that $a \models p_{M_1}$, $b \models q_{M_1}$, but $a \not\subset b$. Let $N_1 \in \mathcal{K}$ containing $M_1 \cup N$. By stationarity, there exists $a'b' \models \text{ga-tp}(ab/N_1)$ such that $a'b' \perp N_1$. Notice that $a' \not\subset b'$, so $a' \not\subset b'$ by Lemma 1.9. But $a' \models p_N$ and $b' \models q_N$, contradicting the fact that $p_N \perp q_N$. \qed

We now expand this definition to orthogonality against models.

Definition 1.20. Let $M, N, M_\ell \in \mathcal{K}$ for $\ell = 0, 1, 2$.

1. Let $p \in \text{ga-S}(N)$. We say that $p$ is orthogonal to $M$, written $p \perp M$, if $p$ is orthogonal to each $q \in \text{ga-S}(M)$.

2. If $M_0 \prec M_1, M_2$, we write that $M_1/M_0 \perp M_2$ if and only if $p \perp M_2$, for every $p \in \text{ga-S}(M_0)$ realized in $M_1$.

We now concentrate on a special kind of types: regular types.

Definition 1.21. A stationary type $p \in \text{ga-S}(M)$ is called regular if for any $N \prec M$ with $p$ based on $N$ and for any $M_1 \in \mathcal{K}$ containing $M$ and $q \in \text{ga-S}(M_1)$ extending $p \upharpoonright N$, either $q = p_{M_1}$ or $q \perp p$.

Lemma 1.22. Let $M \subseteq M_1$. If $p \in \text{ga-S}(M)$ is regular, then $p_{M_1} \in \text{ga-S}(M_1)$ is regular.

Proof. Let $p \in \text{ga-S}(M)$ be regular. Let $N \prec M$ with $p$ based on $N$. Then, $p_{M_1}$ is stationary based on $N$. Let $q \in \text{ga-S}(N_1)$ extend $p \upharpoonright N$. Then either $q = p_{N_1} = (p_{M_1})_{N_1}$ or $q \perp p$. Hence by definition of $\perp$ we have $q \perp p_{M_1}$. This shows that $q$ is regular. \qed

The next axioms guarantee that it is enough to focus on regular types.

Axiom 8 (Existence of Regular types). If $M \subseteq N$ and $M \neq N$, then there exists a regular type $p \in \text{ga-S}(M)$ realized in $N \setminus M$.

Axiom 9 (Perp I). Let $M, N \in \mathcal{K}$ such that $M \prec N$. Let $p \in \text{ga-S}(N)$ be regular. Then $p \perp M$ if and only if $p \perp q$, for every regular type $q \in \text{ga-S}(M)$.

This is to establish connections with the dependence relation and orthogonality.

Axiom 10 (Equivalence). Let $M \in \mathcal{K}$ and let $p, q \in \text{ga-S}(M)$ be regular and let $\vec{b} \notin M$ realize $p$. Then $q$ is realized in $M(\vec{b}) \setminus M$ if and only if $p \not\subset q$.

Note that by Equivalence, the relation $\not\subset$ among regular types (over the same base set) is an equivalence relation.
Lemma 1.23. Let $M_0 \subseteq M \subseteq M' \subseteq N$. Let $p \in \text{ga-S}(M')$ be regular realized in $N \setminus M'$ and $q \in \text{ga-S}(M)$ such that $p \nsubseteq q$. Let $r \in \text{ga-S}(M_0)$ be regular. If $p \perp r$ then $q \perp r$.

Proof. By Lemma 1.22, the types $r_{M'}$ and $q_{M'}$ are regular. By definition, $p \nsubseteq q_{M'}$. If $q \nsubseteq r$, then $q_{M'} \nsubseteq r_{M'}$. By the Parallelism lemma, $p \perp r$ if and only if $p \perp r_{M'}$ and $q \perp r$ if and only if $q_{M'} \perp r_{M'}$. The conclusion follows from the equivalence axiom.

Lemma 1.24 (Primary base). If $M'$ is a primary model over $\bigcup_{\eta \in I} M_{\eta}$, where $(M_{\eta} \mid \eta \in I)$ is an independent system and let $p \in S(M')$ be regular. Then there exists a finite subtree $J \subseteq I$ and a model $M^*$ primary over $\bigcup_{\eta \in J} M_{\eta}$ such that $p$ is based on $M^*$.

Proof. We prove this by induction on $|I|$. Let $(J_i : i < |I|)$ be induced subtrees of $I$ of size less than $|I|$ such that $\bigcup_{i < |I|} J_i = I$. Choose $M_i$ primary over $\bigcup_{\eta \in J_i} M_{\eta}$ such that $(M_i : i < |I|)$ is $\prec$-increasing, continuous and $M' = \bigcup_{i < |I|} M_i$. This is possible by existence of primary models over independent system and the Uniqueness of primary models axiom. By Local Character, there exists $i < |I|$ such that $p$ is based on $M_i$. Since the size of $J_i$ is less than $|I|$, we are done by induction.

Lemma 1.25. Let $p = \text{ga-tp}(a/M)$ be regular and suppose that $p \perp M_1$, with $M_1 \subseteq M$. Then $M(a)/M \perp M_1$.

Proof. By axiom (Perp I) it is enough to show that any regular type $q \in \text{ga-S}(M)$ realized in $M(a) \setminus M$ is orthogonal to any regular type $r$ over $M_1$. But, if $q$ is regular realized in $M(a) \setminus M$, then by Equivalence we must have $q \nsubseteq p$. Since $p \perp M_1$ by assumption, then $p \perp r$. Then, by definition, $q \perp r$ if and only if $q \perp r_{M}$. Hence, we conclude by Equivalence.

Lemma 1.26. Let $M_0 \subseteq M$ and let $\bar{a}_1, \bar{a}_2$ such that $\bar{a}_1 \perp \bar{a}_2$. Suppose that $\text{ga-tp}(\bar{a}_\ell/M) \perp M_0$, for $\ell = 1, 2$. Let $B$ be such that $B \perp M_0$, then $M_0 \bar{a}_1 \bar{a}_2 \perp B$.

Proof. By finite character of independence, it is enough to prove this for finite $B$. Let $\bar{b}$ be finite such that

\[(*) \quad \bar{b} \perp M_0, \quad M\]

First, since $\text{ga-tp}(\bar{a}_2/M) \perp M_0$, (*) implies that

\[(**) \quad \bar{a}_2 \perp \bar{b}. \quad M\]

Thus, by symmetry, we must have $\bar{b} \perp \bar{a}_2$, so $\bar{b} \perp M_2$, where $M_2$ is primary over $M \cup \bar{a}_2$. This shows that $\bar{b} \models \text{ga-tp}(\bar{b}/M)_{M_2}$. By assumption, we have
that

\[ \bar{a}_1 \perp_{\bar{a}_2} M \]

so \( \bar{a}_1 \perp_{M} M_2 \). Then \( \bar{a}_1 \models \text{ga-tp}(\bar{a}_1/M)_{M_2} \). But, \( \text{ga-tp}(\bar{a}_1/M) \perp_{M} \text{ga-tp}(\bar{b}/M) \), so by definition, we must have \( \bar{a}_1 \perp_{M} \bar{b} \). By the first axiom of the independence relation, we have \( \bar{a}_1 \perp_{M_2} \bar{b}\bar{a}_2 \). By transitivity (and dominance) using (**), we obtain \( \bar{a}_1 \perp_{M} \bar{b}\bar{a}_2 \). Hence, by the concatenation property of independence and (†) again, we can derive

\[ \bar{a}_1\bar{a}_2 \perp_{M_0} \bar{b}, \]

which is what we wanted. \( \dagger \)

**Corollary 1.27.** Let \( M \subseteq N \). Let \( \langle A_i \mid i < \alpha \rangle \) be independent over \( N \), such that \( A_i/N \perp_{M} M, \) for each \( i < \alpha \). Let \( B \) be such that \( B \perp_{M} N \). Then

\[ \bigcup \{A_i \mid i < \alpha \} \perp_{M} B. \]

*Proof.* By finite character of independence and monotonicity, we may assume that \( \alpha < \omega \). We prove the statement by induction on \( \alpha \) and use the previous lemma at the successor step. \( \dagger \)

**Corollary 1.28.** Let \( \langle M_\eta \mid \eta \in I \rangle \) be a system satisfying:

1. \( \langle M_\eta \mid \eta^- = \nu, \eta \in I \rangle \) is independent over \( M_\nu \), for every \( \nu \in I \);
2. The type \( \text{ga-tp}(M_\eta/M_{\eta^-}) \perp_{M_{\eta^-}} M_{\eta^-} \), for every \( \eta \in I \).

Then \( \langle M_\eta \mid \eta \in I \rangle \) is an independent system.

*Proof.* By the finite character of independence, we may assume that \( I \) is finite. We prove this statement by induction on \( |I| \). First, notice that if there is no \( \eta \in I \) such that \( \eta^- \) exists, then the result follows from (1). We must show that

\[ M_\eta \perp_{M_{\eta^-}} \bigcup \{M_\nu \mid \eta \not\prec \nu, \nu \in I \}. \]

Choose \( \nu \in I \) of maximal length such that \( \eta \not\prec \nu \). Let

\[ I_1 := \{\rho \mid \eta \not\prec \rho, \nu^- < \rho \text{ and } \rho \neq \nu \}. \]

Then, by (1), the system

\[ \langle M_\rho, M_\nu \mid \rho \in I_1 \rangle, \]

is independent over \( M_{\nu^-} \).

Let

\[ I_2 := \{\rho \mid \eta \not\prec \rho, \nu^- \not\prec \rho \}. \]
By induction hypothesis

\[(**)
M_{\nu^-} \downarrow M_{I_2} M_{\eta^-}.
\]

Hence, by the previous corollary, using (\(*\)), symmetry on (**), and the fact that \(M_\rho/M_{\nu^-} \perp M_{\nu^-}\), for \(\rho \in I_1\) or \(\rho = \nu\), we conclude that

\[(***)
M_{\nu} M_{I_1} \downarrow M_{I_2} M_{\eta^-}.
\]

Now, by induction hypothesis, we must have \(M_{I_2} \downarrow M_{\eta^-}\), so by concatenation, we must have

\[(\dagger)
M_{\eta^-} \downarrow M_{I_1} M_{I_2} M_{\nu^-}.
\]

Now, \(M_{\eta^-} \downarrow M_{\nu^-}\) by monotonicity and induction hypothesis. Therefore, using (\(\dagger\)), transitivity and the definition of \(I_1\) and \(I_2\), we conclude that

\[M_{\eta^-} \downarrow \bigcup \{M_{\nu} \mid \eta \not\in \nu, \nu \in I\}.
\]

We now come to the main definition of this section.

**Definition 1.29.** \(\mathcal{K}\) has NDOP if for every \(M_0, M_1, M_2 \in \mathcal{K}\) such that \(M_1 \perp M_2\), for every \(M'\) primary over \(M_1 \cup M_2\) and for every regular type \(p \in \text{ga-}S(M')\). Either \(p \not\perp M_1\) or \(p \not\perp M_2\).

**Theorem 1.30.** Suppose \(\mathcal{K}\) has NDOP. Let \(M, M_\eta \in \mathcal{K}\), for \(\eta \in I\) be such that \(\langle M_\eta \mid \eta \in I\rangle\) is an independent system and \(M\) is primary over it. Let \(a \in \mathcal{E} \setminus M\) be such that \(\text{ga-}tp(a/M)\) is regular. Then there is \(\eta\) such that \(\text{ga-}tp(a/M) \not\perp M_\eta\).

**Proof.** Let \(p = \text{ga-}tp(a/M)\). Suppose that \(p \perp M_\eta\) for every \(\eta \in I\). By the prime base axiom and parallelism we may assume that \(I\) is finite. We will obtain a contradiction to NDOP by induction on \(|I|\).

If \(I = \{\eta \mid k : k < n\}\), it is obvious because \(\bigcup_{\nu \in I} M_{\nu} = M_\eta\), so by definition of prime, we have \(M' = M_\eta\). But \(p \not\perp p\) by triviality of independence. Therefore, \(p \not\perp M_\eta\) by definition.

Otherwise, there exists \(\nu \in I\) such that both subtrees \(I_1 := \{\eta : \eta \in I \nu \prec \eta\}\) and \(I_2 := \{\eta : \eta \in I \nu \not\prec \eta\}\) are nonempty. By the third axiom on prime models, we can choose \(M_k\) prime over \(\bigcup_{\eta \in I_k} M_\eta\) for \(k = 1, 2\). By induction hypothesis, we have

\[p \perp M_1 \quad \text{and} \quad p \perp M_2.
\]
Furthermore, since $\{M_\eta \mid \eta \in I\}$ is an independent system, we have

$$\bigcup_{\eta \in I_1} M_\eta \perp \bigcup_{\eta \in I_2} M_\eta.$$

Therefore, by the symmetry of independence and dominance, we must have

$$M_1 \perp M_2.$$

But, $M'$ is primary over $M_1 \cup M_2$. This contradicts the fact that $\mathcal{K}$ has NDOP.

An $\omega$-tree is simply a tree of height at most $\omega$.

**Definition 1.31.** We say that $\langle M_\eta, a_\eta \mid \eta \in I \rangle$ is a *decomposition of N over M* if it satisfies the following conditions:

1. $I$ is an $\omega$-tree;
2. $\langle M_\eta \mid \eta \in I \rangle$ is a system with $M_\eta \subseteq N$ for each $\eta \in I$;
3. If $\eta \dashv \dashv$ exists for $\eta \in I$, then $M_\eta/M_\eta \dashv \dashv M_\eta \dashv \dashv$;
4. For every $\nu \in I$ the system $\langle M_\eta \mid \eta \dashv \nu, \eta \in I \rangle$ is independent over $M_\nu$.
5. $M_\emptyset = M$ and $M_\eta$ is primary over $M_\eta \cup a_\eta$;
6. For every $\eta \in I$, the type $\text{ga-tp}(a_\eta/M_\eta \dashv \nu)$ is regular.

We say that $\langle M_\eta, a_\eta \mid \eta \in I \rangle$ is a *decomposition of N* if it is a decomposition of $N$ over $M_\emptyset$ the primary model over the empty set.

Fix $N \in \mathcal{K}$ and $M \prec N$. We can introduce an ordering between decompositions of $N$ over $M$ as follows: We say that

$$\langle M_\eta, a_\eta \mid \eta \in I \rangle \prec \langle N_\eta, b_\eta \mid \eta \in J \rangle$$

if $I \subseteq J$ and for every $\eta \in I$ we have

$$M_\eta = N_\eta, \quad \text{and} \quad a_\eta = b_\eta.$$

It is now easy to show that the set of decompositions of $N$ is inductive: Let $\langle S_i \mid i < \alpha \rangle$ be a chain of decompositions $S_i = \langle M^i_\eta, a^i_\eta \mid \eta \in I^i \rangle$. First, let $I := \bigcup_{i < \alpha} I^i$. Then $I$ is an $\omega$-tree. Hence, we can define the system $S := \langle M_\eta, a_\eta \mid \eta \in I \rangle$, by $M_\eta := M^i_\eta$ if $\eta \in I^i$ and $a_\eta := a^i_\eta$, if $\eta \in I^i$. This is well-defined since $\langle S_i \mid i < \alpha \rangle$ is chain. We need to check that $S$ is a decomposition of $N$. The only nontrivial fact is to check that for every $\nu \in I$ the system

$$\langle M_\eta \mid \eta \dashv \nu, \eta \in I \rangle$$

is independent over $M_\nu$. If it failed, then by finite character, there would be a finite set $F \subseteq I$ such that

$$\langle M_\eta \mid \eta \dashv \nu, \eta \in F \rangle$$

is not independent. By then, there exists $i < \alpha$ such that $F \subseteq I^i$, contradicting the fact that $S_i$ is a decomposition of $N$. 

Recall that we say that a model $N$ is minimal over $A$ if primary models exist over $A$ and if $M(A) \subseteq N$ is primary over $A$, then $N = M(A)$. Note that a decomposition as in the next theorem is called complete.

**Theorem 1.32.** Suppose $\mathcal{K}$ has NDOP. Then for every $M \subseteq N$, there exists $\langle M_\eta, a_\eta \mid \eta \in I \rangle$ a decomposition of $N$ over $M$ such that $N$ is primary and minimal over $\bigcup_{\eta \in I} M_\eta$.

**Proof.** First, notice that the set of decompositions of $N$ over $M$ is not empty. Therefore, by Zorn's Lemma, since the set of decompositions of $N$ over $M$ is inductive, there exists a maximal decomposition

\[
\langle M_\eta, a_\eta \mid \eta \in I \rangle.
\]

By Lemma 1.28, we know that $\langle M_\eta \mid \eta \in I \rangle$ is an independent system. Therefore, by the third axiom for primary models, there exists $M' \subseteq N$ primary over $\bigcup_{\eta \in I} M_\eta$. We will show that $M' = N$. This will show that $N$ is primary and minimal over $\bigcup_{\eta \in I} M_\eta$.

Suppose that $M' \neq N$. Then, by the axiom of existence of regular types, there exists a regular type $p \in \text{ga-S}(M')$ realized in $N \setminus M'$. We are going to contradict the maximality of $\langle M_\eta, a_\eta \mid \eta \in I \rangle$. Since $\mathcal{K}$ has NDOP, by Theorem 1.30, there exists $\eta \in I$ such that $p \not\subseteq M_\eta$. Choose $\eta$ of smallest length such that $p \not\subseteq M_\eta$. By axiom (Perp I), there exists a regular type $q \in \text{ga-S}(M_\eta)$ such that $p \not\subseteq q$. Since $q$ is stationary, we can choose $q_{M'}$ the unique free extension of $q$ to the prime model $M'$. Then, by Lemma 1.22, the type $q_{M'}$ is regular. Since $p \not\subseteq q$ and $p \in \text{ga-S}(M')$, by definition $p \not\subseteq q_{M'}$. By Equivalence, since $p$ is realized in $N \setminus M'$, there exists $a \in N \setminus M'$ realizing $q \nabla M'$. Hence $\text{ga-tp}(a/M') = q_{M'}$ and by choice of $q_{M'}$, this implies that

\[
a \perp M'.
\]

Since $\text{ga-tp}(a/M_\eta)$ is regular and $a \in N \setminus M_\eta$, by the second axiom on primary models, there exists a primary model $M(a) \subseteq N$ over $M_\eta \cup a$. By dominance and (**) we must have

\[
M(a) \perp M'.
\]

Thus, by monotonicity of independence and choice of $M'$, we conclude that

\[
M(a) \perp \bigcup_{M_\eta} \{ M_\nu \mid \nu^- = \eta, \, \nu \in I \}.
\]

But $\{ M_\nu \mid \nu^- = \eta \}$ is independent by definition of decomposition. Thus, (***) and Lemma 1.11 implies that

\[
\{ M_\nu, M(a) \mid \nu^- = \eta, \, \nu \in I \}
\]

is independent over $M_\eta$. Suppose now that $\eta^-$ exists. By choice of $\eta$ we must have $p \perp M_{\eta^-}$. Since $p \not\subseteq \text{ga-tp}(a/M_\eta)$, we must have by Lemma 1.23 and axiom (Perp I) that $\text{ga-tp}(a/M_\eta) \perp M_{\eta^-}$. Hence, by Lemma 1.25, we
must have $M(a)/M_\eta \perp M_\eta$. This shows that we can add $a/M_\eta$ and $M(a)$ to (*) and still have a decomposition of $N$. This contradicts the maximality of (*). Thus $N$ is primary and minimal over $\bigcup_{\eta \in I} M_\eta$.

**Corollary 1.33.** If $\mathcal{K}$ has NDOP and $N \in \mathcal{K}$, then there exists a complete decomposition.

**Proof.** By the previous theorem since by axiom on primary models there exists a primary model over the empty set.

The same proof shows:

**Corollary 1.34.** If $\mathcal{K}$ has NDOP and $N \in \mathcal{K}$ is primary over a decomposition $\langle M_\eta \mid \eta \in I \rangle$ of $N$ over $M$, then $\langle M_\eta \mid \eta \in I \rangle$ is a complete decomposition of $N$ over $M$.

1.1. **Examples.** The abstract decomposition given in this section generalises the known NDOP cases.

There are several classical first order cases. The first one is for $\aleph_0$-saturated models of a totally transcendental theory $T$. A second one is for $\aleph_1$-saturated models of a superstable theory $T$. And finally, for the class of models of a totally transcendental theory $T$. In each case, $\mathcal{C}$ can be taken to be the saturated monster model for $T$. The independence relation is nonforking. **Regular types** in the first two cases are just the regular types in the sense of first order. In the last case, they correspond to strongly regular types. The **primary models** are the $F_{\aleph_0}$-primary models, the $\aleph_1$-primary models (also called $F_{\aleph_0}$) for the second case, and the $F_{\aleph_0}$-primary models in the third case. All the results needed to apply the theorem can be found in [Sh c].

In the nonelementary case, there are two published examples. One in the context of an excellent Scott sentence in $L_{\omega_1,\omega} [GrHa]$. Excellence implies AP and JEP; The model $\mathcal{C}$ can be taken to be any sufficiently large full model. The independence relation is that afforded by the rank. Regular types are the SR types. The **primary models** are the usual primary models and their uniqueness is clear. The existence of primary models follows from excellence (see [Sh87a], [Sh87b]) and the relevant orthogonality calculus can be found in [GrHa]). The other is for the class of locally saturated models of a superstable diagram. There we have amalgamation over sets, so $\mathcal{C}$ can be taken to be strongly homogeneous. The details are in [HySh2].

The aim of the next section is to prove that the axiomatic framework developed in this section holds for the class $\mathcal{K}$ of $(D, \aleph_0)$-homogeneous models of a totally transcendental $D$. As we pointed out in the previous paragraph, we have an abstract elementary class with AP and JEP, and even more: we can work inside a large homogeneous model.

The independence relation is given by the rank; the axioms for independence, the existence of stationary types, the existence of regular types can all be found in [Le1]. The **primary models** are the $D_{\aleph_0}$-primary models; their
uniqueness is clear and their existence over all sets in the totally transcendental case is also proved in [Le1].

This leaves us with the proof of Equivalence, and Dominance. These results are part of what is called Orthogonality Calculus.

2. Orthogonality calculus in finite diagrams

In this section, we work in the context of totally transcendental good diagrams.

Let $T$ be a complete first order theory in a language $L$. A type $\text{tp}(c/A, M)$ is simply the set of first order formulas over $A$ which are true of $c$ in $M$. A diagram $D$ is a set of the form $\{\text{tp}(c/\emptyset, M) : c \in M\}$ for some $M \models T$. Fix a diagram $D$. A set $A$ is a $D$-set, if $\text{tp}(c/\emptyset) \in D$ for each finite $c \in A$. A $D$-model is a model whose universe is a $D$-set. We are interested in the class of $D$-models, i.e. the nonelementary class of models of $T$ omitting, over the empty set, all the types outside $D$.

This leads to the following notion of types:

**Definition 2.1.** $S_D(A) = \{p \in S(A) : Ac \text{ is a } D\text{-set for each } c \models p\}$.

A model $M$ is $\lambda$-homogeneous if whenever $f : M \to M$ is a partial elementary map with $|\text{dom}(f)| < \lambda$ and $a \in M$ then there is an elementary map $g : M \to M$ extending $f$ such that $a \in \text{dom}(g)$. A model $M$ is $(D, \lambda)$-homogeneous if $M$ is $\lambda$-homogeneous and $M$ realizes exactly the types in $D$. Then, if $p \in S_D(A)$ and $A \subseteq M$ has size less than $\lambda$, then $p$ is realized in $M$ if $M$ is $(D, \lambda)$-homogeneous.

We will work inside a large $(D, \bar{\kappa})$-homogenous model $\mathcal{C}$ of size $\bar{\kappa}$, which functions as our monster model. Satisfaction is defined with respect to $\mathcal{C}$, and all sets and models are assumed to be inside $\mathcal{C}$, so all the relevant types are realized in $\mathcal{C}$. The existence of such a model is the meaning of good.

Here is the meaning of stability in this context:

**Definition 2.2.** $D$ is $\lambda$-stable if $|S_D(A)| \leq \lambda$ for each $D$-set of size $\lambda$.

In [Le1], a notion of rank is introduced which is shown to be bounded under $\omega$-stability. $D$ is said to be totally transcendental if the rank is bounded. In the rest of this paper, we assume that $D$ is a totally transcendental diagram. We already established in [Le1] that many of the axioms of the previous section hold for totally transcendental diagrams (notably the properties of the independence relation and the existence of primary models) and facts from [Le1] will be used freely. We will now develop what is referred to as orthogonality calculus for this context and show that the remaining axioms used to obtain an abstract decomposition theorem also hold for the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram $D$.

Notice that homogeneity implies that the notion of $\text{ga-t}\text{p}(\bar{a}/M, N)$ coincides with $\text{tp}(\bar{a}/M, N)$.

The next few lemmas show Dominance.
First, for \(D\)-sets \(A\) and \(B\), we say that \(A \subseteq_{TV} B\), if every \(D\)-type over finitely many parameters in \(A\) realized in \(B\) is realized in \(A\). The subscript TV stands for Tarski-Vaught.

**Lemma 2.3.** Let \(M\) be \((D, \mathcal{N}_0)\)-homogeneous. Suppose \(\bar{a} \subseteq \bar{b}\). Then, for every \(\bar{m} \in M\) the type \(\text{tp}(\bar{b}/\bar{m}\bar{a})\) is realized in \(M\).

**Proof.** By symmetry, \(\bar{b} \subseteq \bar{a}\). Hence, by taking a larger \(\bar{m}\) if necessary, we may assume that \(\text{tp}(\bar{b}/M\bar{a})\) does not split over \(\bar{m}\). By \((D, \mathcal{N}_0)\)-homogeneity of \(M\), we can find \(\bar{b}' \in M\), such that \(\text{tp}(\bar{b}/\bar{m}) = \text{tp}(\bar{b}'/\bar{m})\). We claim that \(\text{tp}(\bar{b}/\bar{m}\bar{a}) = \text{tp}(\bar{b}'/\bar{m}\bar{a})\). If not, there exists a formula \(\varphi(\bar{x}, \bar{m}, \bar{a})\) such that \(\models \varphi[\bar{b}, \bar{m}, \bar{a}]\) and \(\not\models \varphi[\bar{b}', \bar{m}, \bar{a}]\). But, \(\text{tp}(\bar{b}/\bar{m}) = \text{tp}(\bar{b}'/\bar{m})\), so \(\text{tp}(\bar{a}/M\bar{b})\) splits over \(\bar{m}\), a contradiction.

The next lemma is standard. Recall that \(p \in S(A)\) is \(D_{\mathcal{N}_0}^\text{p}\)-isolated if \(p \in S_D(A)\) and is \(\mathcal{P}_{\mathcal{N}_0}^\text{p}\)-isolated.

**Lemma 2.4.** Let \(A, B\) be \(D\)-sets such that \(A \subseteq_{TV} B\). If \(\text{tp}(\bar{c}/A)\) is \(D_{\mathcal{N}_0}^\text{p}\)-isolated, then \(\text{tp}(\bar{c}/A) \vdash \text{tp}(\bar{c}/B)\).

**Proof.** Let \(q(\bar{x}, \bar{a}) \vdash \text{tp}(\bar{c}/A)\), with \(\bar{a} \in A\). Suppose that \(\text{tp}(\bar{c}/A) \not\vdash \text{tp}(\bar{c}/B)\).

Then, there exists \(\bar{b} \in B\) and a formula \(\varphi(\bar{x}, \bar{y})\) such that \(q(\bar{x}, \bar{a}) \cup \varphi(\bar{x}, \bar{b})\) and \(q(\bar{x}, \bar{a}) \cup \neg \varphi(\bar{x}, \bar{b})\) are both realized in \(\mathcal{C}\). By assumption, there exists \(\bar{b}' \in A\) realizing be such that \(\text{tp}(\bar{b}/\bar{a}) = \text{tp}(\bar{c}/\bar{c})\). Hence, by an automorphism fixing \(\bar{a}\) and sending \(\bar{b}\) to \(\bar{b}'\), both \(q(\bar{x}, \bar{a}) \cup \varphi(\bar{x}, \bar{b}')\) and \(q(\bar{x}, \bar{a}) \cup \neg \varphi(\bar{x}, \bar{b}')\) are realized in \(\mathcal{C}\). This contradicts the choice of \(q(\bar{x}, \bar{a})\).

A model is \(M\) is \(D_{\mathcal{N}_0}^\text{p}\)-primary over \(A\) if \(M = A \cup \{a_i : i < \lambda\}\) and \(\text{tp}(a_i/A \cup \{a_j : j < i\})\) is \(D_{\mathcal{N}_0}^\text{p}\)-isolated for each \(i < \lambda\). We denote by \(M(A)\) the \(D_{\mathcal{N}_0}^\text{p}\)-primary model over \(M \cup A\).

**Theorem 2.5 (Dominance).** Let \(M\) be \((D, \mathcal{N}_0)\)-homogeneous and \(A\) be a \(D\)-set. For each \(B\), if \(A \subseteq B\), then \(M(A) \subseteq B\).

**Proof.** By finite character of independence, it is enough to show that if \(\bar{a} \subseteq \bar{b}\), then \(\bar{c} \subseteq \bar{b}\), for each finite \(\bar{c} \in M(\bar{a})\). Let \(\bar{c} \in M(\bar{a})\) be given.

Then \(\text{tp}(\bar{c}/M\bar{a})\) is \(D_{\mathcal{N}_0}^\text{p}\)-isolated. Hence, by assumption and Lemma 2.3, \(\text{tp}(\bar{c}/M\bar{a}) \vdash \text{tp}(\bar{c}/M\bar{a}\bar{b})\). Therefore, \(\bar{c} \subseteq \bar{b}\).

Recall the definition of orthogonality.

**Definition 2.6.** Let \(p \in S_D(B)\) and \(q \in S_D(A)\) be stationary. We say that \(p\) is orthgonal to \(q\), written \(p \perp q\), if for every \(D\)-model \(M\) containing \(A \cup B\) and for every \(a \models p_M\) and \(b \models q_M\), we have \(a \perp b\).

Recall the following technical lemma in [Le1].
Lemma 2.7. Let \( p, q \in S_D(M) \) and \( M \subseteq N \) be in \( \mathcal{K} \). If \( a \perp b \) for every \( a \models q \) and \( b \models p \), then \( a \perp b \) for every \( a \models q_N \) and \( b \models p_N \).

Then, by the previous lemma we can immediately simplify the definition: for \( p, q \in S_D(M) \), we have \( p \perp q \) if and only if \( \bar{a} \perp \bar{b} \) for every \( a \models p \) and \( b \models q \).

The following lemma is also in [Le1].

Lemma 2.8. Let \( M \) be \((D, \mathcal{S}_0)\)-homogeneous. If \( \bar{a} \perp \bar{b} \) and \( \text{tp}(\bar{a}/M\bar{b}) \) is \( D^s_{\mathcal{S}_0} \)-isolated, then \( \bar{a} \subseteq M \).

Lemma 2.9. Let \( \text{tp}(\bar{a}/M\bar{b}) \) be isolated, and \( \text{tp}(\bar{b}/M) \) be regular. Suppose that \( \bar{a} \not\perp \bar{b} \). Then, for any \( \bar{c} \) if \( \bar{a} \perp \bar{c} \), then \( \bar{b} \perp \bar{c} \).

Proof. Suppose that \( \bar{b} \not\perp \bar{c} \). By symmetry, we have that \( \bar{c} \not\perp \bar{b} \). Choose \( q(\bar{x}, \bar{m}, \bar{b}) \subseteq \text{tp}(\bar{c}/M\bar{b}) \) such that

\[
R[q(\bar{z}, \bar{m}, \bar{b})] = R[\text{tp}(\bar{c}/M\bar{b})] < R[\text{tp}(\bar{c}/M)].
\]

Without loss of generality, since \( \bar{a} \not\perp \bar{b} \), we can choose \( p(\bar{x}, \bar{m}, \bar{b}) \subseteq \text{tp}(\bar{a}/M\bar{b}) \) be such that

\[
R[p(\bar{y}, \bar{m}, \bar{a})] = R[\text{tp}(\bar{b}/M\bar{a})] < R[\text{tp}(\bar{b}/M)].
\]

and also

\[
R[p(\bar{b}, \bar{m}, \bar{x})] = R[\text{tp}(\bar{a}/M\bar{b})] < R[\text{tp}(\bar{a}/M)].
\]

Choose \( \bar{c}' \in M \) such that \( \text{tp}(\bar{c}/\bar{m}) = \text{tp}(\bar{c}'/\bar{m}) \). Since \( \bar{a} \perp \bar{c} \), we have in particular that \( \text{tp}(\bar{a}/M\bar{c}) \) does not split over \( \bar{m} \) so that \( \text{tp}(\bar{c}/\bar{m}\bar{a}) = \text{tp}(\bar{c}'/\bar{m}\bar{a}) \). Thus, \( \bar{b} \) realizes the following type

\[(*) \quad p(\bar{y}, \bar{m}, \bar{a}) \cup q(\bar{a}, \bar{m}, \bar{y}) \cup \text{tp}(\bar{b}/\bar{m}).\]

Since \( \text{tp}(\bar{a}/M\bar{b}) \) is isolated, we may assume that \( M(\bar{a}) \subseteq M(\bar{b}) \). Now choose \( \bar{b}' \in M(\bar{a}) \) realizing \((*)\). If \( \bar{b}' \in M \), then \( R[\text{tp}(\bar{a}/M)] < R[\text{tp}(\bar{a}/\bar{m}\bar{b}')] = R[p(\bar{y}, \bar{m}, \bar{x})] \), a contradiction. Hence \( \bar{b}' \not\in M \) and so \( \bar{b}' \not\perp \bar{b} \), by the previous \( M \) lemma. Thus \( \text{tp}(\bar{b}'/M) \) extends \( \text{tp}(\bar{b}/\bar{m}) \) and is not orthogonal to it, thus since \( \text{tp}(\bar{b}/M) \) is regular based on \( \bar{m} \), we must have \( \text{tp}(\bar{b}'/M) = \text{tp}(\bar{b}/M) \).

This is a contradiction, since then, \( \bar{b}' \) realizes \( q(\bar{c}', \bar{m}, \bar{y}) \).

The next corollary is Equivalence.

Corollary 2.10 (Equivalence). Let \( M \in \mathcal{K} \), let \( p, q \in S_D(M) \) be regular, and let \( \bar{b} \not\in M \) realize \( p \). Then \( q \) is realized in \( M(\bar{b}) \setminus M \) if and only if \( p \not\perp q \).
Proof. Let \( \bar{b} \in M \) realize \( p \). Let \( M(\bar{b}) \) be \( D_{N_0}^s \)-primary over \( M \cup \bar{b} \).

Let \( \bar{a} \in M(\bar{b}) \setminus M \). Then \( \text{tp}(\bar{a}/M\bar{b}) \) is \( D_{N_0}^s \)-isolated. If \( p \perp q \), then \( \bar{b} \perp \bar{a} \).

Hence, by symmetry \( \bar{a} \perp \bar{b} \), and so \( \bar{a} \in M \), by Lemma 2.8, a contradiction. For the converse, suppose that \( p \not\perp q \). This implies that there is \( \bar{a} \models q \) such that

\[
\bar{a} \not\perp \bar{b}.
\]

Let \( q(\bar{x}, \bar{m}, \bar{b}) \subseteq \text{tp}(\bar{a}/M\bar{b}) \) be such that

\[
R[q(\bar{x}, \bar{m}, \bar{b})] = R[\text{tp}(\bar{a}/M\bar{b})] < R[q].
\]

Since \( q \) is regular, we may further assume that \( q \) is based on \( \bar{m} \). Thus, the element \( \bar{a} \) realizes the type

\[
(\ast) \quad q(\bar{x}, \bar{m}, \bar{b}) \cup q \upharpoonright \bar{m}.
\]

Since \( M(\bar{b}) \) is in particular \((D, N_0)\)-homogeneous, there is \( \bar{a}' \in M(\bar{b}) \) realizing the type \((\ast)\). Since \( M(\bar{b}) \) is \((D, N_0)\)-primary, we must have that \( \text{tp}(\bar{a}'/M\bar{b}) \) is isolated. Thus, since \( \bar{b} \not\perp \bar{a} \), we must have by the Lemma 2.9 that \( \bar{a}' \not\perp \bar{a} \).

This implies that \( \text{tp}(\bar{a}'/M) \) is an extension of the regular type \( q \upharpoonright \bar{m} \) which is not orthogonal to \( q \). Hence, since \( q \) is regular, we must have \( q = \text{tp}(\bar{a}'/M) \).

This shows that \( q \) is realized (by \( \bar{a}' \)) in \( M(\bar{b}) \).

We encountered Morley sequences when we talked about stationary types in the previous chapter. The definition can be made for any type.

Definition 2.11. Let \( p \in S_D(A) \). We say that \( \langle \bar{a}_i \mid i < \omega \rangle \) is a Morley sequence for \( p \) if

1. The sequence \( \langle \bar{a}_i \mid i < \omega \rangle \) is indiscernible over \( A \);
2. For every \( i < \omega \) we have \( \bar{a}_i \perp A \cup \{ \bar{a}_j \mid j < i \} \).

The next fact was established in the previous chapter.

Fact 2.12. If \( p \in S_D(A) \) is stationary, then there is a Morley sequence for \( p \).

The next theorem is Axiom (Perp I).

Theorem 2.13 (Perp I). Let \( p \in S_D(N) \) be regular, \( M \subseteq N \). Then \( p \perp M \) if and only if \( p \perp q \), for every regular \( q \in S(M) \).

Proof. One direction is obvious. Suppose that \( p \not\perp M \). We will find a regular type \( q \in S_D(M) \) such that \( p \not\perp q \).

Since \( p \) is regular, there exists a finite set \( \bar{f} \subseteq N \) such that \( p \) is regular over \( \bar{f} \). Write \( p(\bar{x}, \bar{f}) \) for the stationary type \( p\bar{f} \). Also, there exists a finite set \( \bar{e} \subseteq M \) such that \( \text{tp}(\bar{f}/M) \) is based on \( \bar{e} \). Since \( p \not\perp M \), there exists a
stationary type \( r \in S(M) \) such that \( p \not\subseteq r \). By monotonicity, we can find \( \bar{a} \models p, \bar{b} \models r_N \) such that \( \bar{a} \not\subseteq \bar{f} \). 

Since \( M \) is \((D, \aleph_0)\)-homogeneous, there exists \( \langle \bar{f}_i \mid i < \omega \rangle \subseteq M \), a Morley sequence for \( \text{tp}(\bar{f}/\bar{e}) \). Let \( p_i := p(\bar{x}, \bar{f}_i)_M \). This is well-defined since \( p(\bar{x}, \bar{f}) \) is stationary and \( \text{tp}(\bar{f}/\bar{e}) = \text{tp}(\bar{f}_i/\bar{e}) \), so \( p(\bar{x}, \bar{f}_i) \) is stationary.

For each \( i < \omega \), we can choose \( M_i \subseteq N \) such that there is an automorphism \( g_i \) with \( g_i(\bar{f}) = \bar{f}_i, g_i(\bar{c}) = \bar{c} \) and \( g_i(M) = M_i \). Since \( p_{M_i} \) is regular and \( p_i = g_i^{-1}(p_{M_i}) \), then

\[
(*) \quad p_i \text{ is regular, for each } i < \omega.
\]

A similar reasoning using an automorphisms sending \( \bar{f} \bar{f}_0 \) to \( \bar{f}_i \bar{f}_j \) shows that

\[
(**) \quad p \perp p_0 \text{ implies } p_i \perp p_j, \text{ for every } i \neq j < \omega.
\]

Finally, using the fact that \( p \not\subseteq r \), we can derive

\[
(***) \quad p_i \not\subseteq r, \text{ for every } i < \omega.
\]

If we show that \( p \not\subseteq p_0 \), then \((*)\) implies the conclusion of the lemma. Suppose, for a contradiction, that \( p \perp p_0 \). By \((***)\) we can find \( \bar{b}' \models r \) and \( \bar{a}_i \models p_i \), such that \( \bar{b}' \not\subseteq \bar{a}_i \) and \( \bar{a}_i \not\subseteq M \), for each \( i < \omega \). Now \((***)\) implies that \( \bar{a}_{j+1} \perp \{\bar{a}_i \mid i \leq j\} \), for every \( j < \omega \). Hence, by \((*)\) and Lemma 2.8, we have \( \bar{a}_{i+1} \not\subseteq M_i \), where \( M_i \) is \( D^g_{\aleph_0} \)-primary over \( M \cup \{\bar{a}_j \mid j < i\} \). Let \( N \) be \( D^g_{\aleph_0} \)-primary over \( M \cup \{\bar{a}_j \mid j < \omega\} \). Since \( \kappa(D) = \aleph_0 \), there exists \( n < \omega \) such that \( \bar{b}' \perp N \). Hence, by monotonicity, \( \bar{b}' \perp \bar{a}_n \). By symmetry over \( M_n \), \( \bar{a}_n \perp M \), \( M_n \models \bar{a}_n \perp \bar{b}' \). But \( \bar{a}_n \perp \{\bar{a}_i \mid i < n\} \), and so \( \bar{a}_n \perp M_n \), by dominance and symmetry. Hence, by transitivity of the independence relation, we have \( \bar{a}_n \perp M \), so \( \bar{b}' \perp \bar{a}_n \), a contradiction.

We now prove two additional lemmas that will be used in the next section.

**Lemma 2.14.** If \( p \in S(M_1) \) is regular, \( p \perp M_0 \), and \( M_1 \perp M_2 \), then \( p \perp M_2 \).

**Proof.** Suppose that \( p \not\subseteq M_2 \). Then, by definition, there exists \( q \in S(M_2) \) such that \( p \not\subseteq q \). By definition, there is \( N \supseteq M_1 \cup M_2 \) such that

\[
(*) \quad p_N \not\subseteq q_N.
\]

We are going to find a type \( q' \in S(M_0) \) such that \( p \not\subseteq q' \).

Since \( p \) and \( q \) are stationary, there exist finite sets \( \bar{c} \subseteq M_1, \bar{d} \subseteq M_2 \), and \( \bar{e} \subseteq M_0 \) such that \( p \) is based on \( \bar{c}, q \) is based on \( \bar{d}, \) and both \( \text{tp}(\bar{c}/M_0) \) and \( \text{tp}(\bar{d}/M_0) \) are based on \( \bar{e} \).
By (**) and finite character, there exist a set $F \subseteq N$, and $\bar{a}, \bar{b}$ such that
\[
\bar{a} \models p_{M_1M_2F}, \quad \bar{b} \models q_{M_1M_2F}, \quad \text{but} \quad \bar{a} \not\models \bar{b}.
\]

By monotonicity, we may assume that $\bar{\alpha}\bar{a}\bar{e} \subseteq F$. Since $\text{tp}(\bar{a}\bar{b}/N)$ is stationary, we may also assume that $\text{tp}(\bar{a}\bar{b}/M_1M_2F)$ is stationary based on $F$. Finally, we may further assume that $R[\text{tp}(\bar{a}/\bar{c})] < R[\text{tp}(\bar{a}/\bar{c}F)]$.

Since $M_0$ is $(D, \aleph_0)$-homogeneous, we can choose $\bar{d}' \in M_0$ such that $\text{tp}(\bar{d}'/\bar{e}) = \text{tp}(\bar{d}/\bar{e})$. By stationarity, we have $\text{tp}(\bar{c}\bar{d}\bar{e}/\emptyset) = \text{tp}(\bar{c}\bar{d}\bar{e}/\emptyset)$. Now choose $F' \subseteq M_1$ such that $\text{tp}(\bar{c}\bar{d}\bar{e}F'/\emptyset) = \text{tp}(\bar{c}\bar{d}\bar{e}F/\emptyset)$. Finally, let $\bar{a}'\bar{b}' \in \mathcal{C}$ such that $\text{tp}(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}F'/\emptyset) = \text{tp}(\bar{a}'\bar{b}'\bar{c}\bar{d}\bar{e}F'/\emptyset)$.

By invariance under automorphism, we have $R[\text{tp}(\bar{a}'/\bar{c})] = R[\text{tp}(\bar{a}'/F')]$ and $R[\text{tp}(\bar{b}'/\bar{d}')] = R[\text{tp}(\bar{b}'/F')]$, since these statements are true without the apostrophe $'$.

Now let $q' := \text{tp}(\bar{b}'/\bar{d}')_{M_0} \in S(M_0)$. Such a type exists since $\text{tp}(\bar{b}'/\bar{d}')$ is stationary. We claim that $p \not\models q'$. Otherwise, by the previous remark, we have $p \perp q'_{M_1}$. Now, let $\bar{a}''\bar{b}'' \models \text{tp}(\bar{a}'\bar{b}'/F')$. We have $\bar{a}'' \models p$, $\bar{b}'' \models q'_{M_1}$ and so $\bar{a}'' \models p_{M_1\bar{b}''}$. But then $R[\text{tp}(\bar{a}'/\bar{c})] = R[\text{tp}(\bar{a}'\bar{b}'F')]$ This contradicts the fact that $\text{tp}(\bar{a}'\bar{b}'/F') = \text{tp}(\bar{a}''\bar{b}'/F')$.

**Lemma 2.15.** Let $p, q \in S_D(M)$ be regular. Let $\bar{a} \not\in M$ realize $p$. If $p \not\models q$, then there exists $\bar{b} \in M(\bar{a}) \setminus M$ realizing $q$ such that $M(\bar{a}) = M(\bar{b})$.

**Proof.** By equivalence, there exists $\bar{b} \in M(\bar{a}) \setminus M$ realizing $q$. By definition of prime, it is enough to show that $\text{tp}(\bar{a}/M\bar{b})$ is $D_{\aleph_0}^s$-isolated.

Let $\bar{c} \in M$ be finite such that $p$ is regular over $\bar{c}$, and write $p(x, \bar{c}) = p[\bar{c}]$. Now, since $\text{tp}(\bar{b}/M\bar{a})$ is $D_{\aleph_0}^s$-isolated, there exists $r_1(\bar{y}, \bar{a})$ over $M$ isolating $\text{tp}(\bar{b}/M\bar{a})$. By a previous lemma, we know that $\bar{a} \not\models \bar{b}$, so let $r_2(\bar{x}, \bar{b})$ witness this. We claim that the following type isolates $\text{tp}(\bar{a}/M\bar{b})$:

\[
(*) \quad p(\bar{x}, \bar{c}) \cup r_1(\bar{b}, \bar{x}) \cup r_2(\bar{x}, \bar{b}).
\]

Let $\bar{a}' \in M(\bar{a})$ realize $(*)$. Then, $\bar{a}' \not\in M$ by choice of $r_1$. Hence, $\bar{a} \not\models \bar{a}'$

so by choice of $p(\bar{x}, \bar{c})$, we have $\text{tp}(\bar{a}'/M) = \text{tp}(\bar{a}/M)$. Thus, $\text{tp}(\bar{a}/M\bar{b}) = \text{tp}(\bar{a}'/M\bar{b})$ using $r_2(\bar{a}', \bar{y})$.

We can now show using the language of Section 1.

**Theorem 2.16.** Let $\mathcal{K}$ be the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram $D$. Let $N \in \mathcal{K}$ have NDOP. Then $N$ has a complete decomposition.

**Proof.** All the axioms of Section 1 have been checked for $\mathcal{K}$.

**Remark 2.17.** Similarly to the methods developed in this section for the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram $D$, we can check all the axioms for the class of $(D, \mu)$-homogeneous models of
a totally transcendental diagram $D$, for any infinite $\mu$. This implies that if $\mathcal{K}$ is the class of $(D, \mu)$-homogeneous models of a totally transcendental diagram $D$ and if $N \in \mathcal{K}$ has NDOP, then $N$ has a complete decomposition (in terms of models of $\mathcal{K}$).

3. DOP IN FINITE DIAGRAMS

Let $\mathcal{K}$ be the class of $(D, \aleph_0)$-homogeneous models of a totally transcendental diagram. In the language of the axiomatic framework, we take $\mathfrak{C}$ to be a large homogeneous model. We say that $\mathcal{K}$ satisfies DOP if $\mathcal{K}$ does not have NDOP. Recall that $\lambda(D)$, the first stability cardinal, is $|D| + |T|$.

**Claim 3.1.** Suppose that $\mathcal{K}$ has DOP. Then there exists $M, M_i, M' \in \mathcal{K}$ for $i = 1, 2$ such that

1. $M_1 \perp M_2$;

2. $M'$ is prime over $M_1 \cup M_2$;

3. $|M'| = \lambda(D)$;

4. $M_i = M(\bar{a}_i)$, for $i = 1, 2$;

5. There exists a regular type $p \in S(M')$ such that $p \perp M_i$, for $i = 1, 2$;

6. The type $p$ is based on $\bar{b}$ and $\text{tp}(\bar{b}/M_1 \cup M_2)$ is isolated over $\bar{a}_1 \bar{a}_2$.

**Proof.** By assumption, there exists $\mathcal{K}$ fails to have NDOP. Then, there exist $M_i \in \mathcal{K}$ for $i \leq 2$ with $M_1 \perp M_2$, there exists $M''$ which is $D_{\aleph_0}^s$-primary $M_0$ over $M_1 \cup M_2$ and there exists a regular type $p \in S(M'')$ such that $p \perp M_i$, for $i = 1, 2$.

Let $\bar{b} \in M''$ be a finite set such that $p$ is based on $\bar{b}$. Let $\bar{a}_i \in M_i$, for $i = 1, 2$ be such that $\text{tp}(\bar{b}/M_1 \cup M_2)$ is $D_{\aleph_0}^s$-isolated over $\bar{a}_1 \bar{a}_2$. Let $M \in \mathcal{K}$, $M \subseteq M_0$ of cardinality $\lambda(D)$ be such that $\bar{a}_1 \perp M_0$. Such a model exists using local character and prime models. Let $M(\bar{a}_i)$ be prime over $M \cup \bar{a}_i$, for $i = 1, 2$. Then, by Dominance, Transitivity, and Monotonicity, we have $M(\bar{a}_1) \perp M(\bar{a}_2)$. By axiom on prime there exists $M' \subseteq M''$ prime over $M M(\bar{a}_1) \cup M(\bar{a}_2)$. We may assume that $B \subseteq M'$. Let $p' = p | M'$. Then $p' \in S(M')$ is regular based on $\bar{b}$ and $p'_{M''} = p$. It remains to show that $p' \perp M(\bar{a}_i)$, for $i = 1, 2$. Let $r \in S(M(\bar{a}_i)$ be regular. Then $r_{M_i}$ is regular by our axiom. Furthermore, by definition, $p' \perp r$ and only if $p' \perp r_{M'}$. By Parallelism, since $M' \subseteq M''$, it is equivalent to show that $p \perp r_{M''}$. But, $r_{M_i} \in S(M_i)$ is regular, $p \perp M_i$, and $r_{M''} = (r_{M_i})_{M''}$. Therefore, by choice of $p$ we have $p \perp r_{M''}$, which finishes the proof.

Let $\mu > \lambda(D)$ be a cardinal (for the following construction, we may have $\mu \geq \lambda(D)$, but the strict inequality is used in the last claim). Let $\langle M_i \mid i < \mu \rangle$ be independent over a model $M \subseteq M_i$. Suppose that $|M_i| = \lambda(D)$. Let $R \subseteq [\mu]^2$ and suppose that $M_s = M(M_i \cup M_j)$, for $s = (i, j)$. Such a model
exist for each \( s \in [\mu]^2 \) by the axioms on prime. Then, by Dominance and the axiom on primes, the following system is independent:

\[
\langle M_i \mid i < \mu \rangle \cup \{ M \} \cup \langle M_s \mid s \in R \rangle.
\]

Hence, there exists a model \( M_R \) prime over \( \bigcup_{i<\mu} M_i \cup \bigcup_{s \in R} M_s \).

Let \( s = (i, j) \) and suppose that there exists a regular type \( p_s \in S(M_s) \) such that \( p_s \perp M_i, p_s \perp M_j \). Let \( I_s \) be a Morley sequence for \( p_s \) of length \( \mu \). (Such a sequence exists since \( \mathcal{E} \) is \( (D, \mu^+)-\)homogeneous. Then, by Dominance, definition of a Morley sequence, and axiom on prime, there exists \( N_s = M_s(I_s) \).

The next claim will allow us to choose prime models over complicated independent systems with some additional properties.

**Claim 3.2.** The system

\[
S_R = \langle M_i \mid i < \mu \rangle \cup \{ M \} \cup \langle N_s \mid s \in R \rangle
\]

is an independent system.

**Proof.** By definition, we must show that \( N_s \downarrow A \), when \( A = \bigcup_{t \in R, t \neq s} N_t \).

By finite character, it is enough to show this for \( R \) finite. We prove this by induction on the cardinality of \( R \). When \( R \) is empty or has at most one element, there is nothing to do. Suppose that \( R = \{ s_i \mid i \leq n \} \cup \{ s \} \).

We show that we can replace \( M_{s_i} \) by \( N_{s_i} \) and \( M_s \) by \( N_s \) and still have an independent system. By (\(*\)), it is enough to show that if \( M_s \downarrow A \), then \( N_s \downarrow A \), for \( A = \bigcup_{i \leq n} N_{s_i} \). Using the axioms of the independence relation, it is enough to show that \( N_s \downarrow A \). By induction hypothesis, we have

\[
N_s \downarrow \bigcup_{M_{s_i} \subset M} N_{s_i} \quad \text{and} \quad N_s \downarrow \bigcup_{M_{s_n} \subset M} N_{s_i}.
\]

Now, either \( s \cap s_n \) is empty so \( M_s \downarrow M_{s_n} \) by (\(*\)) or they extend \( j \) and so \( M_s \downarrow M_{s_n} \), by (\(*\)) again. Since \( M \subseteq M_j \), in either case, \( p_s \perp M_j \), by choice of \( p_j \). Hence \( p_s \perp M_{s_n} \) using Lemma 2.14. By induction hypothesis, there exists \( N' \) a prime model over \( \bigcup_{i \leq n} N_{s_i} \). Hence, by (\(**\)) and Dominance \( N_s \downarrow N' \) and \( N_{s_n} \downarrow N' \). Hence, using again by Lemma 2.14, we have \( p_s \perp N' \). Thus, \( I_s \downarrow N' \) and \( I_s \downarrow N_{s_n} \). Therefore \( I_s \downarrow N' \cup N_{s_n} \). By Dominance \( N_s \downarrow N' \cup N_{s_n} \). We are done by monotonicity. \( \dashv \)

We will now use DOP to construct systems as in the claim.

Let the situation be as in the first claim. Write \( p(\bar{x}, \bar{b}) = p \upharpoonright \bar{b} \). Let \( \langle \bar{a}_1 \bar{a}_2 \alpha < \mu \rangle \) be a Morley sequence for \( \text{tp}(\bar{a}_1 \bar{a}_2 / M) \). Such a Morley sequence
exists by assumption on \( \mathcal{C} \) and stationarity over models. Let \( M_i^\alpha \) be prime over \( M \cup \bar{a}_i^\alpha \), for \( i = 1, 2 \). Such a prime model exists by the axioms. Then \( M_i^\alpha \downarrow M_2^\beta \) for every \( \alpha < \beta \), by Dominance. By axiom on prime there exists \( M^{\alpha \beta} \) prime over \( M_1^\alpha \cup M_2^\beta \). Let \( \bar{b}^{\alpha \beta} \) be the image of \( \bar{b} \) in \( M^{\alpha \beta} \). Let \( p^{\alpha \beta} = p(\bar{x}, \bar{b}^{\alpha \beta})_{M^{\alpha \beta}} \in S(M^{\alpha \beta}) \), which exists and is regular since \( p \) is based on \( \bar{b} \). Thus, \( p^{\alpha \beta} \downarrow M_1^\alpha \) and \( p^{\alpha \beta} \downarrow M_2^\beta \). Let \( I^{\alpha \beta} \) be a Morley sequence of length \( \mu \) for \( p^{\alpha \beta} \). Let \( N^{\alpha \beta} \) be prime over \( M^{\alpha \beta} \cup I^{\alpha \beta} \). Then, for the claim, for each \( R \subseteq [\mu]^2 \), the system

\[
S_R = \{ M \} \cup \langle M_i^\alpha : \alpha < \mu, i = 1, 2 \rangle \cup \langle N^{\alpha \beta} : (\alpha, \beta) \in R \rangle
\]

is an independent system. Hence, there exists \( M_R \) prime over it.

The final claim explains the name of Dimensional Order Property: It is possible to code the relation \( R \) (in particular an order in the following theorem) by looking at dimensions of indiscernibles in a model \( M_R \). Note that the converse holds also, namely that the following property characterises DOP (we do not prove this fact as it is not necessary to obtain the main gap). Recall \( \mu > \lambda(D) \).

**Claim 3.3.** The pair \( (\alpha, \beta) \in R \) if and only if there exists \( \bar{c} \in M_R \) with the property that \( tp(\bar{a}_1 \bar{a}_2 \bar{b} / \emptyset) = tp(\bar{a}_1 \bar{a}_2 \bar{c} / \emptyset) \) and for every prime \( M^* \subseteq M_R \) over \( M \cup \bar{a}_1 \bar{a}_2 \) containing \( \bar{c} \) there exists a Morley sequence for \( p(\bar{x}, \bar{c})_{M^*} \) of length \( \mu \).

**Proof.** If the pair \( (\alpha, \beta) \in R \), then \( p^{\alpha \beta} \) is based on \( \bar{b}^{\alpha \beta} \). Furthermore, \( I^{\alpha \beta} \) is a Morley sequence of length \( \mu \) for \( p^{\alpha \beta} \) in \( M_R \). Let \( M' \) be prime over \( M \cup \bar{a}_1 \bar{a}_2 \) containing \( \bar{b}^{\alpha \beta} \), then \( p(\bar{x}, \bar{b}^{\alpha \beta})_{M'} \) is realized by every element of \( I^{\alpha \beta} \) except possibly \( \lambda(D) \) many. Hence, there exists a Morley sequence of length \( \mu \), since \( \mu > \lambda(D) \).

For the converse, let \( \alpha < \beta < \mu \) be given such that \( (\alpha, \beta) \notin R \). Let \( t = (\alpha, \beta) \). Let \( \bar{c} \subseteq M_R \) finite as in the claim. By using an automorphism, we have that \( tp(\bar{c} / \bar{a}_1 \bar{a}_2) \) isolates \( tp(\bar{c} / M_1^\alpha M_2^\beta) \) and hence there exists \( M_t \subseteq M_R \) prime over \( M_1^\alpha M_2^\beta \) containing \( \bar{c} \). By assumption on \( \bar{c} \), there exists \( I \subseteq M_R \) a Morley sequence for \( p(\bar{x}, \bar{c})_{M_t} \) of length \( \mu \). Let \( N_t \) be prime over \( M_t(I) \), which exists by assumption on prime. By the previous claim, the following system is independent

\[
\{ M \} \cup \langle M_i^\alpha : \alpha < \mu, i = 1, 2 \rangle \cup \langle N^{\alpha \beta} : (\alpha, \beta) \in R \rangle \cup \{ N_t \}.
\]

Thus, in particular \( N_t \downarrow \bigcup_{i<\mu} M_i \cup \bigcup_{s \in R} N_s \), so \( \bar{a} \downarrow \bigcup_{i<\mu} M_i \cup \bigcup_{s \in R} N_s \), for each \( \bar{a} \in I \). By Dominance \( \bar{a} \downarrow M_R \) and so \( \bar{a} \in M_t \). This is a contradiction.

All the technology is now in place to apply the methods of [Sh c] or [GrHa] with the previous claim and to derive:
Theorem 3.4. Suppose that $\mathcal{K}$ has DOP. Then, $\mathcal{K}$ contains $2^\lambda$ nonisomorphic models of cardinality $\lambda$, for each $\lambda > |D| + |T|$.

Theorem 3.5. Suppose that the class of $(D, \mu)$-homogeneous models of a totally transcendental diagram $D$ has DOP. Then, for each $\lambda > |D| + |T| + \mu$ there are $2^\lambda$ nonisomorphic $(D, \mu)$-homogeneous models of cardinality $\lambda$.

4. DEPTH AND THE MAIN GAP

We have now showed that if every model $(D, \mathbb{N}_0)$-homogeneous model of a totally transcendental diagram $D$ has NDOP, then every such model admits a decomposition. We will introduce an equivalence between decompositions, as well as the notion of depth, in order to compute the spectrum function for $\mathcal{K}$. Most of the treatment will be done under the assumption that $\mathcal{K}$ has NDOP.

Definition 4.1. We say that $\mathcal{K}$ has NDOP if every $N \in \mathcal{K}$ has NDOP.

We introduce the depth of a regular type.

Definition 4.2. Let $p \in S_D(M)$ be regular. We define the depth of $p$, written $\text{Dep}(p)$. The depth $\text{Dep}(p)$ will be an ordinal, $-1$, or $\infty$ and we have the usual ordering $-1 < \alpha < \infty$ for any ordinal $\alpha$. We define the relation $\text{Dep}(p) \geq \alpha$ by induction on $\alpha$.

1. $\text{Dep}(p) \geq 0$ if $p$ is regular;
2. $\text{Dep}(p) \geq \delta$, when $\delta$ is a limit ordinal, if $\text{Dep}(p) > \alpha$ for every $\alpha < \delta$;
3. $\text{Dep}(p) \geq \alpha + 1$ if there exists $\bar{a}$ realizing $p$ and a regular type $r \in S_D(M(\bar{a}))$ such that $r \perp M$ and $\text{Dep}(r) \geq \alpha$.

We write:

$\text{Dep}(p) = -1$ if $p$ is not regular;
$\text{Dep}(p) = \alpha$ if $\text{Dep}(p) \geq \alpha$ but it is not the case that $\text{Dep}(p) \geq \alpha + 1$;
$\text{Dep}(p) = \infty$ if $\text{Dep}(p) \geq \alpha$ for every ordinal $\alpha$.

We let $\text{Dep}(\mathcal{K}) = \sup \{ \text{Dep}(p) + 1 \mid M \in \mathcal{K}, p \in S_D(M) \}$. This is called the depth of $\mathcal{K}$.

Lemma 4.3. Let $p \in S_D(M)$ be regular with $\text{Dep}(p) < \infty$. Let $\bar{a} \models p$ with $r \in S_D(M(\bar{a}))$ regular with $r \perp M$. Then $\text{Dep}(r) < \text{Dep}(p)$.

Proof. This is obvious, by definition of depth, if $\text{Dep}(r) = \text{Dep}(p)$ is as above, then $\text{Dep}(p) \geq \text{Dep}(p) + 1$, contradicting $\text{Dep}(p) < \infty$. \(\square\)

Lemma 4.4. Let $p \in S_D(M)$ be regular. If $\text{Dep}(p) < \infty$ and $\alpha \leq \text{Dep}(p)$, then there exists $q$ regular such that $\text{Dep}(q) = \alpha$.

Proof. By induction on $\text{Dep}(p)$. For $\text{Dep}(p) = 0$ it is clear. Assume that $\text{Dep}(p) = \beta + 1$. Let $\bar{a} \models p$ and let $r \in S_D(M(\bar{a}))$ be such that $r \perp M$ and $\text{Dep}(r) \geq \beta$. Then, by the previous lemma, $\text{Dep}(r) = \beta$. Hence, we are done by induction. Assume that $\text{Dep}(p) = \delta$, where $\delta$ is a limit ordinal.
Let \( \alpha < \delta \). Then, \( \text{Dep}(p) > \alpha \) by definition, so there exist \( \bar{a} \models p \) and \( r \in S_D(M(\bar{a})) \) regular such that \( r \perp M \) and \( \text{Dep}(r) \geq \alpha \). By the previous lemma \( \text{Dep}(r) < \text{Dep}(p) \), so we are done by induction.

We first show that the depth respects the equivalence relation \( \equiv \).

**Lemma 4.5.** Let \( p, q \in S_D(M) \) be regular such that \( p \equiv q \). Then \( \text{Dep}(p) = \text{Dep}(q) \).

**Proof.** By symmetry, it is enough to show that \( \text{Dep}(p) \leq \text{Dep}(q) \). We show by induction on \( \alpha \) that \( \text{Dep}(p) \geq \alpha \) implies \( \text{Dep}(q) \geq \alpha \). For \( \alpha = 0 \) or \( \alpha \) a limit ordinal, it is obvious. Suppose that \( \text{Dep}(p) \geq \alpha + 1 \), and let \( \bar{a} \) realize \( p \) and \( r \in S_D(M(\bar{a})) \) be such that \( \text{Dep}(r) \geq \alpha \) and \( r \perp M \). Since \( p \equiv q \), by Lemma 2.15, there exists \( \bar{b} \) realizing \( q \) such that \( M(\bar{a}) = M(\bar{b}) \). This implies that \( \text{Dep}(q) \geq \alpha + 1 \).

**Lemma 4.6.** Suppose \( \mathcal{K} \) has NDOP. Let \( M \subseteq N \), with \( M, N \in \mathcal{K} \). Let \( p \in S_D(M) \) be regular. Then \( \text{Dep}(p) = \text{Dep}(p_N) \).

**Proof.** We first show that \( \text{Dep}(p) \geq \text{Dep}(p_N) \). By induction on \( \alpha \), we show that \( \text{Dep}(p) \geq \alpha \) implies \( \text{Dep}(p_N) \geq \alpha \).

For \( \alpha = 0 \) it follows from the fact that \( p_N \) is regular. For \( \alpha \) a limit ordinal it follows by induction. Suppose \( \text{Dep}(p) \geq \alpha + 1 \). Let \( \bar{a} \) realize \( p \) and \( r \in S_D(M(\bar{a})) \) regular be such that \( \text{Dep}(r) \geq \alpha \) and \( r \perp M \). Without loss of generality, we may assume that \( \bar{a} \downarrow N \). Hence, by Dominance \( M(\bar{a}) \downarrow N \).

Since \( r \perp M \), then Lemma 2.14 implies that \( r \perp N \). By induction hypothesis \( \text{Dep}(r_{N(\bar{a})}) \geq \alpha \). Hence \( \text{Dep}(p_N) \geq \alpha + 1 \).

The converse uses NDOP. We show by induction on \( \alpha \) that \( \text{Dep}(p_N) \geq \alpha \) implies \( \text{Dep}(p) \geq \alpha \). For \( \alpha = 0 \) or \( \alpha \) a limit ordinal, this is clear. Suppose \( \text{Dep}(p_N) \geq \alpha + 1 \). Let \( \bar{a} \) realize \( p_N \). Then \( \bar{a} \downarrow N \), so by Dominance \( M(\bar{a}) \downarrow N \). Consider \( N' \) \( D_{N_0}^s \)-primary over \( M(\bar{a}) \cup N \). We may assume that \( \text{Dep}(q_{N'}) = \text{Dep}(r) \geq \alpha \). Hence, by induction hypothesis, \( \text{Dep}(q) \geq \alpha \). This implies that \( \text{Dep}(p) \geq \alpha + 1 \).

Let \( \lambda(D) = |D| + |T| \). As we saw in Chapter III, if \( D \) is totally transcendental, then \( D \) is stable in \( \lambda(D) \).

**Lemma 4.7.** Let \( \mathcal{K} \) have NDOP. If \( \text{Dep}(\mathcal{K}) \geq \lambda(D)^+ \) then \( \text{Dep}(\mathcal{K}) = \infty \).

**Proof.** Let \( p \) be regular based on \( B \). Let \( M \) be \( D_{N_0}^s \)-primary over the empty set. Then \( |M| \leq \lambda(D) \). By an automorphism, we may assume that \( B \subseteq M \). Then, by Lemma 4.6, we have \( \text{Dep}(p) = \text{Dep}(p \upharpoonright M) \). Thus, since \( |S_D(M)| \leq \lambda(D) \), there are at most \( \lambda(D) \) possible depths. By Lemma 4.4, they form an initial segment of the ordinals. This proves the lemma.
Definition 4.8. The class $\mathcal{K}$ is called *deep* if $\text{Dep}(\mathcal{K}) = \infty$.

The next theorem is the main characterization of deep $\mathcal{K}$. A class $\mathcal{K}$ is deep if and only if a natural partial order on $\mathcal{K}$ is not well-founded. This will be used to construct nonisomorphic models in Theorem 4.23.

**Theorem 4.9.** $\mathcal{K}$ is deep if and only if there exists a sequence 
$$\langle M_i, \bar{a}_i \mid i < \omega \rangle$$

such that

1. $M_0$ has cardinality $\lambda(D)$;
2. $\text{tp}(\bar{a}_i/M_i)$ is regular;
3. $M_{i+1}$ is prime over $M_i \cup \bar{a}_i$;
4. $M_{i+1}/M_i \perp M_{i-1}$, if $i > 0$.

**Proof.** Suppose that $\mathcal{K}$ is deep. Prove by induction on $i < \omega$ that a sequence satisfying (1)–(4) exists and that in addition

5. $\text{Dep}(\text{tp}(\bar{a}_i/M_i)) = \infty$.

This is possible. For $i = 0$, let $M \in \mathcal{K}$ and $p \in S_D(M)$ be regular such that $\text{Dep}(p) \geq \lambda(D)^+ + 1$. Such a type exists since $\mathcal{K}$ is deep. Now, let $B$ be finite such that $p$ is regular over $B$. Let $M_0 \in \mathcal{K}$ contain $B$ of cardinality $\lambda(D)$. Then, since $p = (p \upharpoonright M_0)_M$, we have $\text{Dep}(p \upharpoonright M_0)$ by Lemma 4.6. Let $\bar{a}_0$ realize $p \upharpoonright M_0$. By the previous fact, $\text{Dep}(\text{tp}(\bar{a}_0/M_0)) = \infty$. Now assume that $\bar{a}_i, M_i$ have been constructed. Let $M_{i+1}$ be prime over $M_i \cup \bar{a}_i$. By (5), we must have $\text{Dep}(\text{tp}(\bar{a}_i/M_i)) \geq \lambda(D)^+ + 1$, so there exists $\bar{a}_{i+1}$ realizing $\text{tp}(\bar{a}_i/M_i)$ and a regular type $p_i \in S_D(M_{i+1})$ such that $\text{Dep}(p_i) \geq \lambda(D)^+$ and $p_i \perp M_i$. Let $\bar{a}_{i+1}$ realize $p_i$, then (1)–(5) hold.

For the converse, suppose there exists $\langle M_i, \bar{a}_i \mid i < \omega \rangle$ satisfying (1)–(4). We show by induction on $\alpha$ that $\text{Dep}(\text{tp}(\bar{a}_i/M_i)) \geq \alpha$, for each $i < \omega$. This is clearly enough since then $\text{Dep}(\text{tp}(\bar{a}_0/M_0)) = \infty$. For $\alpha = 0$, this is given by (2), and for $\alpha$ a limit ordinal, this is by induction hypothesis. For the successor case, assume that $\text{Dep}(\text{tp}(\bar{a}_i/M_i)) \geq \alpha$, for each $i < \omega$. Fix $i$. Then by (4) $\text{tp}(\bar{a}_{i+1}/M_{i+1}) \perp M_i$. By (2) $\text{tp}(\bar{a}_{i+1}/M_{i+1})$ is regular and by (3) $M_{i+1} = M_i(\bar{a}_i)$. By induction hypothesis $\text{Dep}(\text{tp}(\bar{a}_{i+1}/M_{i+1})) \geq \alpha$, hence $\text{Dep}(\text{tp}(\bar{a}_i/M_i)) \geq \alpha + 1$ by definition of depth.

Recall the following definition.

**Definition 4.10.** We say that $A$ dominates $B$ over $M$ if for every set $C$, if $A \perp C$ then $B \perp C$.

We rephrase some of the results we have obtained in the following remark.

**Remark 4.11.** For any set $A$, $A$ dominates $M(A)$ over $M$. Thus, if $M \subseteq N$, and $\bar{a} \in N \setminus M$ there always is a model $M'$ such that $\bar{a} \in M' \subseteq N$ and $M'$ is maximally dominated by $\bar{a}$ over $M$, i.e. $M'$ is dominated by $\bar{a}$ over $M$ and every model contained in $N$ strictly containing $M'$ is not dominated by $\bar{a}$ over $M$.  


We introduce triviality. The name comes from the fact that the pregeometry on the set of realizations of a trivial type is trivial.

**Definition 4.12.** A type \( p \in S_D(M) \) is **trivial** if for every \( M', N \in \mathcal{K} \) such that \( M \subseteq M' \subseteq N \) and for every set \( I \subseteq p_{M'}(N) \) of pairwise independent sequences over \( M' \), then \( I \) is a Morley sequence for \( p_{M'} \).

If \( p \) is trivial, \( \bar{a} \models p \) and \( \bar{a} \) dominates \( B \) over \( M \), then we say that \( B/M \) is trivial.

**Remark 4.13.** If \( tp(\bar{a}/M) \) is trivial, then \( M(\bar{a})/M \) is trivial.

The next lemma says essentially that all the regular types of interest are trivial.

**Lemma 4.14.** If \( \mathcal{K} \) has NDOP, then if \( p \in S_D(M) \) is regular with \( \text{Dep}(p) > 0 \), then \( p \) is trivial.

**Proof.** Suppose \( p \in S_D(M) \) is not trivial. Without loss of generality \( \bar{a}_i \) for \( i \leq 2 \) be pairwise independent over \( M \) such that \( \{\bar{a}_i \mid i \leq 2\} \) is not. Since \( \text{Dep}(p) > 0 \), by using an automorphism, we can find \( r \in S_D(M(\bar{a}_0)) \) regular such that \( r \perp M \).

Let \( N = M(\bar{a}_0, \bar{a}_1, \bar{a}_2). \) Let \( M' \subseteq N \) be maximal such that \( \bar{a}_1 \bar{a}_2 \perp M' \).

Thus, we may assume that \( N = M'(\bar{a}_0, \bar{a}_1, \bar{a}_2). \) Since \( \bar{a}_0 \) realizes \( p \), and \( M'/M \perp p \), we have \( \bar{a}_0 \perp M' \). Hence, by Lemma 2.14, we must have \( r \perp M' \).

By the previous remark, choose \( M_i \subseteq N \) maximally dominated by \( \bar{a}_i \) over \( M' \). By choice of \( M_i \) we have \( M_1 \perp M_2 \). Thus, by definition of \( M_i \) and \( M \) NDOP, necessarily \( N \) is \( D_{\aleph_0}^s \)-primary over \( M_1 \cup M_2 \).

Now, since \( M'/M \perp p \), we have \( \bar{a}_0 \bar{a}_i \perp M' \). Hence \( M(\bar{a}_0) \perp M_i \), for \( i = 1, 2 \). By Lemma 2.14, we have \( r_N \perp M_i \) for \( i = 1, 2 \), contradicting NDOP.

The next lemmas are used to calculate the spectrum function.

**Lemma 4.15.** Assume \( \mathcal{K} \) has NDOP. Let \( \langle M_\eta \mid \eta \in J \rangle \) be a complete decomposition of \( N^* \) over \( M \). Let \( I \) be a subtree of \( J \). Then there exists \( N_I \subseteq N^* \) and \( N_\eta \subseteq N^* \) for each \( \eta \in J \setminus I \) such that \( \{N_I \} \cup \{N_\eta \mid \eta \in J \setminus I \} \) is a complete decomposition of \( N^* \) over \( N_I \).

**Proof.** Define \( N_I \subseteq N^* \) and \( N_\eta \subseteq N^* \) for \( \eta \in J \setminus I \) as follows

1. \( N_I \) is \( D_{\aleph_0}^s \)-primary over \( \bigcup \{M_\eta \mid \eta \in I\} \);
2. \( N_I \perp M_I \);
3. \( N_\eta = N_\eta - (M_\eta) \) for \( \eta \in J \setminus I \) and when \( \eta^- \in I \) then \( N_\eta = N_I(M_\eta) \);
4. \( N_\eta \perp \bigcup_{\nu \prec \eta} M_\nu \).

This is easily done and one checks immediately that it satisfies the conclusion of the lemma.
We now define an equivalence relation on decompositions.

**Definition 4.16.** Let \( \langle M_\eta \mid \eta \in I \rangle \) be a complete decomposition of \( N^* \). Define an equivalence relation \( \sim \) on \( I \setminus \{\langle \rangle \} \) by

\[
\eta \sim \nu \quad \text{if and only if} \quad M_\eta/M_\eta^- \nsubseteq M_\nu/M_\nu^-.
\]

By Equivalence, this is indeed an equivalence relation. By the following lemma, any two sequences in the same \( \sim \)-equivalence class have a common predecessor.

**Lemma 4.17.** If \( \langle M_\eta \mid \eta \in I \rangle \) is a decomposition of \( N^* \), then for \( \eta, \nu \in I \setminus \{\langle \rangle \} \) such that \( \eta^- \neq \nu^- \) we have \( M_\eta/M_\eta^- \nsubseteq M_\nu/M_\nu^- \).

**Proof.** Let \( \eta, \nu \in I \setminus \{\langle \rangle \} \) such that \( \eta^- \neq \nu^- \). Let \( u \) be the largest common sequence of \( \eta^- \) and \( \nu^- \). We have \( M_\eta^- \nsubseteq M_\nu^- \), by independence of the \( M_u \) decomposition. By definition \( M_\eta/M_\eta^- \nsubseteq M_\eta^- \). Hence, by Lemma 2.14, we have \( M_\eta/M_\eta^- \nsubseteq M_u \) and also \( M_\eta/M_\eta^- \nsubseteq M_\nu^- \). Therefore \( M_\eta/M_\eta^- \nsubseteq M_\nu/M_\nu^- \).

The next lemma will be used inductively.

**Lemma 4.18.** Let \( \langle M_\eta \mid \eta \in I \rangle \) and \( \langle N_\nu \mid \nu \in J \rangle \) be a complete decompositions of \( N^* \) over \( M \). Let \( I' = \{\eta \in I \mid \eta^- = \langle \rangle \} \) and \( J' = \{\nu \in J \mid \nu^- = \langle \rangle \} \). Then there exists a bijection \( f : I' \to J' \) such that

1. \( f \) preserves \( \sim \)-classes;
2. If \( \eta \in I' \) and \( M_\eta/M \) is trivial then \( M_\eta \nsubseteq N_{f(\eta)} \).

**Proof.** Choose a representative for each \( \mathcal{L} \)-class among the regular types of \( S_D(M) \). Build the bijection by pieces. For each regular \( p \in S_D(M) \), the cardinalities of \( \{\eta \in I \mid M_\eta/M \nsubseteq p\} \) and \( \{\nu \in J \mid N_\nu/N \nsubseteq p\} \) are equal and both equal to the dimension of \( p(N^*) \) by construction. If \( p \) is not trivial, then choose any bijection between the two sets. If \( p \) is trivial, for each \( \eta \in I \) such that \( M_\eta/M \nsubseteq p \) there exists exactly one \( \nu \in J' \) such that \( \overline{M}(\eta) \nsubseteq N_\nu \).

Let \( f \) send each such \( \eta \) to their corresponding \( \nu \). Since there is no relation between \( p \)'s belonging to different equivalence classes, this is enough.

The following quasi-isomorphism will be relevant for the isomorphism type of models.

**Definition 4.19.** Two \( \omega \)-trees \( I, J \) are said to be **quasi-isomorphic**, if there exists a partial function \( f \) from \( I \) to \( J \) such that

1. \( f \) is order-preserving;
2. For each \( \eta \in I \) all but at most \( \lambda(D) \) many successors of \( \eta \) are in \( \text{dom}(f) \);
3. For each \( \nu \in J \) all but at \( \lambda(D) \) many successors of \( \nu \) are in \( \text{ran}(f) \).
A function $f$ as above is called a quasi-isomorphism.

**Theorem 4.20.** Let $\langle M^+_\eta \mid \eta \in I \rangle$ and $\langle M'_\nu \mid \nu \in J \rangle$ be complete decompositions of $N^*$. Then there exists a $\sim$-class preserving quasi-isomorphism from $I$ to $J$.

**Proof.** For each $\eta \in I$, let $I^+_\eta = \{ \nu \in I \mid \nu^- = \eta \}$. We define a partial class preserving function $f_\eta$ from $I^+_\eta$ into $J$ as follows. Then $M^+_\eta$ has cardinality $\lambda(D)$, so we can find $I_0$ and $J_0$ of cardinality at most $\lambda(D)$ such that there exists $N \subseteq N^*$ containing $M^+_\eta$, such that $M$ is $D^8_{\omega_1}$-primary over both $\bigcup \{ M_\nu \mid \nu \in I_0 \}$ and $\bigcup \{ M'_\nu \mid \nu \in J_0 \}$. By Lemma 4.15 and Lemma 4.18, there exists a partial function $f_\eta$ from $I^+_\eta \setminus I_0$ into $J$ satisfying conditions (1) and (2) in Lemma 4.18.

Now let $f = \bigcup_{\eta \in I} f_\eta$ (we let $f()$ map $\langle \rangle$ to $\langle \rangle$). Clearly $f$ is well-defined, since the domains of all the $f_\eta$’s are disjoint. Further, by construction, the condition involving $\lambda(D)$ is satisfied.

It remains to show that $f$ is one-to-one and order preserving. We check order preserving and leave one-to-one to the reader. Let $\eta \prec \nu \in I$ be given. We may assume that $\eta \neq \langle \rangle$. Then, by Lemma 4.14, we have $M^+_{\eta^-}/M^-_{\eta^-}$ is trivial. We are going to compute $f(\eta)$ and $f(\nu)$. Recall that $f(\eta) = f_{\eta^-}(\eta)$.

In the notation of Lemma 4.15 and of the first paragraph, we have

$$\langle N_\zeta : \zeta \in I \setminus I_0 \rangle \cup \{ N \} \quad \text{and} \quad \langle N'_\zeta : \zeta \in J \setminus J_0 \rangle \cup \{ N \},$$

two complete decompositions of $N^*$ over $N$. By Lemma 4.18, we have

$$N^-_{\eta^-} \supset N'_{f(\eta^-)}.\]$$

Then, necessarily $M^\nu/M^-_{\eta^-} \nsubseteq M'_\nu/M^-_{f(\nu^-)}$ and any sequence $\sim$-related to $\nu$ is $\prec$-above $\eta$. Consider the following independent tree

$$\langle N'_\zeta : \zeta \in I \setminus I_0, f^-_\eta(\eta) \not\prec \zeta \rangle \cup \langle N_\zeta : \eta \in I, \eta \prec \zeta \rangle \cup \{ N \}.\]$$

By triviality of $M^\eta/M^-^\eta$, it is a decomposition of $N^*$ over $N$. Hence, by Lemma 4.17 we have $M^\nu/M^-^\nu \perp N^-_{\zeta}/N^-_{\eta^-}$, for each $\zeta \in I \setminus I_0, f^-_\eta(\eta) \not\prec \zeta$. This implies that the $\sim$-class of $f^-_\eta(\nu)$ is above $f^-_\nu(\eta^-)$. Thus, $f$ is order preserving.

In order to construct many nonisomorphic models, we will need a special kind of trees. For an $\omega$-tree $I$ and $\eta \in I$, denote by $I_\eta = \{ \nu \in I \mid \eta \prec \nu \}$. We write $I_\eta \cong I_\nu$ if both trees are isomorphic as trees.

**Definition 4.21.** An $\omega$-tree $I$ is called ample if for every $\eta \in I$, with $\eta^- \in I$, we have

$$|\{ \nu \in I : \nu^- = \eta^- \text{ and } I_\nu \cong I_\eta \}| > \lambda(D).$$

We now state a fact about ample $\omega$-trees. If $I$ is a tree, by definition every $\eta \in I$ is well-founded in the order of $I$. The rank of $\eta$ in $I$ will be the natural rank associated with the well-foundness relation on $\eta$ in $I$. 


Fact 4.22. Let $I$, $J$ be ample trees. Let $f$ be a quasi-isomorphism from $I$ to $J$. Then for each $\eta \in \text{dom}(f)$, the rank of $\eta$ in $I$ is equal to the rank of $f(\eta)$ in $J$.

In the next proof, write $\ell(\eta)$ for the level of $\eta$.

Theorem 4.23. If $\mathcal{K}$ is deep, for each $\mu > \lambda(D)$, there are $2^\mu$ nonisomorphic models of cardinality $\mu$.

Proof. Let $\mu > \lambda(D)$. Since $\mathcal{K}$ is deep by Theorem 4.9, there exists

$$\langle M_i, \bar{a}_i \mid i < \omega \rangle$$

such that

1. $M_0$ has cardinality $\lambda(D)$;
2. $\text{tp}(\bar{a}_i/M_i)$ is regular;
3. $M_{i+1}$ is prime over $M_i \cup \bar{a}_i$;
4. $M_{i+1}/M_i \perp M_{i-1}$, if $i > 0$.

Let $p = \text{tp}(\bar{a}_0/M_0)$. Then $p$ is regular based on a finite set $B$. We will find $2^\mu$ non-isomorphic models of size $\mu$ with $B$ fixed. This implies the conclusion of the theorem since $\mu^{<\kappa_0} = \mu$.

For each $X \subseteq \mu$ of size $\mu$, let $I_X$ be an ample $\omega$-tree with the property that the set of ranks of elements of the first level of $I_X$ is exactly $X$. Such a tree clearly exists ($\mu > \lambda(D)$). Define the following system $\langle M^X_\eta \mid \eta \in I_X \rangle$:

1. $M^X_\emptyset = M_0$;
2. If $\eta_0 < \cdots < \eta_n \in I_X$, then

$$\text{tp}(M^X_{\eta_0} \cdots M^X_{\eta_n}/\emptyset) = \text{tp}(M_{\ell(\eta_0)} \cdots M_{\ell(\eta_n)}/\emptyset).$$

This is easy to do and by choice of $\langle M_i, \bar{a}_i \mid i < \omega \rangle$ this is a decomposition. Let $M_X$ be a $D^X_{\kappa_0}$-primary model over $\bigcup\{M^X_\eta \mid \eta \in I_X\}$. Then $M_X \in \mathcal{K}$ has cardinality $\mu$. By NDOP, $\langle M_i, \bar{a}_i \mid i < \omega \rangle$ is a complete decomposition of $M_X$ over $M_0$.

We claim that for $X \neq Y$ as above, $M_X \not\equiv_B M_Y$. Let $X, Y \subseteq \mu$ of cardinality $\mu$ be such that $X \neq Y$. Suppose $M_X \cong_B M_Y$. Then, by Theorem 4.20, there exists a class-preserving quasi-isomorphism between $I_X$ and $I_Y$. Since $B$ is fixed, the first level of $I_X$ is mapped to the first level of $I_Y$. By the previous fact, we conclude that $X = Y$, a contradiction.

We have shown that deep diagrams have many models. The usual methods (see [Sh c] for example) can be used to compute the spectrum of $\mathcal{K}$ when $\mathcal{K}$ is not deep. Recall that when $\mathcal{K}$ has NDOP but is not deep then $\text{Dep}(\mathcal{K}) < \lambda(D)^+$, by Lemma 4.7.

Theorem 4.24. If $\mathcal{K}$ has NDOP but is not deep, then for each ordinal $\alpha$ with $\kappa_\alpha \geq \lambda(D)$, we have $I(\kappa_\alpha, \mathcal{K}) \leq \sharp_{\text{Dep}(\mathcal{K})}(\kappa_0 + |\alpha|^{2^{\kappa_1}}) < \sharp_{\lambda(D)^+}(\kappa_0 + |\alpha|)$.

This proves the main gap for the class $\mathcal{K}$ of $(D, \kappa_0)$-homogeneous models of a totally transcendental diagram $D$. 
Theorem 4.25 (Main Gap). Let $\mathcal{K}$ be the class of $(D,\aleph_0)$-homogeneous models of a totally transcendental diagram $D$. Then, either $I(\aleph_\alpha, \mathcal{K}) = 2^{\aleph_\alpha}$, for each ordinal $\alpha$ such that $\aleph_\alpha > |T| + |D|$, or $I(\aleph_\alpha, \mathcal{K}) < \beth((|T| + |D|) + (\aleph_0 + |\alpha|))$, for each $\alpha$ such that $\aleph_\alpha > |T| + |D|$.

Proof. If $\mathcal{K}$ has DOP (Theorem 3.4) or has NDOP but is deep (Theorem 4.23), then $\mathcal{K}$ has the maximum number of models. Otherwise, $\mathcal{K}$ has NDOP and is not deep and the bound follows from Theorem 4.24. ⊲

Similar methods using the existence of $D_\mu^s$-prime models for totally transcendental diagrams allow us to prove the main gap for $(D, \mu)$-homogeneous models of a totally transcendental diagram $D$.

Theorem 4.26. Let $\mathcal{K}$ be the class of $(D, \mu)$-homogeneous models of a totally transcendental diagram $D$. Then, either $I(\aleph_\alpha, \mathcal{K}) = 2^{\aleph_\alpha}$, for each ordinal $\alpha$ such that $\aleph_\alpha > |T| + |D| + \mu$, or $I(\aleph_\alpha, \mathcal{K}) < \beth((|T| + |D|) + (\aleph_0 + |\alpha|))$, for each $\alpha$ such that $\aleph_\alpha > |T| + |D| + \mu$.

Finally, similarly to [GrHa] or [Ha], it is possible to show that for $\alpha$ large enough, the function $\alpha \mapsto I(\aleph_\alpha, \mathcal{K})$ is non-decreasing, for the class $\mathcal{K}$ of $(D, \mu)$-homogeneous models of a totally transcendental diagram $D$.

REFERENCES


[Sh 705] Saharon Shelah, Toward classification theory of good $\lambda$-frames and abstract elementary classes. *In preparation*

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A dichotomy theorem for being essentially countable

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ABSTRACT. We introduce a definition to characterize when an orbit equivalence relation is Borel reducible to a countable equivalence relation. As a corollary we obtain a kind of low basis theorem—an orbit equivalence relation which arises from an effective action of a recursively presented Polish group on a recursive Polish space is Borel reducible to a countable Borel equivalence relation if and only if it is reducible to a \( \Delta^1_1 \) countable equivalence relation using a \( \Delta^1_1 \) function. We make some effort to draw out the consequences of the theorem for the model theoretic context, which in turn raises an open problem regarding first order logic.

1. INTRODUCTION

Definition We say that one equivalence relation, \( E \), is Borel reducible to another, \( F \), written \( E \leq_B F \), if there is a Borel function \( \theta \) such that

\[
x_1 E x_2 \iff \theta(x_1)F\theta(x_2).
\]

Definition An equivalence relation is Borel if it is Borel as a subset of the product of the space on which it is defined; it is countable if all its equivalence classes are countable. An equivalence relation on a standard Borel space \( X \) is essentially countable if there is a countable Borel equivalence relation \( E \) to which it is Borel reducible.

Example It is trivially seen that if \( E \) arises from the Borel action of a countable group then necessarily it is Borel and each equivalence class is countable. More surprisingly Kechris has shown in [13] that the orbit equivalence relations arising from Borel actions of locally compact Polish groups are necessarily essentially countable.

In very general terms the essentially countable equivalence relations might be thought of as those where there is some kind of notion of “finite rank”.

Example Let \( \sigma \) be the sentence in \( L_{\omega_1,\omega} \) whose models are exactly the finite rank torsion free abelian groups – that is to say, those abelian, torsion free groups which have some finite sequence \( g_1, g_2, \ldots, g_n \) such that for every \( h \neq 0 \) there is some \( k \) such that \( k \cdot h \) is in the subgroup generated by \( g_1, g_2, \ldots, g_n \). Let \( \text{Mod}(\sigma) \) be the collection of models for \( \sigma \) with underlying set \( \mathbb{N} \) and the usual Borel structure (as described in [10]); let \( \cong_{\text{Mod}(\sigma)} \) be the equivalence relation of isomorphism on

\[\]

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Mod(\sigma). Then \(\cong_{\text{Mod}(\sigma)}\) is essentially countable. In fact, at each \(n\) we can let \(\sigma_n\) be the sentence whose models are exactly the rank \(n\) torsion free abelian groups, and we obtain \(\cong_n (=_{\text{df}} \cong_{\text{Mod}(\sigma_n)}\) Borel reducible to the action of \(\mathbb{Q}^n\) by multiplication on the collection of all subgroups of \((\mathbb{Q}^n, +)\). Indeed recent years have witnessed an outpouring of papers on these equivalence relations – culminating in Simon Thomas’ [15], where it was shown that at every \(n\) one has

\[\cong_n \leq_B \cong_{n+1} .\]

On the other hand it is known the isomorphism relation on countable torsion free abelian groups of infinite rank is not essentially countable in any of its customary Borel structures. (See e.g. [9], §5 [11]). Indeed a natural example of an equivalence relation which is not essentially countable is given by considering the coding of countable sets of reals; that is to say, for \(f_1, f_2 \in \mathbb{R}^\mathbb{N}\) set

\[f_1 F_2 f_2\]

if for all \(n \in \mathbb{N}\) there exists \(m_1, m_2 \in \mathbb{N}\) with

\[f_1(n) = f_2(m_1),\]

\[f_2(n) = f_1(m_2).\]

One certainly finds in [11] a proof that this equivalence relation is not essentially countable and in [9] a proof that isomorphism on general countable torsion free abelian groups Borel reduces \(F_2\) – though in both cases the results are folklore and invoking theorems of these papers would represent a degree of overkill.

**Example** In the natural Borel structure, induced by looking at group operations placed on some fixed countable set, the isomorphism relation on finitely generated groups is essentially countable. (See [10].) This cannot be extended to general countable groups, since, as noted above, the isomorphism relation on countable abelian groups is already \(\leq_B\) above \(F_2\).

A special kind of essentially countable equivalence relation is formed by those which are smooth.

**Definition** An equivalence relation \(E\) is *smooth* if there is a Borel function \(\theta\) from the space on which \(E\) lives to \(\mathbb{R}\) (or, equivalently, any uncountable Polish space) with

\[x_1 Ex_2 \iff \theta(x_1) = \theta(x_2).\]

Not all essentially countable equivalence relations, even among those given by the isomorphism relation on some natural class of structures, are smooth. For instance, §3.1 of [6] gives a gentle proof that \(\cong_1\), isomorphism of rank one torsion free abelian groups, is not smooth. In [10] a model theoretic analysis of smoothness and essential countability is presented.

**Theorem 1.1. (Hjorth, Kechris)** Let \(L\) be a countable language, with the usual Borel structure generated by first order logic.\(^1\) Let \(\sigma \in L_{\omega_1, \omega}\).

1. The following are equivalent:

\(^1\)So that for any \(\psi\) and \(\bar{a}\) a finite sequence from \(\mathbb{N}\), the set of \(M\) which satisfy \(\psi(\bar{a})\) is Borel; this is indeed a standard Borel structure – see [4].
(a) $\cong_{\text{Mod}(\sigma)}$ is smooth; (b) there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \text{Mod}(\sigma)$ we have that

$$\text{Th}_F M,$$

the $F$-theory of $M$, is $\aleph_0$-categorical; (c) there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \text{Mod}(\sigma)$ we have that

$$\text{Th}_F M,$$

the $F$-theory of $M$, is atomic.

(II) The following are equivalent:

(a) $\cong_{\text{Mod}(\sigma)}$ is essentially countable; (b) there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \text{Mod}(\sigma)$ we have some $\bar{a} \in M$ such that

$$\text{Th}_F \langle M, \bar{a} \rangle,$$

the $F$-theory of the expansion of $M$ obtained by adding names for all the elements of the $\bar{a}$ sequence, is $\aleph_0$-categorical; (c) there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1,\omega}$ such that for every $M \in \text{Mod}(\sigma)$ we have some $\bar{a} \in M$ such that

$$\langle M, \bar{a} \rangle,$$

is $F$-atomic.

**Question** Does there exist a first order theory $T$ with $\cong_{\text{Mod}(T)}$ essentially countable but not smooth?

In light of 1.1 the question of whether smoothness and essential countability coincide for first order structures has a purely model theoretic reading. We are asking for a countable first order theory $T$ for which there exists a countable fragment $F$ over which every $M \models T$ has some $\bar{a}$ with

$$\langle M, \bar{a} \rangle$$

$F$-atomic, but for which there is no fragment $F'$ for which every model $M \models T$ is $F'$-atomic; notice that this purely model theoretic formulation makes no explicit reference to countable models. In practically all the standard cases, say for instance the kinds of first order theories one finds in [2], it is easily determined that $\text{Mod}(T)$ is either smooth or not essentially countable.

While I would conjecture that an example of a countable first order theory with $\cong_{\text{Mod}(T)}$ essentially countable but non-smooth should exist, but simply be very hard to conceive, it does pay to seriously consider the possibility that there may be a structure theorem which denies this. This would represent some kind of basic obstruction or dichotomy theorem for $L_{\omega_1,\omega}$ equivalence relations which is entirely missing for $L_{\omega_1,\omega}$. Here it is worth pointing out that any such dichotomy theorem could not be proved using purely Baire category methods.

**Example** Let $G$ be a countable infinite group and $L$ the language which contains function symbols $F_g$ for each $g \in G$ along with unary predicates $(U_n)_{n \in \mathbb{N}}$. Let $T$ be the first order theory which asserts that the composition of the functions exactly mimics the group (so for each $g_1, g_2$ we have an axiom asserting $F_{g_1} \circ F_{g_2} = F_{g_1g_2}$), the identity element acts trivially ($\forall a(F_1(a) = a)$), and that otherwise the action of $G$ is free (for each $g \neq 1$ we have $\forall a(F_g(a) \neq a)$). We can equip with $\text{Mod}(T)$, the space of $T$-structures with underlying set $\mathbb{N}$, the usual topological structure, under which for every quantifier free $\psi(\cdot)$ and $\bar{a}$ we have

$$\{ M : M \models \psi(\bar{a}) \}$$
becomes clopen. In this topology it is a Polish space, and it follows easily from the usual kinds of arguments that on no comeager set $C$ is

$$\cong |C|$$

smooth. (Compare the arguments from say §3.1[6].) A failure to be essentially countable is not yet detectable at the level of Baire category. If we take $C$ to be the comeager set of models on which the group action is transitive, that is to say, the structure satisfies the $L_{\omega_1,\omega}$ sentence

$$\forall a,b \bigvee_{g \in G} (F_g(a) = b),$$

then we have that

$$\cong |C|$$

is essentially countable. This can be observed by (II)(c) 1.1 since on this set we have that any $M \in C$ is the functional closure of any $a \in M$. However the isomorphism relation of the theory is not essentially countable. One way to see this is actually to use forcing, starting with a fragment $F$ in the ground model and showing its failure to achieve (II)(c) in a suitable generic extension. We begin with an action of $G$ on $\mathbb{N}$ which is free but has infinitely many orbits and interpret the function symbols $(F_g)_{g \in G}$ accordingly. Then we force to obtain mutually Cohen generic $(x_a)_{a \in \mathbb{N}}$ elements of $2^\mathbb{N}$; for each $x_a$ we define the indicated pattern of unary predicates at $a$, with

$$U_n(a) \iff x_a(n) = 1.$$

It is a routine exercise in forcing to see that if $a_0, a_1, \ldots, a_\ell = \bar{a}$ is a finite sequence from $\mathbb{N}$ then

$$\text{Th}_F(M, \bar{a})$$

can be uniformly defined in

$$V[\{x_{F_g(a_i)} : i \leq \ell, g \in G\}$$

from $F$ and $(x_{F_g(a_i)})_{i \leq \ell, g \in G}$; then considering some $b \notin \{F_g(a_i) : i \leq \ell, g \in G\}$ we have that the $F$-type of $x_b$ will not be in $V[\{x_{F_g(a_i)} : i \leq \ell, g \in G\}]$, and hence certainly not $F$-atomic over $\text{Th}_F(M, \bar{a})$.

In this note we introduce a new definition to analyze essential countability.

**Definition** Let $G$ be a Polish group acting continuously on a Polish space $X$. We then say that $X$ is a *stormy* $G$-space if for every non-empty, open $V \subset G$ and $x \in X$ we have that the natural map

$$g \mapsto g \cdot x$$

from $V$ to $V \cdot x$ is not an open function.

For future reference note that this definition has a kind of stability and upward persistence.

**Lemma 1.2.** If $X$ is a stormy Polish $G$-space with respect to the topology $\tau$, then for any stronger Polish $G$-space topology $\tau_0 \supset \tau$ we can find a $G_{\delta}(X, \tau)$ subspace $X_0 \subset X$ such that $(X_0, \tau_0)$ is still a stormy Polish $G$-space.
Proof. It is an entirely classical fact that we can find an $X_1$ which is a $G_δ$ subspace of $(X, τ)$ such that the induced inclusion map

$$(X_1, τ) ↪ (X, τ_0)$$

is continuous; this follows since any Borel map is continuous on a dense $G_δ$ (see [12]). Then we let

$$X_0 = \{ x ∈ X : \forall^* g ∈ G(g \cdot x ∈ X_1) \},$$

the set of points for which a comeager collection of group elements place us in $X_1$. Using the continuity of the action of $G$ on $(X, τ_0)$ we still have that the induced map

$$(X_0, τ) ↪ (X, τ_0)$$

continuous; so the topology $τ$ on $X_0$ equals $τ_0$ on $X_0$. So then the structure of the definition of stormy, with its insistence that something happens at every orbit, guarantees that

$$(X_0, τ_0)$$

is still stormy. □

An especially important instance of this definition is provided by model theory. Here let us consider that $S_∞$ acts in the usual manner on the space of all models of some infinitary sentence $σ ∈ L_{ω_1, ω}$; cofinally many of the Polish topologies are of the form $τ_F$ for some countable fragment $F$ — that is to say, they have a basis consisting of sets of the form

$$\{ M : M ⊨ \varphi(\vec{a}) \}$$

for some $φ ∈ F$, $\vec{a} ∈ N^{<ω}$. (Again, see [10]). Here we may assume without loss of generality that we restrict ourselves to open sets $V$ of the form

$$\{ π ∈ S_∞ : ∀i ≤ ℓ(π(a_i) = a_i) \}$$

for some chosen $\vec{a} = a_0, a_1, ... a_ℓ$ in $N$. Then the action of $S_∞$ on $⟨ \text{Mod}(σ), τ_F⟩$ is stormy if and only if for every $M ⊨ σ$ there is no $\vec{a} ∈ M$ with $(M, \vec{a})$ being $F$-atomic. Building on the methods of [8]:

**THEOREM 1.3.** Let $G$ be a Polish group and $X$ a Polish $G$-space with the induced orbit equivalence relation $E^X_G$. Then exactly one of the following holds:

(I) $E^X_G$ is essentially countable;

(II) there is a stormy Polish $G$-space $Y$ and a continuous $G$-embedding

$$ρ : Y → X.$$

Note in (II) that since $ρ$ is one to one, the image $ρ[Y]$ is Borel in $X$. Thus we may without loss of generality assume that $Y$ is a Borel invariant subset of $X$ and that we have found a Polish topology on $Y$ which refines the original $X$-topology under which the action remains continuous and becomes stormy. I have no concrete instances of an equivalence relation whose essential countability can only be affirmed or refuted by appeal to 1.3. However it can be said that the argument is effective; it uses the “geometry” of $Σ^1_1$ sets in the now usual way. In particular the following represents a corollary of the method, which would presumably have not been obvious hitherto.
THEOREM 1.4. Let $G$ be a recursive Polish group and $X$ a recursive Polish $G$-space with the induced orbit equivalence relation $E^X_G \in \Delta^1_1$. If $E^X_G$ is essentially countable, then there is a countable $\Delta^1_1$ equivalence relation on an effective Polish space to which $E^X_G$ is reducible by a $\Delta^1_1$ function.

It might also be worth thinking about the model theoretic consequences of this theorem. Again, for a logician the isomorphism relations on classes of countable structures, while not quite Borel, comprise the central examples to which all others should be compared.

In the next corollary we bear in mind that $\equiv_{\text{Mod}(\sigma)}$ is Borel if and only if there is some countable ordinal which bounds the Scott heights of models of $\sigma$. This result from [1] makes possible to state the following corollary in largely model theoretic terminology.

COROLLARY 1.5. Let $\sigma \in L_{\omega_1,\omega}$ for some countable language $L$. Assume that there is some countable ordinal which bounds the Scott heights of the models of $\sigma$. Then exactly one of the following holds.

(I) there is a countable fragment $F \subseteq L_{\omega_1,\omega}$ such that for every $M \in \text{Mod}(\sigma)$ we have some $\bar{a} \in M$ such that

$$\langle M, \bar{a} \rangle,$$

is $F$-atomic;

(II) there is a countable fragment $F$ and some $\varphi \in L_{\omega_1,\omega}$ for which there exists a model of $\varphi \land \psi$, such that for every

$$M \models \varphi$$

there is no $\bar{a} \in M$ with

$$\langle M, \bar{a} \rangle$$

$F$-atomic.

To see that the corollary indeed follows from the theorem we observe first of all that (I) 1.5 is a direct restatement of (I) 1.3 in light of 1.1. This leaves (II). Note at this point we actually obtain in applying 1.3 a Borel $S_\infty$-invariant Borel subspace $Y$ and a Polish topology $\tau_Y$ on $Y$ under which the action becomes stormy. We may find a countable fragment $F$ such that $\tau_F \supset \tau_Y$ by [14], and then by 1.2 find a $Y_0 \subseteq Y$ so that $(Y_0, \tau_F)$ is a stormy Polish $S_\infty$-space; membership in $Y_0$, by [14] again, is given by some sentence $\varphi \in L_{\omega_1,\omega}$.

2. SETTING SAIL

In all the lemmas below we assume that $G$ is a Polish group and $X$ is a Polish $G$-space. $E^X_G$ is the orbit equivalence relation. We assume there are fixed countable bases for $X$ and $G$ and that the basis for $G$ contains a neighborhood basis of symmetric sets around $1_G$.

LEMMA 2.1. Suppose for every $x \in X$ there is a non-empty open $V \subseteq G$ with

$$V \to V \cdot x$$

$$g \mapsto g \cdot x$$

an open mapping. Then $E^X_G$ is essentially countable.
Proof. Let us say that a non-empty basic open set $V \subset G$ is $x$-safe if for all $W \subset V$ open, non-empty, $W \cdot x$ includes a non-empty relatively open subset of $\overline{V \cdot x}$. Our assumption in the lemma yields that for each $x \in X$ there is such an $x$-safe point. Note moreover that any non-empty open subset of an $x$-safe set is again $x$-safe. If $V$ is $x$-safe and $U \subset X$ is a non-empty basic open set, then we say that $U$ is a $V \cdot x$-harbor if there is some symmetric basic open neighborhood $W$ of $1_G$ such that

$$\forall h \in W \forall y \in U \cap \overline{Vh \cdot x} (y \in \overline{V \cdot x}).$$

Claim(I): For each $x$ there are $U$ and $V$ as above.

Proof of claim: Choose open $\hat{V}$ with

$$\hat{V} \mapsto \hat{V} \cdot x$$

an open map. Let $g_0 \in \hat{V}$. Appealing to the continuity of the group operations we may choose $V \subset \hat{V}$ an open neighborhood of $g_0$ and $W$ a symmetric open neighborhood of $1_G$ such that

$$VW \subset \hat{V}.$$

$V$ is clearly $x$-safe, since the mapping

$$V \mapsto V \cdot x$$

is open. Let us choose $U$ basic open with

$$\overline{V \cdot x} \supset U \cap \overline{\hat{V} \cdot x}.$$

Then we have for all $h \in W$ and $y \in U \cap \overline{Vh \cdot x}$

$$y \in U \cap \overline{\hat{V} \cdot x}$$

$$\therefore y \in \overline{V \cdot x},$$

just as required. \quad \text{(Claim(I)\square)}

For each $x$, the collection of such $(U, V)$ as above is uniformly $\Pi_1^1$ in $x$. The bases from which they are chosen are countable. Hence we may assign

$$x \mapsto (U_x, V_x)$$

in a Borel manner so that each $V_x$ is $x$-safe and each $U_x$ is a $V_x \cdot x$-harbor. We can then define a Borel function

$$X \to \mathcal{F}(X),$$

from $X$ to the standard Borel space of closed subsets of $X$, given by

$$x \mapsto U_x \cap \overline{V_x \cdot x}.$$
Proof of claim: Otherwise we could find an uncountable set \((y_\alpha)_{\alpha \in \aleph_1}\) of elements of \([x]_G\) with each \(y_\alpha\) having the same \(U_{y_\alpha} = U, V_{y_\alpha} = V\) and the same \(W\) witnessing that \(U\) is a \(V_{\cdot y_\alpha}\)-harbor, and yet
\[
U \cap \overline{V_{\cdot y_\alpha}} \neq U \cap \overline{V_{\cdot y_\beta}}
\]
for \(\alpha \neq \beta\). Considering the separability of the group \(G\) we can find two \(y, y'\) in this uncountable set with \(y \in W_{y'}\) and \(y' \in W_y\). Then for all \(z \in U \cap \overline{V_{\cdot y}} \subset U \cap \overline{V_{\cdot W_{\cdot y'}}}\) we must have by assumptions on \(U\) and \(V\) that \(z \in U \cap \overline{V_{\cdot y'}}\), and conversely. Thus
\[
U \cap \overline{V_{\cdot y}} = U \cap \overline{V_{\cdot y'}}
\]
after all. \(\square\)

Claim(III): If \(V\) is \(x\)-safe, then \(V \cdot x\) is comeager in \(\overline{V_{\cdot x}}\).

Proof of claim: Suppose not. Then there is some open \(U \subset X\) with \(V \cdot x\) meager in \(U \cap \overline{V_{\cdot x}}\). And then we can find some open \(V_0 \subset V\) and open \(U_0 \subset U\) with \(V_0 \cdot x\) dense in \(U_0 \cap \overline{V_{\cdot x}}\); which would place us in the situation that \(V_0\) is now our \(x\)-safe set and \(V_0 \cdot x\) is meager in \(U_0 \cap \overline{V_0 \cdot x}\). But then if we cover \(V_0 \cdot x\) by countably many closed sets, \(F_0, F_1, ..., F_n, ...\), each of which is nowhere dense in \(U_0 \cap \overline{V_0 \cdot x}\), then there must be some open non-empty \(W \subset V_0\) with \(W \cdot x \subset F_n\), at some \(n \in \mathbb{N}\). And so \(W \subset V_0 \subset V\) contradicts the definition of safety. \(\square\)

This lemma has as an immediate corollary a result previously proved by Alexander Kechris by other means:

**Corollary 2.2.** (Kechris) If \(G\) is a locally compact Polish group and \(X\) is a Polish \(G\)-space, then \(E^X_G\) is essentially countable.

Proof. Fix \(x \in X\) and consider some non-empty precompact \(V \subset G\). \(\overline{V}/G_x\) is compact\(^2\), and hence the natural map
\[
\overline{V}/G_x \rightarrow \overline{V_{\cdot x}}
\]
maps closed sets to closed sets and is one-to-one and onto, thus it is open. In particular
\[
V/G_x \rightarrow V_{\cdot x}
\]
and
\[
V \rightarrow V_{\cdot x}
\]
are both open, and we fall under the case assumptions of 2.1. \(\square\)

**Lemma 2.3.** If \(E^X_G\) is essentially countable, then for a comeager collection of \(x \in X\) there exists \(V \subset G\) open and non-empty with
\[
V \rightarrow V_{\cdot x}
\]
\[
g \mapsto g \cdot x
\]
open.

\(^2\)Here as elsewhere we follow the notation of [1]; so \(G_x\) is the stabilizer of the point \(x\): that is to say, the subgroup \(\{g \in G : g \cdot x = x\}\).
Proof. Let \( \theta : X \to Z \) be a Borel function reducing \( E^X_G \) to some countable equivalence relation on a Polish space \( Z \). Following 4.1 [3] (or see §18 of [12], along with references tracing back to earlier work by Burgess) we have that the range of \( \theta \) is Borel and there is a Borel function

\[ \rho : \text{Ran}(\theta) \to X \]

such that for all \( z \in \text{Ran}(\theta) \) we have \( \theta(\rho(z)) = z \). For each \( x \in X \) let

\[ p_x = \rho(\theta(x)); \]

thus this \( x \mapsto p_x \) is a Borel function which in effect "selects" a countable set from each orbit. Following 7.1.2 of [1] (the orbit equivalence relation is Borel, and thus too is the stabilizer function \( x \mapsto G_x \) from \( X \) to \( \mathcal{F}(X) \)) we may in a Borel manner choose for each \( x \) a group element \( g_x \) with

\[ g_x \cdot x = p_x. \]

For the moment let us say that a point \( y \in X \) is good if there is some basic open neighborhood \( W \) of \( 1_G \) such that \( \forall^* h \in W(p_y = p_{h \cdot y}) \).

**Claim:** The set of good points is comeager.

**Proof of claim:** Otherwise use Kuratowski-Ulam to find some \( y \in X \) and open non-empty \( V \subseteq G \) such that \( \forall^* g \in V(g \cdot y \text{ is not good}) \). Then since \( \{ p_{y'} : y' \in E^X_G y \} \) is countable, we may find some single \( y_0 \) and non-empty open \( W \subseteq V \) such that

\[ \forall^* g \in W(p_{g \cdot y} = y_0). \]

Then we find some basic open neighborhood \( \hat{W} \) of \( 1_G \) and \( g_0 \in G \) with \( \hat{W}g_0 \subseteq W \); after possibly nudging \( g_0 \) slightly, we may assume it falls into the relatively comeager sets \( \{ g \in W : g \cdot y \text{ not good} \} \) and \( \{ g \in W : p_{g \cdot y} = y_0 \} \). But this gives

\[ \forall^* g \in \hat{W}(p_{g \cdot (g_0 \cdot y)} = y_0 = p_{g_0 \cdot y}), \]

and hence \( g_0 \cdot y \) must be good after all. \( \square \)

Let \( (W_\ell)_{\ell \in \mathbb{N}} \) be a neighborhood basis at \( 1_G \). We may apply the above claim to choose a Borel function

\[ \gamma : X \to \mathbb{N} \]

such that

\[ \forall^* y \in X \forall^* h \in W_{\gamma(y)}(p_y = p_{h \cdot y}). \]

Finally choose a comeager set \( C \subseteq X \) such that

(a) \( \gamma|_C \) is continuous on \( C \); (b) \( x \mapsto g_x \) and \( x \mapsto p_x \) are continuous on \( C \); (c) for all \( x \in C \) we have \( \forall^* g \in W_{\gamma(x)}(p_x = p_{g \cdot x}). \)

Let \( x \in X \) be such that \( \forall^* g \in G(g \cdot x \in C) \). We can then find some non-empty open set \( \hat{V} \subseteq G \) and a point \( x_0 \in [x]_G \) such that

\[ \forall^* g \in \hat{V}(p_{g \cdot x} = x_0). \]
We then find open $U_0$ having non-empty intersection with $\hat{V} \cdot x$ such that for some single $\ell_0$

$$\forall y \in U_0 \cap C(\gamma(y) = \ell_0).$$

Finally we pass to $V = \{ g \in \hat{V} : g \cdot x \in U_0 \}$. I now want to show that $g \mapsto g \cdot x$ is open from $V$ to $V \cdot x$. So consider some $y \in V \cdot x$ and $V_0$ an open neighborhood of $1_G$; I want to show that there is an open $U_1$ containing $y$ such that for all $y_1 \in V \cdot x \cap U_1$ we have $y_1 \in V_0 \cdot y$. First choose $V_1$ an open neighborhood of $1_G$ with

$$(V_1)^{-1} = V_1,$$

$$(V_1)^4 \subset V_0,$$

$$V_1 \cdot y \subset V \cdot x,$$

$$V_1 \subset W_{\ell_0}.$$ 

And then take $g_0 \in V_1$ with

$$y' = g_0 \cdot y \in C$$

and

$$p_{y'} = x_0.$$ 

And then let $U' \subset U_0$ be a basic neighborhood of $y'$ such that for all $\hat{y} \in U' \cap C$ we have

$$g_{\hat{y}} \in g_{y'} V_1.$$ 

I claim that $U_1 = U_0 \cap (U')^\Delta V_1$ succeeds. For this purpose, let us suppose $y_1 \in V \cdot x \cap U_1$; we can choose $g_1 \in V_1$ with

$$y_2 =_{df} g_1 \cdot y_1 \in V \cdot x \cap U_1 \cap C$$

and $p_{y_2} = x_0$. We may then appeal to the definition of $U_1$ to find $g_2 \in V_1$ with

$$y_2' =_{df} g_2 \cdot y_2 \in U' \cap C.$$ 

Since $y_2 \in U_0 \cap C$ and $g_2 \in V_1 \subset W_{\ell_0}$ we can appeal to the assumption on $U_0$ to find a sequence of group elements $(\epsilon_n)_{n \in \mathbb{N}}$ which approach $1_G$ and have the property that

$$p_{\epsilon_n \cdot y_2} = p_{y_2} = x_0$$

and each $\epsilon_n \cdot y_2' \in C$; since the function $\hat{y} \mapsto p_{\hat{y}}$ is continuous on $C$ we must have $p_{y_2'}$ equals $x_0$ as well. By assumption on $U'$ we may find some $h \in V_1$ with

$$g_{y_2'} = y' \cdot h.$$ 

But this gives

$$g_{y_2'} \cdot y_2' = x_0 = g_{y'} \cdot y'$$

$$\therefore g_{y'} h \cdot y_2' = g_{y'} \cdot y'$$

$$\therefore h \cdot y_2' = y'$$

$$\therefore h g_2 g_1 \cdot y_1 = y' = g_0 \cdot y$$

$$\therefore y_1 = g_1^{-1} g_2^{-2} h^{-1} g_0 \cdot y \in (V_1)^4 \cdot y \subset V_0 \cdot y$$
thereby concluding the proof. □

**Definition** Let $G$ be a Polish group and $X$ a Polish $G$-space. We say that $X$ is a *stormy* $G$-space if

$$ V \to V \cdot x $$

$$ g \mapsto g \cdot x, $$

as $V$ ranges over non-empty open subsets of the group and $x$ ranges over points in $X$, is never an open mapping.

**Proposition 2.4.** If $E_X^G \leq_B E$, and $X$ is a stormy Polish $G$-space, then $E$ is not essentially countable.

Proof. This is a corollary of 2.3. □

In the remainder of the note we wish to start working towards a kind of converse to this proposition. If $E$ is induced by continuous action of a Polish group $G$, then $E$ is essentially countable if and only if there is no stormy Polish $G$-space $Y$ and continuous $G$-embedding from $Y$ to $X$. It might be worthwhile first to pause in order to point out a couple of facts in the neighborhood of this definition.

**Example** It is not hard to see that if the canonical map from $V$ to $V \cdot x$ is open, then $V \cdot x$ is comeager in its closure. This was implicitely proved at claim(III) of 2.1. The reverse implication fails. For instance let $G = S_\infty$, the infinite symmetric group, and consider the space $B_2$, consisting of all one-to-one sequences $\vec{x} = (x_0, x_1, \ldots) \in ([0, 1])^N$. The responsible group acts by permutation of the indices in the obvious manner:

$$(\pi \cdot \vec{x})_n = x_{\pi^{-1}(n)}. $$

Consider the point $\vec{y} = (y_0, y_1, \ldots)$ defined by

$$ y_0 = 0, $$

but

$$ y_n = \frac{1}{n} $$

for $n > 0$. We can then let

$$ V_0 = \{ \pi \in S_\infty : \pi(0) = 0 \}, $$

and for $k > 0$ we let

$$ V_k = \{ \pi \in S_\infty : \pi(k) = 0, \pi(0) = k \}. $$

It is not hard to see that at each $k > 0$ we have $V_k \cdot \vec{y}$ comeager in its closure; and thus for

$$ V = \bigcup_{k \geq 0} V_k $$

we have that $V \cdot \vec{y}$ comeager in $\overline{V \cdot \vec{y}}$. In fact the set $V \cdot \vec{y}$ is a $G_\delta$. A point $\vec{x}$ is in $V \cdot \vec{y}$ if and only if

(i) $\forall n, m \exists \ell(|x_\ell - 1/n| < 1/m);$ 
(ii) $\forall n, \ell_1, \ell_2 (\ell_1 \neq \ell_2 \Rightarrow (|x_{\ell_1} - 1/n| \geq (n+1)^{-2} \lor |x_{\ell_2} - 1/n| \geq (n+1)^{-2});$ 
(iii) $\forall \ell, m \exists n(|x_\ell - 1/n| < 1/m);$ 
(iv) $\forall m \exists k \neq 0((|x_0| < 1/m) \lor (|x_k| < (k+1)^{-2} \lor |x_0 - 1/k| < (m(k+1))^{-2})).$
However the map

\[ V \to V \cdot \bar{y} \]

\[ \pi \mapsto \pi \cdot \bar{y} \]

is not open, since \( V_0 \cdot \bar{y} \) is contained in a closed nowhere dense subset of \( \overline{V \cdot \bar{y}} \).

**Question** Suppose that for every \( x \in X \) there is some \( V \) such that \( V \cdot x \) is non-meager in \( \overline{V \cdot x} \). Must \( E_G^X \) be essentially countable? (must the hypotheses of 2.1 hold?)

**Proposition 2.5.** Let \( G \) be a Polish group acting continuously on a Polish space \( X \). Let \( B \) be a countable basis for \( G \). Suppose

\[ \forall^* x \in X \forall W \in B( W \neq 0 \Rightarrow W \cdot x \text{ is meager in } \overline{W \cdot x}). \]

Then \( E_G^X \) is not essentially countable.

**Proof.** First of all we can take a dense \( G_\delta \) set \( C \subset X \) such that for all \( x \in C \) and all \( W \in B, W \neq 0 \) we have \( W \cdot x \) is meager in \( \overline{W \cdot x} \). Then we can let

\[ X_0 = C^*G, \]

the Vaught transform of \( C \). This is then an invariant \( G_\delta \) set, and it suffices to show \( E_G^{X_0} \) is not essentially countable. For this it suffices to observe the following claim:

**Claim:** \( X_0 \) is a stormy Polish \( G \)-space.

**Proof of Claim:** Fix \( x \in X_0 \) and \( V \subset G \) open. We may then choose \( \epsilon \in G \) with \( \epsilon \cdot x \in C \). Then we may choose some non-empty \( W \in B \) with \( W\epsilon \subset V \). \( W \cdot (\epsilon \cdot x) \) is meager in \( \overline{W \cdot (\epsilon \cdot x)} \), and thus the map

\[ W \to W \cdot (\epsilon \cdot x) \]

\[ g \mapsto g\epsilon \cdot x \]

is not open, and hence

\[ W\epsilon \to W\epsilon \cdot x \]

\[ g\epsilon \mapsto g\epsilon \cdot x \]

is not open, and thus, since \( W\epsilon \subset V \),

\[ V \to V \cdot x \]

\[ g \mapsto g \cdot x \]

is also not an open mapping. \((\text{Claim})\)

**Example** There is a canonical example of a Polish group giving rise to an orbit equivalence relation which is not essentially countable. The orbit equivalence relation, \( F_2 \), corresponds in some rough sense to countable sets of reals. It arises on the \( B_2 \), the space of one-to-one sequences of reals in the unit interval, as a result of the natural action of \( S_\infty \) permuting the indices, as introduced in the previous example. There are several ways to prove that this equivalence relation is not essentially countable. One can even use forcing and give a metamathematical proof. However the most direct method, as sketched in the exercises from chapter 2 of [6],
is in fact to show that for $\mathcal{B}$ the canonical basis of $S_\infty$, we have for each $W \in \mathcal{B}$ non-empty
\[ \forall^* \vec{x} \in B_2(W \cdot \vec{x} \text{ is meager in } \overline{W \cdot \vec{x}}), \]
and then to combinatorially verify that no function continuous on a comeager set can reduce $F_2$ to a countable equivalence relation. The interest of 2.5 is to show that this "standard proof" fits naturally into a more general context.

3. ORGANIZING THE CASES

We from now on assume that $G$ is a Polish group, which is recursive as a Polish space, with the group operations of multiplication and inversion both recursive and continuous. We assume that $X$ is a recursive Polish space, equipped with an action of $G$ on $X$ which is recursive and continuous as a function
\[ G \times X \rightarrow X. \]
We further assume $E_X^G \in \Delta_1^1$. We wish to work towards showing that either $E_X^G$ may be reduced in to a countable Borel equivalence relation or we may continuously and in an action respectful manner embed into $X$ a stormy Polish $G$-space. In the first case one can in fact obtain that $X$ is reducible to a countable Borel equivalence relation which is actually $\Delta_1^1$ using a function which is actually $\Delta_1^1$; for clarity of exposition we will not stress the various small calculations which must be checked to obtain this extra effectivity, but instead push on with the proof of the main result at 1.3. At the very end we make a few remarks about how it may be refined. In general of course the Polish group, the Polish space, and the action of the group on the space might only be recursive in $z$, for some parameter $z \in \mathbb{N}^\mathbb{N}$. The proof of this more general case is a relativized version of the specific one we consider below. We assume that $G$ and $X$ come equipped with some suitably recursive countable bases in the sense of [8]. At some point it will be helpful to make some simplifying assumptions: We ask that the basis for $G$ be closed under finite intersection and inversion; that the basic open sets in $X$ and $G$ be closed under translation by elements from the subgroup $G_0$ consisting of recursive elements of $G$.

**Definition** For $A \subset X$ be $\Sigma_1^1$, $W_0, V_0 \subset G$ basic open with $1_G \in V_0$, $V_0^{-1} = V_0$, we say that $(A, W_0)$ is $V_0$-damming if for all $V_1 \subset G$ with $1_G \in V_1$, $V_1^{-1} = V_1$, all $x \in A^{\Delta W_0}$, and $y_0 \in V_0 \cdot x$, there is $B \in \Delta_1^1$ and there is $W_1 \subset G$ basic open, with
\[ y_0 \in B^{\Delta W_1} \]
such that
\[ \forall y_1, y_2 \in V_0 \cdot x \cap B^{\Delta W_1} (y_1 \in V_1 \cdot y_2). \]

The property of being $V_0$-damming is $\Pi_1^1$ in the indices. The point is that if $E_X^G$ is $\Delta_1^1$ then it follows from [1] ¶7.1 that the stabilizer function $x \mapsto G_x$ is $\Delta_1^1$, in some appropriately understood effective presentation of the collection of closed subsets of $G$. From this we obtain that the set of $(y_1, y_2, V_1)$ such that
\[ y_1 \in V_1 \cdot y_2 \]
is $\Delta_1^1$. This granted, it all routinely unravels to the level of $\Pi_1^1$. Thus by $\Pi_1^1$-reflection\(^3\), if $(A, W_0)$ is $V_0$-damming then we may find some $\Delta_1^1$ set $D \supset A$ with $(D, W_0)$ again $V_0$-damming. We are here confronted with a split in cases.

\(^3\)See [5].
**Definition** We let $X_{\text{bad}}$ be the set of all $x \in X$ such that for all $A \in \Sigma^1_1$, all basic open $W_0, V_0 \subset G$ with $1_G \in V_0$, $V_0^{-1} = V_0$, if $x \in A^{\Delta^0_0}$ then $(A, W_0)$ is not $V_0$-damming.

The set $X_{\text{bad}}$ is $\Sigma^1_1$ by the reflection argument indicated above. From the structure of the definitions one can see that $X_{\text{bad}}$ is invariant under the action of $G_0$: if $g_0 \cdot x \in A^{\Delta^0_0}$ for some $g_0 \in G_0$, some $V_0$-damming $(A, W_0)$, then $x \in A^{\Delta^0_0 g_0}$ and $(A, W_0 g)$ is $g_0^{-1} V_0 g_0$-damming. But then we see that it must be invariant under the entire action of $G$, since if $g_0 \cdot x \in A^{\Delta^0_0}$ then for any $g$ sufficiently close to $g_0$ we have $g \cdot x \in A^{\Delta^0_0}$. Thus $X_{\text{bad}}$ is invariant under the action of the entire group $G$. Whether this set $X_{\text{bad}}$ is empty provides the split in cases.

**Notation** From now on let us understand that

$$V_0, V_1, V_2, ..., V, \tilde{V}, V', W, ..., W_0, W_1, W_2...$$

and such like always refer to basic open subsets of the group $G$.

**4. Case (1): $X_{\text{bad}}$ is Empty**

**Claim 4.1.** For all $x \in X$ there exists $V_0$ non-empty such that for all $y \in V_0 \cdot x$ and $V_1$ symmetric containing $1_G$, we have some $B \in \Delta^1_1$ and $W \subset G$ such that

$$y \in B^{\Delta^0_0},$$

$$\forall y_1, y_2 \in B^{\Delta^0_0} \cap V_0 \cdot x (y_1 \in V_1 \cdot y_2).$$

Proof. The case assumption implies that every such $x$ will be included in some $A^{\Delta^0_0}$ which necessitates some $V_0$ to succeed.

But then we may find a countable algebra $B$ of $\Delta^1_1$ subsets of $X$ with the following properties:

(i) $B$ is closed under finite boolean operations;
(ii) $B$ is closed under translation by elements in $G$;
(iii) for any $B \in B$ and $V$ basic open we have

$$C^{\Delta^0_0}, C^{*V} \in B;$$

(iv) $\forall x \in X \exists V_0 \neq 0 \forall y \in V_0 \cdot x \forall V_1$, if $V_1$ is a symmetric neighborhood of $1_G$ then there exists $B \in B$ and $W \subset G$ such that

$$y \in B^{\Delta^0_0} \cap \forall y_1, y_2 \in B^{\Delta^0_0} \cap V_0 \cdot x (y_1 \in V_1 \cdot y_2);$$

(v) we may decompose $B$ into some directed union

$$B = \bigcup_{\alpha \in \delta} B_\alpha,$$

some $\delta \leq \omega^k_1$, such that each element of $B_0$ is either closed or open, and for $\gamma > 0$ we have that each element in $B_\gamma$ is either the countable union or intersection of elements in

$$\bigcup_{\alpha < \gamma} B_\alpha.$$

**Definition** We let $\tau_{*B}$ be the topology generated by taking as a basis all sets of the form $B^{\Delta^0_0}$ where $B \in B$ and $V \subset G$ basic open.

**Claim 4.2.** $(X, \tau_{*B})$ is a Polish $G$-space giving rise to the same Borel structure as the original topology on $X$. 


Proof. (v) implies by an appeal to transfinite induction on \( \delta \) that there is a Polish topology having \( B \) as its basis. This granted, the claim follows from the Becker-Kechris theorem on changing topologies from chapter 5 of [1]. \( \square \)

**Claim 4.3.** For each \( x \in X \) there is some \( V \subset G \) non-empty such that

\[ g \mapsto g \cdot x \]

is an open mapping from \( V \) to \((V \cdot x, \tau_{*, B})\).

Proof. For any \( x \) we appeal to the listed properties of \( B \); \( V = V_0 \) from (iv) succeeds by the definition of the topology \( \tau_{*, B} \).

But then by 2.1 we have that \( E_G^X \) is essentially countable. \( \square \)

5. **Case(2):** \( X_{bad} \) is Non-Empty

Following [8], we let

\[ X_{low} = \{ x \in X : \forall^* g \in G (\omega_1^{ck}(g \cdot x) = \omega_1^{ck}) \} \]

and consider the topology \( \tau_{\infty}^* \) whose basis consists of all sets of the form

\[ X_{low} \cap A^{AW}, \]

\( A \in \Sigma^1 \) in \( X, W \) basic open in \( G \). As shown there, this is a Polish topology, and in fact \((X_{low}, \tau_{\infty}^*)\) is a Polish \( G \)-space in the action inherited from \( X \). As at [8], we will use the following characterization of when \( \omega_1^{ck}(x) = \omega_1^{ck} \): \( x \) is low exactly when for every \( \Sigma^1 \) set \( A \), either \( x \in A \) or there is a \( \Sigma^1 \) set \( B \) with \( x \in B \) and \( A \cap B = 0 \).

**Definition** We let \( X_{bad} = X_{low} \cap X_{bad} \), equipped with the subspace topology induced by \( \tau_{\infty}^* \).

This \( X_{bad} \) is the intersection of two invariant sets, and hence \( G \)-invariant. It is relatively open in \( X_{low} \), and hence again a Polish \( G \)-space. We want to show that it is in fact a stormy Polish \( G \)-space. So fix any \( x \in X_{bad} \). If one point in the orbit of \( x \) fails to have the natural map \( V \to V \cdot x \) open for all \( V \subset G \) then so does every point in the orbit of \( x \). Thus after possibly nudging \( x \) by a sufficiently generic group element \( g \in G \) we may assume first of all that

\[ \omega_1^{ck}(x) = \omega_1^{ck} \]

and secondly that for all \( A \in \Sigma_1 \), if \( x \in A \) then there is a basic open neighborhood \( W \) of \( l_G \) such that

\[ x \in A^{*W}. \]

The assumption that \( x \in X_{bad} \) implies that for any \( A_0 \in \Sigma^1_1, W_0, V_0 \subset G \), with \( l_G \in V_0, V_0^{-1} = V_0, x \in A_0^{AW} \) there exists \( V_1 \) with \( l_G \in V_1, V_1^{-1} = V_1 \), and there exist \( x' \in A_0^{AW}, \ y' \in V_1 \cdot x' \) such that for all \( B \in \Delta^1, W_1 \subset G \),

\[ y' \in B^{AW} \Rightarrow (\exists y_1', y_2' \in B^{AW} \cap V_0 \cdot x'(y_1' \notin V_1 \cdot y_2)). \]

**Claim 5.1.** For all \( V \subset G \) with \( l_G \in V, V^{-1} = V \), there exist \( V_1, \) with \( l_G \in V_1, \) and \( y \in V \cdot x \) such that for all \( B \in \Delta^1, W_1 \subset G \),

\[ y \in B^{AW} \Rightarrow (\exists y_1, y_2 \in B^{AW} \cap V \cdot x(y_1 \notin V_1 \cdot y_2)). \]
Proof. Otherwise the characteristic property of low reals implies that there is a \( \Sigma^1_1 \) set \( A \) containing \( x \) such that for all \( x' \in A \), for all \( y' \in V \cdot x' \), for all \( V_1 \) containing \( 1_G \), there exists \( B \in \Delta^1_1 \), \( W_1 \subset G \) such that
\[
y' \in B^{\Delta^W_1},
\]
\[\forall y_1', y_2' \in V \cdot x' \cap B^{\Delta^W_1} (y_1' \in V_1 \cdot y_2').\]
Choose \( V_0 \) containing \( 1_G \) with \( V_0 \cap V, V_0^{-1} = V_0 \). Choose \( W_0 \subset V_0 \cap W \) such that
\[x \in A^{\cdot W_0} \subset A^{\Delta^W_0}.
\]
Then for any \( \hat{x} \in A^{\Delta^W_0} \) and any \( y' \in V_0 \cdot \hat{x} \) and \( V_1 \) with \( 1_G \in V_1 \), \( V_1^{-1} = V_1 \), we may choose
\[y' \in W_0 \cdot \hat{x} \cap A \subset V_0 \cdot \hat{x} \cap A.
\]
We have \( y' \in V_0 V_0^{-1} \cdot x' \subset V \cdot x' \), and so assumption on \( A \) yields some \( B^{\Delta^W_1} \) containing \( y' \) such that
\[\forall y_1', y_2' \in V \cdot x' \cap B^{\Delta^W_1} (y_1' \in V_1 \cdot y_2').
\]
But now note that for all \( y_1', y_2' \in B^{\Delta^W_1} \cap V_0 \cdot \hat{x} \) we have
\[y_1', y_2' \in V_0 V_0^{-1} \cdot x' \subset V \cdot x'
\]
\[\therefore y_1' \in V_1 \cdot y_2'.
\]
Thus \( A_0 \) and \( W_1 \) as given here contradict the assumption on \( x \). \( \square \)

The next claim states that we can obtain the same result even for \( B \in \Sigma^1_1 \). Since our topology is based around the \( \Sigma^1_1 \) sets, not just the \( \Delta^1_1 \), this is critical.

**Claim 5.2.** For all \( V \subset G \) with \( 1_G \in V \), \( V^{-1} = V \), there exist \( V_1 \), with \( 1_G \in V_1 \), and \( y \in V_1 \cdot x \) such that for all \( B \in \Sigma^1_1 \), \( W_1 \subset G \),
y \in B^{\Delta^W_1} \Rightarrow (\exists y_1, y_2 \in B^{\Delta^W_1} \cap V \cdot x(y_1 \notin V_1 \cdot y_2)).

Proof. Fix \( V \). Choose \( y \) and \( V_1 \) as in the last claim. Since the property indicated for \( y \) is \( \Sigma^1_1(x) \), we may assume
\[\omega^\ck(x, y) = \omega^\ck(x) = \omega^\ck_1
\]
by the Gandy low basis theorem. With this extra assumption on \( y \) we will prove that \( y \) satisfies the conclusion of the stronger claim presently before us. For this purpose consider some \( B \in \Sigma^1_1 \) and \( W_1 \subset G \). Let
\[f_B : X \to LO
\]
be a rank, assigning to each \( z \in X \) a linear ordering \( f_B(z) \subset \mathbb{N} \times \mathbb{N} \) in a \( \Delta^1_1 \) manner, such that \( f_B(z) \) is a well ordering if and only if \( z \notin B \). For each \( \alpha < \omega^\ck_1 \) we let \( B^{(\alpha)}(\alpha) \) be the set of \( z \in X \) such that \( (\alpha, \varepsilon) \) allows an order preserving injection into \( (\mathbb{N}, f_B(z)) \). These sets are all \( \Delta^1_1 \). One possibility is that for some \( \alpha < \omega^\ck_1 \) we have \( y \notin (B^{(\alpha)})^{\Delta^W_1} \). But then if we let \( B \) be the set of \( z \in B \) such that \( (\mathbb{N}, f_B(z)) \) does not contain \( \alpha \) in its wellfounded part, then we have that \( B \) is a \( \Delta^1_1 \) set included in \( B \) and \( y \in B^{\Delta^W_1} \); and then we are immediately done by the assumptions on \( y \).

The other possibility is that for each \( \alpha < \omega^\ck_1 \) we have \( y \in (B^{(\alpha)})^{\Delta^W_1} \), and hence appealing to assumptions on \( y \) we may find \( y_1^\alpha, y_2^\alpha \in (B^{(\alpha)})^{\Delta^W_1} \) with \( y_1^\alpha \notin V_1 \cdot y_2^\alpha \).
But then appealing to boundedness we can find an illfounded linear ordering \((\mathbb{N}, <^*)\) such that there exists \(y_1^\infty, y_2^\infty\) with
\[
\exists^* g \in W_1((\mathbb{N}, <^*) \text{ embeds into } f_B(g \cdot y_1^\infty)), \\
\exists^* g \in W_1((\mathbb{N}, <^*) \text{ embeds into } f_B(g \cdot y_2^\infty)), \\
y_1^\infty \notin V_2 \cdot y_2^\infty.
\]
But if \((\mathbb{N}, <^*)\) embeds into \(f_B(g \cdot y_i^\infty)\) then \(f_B(g \cdot y_i^\infty)\) is illfounded and so \(g \cdot y_i^\infty \in B\). Thus we are done.

But since this last claim holds for any \(x \in X_{\bullet \text{bad}}\), we have that the action of \(G\) on \(X_{\bullet \text{bad}}\) is indeed stormy.

6. Remarks on Effectivity

Little effort was made in case(1) to indicate how we can produce a lightfaced reduction to a lightfaced \(\Delta_1^1\) equivalence relation. On the surface, 2.1 is phrased so that it only looks like a bold faced result and even the statement of Kechris’ criterion at 5.2 of [7] is only as a boldfaced result. First of all, inspecting the proof of Kechris’ criterion one actually obtains:

**Proposition 6.1.** (Kechris) Let \(E\) be a \(\Delta_1^1\) equivalence relation on a recursive Polish space \(\mathcal{X}\) and
\[
\theta : \mathcal{X} \rightarrow \{0, 1\}^\mathbb{N}
\]
a \(\Delta_1^1\) function such that:
(i) \(\theta\) has countable image on each equivalence class;
(ii) \(\theta\) maps distinct equivalence classes to disjoint sets.
Then there is a \(\Delta_1^1\) equivalence relation \(F\) on \(\{0, 1\}^\mathbb{N}\) all of whose equivalence classes are countable such that \(\theta\) witnesses \(E \leq_B F\).

We are still faced with the brutishness of \(B\) and boldfacedness of 2.1. These problems are both dealt with in the same way. The main claim is this:

**Claim 6.2.** In the proof from case(1), we may assume that there is a \(D \subset X \times \mathbb{N}\) in \(\Delta_1^1\) such that each \(B \in B\) has the form
\[
B = D_k = \{x : (x, k) \in D\}
\]
for some suitable \(k\).

This claim is proved by the boundedness theorem for \(\Sigma_1^1\) subsets of \(\omega_1^{ck}\); in some sense, we can “bound” all of the needed \(\Delta_1^1\) sets in some manageable object. To see this recall that every \(\Delta_1^1\) set has the form
\[
\{y : L_\alpha[y] \models \varphi(y)\}
\]
for some \(\alpha < \omega_1^{ck}\) and formula \(\varphi(\cdot)\) of set theory. If \(V_x \subset G\) satisfies the statement of 4.1, then this is a \(\Pi_1^1\) property on the basic open set \(V\). If \(x \in X, y \in V_x \cdot x\), then the statement that \(W_0 \subset G\) and
\[
B = \{y : L_\alpha[y] \models \varphi(y)\}
\]
satisfy the assumption of 4.1 is \(\Pi_1^1(x, y)\). Thus we have in effect a \(\Pi_1^1\) function from \(X\) to the basic open sets in \(G\), and then composing a \(\Pi_1^1\) function from \(X \times X\) to
\[
\omega_1^{ck} \times \mathcal{L}(\varepsilon).
\]
\( \Pi_1^1 \) functions into the natural numbers are always \( \Delta_1^1 \). Thus by boundedness we can assume that there is a single \( \delta < \omega_1^{ck} \) such that we can always choose \( \alpha \leq \delta \). In this way, granting certain routine technical details about effectivizing the enumeration, we can prove the claim. But then if \( (\hat{V}_t)_{t \in \mathbb{N}} \) is a basis for \( G \), we have a \( \Pi_1^1 \), and hence \( \Delta_1^1 \), function

\[
X \rightarrow \mathbb{N}
\]

\[
x \mapsto n_x
\]

assigning to each \( x \) some \( n_x \) such that \( \hat{V}_{n_x} \) fulfills 4.1. Then we can let \( (m_x, k_x) \) be chosen in a \( \Delta_1^1 \) way so that

\[
U_x = (D_{k_x})^{\Delta \hat{V}_{m_x}}
\]

along with \( V_x = \hat{V}_{n_x} \) completes the construction of 2.1. We can then finish by letting \( \theta(x) \) be the characteristic function of the following set of natural numbers:

\[
\{2^k3^n : V_x \cdot x \cap U_x \cap (D_k)^{\Delta \hat{V}_n} \neq 0 \};
\]

in this manner we have assigned to \( x \) in a \( \Delta_1^1 \) way an element \( \theta(x) \) such that

\[
\theta(x_1) = \theta(x_2)
\]

if and only if

\[
U_{x_1} \cap V_{x_1} \cdot x = U_{x_2} \cap V_{x_2} \cdot x.
\]

By the proof of 2.1, this function satisfies Kechris’ criterion.

References


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Mad Families Are Small

Taneli HUUSKONEN and Yi ZHANG

ABSTRACT. We consider several kinds of almost disjoint families in \( ^N\mathbb{N} \), namely, eventually different families in \( ^N\mathbb{N} \), almost disjoint families of permutations in \( Sym(\mathbb{N}) \), and cofinitary permutation groups in \( Sym(\mathbb{N}) \). We show that each of these families is meager and not of positive measure.

1. Introduction

Let \( x, y \) be two different countably infinite sets. We say that \( x, y \) are almost disjoint (a.d.) iff \( x \cap y \) is finite. A set \( F \) is an a.d. family iff \( x \) is countably infinite for every \( x \in F \) and \( x, y \) are a.d. for any two different \( x, y \in F \). In this paper, we shall study several kinds of almost disjoint families in \( ^N\mathbb{N} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \} \). Let us first give the definition of them, then we shall state several results about these families which motivated the research of this paper.

We write \( ^{<N}\mathbb{N} \) for the set \( \bigcup_{n \in \mathbb{N}} ^n\mathbb{N} \), where we follow the set theoretical convention \( n = \{ 0, 1, \ldots, n - 1 \} \) for \( n \in \mathbb{N} \). Let \( [s] = \{ f \in ^N\mathbb{N} \mid s \subseteq f \} \) where \( s \in ^{<N}\mathbb{N} \). By \( Sym(\mathbb{N}) \) we denote the set of permutations on \( \mathbb{N} \).

DEFINITION 1.1. Two different functions \( f, g \in ^N\mathbb{N} \) are eventually different (e.d.) iff \( f \cap g \) is finite, i.e.,

\[ |\{ n \in \mathbb{N} \mid f(n) = g(n) \}| < \omega. \]

\( E \subseteq ^N\mathbb{N} \) is an e.d. family of functions iff \( f, g \) are e.d. for any different \( f, g \in E \).

The following results can be easily shown.

i) If \( A \subseteq ^N\mathbb{N} \) is a maximal eventually different (m.e.d.) family, then \( A \) is not countable.

ii) There exists an e.d. family of cardinality \( \mathfrak{c} \), the size of the continuum.

During the past decades, a plethora of so-called cardinal invariants of the continuum have been investigated. Cardinal invariants are cardinals defined as the smallest size of a set of reals with certain properties. So it is natural to introduce the following number:

\[ a_e = \text{ the least } \lambda \text{ such that there exists a m.e.d. family } E \subseteq ^N\mathbb{N} \text{ with } |E| = \lambda. \]

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Note. The study of the families of eventually different functions in $\mathbb{N}^\mathbb{N}$ was considered by E. van Douwen and A. Miller (see, e.g., [vD], [M], [M1]). The number $\alpha_e$ was suggested to Y. Zhang by B. Velickovic (see, e.g., [Z]). For simplicity, we will just use a.d. when we talk about e.d. in the rest of this paper. The word “mad” will stand for “maximal almost disjoint.”

Now, let us give a brief introduction to cofinitary permutation groups.

**Definition 1.2.** A permutation $g \in \text{Sym}(\mathbb{N})$ is cofinitary iff $g$ has only finitely many fixed points. A group $G \leq \text{Sym}(\mathbb{N})$ is cofinitary iff every non-identity element is cofinitary.

It is easily seen that $G \leq \text{Sym}(\mathbb{N})$ is cofinitary iff $G$ is both an almost disjoint set of permutations and a group. For a discussion of different aspects of cofinitary groups, the reader can consult the well written survey paper by P. J. Cameron [C]. Since the union of a chain of cofinitary permutation groups is cofinitary, Zorn’s Lemma implies that maximal cofinitary groups exist, and indeed any cofinitary group is included in a maximal one. The following theorem was proved independently by S. A. Adeleke and J. K. Truss (see, e.g., [A], [T] and [T1] for details).

**Theorem 1.3.** If $G \leq \text{Sym}(\mathbb{N})$ is a maximal cofinitary group, then $G$ is not countable.

P. Neumann showed (see, e.g., [C, Proposition 10.4])

**Theorem 1.4.** There exists a cofinitary group of cardinality $\mathfrak{c}$, the size of the continuum.

The two theorems motivated P. Cameron (see, e.g., [C]) to ask the following question:

**Question 1.5.** If the continuum hypothesis (CH) fails, is it possible that there exists a maximal cofinitary group $G$ such that $|G| < \mathfrak{c}$?

In [Z1, Theorem 1.5] this problem was solved by:

**Theorem 1.6.** Assume $\kappa$ is a fixed uncountable cardinal such that $\aleph_1 \leq \kappa \leq \lambda$, and $cf(\lambda) > \aleph_0$. Then it is consistent with ZFC that there exists a maximal cofinitary group $G \leq \text{Sym}(\mathbb{N})$ such that $|G| = \kappa$ and $\mathfrak{c} = \lambda$.

Thus it is natural to define the following number.

$\alpha_g = \text{the least } \lambda \text{ such that there exists a maximal cofinitary group } G \leq \text{Sym}(\mathbb{N}) \text{ with } |G| = \lambda$.

**Note.** A cofinitary permutation group $G$ is also an a.d. family in $\text{Sym}(\mathbb{N})$, i.e.,

$$\{n \in \mathbb{N} \mid f(n) = g(n)\}$$

is finite for any $f \neq g$ in $\text{Sym}(\mathbb{N})$. One can easily prove the results corresponding to Theorems 1.3, 1.4, 1.6 for mad permutation families in $\text{Sym}(\mathbb{N})$. S. Thomas introduced the following number:

$\alpha_p = \text{the least } \lambda \text{ such that there exists a mad permutation family } A \text{ in } \text{Sym}(\mathbb{N}) \text{ with } |A| = \lambda$.

A number of interesting results about $\alpha_e, \alpha_g, \alpha_p$ have been achieved. For example, in [BSZ], the following result has been proved.
Theorem 1.7. $a_e, a_g, a_p \geq \text{Non}(M)$, where $\text{Non}(M)$ is the cardinality of the smallest non-meager subset of reals.

As a corollary of Theorem 1.7, we also know the following:

Corollary 1.8. It is consistent with ZFC that $a_e, a_g, a_p > \text{Non}(N)$, where $\text{Non}(N)$ is the cardinality of the smallest non-measure 0 set of reals.

However, we should notice that all these results above are interesting only when CH fails. The proofs of results 1.7 and 1.8 do not give any indication as to whether or not those mad families are meager or null in the appropriate context. So, it is natural to ask the following question:

Question 1.9. Regardless of CH, are those mad families we discussed above small or big? E.g., are they meager or null in the appropriate context?

We shall answer this question in the next two sections of this paper.

2. Mad families are meager

For $\mathbb{N}$ and its subspaces, we will consider the so-called natural topology, whose basic open sets are those of the form

$$[s] = \{ f \in \mathbb{N} \mid s \subseteq f \}$$

for $s \in \mathcal{P}^\mathbb{N}$. The following metric $d$ induces this topology:

$$d(f, g) = \begin{cases} 0, & f = g, \\ 2^{-m(f, g)}, & f \neq g, \end{cases}$$

where $m(f, g)$ is the smallest $m$ such that $f(m) \neq g(m)$.

If $X$ is a topological space, and $A \subseteq X$, we say that $A$ is nowhere dense in $X$, if $X \setminus A$ contains a dense open set and $A$ is meager in $X$ if $A$ is a countable union of nowhere dense sets. Clearly, $A \subseteq X$ is nowhere dense iff for every basic open $B \subseteq X$, there is an open $C \subseteq B$ such that $A \cap C = \emptyset$. So, to say a set is nowhere dense is a way to say that this set is small.

Definition 2.1. Let $X \subseteq \mathbb{N}$. We say that $f \in X$ is lonely if there exists a non-empty open set $U \subseteq X$ such that for all $g \in U$, the set $\{ m \in \mathbb{N} \mid f(m) = g(m) \}$ is finite.

Lemma 2.2. Let $X \subseteq \mathbb{N}$, and let $A$ be an a.d. family in $X$ such that not all elements of $A$ are lonely in $X$. Then $A$ is meager in $X$.

Proof. Let $f \in A$ be non-lonely. For $n \in \mathbb{N}$, $A_n = \{ g \in A \mid g(m) \neq f(m) \text{ if } m > n \}$. Since $A$ is a.d., $A = \bigcup_{n \in \mathbb{N}} A_n$. Moreover, let $s \in \mathcal{P}^\mathbb{N}$ be such that $[s] \cap A_n \neq \emptyset$. Since $f$ is not lonely in $X$, there exists a $g \in [s] \cap X$ such that $f(m) = g(m)$ for infinitely many $m$. Let $m > \max(n, |s|)$ be such that $f(m) = g(m)$, and let $t(0) = g(0), t(1) = g(1), \ldots, t(m) = g(m)$. Then $[t] \subseteq [s]$ and $[t] \cap A_n = \emptyset$. Hence $A_n$ is nowhere dense for all $n \in \mathbb{N}$, and therefore, $A$ is meager.

It is easy to see that neither $\mathbb{N}$ nor $\text{Sym}(\mathbb{N})$ has any lonely elements, so the above lemma, when applied to them, yields the following theorem.
Theorem 2.3. (1). Every mad family in $^*\mathbb{N}$ is meager in $^*\mathbb{N}$.
(2). Every mad permutation family in $\text{Sym}(\mathbb{N})$ is meager in $\text{Sym}(\mathbb{N})$.
(3). Every cofinitary permutation group is meager in $\text{Sym}(\mathbb{N})$.

3. Mad families cannot be big anyway

Before we prove our results, we first introduce some definitions, most of which are well known.

We equip $\mathbb{N}$ with the following measure $\mu_0$:
$$\mu_0(S) = \Sigma_{k \in S} 2^{-k-1}$$
for every $S \subseteq \mathbb{N}$. Clearly, $\mu_0$ is a measure on $\mathbb{N}$ and $\mu_0(\mathbb{N}) = 1$. Let $\mu$ be the product measure on $^*\mathbb{N}$, i.e., define
$$\mu(\Pi_{n \in \mathbb{N}} S_n) = \Pi_{n \in \mathbb{N}} \mu_0(S_n),$$
where $S_n \subseteq \mathbb{N}$ for all $n \in \mathbb{N}$, and extend $\mu$ to the corresponding $\sigma$-algebra. Moreover, if $\mu(U) = 0$, $\mu(V)$ is defined, and $V \Delta W \subseteq U$ for some $U, V, W \subseteq ^*\mathbb{N}$, then we define $\mu(W) = \mu(V)$. Here $V \Delta W$ denotes the symmetric difference of $V$ and $W$, i.e., $(V \setminus W) \cup (W \setminus V)$.

Let $I$ be the set of one-to-one functions from $\mathbb{N}$ to $\mathbb{N}$. We define a function $H : I \rightarrow ^*\mathbb{N}$ as follows. For a finite $A \subseteq \mathbb{N}$, let $h_A$ be the unique order-preserving bijection from $\mathbb{N} \setminus A$ to $\mathbb{N}$. Let then $f \in I$ and let $n \in \mathbb{N}$. We define
$$H(f)(n) = h_{f[n]}(f(n)).$$
Clearly for any $f, g \in I$ such that $f \upharpoonright n = g \upharpoonright n$, it holds that $H(f)(n) = H(g)(n)$ iff $f(n) = g(n)$. Thus $H$ is an isometry and hence, in particular, one-to-one, and for any $f \in ^*\mathbb{N}$, the inverse image $H^{-1}(f)$ is easy to construct recursively. So, $H$ is a homeomorphism. We define a measure $\mu_I$ on $I$ as the measure induced from $^*\mathbb{N}$ by $H$, i.e., $\mu_I(A) = \mu(H[A])$ for every $A \subseteq I$ such that the right-hand side is defined.

Now, we shall prove that $I$ is small in $^*\mathbb{N}$ and $\text{Sym}(\mathbb{N})$ is large in $I$, in terms of both measure and category. The following lemma is a handy tool.

Lemma 3.1. Let $A \subseteq ^*\mathbb{N}$, $U \subseteq \mathbb{N}$ and $m, n \in \mathbb{N}$ be such that $U$ is infinite and for all $f \in A$, $i \in U$ it is not the case that $f(i) = m$ and $f(i + 1) = n$. Then $A$ is nowhere dense and $\mu(A) = 0$.

Proof. Let $s \in ^{<\mathbb{N}}$. Choose an index $i \in U \setminus \text{dom}(s)$. Let $t \in ^{<\mathbb{N}}$ be an extension of $s$ such that $t(i) = m$ and $t(i + 1) = n$. Then $[t] \subseteq [s]$ and $[t] \cap A = \emptyset$. Thus $A$ is nowhere dense.

Let then $\varepsilon > 0$, and choose a finite $U' \subseteq U$ such that
$$(1 - 2^{-m-n-2})U' < \varepsilon$$
and there is no $i \in \mathbb{N}$ such that both $i \in U'$ and $i + 1 \in U'$. For $i \in U'$, let $B_i$ be the set of $f \in ^*\mathbb{N}$ such that it is not the case that $f(i) = m$ and $f(i + 1) = n$, and let $B = \cap_{i \in U'} B_i$. It is easy to see that
$$\mu(B_i) = 1 - \mu_0(\{m\})\mu_0(\{n\}) = 1 - 2^{-m-n-2}$$
for each $i \in U'$, and consequently
$$\mu(B) = \mu(B_i)U' = (1 - 2^{-m-n-2})U' < \varepsilon.$$
Moreover, $A \subseteq B$. Thus, for every $\varepsilon > 0$, there is $B \subseteq \mathbb{N}$ such that $A \subseteq B$ and $\mu(B) < \varepsilon$, and therefore $\mu(A) = 0$. 

Let $K = \{f \in \mathbb{N} | f(m) = 0 \text{ for finitely many } m \text{ only}\}$.

**Lemma 3.2.** $K$ is meager in $\mathbb{N}$ and $\mu(K) = 0$.

**Proof.** Clearly $K = \bigcup_{i \in \mathbb{N}} K_i$, where

$$K_i = \{f \in \mathbb{N} | f(j) \neq 0 \text{ for } j \geq i\}.$$

Apply the previous lemma to each $K_i$ with $m = n = 0$, $U = \{j \in \mathbb{N} | j \geq i\}$.

**Corollary 3.3.** $\mu(I) = 0$.

**Proof.** Clearly $I \subseteq K$.

The lemma would also show that $I$ is meager, but it is just as easy to see directly that $I$ is actually nowhere dense.

Let then $S = I \setminus \text{Sym}(\mathbb{N})$.

**Lemma 3.4.** $H[S] = K$.

**Proof.** Let $g \in I$, and let $f = H(g)$. For $m \in \mathbb{N}$, let $h(m) = \min(\mathbb{N} \setminus \{g(0), g(1), \ldots, g(m-1)\})$. The function $h$ is nondecreasing, and $h(m + 1) > h(m)$ iff $g(m) = h(m)$ iff $f(m) = 0$. Now $g \in S$ iff $g$ is not onto iff $h$ is bounded iff $f$ has only finitely many zeroes iff $f \in K$.

**Corollary 3.5.** $S$ is meager in $I$, and $\mu_I(S) = 0$.

Thus with respect to measure and category, we may consider $I$ and $\text{Sym}(\mathbb{N})$ to be the same. Formally, concerning category, the following conditions are equivalent for any set $X \subseteq I$:

1. $X$ is meager in $I$.
2. $X \cap \text{Sym}(\mathbb{N})$ is meager in $I$.
3. $X \cap \text{Sym}(\mathbb{N})$ is meager in $\text{Sym}(\mathbb{N})$.

As to measure, we simply define measure on $\text{Sym}(\mathbb{N})$ as the restriction of $\mu_I$. Then clearly $\mu_I(X) = 0$ iff $\mu_I(X \cap \text{Sym}(\mathbb{N})) = 0$ whenever $X \subseteq I$.

Now we prove that the mad families which we deal with cannot have positive measure.

**Lemma 3.6.** Let $A \subseteq \mathbb{N}$ be almost disjoint. Then $A$ is either non-measurable or has measure 0.

**Proof.** Assume towards a contradiction that $\mu(A) > 0$. For $f \in \mathbb{N}$, $n \in \mathbb{N}$, let $g_{f,n}$ be the following function:

$$g_{f,n}(m) = \begin{cases} f(m), & m \neq n, \\ f(m) + 1, & m = n, \end{cases}$$

and let $C_{A,n} = \{g_{f,n} | f \in A\}$. Then $\mu(C_{A,n}) = \mu(A)/2 > 0$, and $C_{A,n} \cap C_{A,m} = \emptyset$ for $m \neq n$. This is a contradiction, since $\mu(\mathbb{N}) = 1$. 

Lemma 3.7. Let $A \subseteq \text{Sym}(\mathbb{N})$ be almost disjoint. Then $A$ is either non-measurable or has measure 0.

Proof. For $f \in \text{Sym}(\mathbb{N})$, $n \in \mathbb{N}$, let $g_{f,n}$ be the following function:

$g(m) = f(n+1)$, if $m = n$,

$g(m) = f(n)$, if $m = n+1$,

$g(m) = f(m)$, otherwise.

Then $H(g)(i) = H(f)(i)$ whenever $i < n$ or $i > n + 1$, and

$|H(g)(i) - H(f)(2n + 1 - i)| \leq 1$

for $i = n, n + 1$. Let then

$C_{A,n} = \{g_{f,n} \mid f \in A\}$.

Then $\mu_I(C_{A,n}) \geq \varepsilon/4$, and $C_{A,n} \cap C_{A,m} = \emptyset$ for $m \neq n$. This is a contradiction, since $\mu_I(\text{Sym}(\mathbb{N})) = 1$. \hfill \Box

As an easy corollary, we know that any cofinitary permutation group on $\mathbb{N}$ is either measure 0 or non-measurable. Thus none of these mad families which we study can have positive measure.

We do not know whether there can exist a non-measurable mad family in $\mathbb{N}\mathbb{N}$ or $\text{Sym}(\mathbb{N})$. However, for each mad family in $\mathbb{N}\mathbb{N}$, we can construct a mad family of the same cardinality and of measure 0.

For $f : \mathbb{N} \to \mathbb{N}$, let $W(f)$ be the function $g : \mathbb{N} \to \mathbb{N}$ such that

$g(n) = f([n/2])$.

It is easy to see that $W : \mathbb{N}\mathbb{N} \to \mathbb{N}\mathbb{N}$ is one-to-one.

Proposition 3.8. Let $A$ be a mad family in $\mathbb{N}\mathbb{N}$. Then $W[A]$ is a mad family in $\mathbb{N}\mathbb{N}$ satisfying $|W[A]| = |A|$ and $\mu(W[A]) = 0$.

Proof. Let $f, g \in A$, and let $n \in \mathbb{N}$ be such that $f(m) \neq g(m)$ for $m > n$. Clearly $W(f)(m) \neq W(g)(m)$ for $m > 2n + 1$. Thus $W[A]$ is almost disjoint. Moreover, if $f \in \mathbb{N}\mathbb{N}$, define $g \in \mathbb{N}\mathbb{N}$ as follows:

$g(n) = f(2n)$.

Since $A$ is maximal, there is some $h \in A$ such that $g(m) = h(m)$ for infinitely many $m$. Then $f(m) = W(h)(m)$ for infinitely many $m$. Thus $W[A]$ is maximal. Applying Lemma 3.1 with $m = 0, n = 1$ and $U = 2\mathbb{N}$, we can easily see that $W[A]$ has measure 0 in $\mathbb{N}\mathbb{N}$. \hfill \Box
References


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Random logarithm and homogeneity

Tapani Hyttinen

ABSTRACT. In this paper we study the class \( \mathcal{R}^e \) of algebraically closed fields \( F \) of characteristic 0 which in addition contain an epimorphism \( \exp : (F, +) \to (F - \{0\}, \cdot) \) such that the kernel of \( \exp \) is \( \pi \mathbb{Z} \) for some transcendental element \( \pi \) and such that Schanuel's conjecture holds. We show that if we expand the models of the class by a suitable inverse of \( \exp \), then the class becomes homogeneous. Using model theoretic knowledge of homogeneous classes of models, the structure of the class of expansions is classified (supersimple with dop). Finally, it is shown that this tells a lot about the structure of the original class \( \mathcal{R}^e \), in particular, we prove a non-structure theorems for the class \( \mathcal{R}^e \). E.g. we show that there are epimorphisms \( \exp_\alpha : (\mathbb{C}, +) \to (\mathbb{C} - \{0\}, \cdot) \), \( \alpha < 2^{(2^\omega)} \), such that the kernel of \( \exp_\alpha \) is \( 2\pi i \mathbb{Z} \), \( (\mathbb{C}, +, \cdot, \exp_\alpha) \) satisfies Schanuel's conjecture and for \( \alpha \neq \beta \), \( (\mathbb{C}, +, \cdot, \exp_\alpha) \not\cong (\mathbb{C}, +, \cdot, \exp_\beta) \). It follows that even if one can prove Schanuel's conjecture, this alone does not fix the structure of \( (\mathbb{C}, +, \cdot, \exp) \). We show also that the picture does not change if, in addition, one proves that \( (\mathbb{C}, +, \cdot, \exp) \) is existentially closed in some reasonable sense.

In [Z2], B. Zilber studied model theoretic properties of (very) weak structures that arise from the structure \( (\mathbb{C}, +, \cdot, \exp) \). He showed that the class of these structures is not homogeneous, although apart from this, the class behaves nicely. In this paper we study the possibilities of seeing the full structure \( (\mathbb{C}, +, \cdot, \exp) \) as a member of a homogeneous class of structures, i.e. as a member of a class of structures that has a homogeneous universal domain (see Section 1 for the precise definition).

Let \( \mathcal{R}^e \) be the class of algebraically closed fields \( F \) of characteristic 0 which in addition contain an epimorphism \( \exp : (F, +) \to (F - \{0\}, \cdot) \) such that the kernel of \( \exp \) is \( \pi \mathbb{Z} \) for some transcendental element \( \pi \) and such that Schanuel's conjecture holds. In [Z1], Zilber pointed out that Schanuel's conjecture gives us a Hrushovski style predimension that seems to be an effective tool in studying the class \( \mathcal{R}^e \). It also suggests that using the argument from [H1] Example 3.3, one can show that \( \mathcal{R}^e \) is homogeneous. This allows us to use the machinery from homogeneous model theory to study the class. However, one soon notices that problems arise. It seems

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that some kind of logarithm (i.e. inverse for the function \( \exp \)) is needed to push the argument through. Also the proof of the result from [Z2] mentioned above suggests this.

But if one adds any particular logarithm to \((\mathbb{C}, +, \cdot, \exp)\), like the principal one, one makes the reals definable and so also the metric is definable etc. This changes the structure too much. So one should add a logarithm that increases the expressive power minimally. These kinds of problems arise regularly in descriptive finite model theory, there is a standard technique that often solves the problem. Instead of adding some particular function, one adds a random one. In our case this idea seems a good one, intuitively. The problem is that it is nonsense: what does it mean to make randomly arbitrarily many choices so that each choice is made from a countable set with equal probability between the elements of the set?

Below we will extract some properties of the 'intuitive' random logarithm, show that the properties can be realized and show that if we add such a function to the models from \( \mathcal{R}^e \), we get a homogeneous class \( \mathcal{R} \) of structures. In the last two sections we study the structure of the original class \( \mathcal{R}^e \) of models. In particular, in the last section a non-structure theorem is proved for \( \mathcal{R}^e \).

We want to point out that our random logarithm can also be used to make various modifications of \( \mathcal{R}^e \) homogeneous. For example, if we drop the requirement that \( \exp \) satisfies Schanuel's conjecture from the definition of \( \mathcal{R}^e \) we get a class that becomes homogeneous by adding the random logarithm to the models (it is unknown whether the class is homogeneous without the logarithm).

We wish to point out also that our proofs do not assume (that \((\mathbb{C}, +, \cdot, \exp)\) satisfies) Schanuel's conjecture. For this paper, the only consequence of the failure of Schanuel's conjecture is that then there is no function log such that

\[
(\mathbb{C}, +, \cdot, \exp, \log, 2\pi i, \mathbb{Z}, \mathbb{Q}, q)_{q \in \mathbb{Q}}
\]

belongs to our class of structures.

This work can be seen as an attempt to explore the boundaries of the homogeneous framework. In particular we will give a non-trivial example of a homogeneous class of structures which, in addition, arises from 'core mathematics'. Also this work can be seen to suggest one possibility of what the structure \((\mathbb{C}, +, \cdot, \exp)\) looks like. Since very little is known about the structure, it is not even known whether it has a non-trivial automorphism (there are two trivial ones), one can not hope to prove a theorem that describes the structure; at least not, one assumes, by using only methods from model theory.

We assume that the reader is familiar with the basic techniques from stability theory as well as with the basic ideas behind homogeneous model theory, see e.g. [S1].

1. Homogeneous class of structures

In this section we explain what we mean by a homogeneous class of structures. Our notion of a homogeneous class of structures is from [H1] and it is somewhat more general than the usual one (the class of all models of a good finite diagram, see [S4]). However, modulo the current state of homogeneous model theory, there is no difference between the two notions.

Let \( \mathcal{R} \) be a collection of \( \tau \)-models for a fixed similarity type \( \tau \). We say that a collection \( L \) of first-order \( \tau \)-formulas is regular for \( \mathcal{R} \) if the following holds:
(i) $L$ contains all atomic $\tau$-formulas,
(ii) $L$ is closed under freely substituting free variables by terms,
(iii) $L$ is closed under negation,
(iv) if for $i < j < \alpha$, $\mathfrak{A}_i \in \mathfrak{K}$ is an $L$-elementary submodel of $\mathfrak{A}_j \in \mathfrak{K}$ (see below), then for all $i < \alpha$, $\mathfrak{A}_i$ is $L$-elementary submodel of $\bigcup_{i < \alpha} \mathfrak{A}_i$.

We will not distinguish notationally finite sequences from elements, e.g. $a \in A$ means that $a$ is a finite sequence of elements from $A$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models and $f$ a partial map from $\mathfrak{A}$ to $\mathfrak{B}$. We say that $f$ is $L$-elementary iff for all formulas $\phi(x) \in L$ and all sequences $a \in \text{dom}(f)$ of the same length as $x$, $\mathfrak{A} \models \phi(a)$ iff $\mathfrak{B} \models \phi(f(a))$. We say that $\mathfrak{A}$ is an $L$-elementary submodel of $\mathfrak{B}$ and write $\mathfrak{A} \prec_L \mathfrak{B}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $id_{\mathfrak{A}}$ is $L$-elementary. Also the notion of an $L$-type $t_L(a/A)$ of a finite sequence $a \in \mathfrak{A}$ over a set $A \subseteq \mathfrak{A}$ is defined as usual.

Let $\mathfrak{K}$ be a class of $\tau$-models for fixed similarity type $\tau$. Suppose $\mathfrak{K}$ is closed under isomorphisms and contains arbitrary large (infinite) structures. We say that $\mathfrak{K}$ is homogeneous if for some regular collection $L$ of formulas for $\mathfrak{K}$ the following holds:

- AP (L-amalgamation property): Assume $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$ and for all $L$-formulas $\phi(x)$ and sequences $a \in \mathfrak{A} \cap \mathfrak{B}$ (possibly empty) we have $\mathfrak{A} \models \phi(a)$ iff $\mathfrak{B} \models \phi(a)$. Then there are a model $\mathfrak{C} \in \mathfrak{K}$ and a function $f : \mathfrak{A} \cup \mathfrak{B} \rightarrow \mathfrak{C}$ such that $f \upharpoonright \mathfrak{A}$ and $f \upharpoonright \mathfrak{B}$ are $L$-elementary.
- sI (Strong L-inductivity): Assume that for all $\beta < \alpha$, $\mathfrak{A}_\beta \in \mathfrak{K}$ and that for all $\gamma < \beta < \alpha$, $\mathfrak{A}_\gamma \prec_L \mathfrak{A}_\beta$. Then $\mathfrak{A} = \bigcup_{\beta < \alpha} \mathfrak{A}_\beta \in \mathfrak{K}$ (and for all $\beta < \alpha$, $\mathfrak{A}_\beta \prec_L \mathfrak{A}$).
- Co (L-Completeness): For all $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$ and sentences $\phi \in L$, $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.

Such a class of structures is called homogeneous because the Fraïssé-construction immediately gives the following:

**Proposition 1.1.** Suppose $\mathfrak{K}$ is a homogeneous class of models witnessed by $L$. Then for all cardinals $\kappa$, there is a model $\mathfrak{A} \in \mathfrak{K}$ of power $\geq \kappa$ such that

(i) $\mathfrak{A}$ is strongly $L$-$\kappa$-homogeneous i.e. for all $L$-elementary partial maps $f : \mathfrak{A} \rightarrow \mathfrak{A}$ the following holds: if $\text{dom}(f)$, $\text{rng}(f)$ are of power $< \kappa$ then there is an automorphism $g$ of $\mathfrak{A}$ such that $f \subseteq g$.

(ii) $\mathfrak{A}$ is $L$-$\kappa$-universal for $\mathfrak{K}$ i.e. for all models $\mathfrak{C} \in \mathfrak{K}$ of power $\leq \kappa$ there is an $L$-elementary function $f : \mathfrak{C} \rightarrow \mathfrak{A}$.

The following lemma is the main ingredient behind the nice behaviour of the homogeneous classes of models.

**Lemma 1.2.** Suppose $L$ is a collection of first-order formulas closed under negation and $\mathfrak{M}$ is a (strongly) $L$-$\kappa$-homogeneous model of power $\geq \kappa$. Let $A \subseteq \mathfrak{M}$ be of power $< \kappa$ and $p$ a complete $L$-type over $A$. If for all finite $B \subseteq A$, $p \upharpoonright B$ is realized in $\mathfrak{M}$, then $p$ is realized in $\mathfrak{M}$.

**Proof.** Easy. \qed

## 2. Basic definitions with some observations

We start by defining a class of structures that is not our 'target' class of structures. The class is needed to provide a platform for carrying out some constructions we will use in the proofs later. In the constructions we need to work with partial interpretations for exp. However, later we want exp to be a function symbol. This
simplifies the definition of our collection \( L \) of formulas as well as the proof of \( L \)-amalgamation.

We let \( \tau \) be the similarity type which consists of the following symbols: a constant \( \pi \) and for \( q \in \mathbb{Q} \), a constant \( c_q \), unary predicates \( Z \) and \( Q \), a binary predicate \( Exp \) and binary functions \( + \) and \( \cdot \). We let \( \mathcal{R}^v \) be the class of all \( \tau \)-models \( \mathfrak{A} \) with the following properties:

1. \( F_{\mathfrak{A}} = \mathfrak{A} \upharpoonright \{+, \cdot\} \) is an algebraically closed field of characteristic 0 and
2. \( \mathfrak{A} \models \frac{Q(a)}{a} \) if \( a \) is a rational element of \( F_{\mathfrak{A}} \) and \( \mathfrak{A} \models Z(a) \) if \( a \) is an integer in \( F_{\mathfrak{A}} \),
3. the interpretation of \( \pi \) is a transcendental element of \( F_{\mathfrak{A}} \) and for \( q \in \mathbb{Q} \), the interpretation of \( c_q \) is the rational \( q \) as calculated in \( F_{\mathfrak{A}} \) (we will not distinguish \( c_q \) and \( q \)).

The interpretation of the predicate \( Exp \) satisfies the following:

1. if for all \( i < 3 \), \( \mathfrak{A} \models Exp(a_i, b_i) \) and \( \mathfrak{A} \models a_0 + a_1 = a_2 \), then \( \mathfrak{A} \models b_0 \cdot b_1 = b_2 \),
2. if \( \mathfrak{A} \models Exp(a, b) \) then for all \( c \in \mathfrak{A} \), \( \mathfrak{A} \models Exp(c, b) \) if \( \mathfrak{A} \models \exists x (Z(x) \land c = a + x \cdot \pi) \),
3. \( \mathfrak{A} \models Exp(\pi, 1) \land \forall x_0 \forall x_1 \forall x_2 ((Exp(x_0, x_1) \land Exp(x_0, x_2)) \rightarrow x_1 = x_2) \),
4. if \( i < 2 \), \( \mathfrak{A} \models Exp(a_i, b_i) \land \exists q (q_i) \), then there is \( b_2 \in \mathfrak{A} \) such that \( \mathfrak{A} \models Exp(q_0 \cdot a_0 + q_1 \cdot a_1, b_2) \).

Suppose that \( \mathfrak{A} \in \mathcal{R}^v \) and \( X \subseteq \mathfrak{A} \). By \( \dim(X) \) we mean the dimension of \( X \) as a subset of a vector space over \( \mathbb{Q} \) (\( \mathfrak{A} \) can be seen as a vector space over \( \mathbb{Q} \) the natural way). Notice that it is first-order expressible in \( \mathfrak{A} \) (with a negation of an existential formula) that \( \dim(X) = |X| \) for sets \( X \) of fixed finite size. By \( \deg(X) \) we mean the transcendence degree of \( X \) as a subset of \( F_{\mathfrak{A}} \). For \( X \subseteq \mathfrak{A} \), by \( Exp(X) \) we mean the set of all \( b \in \mathfrak{A} \) such that for some \( a \in X \), \( \mathfrak{A} \models Exp(a, b) \). Similarly, for an element \( a \in \mathfrak{A} \), by \( Exp(a) \) we mean the unique \( b \in \mathfrak{A} \) such that \( \mathfrak{A} \models Exp(a, b) \) (if exists). By \( dom(Exp) = dom_{\mathfrak{A}}(Exp) \) we denote the set of all \( a \in \mathfrak{A} \) such that for some \( b \in \mathfrak{A} \), \( \mathfrak{A} \models Exp(a, b) \).

We let \( \mathcal{R}^s \) be the class of all \( \mathfrak{A} \in \mathcal{R}^v \) such that

1. \( \delta(X) = \deg(X \cup \{1, \pi\} \cup Exp(X \cup \{1, \pi\})) - \dim(X \cup \{1, \pi\}) \geq 0 \).

The properties of this kind Hrushovski style predimension \( \delta \) are well-known, see e.g. [BS] (and notice that in the definition of \( \delta \), \( \dim(X \cup \{1, \pi\}) \) is a dimension that is based on a modular pregeometry). We assume that the reader is familiar with these properties i.e. we will use the properties regularly and usually without proving them.

Suppose \( \mathfrak{A} \in \mathcal{R}^s \). If \( B \) is a subset of \( dom_{\mathfrak{A}}(Exp) \) and \( \dim(B) \) is finite, then \( \delta(B) \) is defined as in (3) above. If \( C \subseteq B \), then by \( \delta(B/C) \) we mean \( \delta(B) - \delta(C) \). We say that \( B \) is a relevant subset of \( \mathfrak{A} \) if \( B \subseteq dom_{\mathfrak{A}}(Exp) \), \( B \) contains the rationals and \( \pi \), \( B \) is closed under \( + \) and for all \( b \in B \) and rationals \( q, q \cdot b \in B \). We say that a relevant set \( B \subseteq \mathfrak{A} \) is closed (in \( \mathfrak{A} \)) if for all relevant \( D \subseteq \mathfrak{A} \) which have finite dimension, \( \delta(D/B \cap D) \geq 0 \).

Notice that if \( B \subseteq dom_{\mathfrak{A}}(Exp) \) is a subset of the relevant set \( A \) generated by finite \( X \) (i.e. \( A \) is the least relevant set which contains \( X \)) and also \( B \) generates \( A \), then \( \delta(B) = \delta(X) \).

**Lemma 2.1.** Suppose \( \mathfrak{A} \in \mathcal{R}^s \). Then there is \( \mathfrak{B} \in \mathcal{R}^s \) such that

1. \( \mathfrak{A} \) is a substructure of \( \mathfrak{B} \),
(ii) \( \text{dom}_\mathfrak{A}(\text{Exp}) \) is closed in \( \mathfrak{B} \),
(iii) for all \( a \in \mathfrak{B} \) there is \( b \in \mathfrak{B} \) such that \( \mathfrak{B} \vDash \text{Exp}(a, b) \),
(iv) for all \( b \in \mathfrak{B} - \{0\} \) there is \( a \in \mathfrak{B} \) such that \( \mathfrak{B} \vDash \text{Exp}(a, b) \).

**Proof.** Clearly it is enough to prove the Claims 2.1.1 and 2.1.2 below.

**Claim 2.1.1.** If \( a \in \mathfrak{A} - \text{dom}_\mathfrak{A}(\text{Exp}) \), then there is \( \mathfrak{B} \in \mathbb{R}^s \) such that (i) and (ii) from Lemma 2.1 hold and \( a \in \text{dom}_\mathfrak{B}(\text{Exp}) \).

**Proof.** Choose \( \mathfrak{B}' \) so that (i) from Lemma 2.1 holds,
\[
\text{dom}_{\mathfrak{B}'}(\text{Exp}) = \text{dom}_\mathfrak{A}(\text{Exp})
\]
and there is \( b \in \mathfrak{B}' \) such that \( b \) is transcendental over \( \mathfrak{A} \). For \( q \in \{ x \in \mathbb{Q} \mid x > 0 \} \), pick elements \( b_q \in \mathfrak{B}' \) so that \( b_1 = b \) and for all \( q \in \{ x \in \mathbb{Q} \mid x > 0 \} \) and \( n \in \mathbb{N} - \{0\} \), \( (b_q)^n = b_{q^n} \). Then we get \( \mathfrak{B} \) by extending \( \text{Exp} \) so that \( \text{dom}_\mathfrak{B}(\text{Exp}) \) is the relevant set generated by \( \text{dom}_\mathfrak{A}(\text{Exp}) \cup \{a\} \) and for all \( c \in \text{dom}_\mathfrak{A}(\text{Exp}) \) and non-zero rational \( q, \mathfrak{B} \vDash \text{Exp}(c + q \cdot a, d) \) iff either
\[
q > 0 \text{ and } \mathfrak{B} \vDash e \cdot b_q = d
\]
or
\[
q < 0 \text{ and } \mathfrak{B} \vDash e \cdot b_q^{-1} = d,
\]
where \( e \) is such that \( \mathfrak{B} \vDash \text{Exp}(c, e) \). It is easy to see that this \( \mathfrak{B} \) satisfies (i) and (ii) and that \( \mathfrak{B} \in \mathbb{R}^s \). (E.g. Suppose that \( e \cdot b_q = e' \cdot b_q' \), \( e, e' \in \mathfrak{A} \). Then \( e/e' = b_q'/b_q \) and since if \( q \neq q' \), then \( b_q'/b_q \) is transcendental over \( \mathfrak{A} \), we get that \( q = q' \) and \( e = e' \). Thus \( \mathfrak{B} \) satisfies (2)(b).)

**Claim 2.1.2.** If \( a \in \mathfrak{A} - \text{Exp}(\text{dom}_\mathfrak{A}(\text{Exp})) \), then there is \( \mathfrak{B} \in \mathbb{R}^s \) such that (ii) and (iii) from Lemma 2.1 hold and \( a \in \text{Exp}(\text{dom}_\mathfrak{B}(\text{Exp})) \).

**Proof.** This can be proved exactly as Claim 2.1.1 as soon as we have shown that for all positive integers \( z \), \( a^z \notin \text{Exp}(\text{dom}_\mathfrak{A}(\text{Exp})) \). Suppose not, say \( a^z = b \) and \( \mathfrak{A} \vDash \text{Exp}(c, b) \). For all integers \( 0 \leq n < z \), let \( c_n = c/z + n \cdot \pi/z \). Let \( b_n \) be such that \( \mathfrak{A} \vDash \text{Exp}(c_n, b_n) \). Clearly \( (b_n)^z = b \). Also for \( n \neq m \), \( b_n \neq b_m \). Since there are only \( z \) many \( d \) such that \( d^z = b \), for some \( n \), \( a = b_n \), a contradiction. (For (2)(b) notice that if \( \text{Exp}(e + q \cdot b) = 1 \), then \( a_q = \text{Exp}(-e) \) and thus for some integer \( z \), \( a^z \in \text{Exp}(\text{dom}_\mathfrak{A}(\text{Exp})) \) (if \( q \neq 0 \)), a contradiction.)

A structure \( \mathfrak{B} \in \mathbb{R}^s \) is called full if it satisfies (iii) and (iv) from Lemma 2.1.

Let \( \mathfrak{A} \in \mathbb{R}^s, a \in \mathfrak{A} \) and \( A, B \subseteq \mathfrak{A} \). We say that \( a \) is independent from \( B \) over \( A \) and write \( a \upharpoonright_A B \) if \( a \) is independent from \( B \) over \( A \) in \( F_{\mathfrak{A}} \) in the sense of non-forking. If we consider \( F_{\mathfrak{A}} \) as a vector space over \( \mathbb{Q} \), we talk about linear independence.

**Lemma 2.2.** Suppose \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathbb{R}^s \) are such that
(i) \( \mathfrak{C} = \mathfrak{A} \cap \mathfrak{B} \) is a substructure of both \( \mathfrak{A} \) and \( \mathfrak{B} \),
(ii) \( \mathfrak{C} \) is full,
(iii) \( \mathfrak{C} = \text{dom}_\mathfrak{E}(\text{Exp}) \) is closed both in \( \mathfrak{A} \) and in \( \mathfrak{B} \).
Then there is full \( \mathfrak{D} \in \mathbb{R}^s \) such that \( \mathfrak{A} \) and \( \mathfrak{B} \) are substructures of \( \mathfrak{D} \) and both \( \text{dom}_\mathfrak{A}(\text{Exp}) \) and \( \text{dom}_\mathfrak{B}(\text{Exp}) \) are closed in \( \mathfrak{D} \).
Proof. By Lemma 2.1, we may assume that also both $\mathfrak{A}$ and $\mathfrak{B}$ are full. Also by Lemma 2.1, it is enough to find $\mathcal{D} \in \mathcal{R}^u$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are substructures of $\mathcal{D}$ and

(a) the property (3) from the definition of $\mathfrak{R}^u$ holds,
(b) $\mathfrak{A} = \text{dom}_{\mathfrak{A}}(\text{Exp})$ is closed in $\mathcal{D}$,
(c) $\mathfrak{B} = \text{dom}_{\mathfrak{B}}(\text{Exp})$ is closed in $\mathcal{D}$.

We let $F_\mathfrak{A}$ be the free amalgamation of $F_\mathfrak{A}$ and $F_\mathfrak{B}$ over $F_\mathfrak{C}$ (i.e. the algebraic closure of the tensor product over $\mathfrak{C}$). This determines $\mathcal{D}$ excluding the interpretation of $\text{Exp}$. The relation $\text{Exp}$ is defined to be the extension of $\text{Exp}_{\mathfrak{A}}^u \cup \text{Exp}_{\mathfrak{B}}^u$ such that $\text{dom}_{\mathcal{D}}(\text{Exp})$ is the relevant subset of $\mathcal{D}$ generated by $\text{dom}_{\mathfrak{A}}(\text{Exp}) \cup \text{dom}_{\mathfrak{B}}(\text{Exp})$ and for all $a \in \text{dom}_{\mathfrak{A}}(\text{Exp})$ and $b \in \text{dom}_{\mathfrak{B}}(\text{Exp}),$

$\mathcal{D} \models \text{Exp}(a + b, c)$ iff $\mathcal{D} \models \text{Exp}(a) \cdot \text{Exp}(b) = c.$

Then it is easy to check that $\mathcal{D} \in \mathfrak{R}^u$ and clearly $\mathfrak{A}$ and $\mathfrak{B}$ are substructures of $\mathcal{D}$. We are left to prove that (a)-(c) hold.

A simple calculation shows that (b) implies (a) and by symmetry, (b) implies (c). Thus it suffices to prove (b). For a contradiction, assume that $D$ is a relevant subset of $\text{dom}_{\mathcal{D}}(\text{Exp})$ such that $\delta(D/A) < 0$, where $A = D \cap \mathfrak{A}$. Then for all relevant subsets $A'$ of $\mathfrak{A}$ with finite dimension, if $A \subseteq A'$, then $\delta(D \cup A'/A') < 0$. Thus we may assume the following (keeping in mind that $\mathfrak{A}$ is full):

(d) $A \cup \text{Exp}(A)$ is independent from $\mathfrak{B}$ over $(A \cup \text{Exp}(A)) \cap \mathfrak{C},$
(e) for all $d \in D$ there are $a \in A$ and $b \in \mathfrak{B}$ such that $d = a + b$.

Let $d_i, i < n$, be a linear base for $D$. Let $a_i$ and $b_i$ be as given by (e) for $d_i$. In addition choose these so that if $d_i \in \mathfrak{C}$, then $a_i = 0$.

Then

$$\delta(\{b_0, ..., b_{n-1}\} \cup A/A) < 0.$$

Clearly we may assume that in addition $B \cap \mathfrak{C} = A \cap \mathfrak{C}$, where $B \subset \mathfrak{B}$ is the relevant set generated by $\{b_0, ..., b_{n-1}\}$. Then by (d), (e) contradicts the assumption that $\text{dom}_{\mathfrak{C}}(\text{Exp})$ is closed in $\mathfrak{B}$.

By a canonical $s$-structure $\mathfrak{A}_C^s$, we mean the obvious structure we get from the complex numbers ($\pi$ is interpreted as $2\pi i$, of course). Clearly, $\mathfrak{A}_C^s \in \mathfrak{R}^u$ but it is not known whether $\mathfrak{A}_C^s$ satisfies (3) or not (Schanuel's conjecture is an open question).

Now we turn to define our target class $\mathfrak{R}$. We let $\rho$ be the similarity type be which consists of the following symbols: constant $\pi$ and for $q \in \mathbb{Q}$, constant $c_q$, unary predicates $Z$ and $Q$, unary functions $\exp$ and $\log$ and binary functions $+$ and $\cdot$. We let $\mathcal{R}^{-3}$ be the class of all $\rho$-models $\mathfrak{A}$ such that the following holds:

1. $F_\mathfrak{A} = \mathfrak{A} \upharpoonright \{+, \cdot\}$ is an algebraically closed field of characteristic 0.
2. $\mathfrak{A} \models Q(a)$ iff $a$ is a rational element of $F_\mathfrak{A}$ and $\mathfrak{A} \models Z(a)$ iff $a$ is an integer in $F_\mathfrak{A}$.
3. The interpretation of $\pi$ is a transcendental element of $F_\mathfrak{A}$ and for $q \in \mathbb{Q}$, the interpretation of $c_q$ is the rational $q$ as calculated in $F_\mathfrak{A}$ (again we will not distinguish $c_q$ and $q$).
4. $\exp$ is a homomorphism from $(F_\mathfrak{A}, +)$ onto $(F_\mathfrak{A} - \{0\}, \cdot)$ with kernel $\pi Z$.
5. $\mathfrak{A} \models \forall x(\exp(\log(x)) = x) \land \log(0) = 0$.

Clearly every $\mathfrak{A} \in \mathcal{R}^{-3}$ determines a unique structure $\mathfrak{A}^s \in \mathfrak{R}^u$ (in fact, it suffices that $\mathfrak{A}$ is a $\rho - \{\log\}$-model satisfying (1)-(4)). We let $\mathcal{R}^{-2}$ be the class of all $\mathfrak{A} \in \mathcal{R}^{-3}$ such that

6. $\mathfrak{A}^s \in \mathfrak{R}^u$ (i.e. those which satisfy Schanuel's conjecture).
The structures $\mathfrak{A}^*$ allow us to transfer the concepts defined for models from $\mathfrak{K}^*$ to models from $\mathfrak{K}^{-2}$. In particular, we have the notion of predimension $\delta$ and the notion of a relevant set for models $\mathfrak{A} \in \mathfrak{K}^{-2}$.

We let $\mathfrak{K}^-$ be the set of all $\mathfrak{A} \in \mathfrak{K}^{-2}$ such that the following holds:

(7) for all relevant sets $A \subseteq \mathfrak{A}$ with finite dimension and rationals $q \neq 1$ and $r$, there are at most finitely many $a, b \in A - \pi Q$ such that

$$\mathfrak{A} \models (b = q \cdot a + r \cdot \pi) \land (\log(\exp(a)) = a) \land (\log(\exp(b)) = b).$$

This property is the key technical ‘trick’ behind the proof of homogeneity of our class of structures. The point here is the following: The difficult step in the homogeneity proof is Claim 2.8.3. There a certain type $p = t_L(a/\mathfrak{A} \cap \mathfrak{B})$ is considered. Using the property (7), we can choose $a$ so that there is a formula $\phi$ such that

(i) there are only finitely many sequences that satisfy $\phi$,  
(ii) the sequence $a$ satisfies $\phi$.

With this we can see that $a$ can be chosen so that $p$ is ‘principal’. This is what is needed in the proof of the claim.

**Lemma 2.3.** Suppose $\mathfrak{A}$ is a $(\rho - \{\log\})$-model such that it satisfies (1)-(4) above, $B$ and $C$ are subsets of $\mathfrak{A}$ such that either $B = C = \emptyset$ or both are relevant and $\pi Q \subseteq B \cap C$ and we have a function $\log^*$ defined on $\exp(B) \cup \exp(C)$ such that for all $a \in \exp(B) \cup \exp(C)$, $\exp(\log^*(a)) = a$ and (7) holds for $\log^*$ for all relevant $A$ such that $A$ has finite dimension and $A \subseteq B$ or $A \subseteq C$. Then there is a function $\log : \mathfrak{A} \to \mathfrak{A}$ such that

(i) $\log \upharpoonright (\exp(B) \cup \exp(C)) = \log^*$,  
(ii) if $\mathfrak{A}^+$ is the model one gets by adding log to $\mathfrak{A}$, then $\mathfrak{A}^+$ satisfies condition (7) and so if in addition $\mathfrak{A}$ satisfies (6), then $\mathfrak{A}^+ \in \mathfrak{K}^-.$

**Proof.** Clearly we may assume that $\pi Q \subseteq B \cap C$ (log $\upharpoonright \exp(\pi Q)$ is easy to define, since (5) is the only requirement). Choose a linear base $D = \{d_i | i < \alpha\}$ for $\mathfrak{A}$ so that $d_0 = \pi$, $D \cap B$ is a base for $B$ and $D \cap C$ is a base for $C$. If $X = \{x_0, ..., x_n\} \subseteq D - \{d_0\}$, then by $\mathfrak{A}_X$ we denote the set of all $a \in \mathfrak{A}$ such that there are non-zero $q_0, ..., q_n \in Q$ and $r \in Q$ (possibly zero) such that $a = q_0 \cdot x_0 + ... + q_n \cdot x_n + r \cdot \pi$.

**Claim 2.3.1.** There is a function $\log : \mathfrak{A} \to \mathfrak{A}$ such that

(a) $\log \upharpoonright (\exp(B) \cup \exp(C)) = \log^*$,  
(b) for all $a \in \mathfrak{A}$, $\log(\exp(a)) = a$,  
(c) for all finite $X \subseteq D$ and rationals $q \neq 1$ and $r$, there are at most finitely many $a, b \in \mathfrak{A}_X - \pi Q$ such that $b = q \cdot a + r \cdot \pi$, $\log(\exp(a)) = a$ and $\log(\exp(b)) = b$.

**Proof.** Easy diagonalization argument. □

It suffices to prove that log from Claim 2.3.1 satisfies the condition (7): Suppose not. Let $A$, $q$ and $r$ witness this. Then $q \neq 0$. Since every relevant $A' \supseteq A$ with finite dimension still witnesses the failure of (7), we may assume that $A \cap D$ is a base for $A$. Clearly,

(*) for all $X \subseteq (A \cap D) - \{\pi\}$, if $a \in \mathfrak{A}_X$, then $q \cdot a + r \cdot \pi \in \mathfrak{A}_X$.

By this and the pigeon-hole principle, there is $X \subseteq (A \cap D) - \{\pi\}$ such that for infinitely many $a, b \in \mathfrak{A}_X - \pi Q$, $b = q \cdot a + r \cdot \pi$, $\log(\exp(a)) = a$ and $\log(\exp(b)) = b$. This contradicts Claim 2.3.1 (c). □
By Lemma 2.3 there is a function log such that
\[
\mathfrak{A}_C = (\mathbb{C}, +, \cdot, \exp, \log, 2\pi i, \mathbb{Q}, q)_{q \in \mathbb{Q}}
\]
satisfies (1)-(5) and (7). We fix such a function and call \( \mathfrak{A}_C \) the canonical structure. Notice that \( \mathfrak{A}_C^* = \mathfrak{A}_C^* \).

For all \( \rho \)-models \( \mathfrak{A} \) which satisfy (1)-(5), we let \( P(\mathfrak{A}) \) be the least substructure of \( \mathfrak{A} \) such that \( F_{P(\mathfrak{A})} \) is algebraically closed. We fix some countable \( \rho \)-model \( \mathfrak{A}_{pr} \) so that for some \( \mathfrak{A} \in \mathfrak{R}^{-} \), \( \mathfrak{A}_{pr} = P(\mathfrak{A}) \) (by Lemmas 2.1 and 2.3, \( \mathfrak{R}^{-} \neq \)). If Schanuel's conjecture holds, the natural choice for \( \mathfrak{A}_{pr} \) is \( P(\mathfrak{A}_C) \). Then we let \( \mathfrak{R} \) be the class of all \( \mathfrak{A} \in \mathfrak{R}^{-} \) such that

(8) \( P(\mathfrak{A}) \) is isomorphic to \( \mathfrak{A}_{pr} \).

We want to point out that one can define \( \log \upharpoonright \mathfrak{A}_{pr} \) anyway one wants (i.e. it does not need to satisfy (7)) and still the class is homogeneous if in the requirement (7), \( \pi Q \) is replaced by \( P(\mathfrak{A}) \).

3. Homogeneity of \( \mathfrak{R} \)

In this section we show that \( \mathfrak{R} \) is a homogeneous class of models. The proof uses ideas from [H1] Example 3.3.

We start by defining a collection \( L \) of first-order formulas. These formulas express roughly the following: Let \( \mathfrak{A} \in \mathfrak{R} \) and \( a_0, \ldots, a_n \in \mathfrak{A} \). Let \( \mathfrak{B} \supseteq \{a_0, \ldots, a_n\} \) be the least substructure of \( \mathfrak{A} \) such that it is algebraically closed as a subset of \( F_{\mathfrak{B}} \) and closed as a relevant subset of \( \mathfrak{A} \) (such \( \mathfrak{B} \) exists). Then each (positive) formula (in \( n+1 \) free variables) from \( L \) state about \( (a_0, \ldots, a_n) \) that there is some finite sequence of elements in \( \mathfrak{B} \) and describe this sequence up to (finite) quantifier free information together with complete information on linear dependencies (and independencies). The linear dependencies can be described since we can quantify over the rationals.

Suppose \( \mathfrak{A} \in \mathfrak{R} \). We say that relevant \( C \subseteq \mathfrak{A} \) is \textit{unclosed} in relevant \( B \subseteq \mathfrak{A} \) with finite dimension if for all relevant \( D \) the following holds: if \( C \subseteq D \subseteq B \) and \( D \neq B \), then \( \delta(B/D) < 0 \). We say that relevant \( B \) is an \textit{intrinsic closure} of relevant \( A \) with finite dimension if \( A \) is unclosed in \( B \) and \( B \) is closed. If \( A \) and \( B \) are just subsets of \( \mathfrak{A} \) of finite dimension, then these notions are defined for them by looking the relevant subsets generated by them.

\textbf{Lemma 3.1.}

(i) If \( A \subseteq \mathfrak{A} \in \mathfrak{R} \) is relevant and of finite dimension, then the intrinsic closure of \( A \) exists and is of finite dimension.

(ii) Suppose \( B, C, B' \) and \( C' \) are relevant subsets of \( \mathfrak{A} \in \mathfrak{R} \) with finite dimension, \( B \subseteq C \), \( B' \) is an intrinsic closure of \( B \) and \( C' \) is an intrinsic closure of \( C \). Then \( B' \subseteq C' \). In particular, if \( B = C \), then \( B' = C' \) i.e. the intrinsic closure is unique.

\textbf{Proof.} (i): Let \( B \subseteq \mathfrak{A} \) be such that

(a) \( A \subseteq B \) and \( B \) is relevant and of finite dimension,
(b) \( \delta(B) \) is minimal among those that satisfy (a).

In addition choose \( B \) so that

(c) \( \dim(B) \) is minimal among those which satisfy (a) and (b).

Clearly \( B \) is an intrinsic closure of \( A \).

(ii): Immediate by the definition of intrinsic closure. \(\square\)
We say that a first-order formula $\phi(x_0, ..., x_n, ..., x_{n+k})$ is $n$-unclosed if the following holds:

(i) $\phi = \psi \land \theta$, where $\psi$ is quantifier free and $\theta$ is a universal formula which describes the linear dependencies within $\pi, 1, x_0, ..., x_{n+k}$. In particular, $\theta$ fixes the dimension of $\{\pi, 1, x_0, ..., x_{n+k}\}$.

(ii) There are $A \in R$ and $a_0, ..., a_{n+k} \in A$ such that $A \models \phi(a_0, ..., a_{n+k})$ and among $\pi, 1, a_0, ..., a_{n+k}, \exp(1), \exp(a_0), ..., \exp(a_{n+k})$ there are no other algebraic dependencies than those explicitly stated in $\psi$ (i.e. the transcendence degree is maximal). Such a sequence is called generic for $\phi$.

(iii) If $a_0, ..., a_{n+k} \in A \in R$ and $b_0, ..., b_{n+k} \in B \in R$ are generic for $\phi$ then the subfields generated by $\{\pi, 1, a_0, ..., a_{n+k}, \exp(1), \exp(a_0), ..., \exp(a_{n+k})\}$ and by $\{\pi, 1, b_0, ..., b_{n+k}, \exp(1), \exp(b_0), ..., \exp(b_{n+k})\}$ are isomorphic.

(iv) If $a_0, ..., a_{n+k} \in A \in R$ is generic for $\phi$ then $\{\pi, 1, a_0, ..., a_n\}$ is unclosed in $\{\pi, 1, a_0, ..., a_{n+k}\}$.

Notice that if $a_0, ..., a_{n+k} \in A \in R$ are distinct elements, and $\{\pi, 1, a_0, ..., a_n\}$ is unclosed in $\{\pi, 1, a_0, ..., a_{n+k}\}$, then there is an un-closed formula $\phi$ such that $(a_0, ..., a_{n+k})$ is generic for $\phi$.

**Lemma 3.2.**

(i) If $\phi(x_0, ..., x_n, ..., x_{n+k})$ is an $n$-unclosed formula, then for all $0 \leq p \leq k$, $\phi(x_0, ..., x_n, ..., x_{n+k})$ is an $(n+p)$-unclosed formula.

(ii) Suppose $A, B \in R$, $A \cap B$ is a relevant subset of $A$ and of $B$ and for all $n$-unclosed $\phi(x_0, ..., x_{n+k})$, $n < \omega$, and $a \in A \cap B$, $A \models \exists x_{n+1}...\exists x_{n+k}\phi(a)$ iff $B \models \exists x_{n+1}...\exists x_{n+k}\phi(a)$.

Let $A \subseteq A$ be the intrinsic closure of $B = A \cap B$ and let $a_i \in A$, $i < n + k$, be such that $\{a_0, ..., a_n\}$ generate $B$, $\{a_0, ..., a_{n+k}\}$ generate $A$ and for all $m \leq k$, $a_{n+m} \notin B$. If $\phi(x_0, ..., x_{n+k})$ is such an $n$-unclosed formula that $(a_0, ..., a_{n+k})$ is generic for $\phi$, then

(a) for all $b_{n+1}, ..., b_{n+k} \in A$, if $A \models \phi(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$, then $(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ is generic for $\phi$.

(b) for all $b_{n+1}, ..., b_{n+k} \in B$, if $B \models \phi(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$, then $(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ is generic for $\phi$.

**Proof.** (i): Immediate by the definitions.

(ii): We note first that by basic properties of $\delta$,

(*) if $C \subseteq A$ is relevant, has finite dimension and $B \subseteq C$, then $\delta(C) \geq \delta(A)$

(if $D$ is the relevant set generated in $A$ by $C \cup \{a_{n+1}, ..., a_{n+k}\}$, then $\delta(D/C) \leq 0$).

Since $\phi$ decides the linear dependencies within $a_0, ..., a_n, b_{n+1}, ..., b_{n+k}$, the existence of a sequence that witnesses the failure of (a) would contradict (*). Similarly, if there is a sequence that witnesses the failure of (b), then there is $D \subseteq B$ such that $\delta(D) < \delta(A)$. But then by the assumptions, there is $C \subseteq A$ such that $\delta(C) < \delta(A)$ and $B \subseteq C$ which as before contradicts (*).

We say that $\theta(x_0, ..., x_{n+k})$ is $n$-algebraic if $\theta$ is quantifier free and for all $0 < i \leq k$, $\theta$ implies $x_{n+i}$ is algebraic over $\{x_0, ..., x_{n+i-1}\}$ or $x_{n+i} = \exp(x)$ for some $x \in \{x_0, ..., x_{n+i-1}\}$ or $x_{n+i} = \log(x)$ for some $x \in \{x_0, ..., x_{n+i-1}\}$.

We define the class of extended $n$-unclosed formulas, $n \in \mathbb{N}$, as follows:

(a) every $n$-unclosed formula is extended $n$-unclosed formula,

(b) $\phi = \exists x_{n+1}...\exists x_{n+k}\theta(x_0, ..., x_{n+k})$ in an extended $n$-unclosed formula if $\theta$ is $n$-algebraic,
(c) if $\phi(x_0, \ldots, x_{n+k})$ is an extended $n$-unclosed formula, $\psi(x'_0, \ldots, x'_{m+p})$ is an extended $m$-unclosed formula and no $x_i$, $i \leq n + k$, appears in $\psi$ then

$$\phi(x_0, \ldots, x_{n+k}) \land \exists x'_{m+1} \ldots \exists x'_{m+p} \psi(t_0, \ldots, t_m, x'_{m+1}, \ldots, x'_{m+p})$$

in an extended $n$-unclosed formula, where each $t_i = t_i(x_0, \ldots, x_{n+k})$ is a term.

We let $L^c$ be the class of all formulas $\phi(x_0, \ldots, x_m)$ of the form

$$\phi(x_0, \ldots, x_m) = \exists x'_{n+1} \ldots \exists x'_{n+k} \psi(t_0, \ldots, t_n, x'_{n+1}, \ldots, x'_{n+k}),$$

where $\psi = \psi(x'_0, \ldots, x'_{n+k})$ is an extended $n$-unclosed formula, $n \in \mathbb{N}$, and each $t_i = t_i(x_0, \ldots, x_m)$ is a term and no $x_i$, $i \leq m$, appears in $\psi$.

Finally, we let

$$L = L^c \cup \{-\phi \mid \phi \in L^c\}.$$ 

**Lemma 3.3.** $L$ is regular for $\mathcal{A}$.

**Proof.** It is routine to check that $L$ satisfies the requirement (iv). The other requirements are explicit in the definition of $L$. □

**Lemma 3.4.** Suppose $\mathcal{A}, \mathcal{B} \in \mathcal{K}$. Then $\mathcal{A} \prec_L \mathcal{B}$ iff $\mathcal{A}$ is a substructure of $\mathcal{B}$ and $\mathcal{A}$ is closed in $\mathcal{B}$.

**Proof.** Follows easily from the basic properties of $\delta$, especially Lemma 3.1, and the definition of $L$. □

The following result is the main theorem of this paper.

**Theorem 3.5.** $\mathcal{K}$ is $L$-complete and strongly $L$-inductive and it has the $L$-amalgamation property.

We will prove the theorem in a series of lemmas.

**Lemma 3.6.** $\mathcal{K}$ is $L$-complete.

**Proof.** Suppose $\mathcal{A}, \mathcal{B} \in \mathcal{K}$. By the requirement (8) in the definition of $\mathcal{K}$, there is an isomorphism $f : P(\mathcal{A}) \rightarrow P(\mathcal{B})$. Clearly it suffices to prove that as a partial function from $\mathcal{A}$ to $\mathcal{B}$, $f$ is $L$-elementary. By Lemma 3.4, it suffices to prove that for every $\mathcal{C} \in \mathcal{K}$, $P(\mathcal{C})$ is a closed subset of $\mathcal{C}$. But this is clear since for every finite $X \subseteq P(\mathcal{C})$, there is finite $Y \subseteq P(\mathcal{C})$ such that $X \subseteq Y$ and $\delta(Y) = 0$. □

**Lemma 3.7.** Suppose $\mathcal{A} \in \mathcal{K}$, $a_0, \ldots, a_{n+k} \in \mathcal{A}$, $A$ the relevant subset of $\mathcal{A}$ generated by $\{a_0, \ldots, a_n\}$ and $B$ the relevant subset of $\mathcal{A}$ generated by $\{a_0, \ldots, a_{n+k}\}$. If $B$ is the intrinsic closure of $A$, then there is an $n$-unclosed formula $\phi$ such that

(i) $(a_0, \ldots, a_{n+k})$ is generic for $\phi$ (in particular, $\mathcal{A} \models \phi(a_0, \ldots, a_{n+k}))$,

(ii) for all $\mathcal{B} \in \mathcal{K}$ and $b_0, \ldots, b_n \in \mathcal{B}$, there are at most finitely many $b_{n+1}, \ldots, b_{n+k} \in \mathcal{B}$ such that $(b_0, \ldots, b_{n+k})$ is generic for $\phi$,

(iii) for all $\mathcal{B} \in \mathcal{K}$ and $b_0, \ldots, b_{n+k} \in \mathcal{B}$, if $(b_0, \ldots, b_{n+k})$ is generic for $\phi$, then $\{b_0, \ldots, b_n\}$ is unclosed in $\{b_0, \ldots, b_{n+k}\}$.
PROOF. If needed, we can replace $a_i$ by any $a_i + z \cdot \pi$, $z \in Z$, since these are interdefinable and generate the same relevant set (even $\exp(a_i) = \exp(a_i + z \cdot \pi)$). So we may assume that for all $i \leq n + k$, $\log(\exp(a_i)) = a_i$. For all $i \leq n + k$ such that $a_i \not\in \pi Q$, choose some $q_i, r_i \in Q$ so that $q_i \neq 0, 1$ and 

(*) $\log(\exp(q_i \cdot a_i + r_i \cdot \pi)) = q_i \cdot a_i + r_i \cdot \pi$.

Let $\phi$ be an $n$-unclosed formula such that (i) holds and for all $i \leq n + k$, if $a_i \in \pi Q$, then $\phi$ expresses that $x_i = q \cdot \pi$ for some $q \in Q$ and otherwise $\phi$ expresses (*) and that $\log(\exp(x_i)) = x_i$. It is clear that (iii) holds. We prove that (ii) holds.

Let $\mathfrak{B} \in \mathfrak{K}$ and $b_0, ..., b_n \in \mathfrak{K}$. Let $A'$ be the relevant set generated by $\{b_0, ..., b_n\}$. Let $B'$ be the intrinsic closure of $A'$. By the property (7) and the finite dimension of $B'$, it suffices to show that if $b_{n+1}, ..., b_{n+k} \in \mathfrak{B}$ are such that $(b_0, ..., b_{n+k})$ is generic for $\phi$, then for all $n < i \leq n + k$, $b_i \in B'$. This is clear by (iii) and the basic properties of $\delta$.

\[ \square \]

**LEMMA 3.8.** $\mathfrak{K}$ has the L-amalgamation property.

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in the definition of AP. Clearly (by 'moving' the universe of $\mathfrak{B}$) we may assume that

(*) there is no partial L-elementary map $f : \mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{A} \cap \mathfrak{B} \subseteq \text{dom}(f) \neq \mathfrak{A} \cap \mathfrak{B}$ and $f \upharpoonright (\mathfrak{A} \cap \mathfrak{B}) = id$.

**CLAIM 3.8.1.** $\mathfrak{A} \cap \mathfrak{B}$ contains the interpretations of the constants from $\rho$.

PROOF. Immediate by Lemma 3.3 and the assumptions on $\mathfrak{A}$ and $\mathfrak{B}$.

**CLAIM 3.8.2.** $\mathfrak{A} \cap \mathfrak{B}$ is a substructure of $\mathfrak{A}$ (and of $\mathfrak{B}$).

PROOF. Immediate by Lemma 3.3.

**CLAIM 3.8.3.** $\mathfrak{A} \cap \mathfrak{B}$ is closed in $\mathfrak{A}$ (and in $\mathfrak{B}$).

PROOF. Suppose not. By Lemmas 3.7 and 3.3 (ii), choose $a_0, ..., a_{n+k} \in \mathfrak{A}$, $0 < k, n < \omega$ and $n$-unclosed $\theta$ so that if $A$ is the relevant subset of $\mathfrak{A}$ generated by $\{a_0, ..., a_n\}$ and $B$ is the relevant subset of $\mathfrak{A}$ generated by $\{a_0, ..., a_{n+k}\}$, then

(i) $B \neq A$ is the intrinsic closure of $A$, $A = B \cap \mathfrak{B}$ (and so $A$ is closed in $\mathfrak{A} \cap \mathfrak{B}$)

(ii) the elements $a_0, ..., a_{n+k}$ are linearly independent,

(iii) there are at most finitely many $b_{n+1}, ..., b_{n+k} \in \mathfrak{A}$ ($b_{n+1}, ..., b_{n+k} \in \mathfrak{B}$) such that $\mathfrak{A} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ ($\mathfrak{B} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$),

(iv) for all $b_{n+1}, ..., b_{n+k} \in \mathfrak{A}$ ($b_{n+1}, ..., b_{n+k} \in \mathfrak{B}$), if

$\mathfrak{A} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ ($\mathfrak{B} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$),

then $\{a_0, ..., a_n\}$ is unclosed in $\{a_0, ..., a_n, b_{n+1}, ..., b_{n+k}\}$. By Lemma 3.1, it is easy to see that

(v) if $b_{n+1}, ..., b_{n+k} \in \mathfrak{A}$ ($\in \mathfrak{B}$) and

$\mathfrak{A} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$ ($\mathfrak{B} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$),

then $b_i \not\in \mathfrak{A} \cap \mathfrak{B}$ for any $k < i \leq n + k$.

For $b_{n+1}, ..., b_{n+k} \in \mathfrak{A}$ such that $\mathfrak{A} \models \theta(a_0, ..., a_n, b_{n+1}, ..., b_{n+k})$, we let

$\Phi(b_{n+1}, ..., b_{n+k})$
be the set
\[
\{ \exists y_0 \ldots \exists y_r \phi(a_0, \ldots, a_n, c_0, \ldots, c_p, x_{n+1}, \ldots, x_{n+k}, y_0, \ldots, y_r) \mid \\
\exists x_{n+1} \ldots \exists x_{n+k} \exists y_0 \ldots \exists y_r \phi \in L^e, \\
c_0, \ldots, c_p \in \mathfrak{A} \cap \mathfrak{B}, \\
\mathfrak{A} \models \exists y_0 \ldots \exists y_r \phi(a_0, \ldots, a_n, c_0, \ldots, c_p, b_{n+1}, \ldots, b_{n+k}) \}.
\]

By (iii), we can choose the elements \(a_0, \ldots, a_{n+k}\) so that in addition

(vi) \(\Phi(a_{n+1}, \ldots, a_{n+k})\) is maximal.

By Lemma 3.1 and the definition of \(L^e\),

(vii) for all \(\psi_i \in \Phi(a_{n+1}, \ldots, a_{n+k})\), \(i < s < \omega\), there is \(\psi \in \Phi(a_{n+1}, \ldots, a_{n+k})\) such that \(\models \forall x_{n+1} \ldots \forall x_{n+k} (\psi \rightarrow \wedge_{i < s} \psi_i)\).

Then by (iii) and (vii) we can find \(b_{n+1}, \ldots, b_{n+k} \in \mathfrak{B}\) so that for all formulas \(\phi(x_0, \ldots, x_{m+k}) \in L^e\) and \(c_0, \ldots, c_m \in \mathfrak{A} \cap \mathfrak{B}\),

(a) if \(\mathfrak{A} \models \phi(c_0, \ldots, c_m, a_{n+1}, \ldots, a_{n+k})\), then \(\mathfrak{B} \models \phi(c_0, \ldots, c_p, b_{n+1}, \ldots, b_{n+k})\).

By repeating this argument, it is easy to see that for all formulas \(\phi(x_0, \ldots, x_{m+k}) \in L^e\) and \(c_0, \ldots, c_m \in \mathfrak{A} \cap \mathfrak{B}\),

(b) if \(\mathfrak{B} \models \phi(c_0, \ldots, c_m, b_{n+1}, \ldots, b_{n+k})\), then \(\mathfrak{A} \models \phi(c_0, \ldots, c_p, a_{n+1}, \ldots, a_{n+k})\).

Items (a) and (b) together mean that there is a partial \(L\)-elementary map \(h: \mathfrak{A} \rightarrow \mathfrak{B}\) such that \(\mathfrak{A} \cap \mathfrak{B} \subseteq \text{dom}(f)\) and by (v), \(A \neq \mathfrak{A} \cap \mathfrak{B}\). This contradicts (*)..

\[\square\]

**Claim 3.8.4.** \(\mathfrak{A} \cap \mathfrak{B}\) is an algebraically closed subset of \(F_\mathfrak{A}\).

**Proof.** Let \(\mathfrak{A}^* \subseteq \mathfrak{A}\) be the least submodel of \(\mathfrak{A}\) such that it is algebraically closed as a subset of \(F_\mathfrak{A}\), and \(\mathfrak{A} \cap \mathfrak{B} \subseteq \mathfrak{A}^*\). \(\mathfrak{B}^* \subseteq \mathfrak{B}\) is defined similarly. Let \(L^a\) be the class of all formulas \(\phi\) such that

(i) \(\phi = \exists x_{n+1} \ldots \exists x_{n+k} \theta(x_0, \ldots, x_{n+k})\), where \(\theta\) is \(n\)-algebraic (i.e. \(\theta\) is quantifier free and implies that for all \(0 < i \leq k\), \(x_{n+i}\) is algebraic over \(\{x_0, \ldots, x_{n+i-1}\}\) or \(x_{n+i} = \exp(x)\) for some \(x \in \{x_0, \ldots, x_{n+i-1}\}\) or \(x_{n+i} = \log(x)\) for some \(x \in \{x_0, \ldots, x_{n+i-1}\}\).

Let \(f: \mathfrak{A} \rightarrow \mathfrak{B}\) be a partial function such that

(a) \(\mathfrak{A} \cap \mathfrak{B} \subseteq \text{dom}(f)\) and \(f \upharpoonright (\mathfrak{A} \cap \mathfrak{B}) = \text{id}\),

(b) \(f\) is \(L^a\)-elementary,

(c) \(\text{dom}(f) \subseteq \mathfrak{A}^*\),

(d) \(f\) is maximal among those that satisfy (a)-(c).

**Subclaim 3.8.4.1.** \(\text{dom}(f) = \mathfrak{A}^*\) and \(\text{rng}(f) = \mathfrak{B}^*\).

**Proof.** By the definition of \(L^e\), \(\text{id}: \mathfrak{A} \cap \mathfrak{B} \rightarrow \mathfrak{A} \cap \mathfrak{B}\) satisfies (a)-(c). But then one can prove that \(\text{dom}(f) = \mathfrak{A}^*\) essentially as the related claim was proved in the proof of Claim 3.8.3. By the definition of \(\mathfrak{B}^*\), this implies that \(\text{rng}(f) = \mathfrak{B}^*\). \(\square\)

So by Subclaim 3.8.4.1, \(f\) is an isomorphism between \(\mathfrak{A}^*\) and \(\mathfrak{B}^*\). By (*) and Lemma 3.4 (and symmetry) it suffices to prove the following subclaim.

**Subclaim 3.8.4.2.** \(\mathfrak{A}^*\) is closed in \(\mathfrak{A}\).

**Proof.** Let \(A\) be a relevant subset of \(\mathfrak{A}\) with finite dimension. Let \(B = A \cap \mathfrak{B}\) and \(C = A \cap \mathfrak{A}^*\). We need to prove that \(\delta(A/C) \geq 0\). By the basic properties of \(\delta\), it suffices to prove this for any relevant \(A' \supseteq A\) such that \(A'\) has finite dimension and is the relevant subset of \(\mathfrak{A}\) generated by \(A \cup C'\) for some relevant \(C' \subseteq \mathfrak{A}^*\) such that \(C \subseteq C'\). By the definition of \(\mathfrak{A}^*\), we can find \(C'\) so that it is contained in the
least submodel $C$ of $A$ such that $C' \cap B \subseteq C$ and $C$ is algebraically closed in $F_A$. So we may also assume that $\delta(C/B) = 0$ ($\delta(C/B) \geq 0$ follows from Claim 3.8.3).

Now

$$\delta(A/C) = \delta(A/B) - \delta(C/B).$$

Since by Claim 3.8.3, $\delta(A/B) \geq 0$, $\delta(A/C) \geq 0$. □

By Claims 3.8.3 and 3.8.4, $A \cap B \in \mathcal{K}$ is a closed substructure of both $A$ and $B$. So by Lemmas 2.2 and 2.3, there is $C \in \mathcal{K}$ such that both $A$ and $B$ are closed substructures of $C$ i.e. $C$ and $id : A \cup B \rightarrow C$ are as required in the $L$-amalgamation property. □

**Lemma 3.9.** $\mathcal{K}$ is strongly $L$-inductive.

**Proof.** Immediate by the definition of $\mathcal{K}$ (all the requirements are local). □

This finishes the proof of Theorem 3.5.

**Conclusion 3.10.** For all cardinals $\kappa$, there is a model $A \in \mathcal{K}$ of power $\geq \kappa$ such that $A$ is strongly $L$-$\kappa$-homogeneous and $L$-$\kappa$-universal for $\mathcal{K}$. Furthermore, $A$ can be chosen so that, in addition, the following holds: If $g : scl(A) \uparrow (\rho - \{log\}) \rightarrow scl(B) \uparrow (\rho - \{log\})$ is an isomorphism and $|A| < \kappa$, then there is an automorphism $f$ of $\mathcal{M} \uparrow (\rho - \{log\})$ such that $g \subseteq f$.

### 4. On independence in $\mathcal{K}$

In this section we will study mainly the meaning and behaviour of dividing in the class $\mathcal{K}$, see e.g. [BL].

Let $\kappa$ be a relatively large cardinal and $\mathcal{M} \in \mathcal{K}$ be a structure as given by Conclusion 3.10, i.e. $\mathcal{M}$ is strongly $L$-$\kappa$-homogeneous and $L$-$\kappa$-universal for $\mathcal{K}$. We will use concepts and notation from [S1], [S2] and [BL] freely (in particular, $A \subseteq \mathcal{M}$ means that $A$ is a subset of $\mathcal{M}$ and $|A| < \kappa$ or even $|A| \ll \kappa$ if needed). Notice that for all $a \in \mathcal{M}$ and $A \subseteq \mathcal{M}$ $t_L(a/A)$ determines the first-order type $t(a/A)$ of $a$ over $A$.

In [S5], stability of a structure $\mathcal{A}$ is defined: $\mathcal{A}$ is $\lambda$-stable if $|\{t(a/A) | a \in \mathcal{A}\}| \leq \lambda$, for all $A \subseteq \mathcal{A}$ of power $\leq \lambda$. We say that $\mathcal{M}$ is unstable if it is not $\lambda$-stable for any infinite $\lambda < \kappa$.

**Remark 4.1.** $\mathcal{M}$ is unstable.

**Proof.** Let $\lambda$ be an infinite cardinal and $\mathcal{A}$ an $L$-elementary submodel of $\mathcal{M}$ of power $\lambda$. Let $a_i \in \mathcal{A}$, $i < \lambda$, be distinct elements such that for all $i < \lambda$, $a_i \not= \pi$ and $\{\pi\} \cup \{a_i | i < \lambda\}$ is a linear base for $\mathcal{A}$. If $f$ is a function from $\lambda$ to $\{1, 2\}$, then by $a_i^f$ we denote the element $f(i)\pi + a_i \in \mathcal{A}$. By $p^f$ we denote the type $(\log(\exp(x + a_i^f)) = x + a_i^f | i < \lambda)$. By the proof of Lemma 2.3, $p^f$ is realized in $\mathcal{M}$ for any $f : \lambda \rightarrow \{1, 2\}$. □

Now Remark 4.1 implies that $\mathcal{K}$ has all the non-structure properties one can think of. However, since all these properties arise from the added 'random' function, this is not very interesting. Also the class may be simple (it is as we will see) and
as we know from the elementary case, from such classes one may obtain a lot of structural information. So we will try to do better. In a sense, we look at how these structures behave if we remove the random function from the structures. (However, we do use the fact that the function can be added making the behaviour of the class nice.)

Below we define some concepts for models from $\mathcal{R}$. As before, this defines the same concepts for $\mathcal{M}$.

**Definition 4.2.** Suppose $\mathcal{C} \in \mathcal{R}$, $a \in \mathcal{C}$ and $A, B \subseteq \mathcal{C}$.

(i) By $\text{scl}(A) = \text{scl}_A(A)$ (small closure operator, see [BL]; in [S1] $\text{scl}$ is denoted by $\text{acl}$) we denote the least submodel $\mathfrak{A} \supseteq A$ of $\mathcal{C}$ such that $\mathfrak{A}$ is algebraically closed as a subset of $\mathcal{C}$, is closed as a relevant subset of $\mathcal{C}$ and for all elements $b, c \in \mathcal{C}$, if $\mathcal{C} \models \exp(b, c)$, then $b \in \mathfrak{A}$ iff $c \in \mathfrak{A}$ (i.e. in $\mathcal{M}$, the least submodel of $\mathcal{M}$ which contains all the intrinsic closures of finite subsets of $A$ and which is algebraically closed as a subset of $F_{\mathcal{M}}$, see the proof of Subclaim 3.8.4.2).

(ii) We define $R(a/A) \in \mathbb{N}$ as follows: If $a \in \text{scl}(A)$, then $R(a/A) = 0$ and otherwise $R(a/A)$ is the least $n$ such that there is $C \subseteq \text{scl}(A \cup a)$ for which the following holds:

(a) $C \subseteq \text{scl}(A \cup a)$ is a relevant set and has finite dimension,
(b) $a \in C$,
(c) $C \cap \text{scl}(A)$ is closed,
(d) $n = \delta(C/C \cap \text{scl}(A)) + 1$.

If $C$ satisfies (a)-(c) and $\delta(C/C \cap \text{scl}(A)) + 1 = R(a/A)$, then we say that $C$ is an $R$-witness for $(a, A)$.

(iii) We write $a \downarrow^d_A B$ if $\text{scl}(a \cup A) \downarrow^d_{\text{scl}(A)} \text{scl}(A \cup B)$ and the relevant set generated by $\text{scl}(a \cup A) \cup \text{scl}(A \cup B)$ is closed. For (possibly infinite) sets $C \subseteq \mathcal{C}$, $C \downarrow^d_A B$ is defined similarly.

Notice that for all elements $a \in \mathcal{M}$ and $A \subseteq \mathcal{M}$, $0 \leq R(a/A) \leq 2$. Notice also that $\text{scl}(A) = \bigcup \{ \text{scl}(B) \mid B \subseteq A \text{ finite} \}$.

Our goal is to prove that $t(a/B)$ does not divide over $A \subseteq B$ iff $a \downarrow^d_A B$.

**Lemma 4.3.**

(i) For all $a \in \mathcal{M}$ and $A \subseteq B \subseteq \mathcal{M}$, $a \downarrow^d_A B$ iff $\text{scl}(a \cup A) \cap \text{scl}(B) = \text{scl}(A)$ and $R(a/B) = R(a/A)$.

(ii) For all $a \in \mathcal{M}$ and $B \subseteq \mathcal{M}$ there is countable (or finite) $A \subseteq B$ such that $a \downarrow^d_A B$.

(iii) For all $a \in \mathcal{M}$ and $A \subseteq B \subseteq \mathcal{M}$ there is $b \in \mathcal{M}$ such that $t(b/A) = t(a/A)$ and $b \downarrow^d_A B$.

(iv) Suppose $A \subseteq B \subseteq C \subseteq \mathcal{M}$. Then $a \downarrow^d_A C$ iff $a \downarrow^d_A B$ and $a \downarrow^d_B C$.

(v) For all $A \subseteq \mathcal{M}$ and $a, b \in \mathcal{M}$, $a \downarrow^d_A b$ iff $b \downarrow^d_A a$.

(vi) For all $A \subseteq B \subseteq \mathcal{M}$ and $C \subseteq \mathcal{M}$, $C \downarrow^d_A B$ iff for all (finite sequences) $c \in C$, $C \downarrow^d_A B$.

**Proof.** (i): "⇒": Clearly it suffices to prove that $R(a/B) = R(a/A)$. For this, let $C$ be the relevant set generated by $\text{scl}(A \cup a) \cup \text{scl}(B)$, $D$ an $R$-witness for $(a, A)$ and $E$ an $R$-witness for $(a, B)$. Since $C$ is closed, $\delta(E \cap C/E \cap \text{scl}(B)) \leq \delta(E/E \cap \text{scl}(B))$. Thus we may assume that $C \subseteq E$. As in the proof of Lemma 2.2, we can find $E' \subseteq C$ such that $E \subseteq E'$, $\delta(E'/E' \cap \text{scl}(B)) \leq \delta(E/E \cap \text{scl}(B))$ and $\delta(E'/\text{scl}(A \cup a)/E' \cap \text{scl}(A)) \leq \delta(E'/E' \cap \text{scl}(B))$. But then $\delta(D/D \cap \text{scl}(A)) \leq \delta(E/E \cap \text{scl}(B))$.\hfill\pagebreak
"\( \leq \)". Let \( C \) be the relevant set generated by \( \text{scl}(A \cup a) \cup \text{scl}(B) \) and \( D \) an \( R \)-witness for \( (a, A) \). Since \( R(a/A) = R(a/B) \), \( D \) is an \( R \)-witness for \( (a, B) \). Thus if \( E \) is the relevant set generated by \( \text{scl}(B) \cup D \), \( E \) is closed. As in the proof of Lemma 2.1, one can see that every \( C' \subseteq C \) with finite dimension is contained in \( C'' \subseteq C \) such that \( C'' \) has finite dimension, \( C' \subseteq C'' \) and \( \delta(C''/C'' \cap E) = 0 \). As before, this implies that \( C \) is closed.

Let \( D \) be as above and \( E \) a relevant set generated by \( D \cup \text{scl}(A) \). Again, for every \( C \subseteq \text{scl}(A \cup a) \) with finite dimension, there is \( C' \subseteq \text{scl}(A \cup a) \) such that \( C' \) has finite dimension, \( C \subseteq C' \) and \( \delta(C'/C' \cap E) = 0 \). With this, since \( \text{scl}(B) \) and \( \text{scl}(A \cup a) \) are linearly independent over \( \text{scl}(A) \), a simple calculation shows that if \( \text{scl}(A \cup a) \n \text{scl}(B) \), then \( R(a/B) < R(a/A) \).

(ii): By (i), just choose countable \( A \subseteq B \) such that for some \( R \)-witness \( C \) for \( (a, B), C \subseteq \text{scl}(A \cup a) \) and \( \text{scl}(C \cup A) \cap \text{scl}(B) = \text{scl}(A) \).

(iii): Let \( A = (\text{scl}(B))^s \in \mathfrak{R}^s \) and let \( B = \text{scl}(A \cup a)^s \in \mathfrak{R}^s \). Let \( \mathfrak{C} \) be the 'free' amalgamation of \( A \) and \( B \) over \( \text{scl}(A) \) from the proof of Lemma 2.2. By Lemma 2.3, there is \( \mathfrak{D} \in \mathfrak{R} \) such that \( \mathfrak{S}^* \equiv \mathfrak{C} \) and both \( \text{id} : \text{scl}(B) \to \mathfrak{D} \) and \( \text{id} : \text{scl}(A \cup a) \to \mathfrak{D} \) are \( L \)-elementary. By the universality and the homogeneity of \( \mathfrak{M} \), there is \( L \)-elementary \( g : \mathfrak{D} \to \mathfrak{M} \) such that \( g \circ (f \text{scl}(B)) = \text{id} \). Then a simple predimension calculation (as in the proof Lemma 2.2) shows that \( b = g(f(a)) \) is as required.

(iv): It is easy to see that \( c \downarrow^d E \) iff \( c \downarrow^d \text{scl}(D) \text{scl}(E) \). Thus we may assume that \( A = \text{scl}(A), B = \text{scl}(B) \) and \( C = \text{scl}(C) \). Then the claim is clear by (i).

(v): Immediate by the definition of \( \downarrow^d \).

(vi): Follows easily from the definition of \( \downarrow^d \).

Now we borrow ideas from [Z1].

**Definition 4.4.** Suppose \( H \) is an algebraically closed subset of \( \mathfrak{F}_\mathfrak{M} \).

(i) We say that \((x_0, \ldots, x_m)\) is multiplicatively independent over \( H \) if for all integers \( n_i, i \leq m \), if some \( n_i \neq 0 \), then \( x_0^{n_0} \cdot \ldots \cdot x_m^{n_m} \notin H \).

(ii) Let \( W^{1/n}, n \in \mathbb{N} - \{0\} \), be irreducible algebraic varieties over \( H \) in \( m + 1 \) variables. We say that the sequence is coherent if for all \( n, k, (a_0, \ldots, a_m) \mapsto (a_0^n, \ldots, a_m^n) \) maps \( W^{1/nk} \) onto \( W^{1/k} \).

(iii) We say that \((a_0^{1/n}, \ldots, a_m^{1/n})\), \( n \in \mathbb{N} - \{0\} \), is coherent sequence if for all \( n, k \) and \( i \leq m \), \((a_i^{1/nk})^n = a_i^{1/k} \).

(iv) By a multipliciative closure \( M(A) \) of \( A \) we mean the least set which contains \( A \) and is closed under multiplication, inverses and (all) roots.

The following theorem is proved in [Z1].

**Theorem 4.5.** Suppose \( H \) is algebraically closed and \( W^{1/n} \) is a coherent sequence of irreducible algebraic varieties over \( H \) in \( m + 1 \) variables. Then there is \( n_0 \) such that if \((a_0^{1/n}, \ldots, a_m^{1/n})\) is a coherent sequence, \((a_0^{1/1}, \ldots, a_m^{1/1}) \in W^{1/1} \) is generic and multiplicatively independent over \( H \) and \((a_0^{1/n_0}, \ldots, a_m^{1/n_0}) \in W^{1/n_0} \), then for all \( n, (a_0^{1/n}, \ldots, a_m^{1/n}) \in W^{1/n} \).

**Lemma 4.6.** Suppose \( A = \text{scl}(A) \subseteq B = \text{scl}(B) \subseteq \mathfrak{M}, |B| = \omega, a \downarrow^d B, b \downarrow^d A B \) and that there is an isomorphism

\[
g : \text{scl}(A \cup a) \uparrow (\rho - \{\log\}) \to \text{scl}(A \cup b) \uparrow (\rho - \{\log\})
\]
such that $g \upharpoonright A = id$ and $g(a) = b$. Then there is an isomorphism

$$f : \text{scl}(B \cup a) \upharpoonright (\rho \setminus \{\log\}) \to \text{scl}(B \cup b) \upharpoonright (\rho \setminus \{\log\})$$

such that $f \upharpoonright B = id$ and $f(a) = b$.

**Proof.** It suffices to construct by induction on $i < \omega$ sequences $(x_0, \ldots, x_{n_i}) \in \text{scl}(B \cup a)$, $(y_0^{1/\ell}, \ldots, y_{n_i}^{1/\ell}) \in \text{scl}(B \cup a)$ and $(z_0^{1/\ell}, \ldots, z_{n_i}^{1/\ell}) \in \text{scl}(B \cup b)$ such that:

(a) Let $f_i$ be such that $\text{dom}(f_i) = B \cup \{y_j^{1/\ell} \mid l \in \mathbb{N} \setminus \{0\}, j \leq m_i\}$, $f_i \upharpoonright B = id$ and $f_i(y_j^{1/\ell}) = z_j^{1/\ell}$. Then $f_i$ is an elementary map in $F_{\mathbb{M}}$.

(b) $\{\exp(x_j) \mid j \leq n_i\} \subseteq \{y_j^{1/\ell} \mid j \leq m_i\} \subseteq \{\exp(x_j), x_j \mid j \leq n_i\}$ By $M_i$, we denote the set $M(B \cup \{y_j^{1/\ell} \mid j \leq m_i\}$ and require that for all $j \leq n_i$, $x_j \in M_i$.

(c) $(x_0, \ldots, x_{n_i})$ is linearly independent over $B$ and $(y_0^{1/\ell}, \ldots, y_{n_i}^{1/\ell})$ is a coherent sequence and $(y_0^{1/\ell}, \ldots, y_{n_i}^{1/\ell})$ is multiplicatively independent over $B$ (by (a) the same holds for $(z_0^{1/\ell}, \ldots, z_{n_i}^{1/\ell})$).

(d) If $g_i$ is an elementary map in $F_{\mathbb{M}}$ such that $\text{dom}(g_i) = M_i$ and $f_i \subseteq g_i$, then for all $j \leq n_i$ and $l$, $g_i(\exp((1/l) \cdot x_j)) = \exp((1/l) \cdot g_i(x_j))$ and if $x$ is in the relevant set generated by $B \cup \{x_j \mid j \leq n_i\}$ and $\exp(x) = y_k^{1/\ell}$ for some $k \leq m_i$, then for some $p \leq n_i$, $\exp((1/l) \cdot x_p) = y_k^{1/\ell}$, for all $l$.

(e) The relevant set generated by $A \cup \{x_0, \ldots, x_{n_0}\}$ contains the $R$-witness for $(a, A)$ and for all $i < \omega$, the relevant set generated by $B \cup \{x_0, \ldots, x_{n_i}\}$ is closed as well as the relevant set generated by $B \cup \{g_i(x_0), \ldots, g_i(x_{n_i})\}$ where $g_i$ is as in (d) above.

(f) For all $j \leq m_i$, there is a $c$ such that for all $l$, $y_j^{1/\ell} = \exp((1/l) \cdot c)$ and similarly for $z_j^{1/\ell}$.

(g) The relevant set generated by $B \cup \{x_j \mid j < \omega\}$ is $M(B \cup \{y_j^{1/\ell} \mid j < \omega\}) = \text{scl}(B \cup a)$.

We start by an observation.

**Claim 4.6.1.** For all $i$, $g_i$ from (d) exists and is unique.

**Proof.** The existence follows from (a) and the fact that $F_{\mathbb{M}}$ is saturated. Uniqueness is clear since $f_i \upharpoonright B = id$ and $B$ contains all the roots of unity.

We construct now the sequences by induction on $i$: By the assumptions, the case $i = 0$ is easy.

Any reasonable bookkeeping shows that (a)-(g) can be satisfied if we can do the following three extensions. By $X_i$ we denote the relevant set generated by $B \cup \{x_j \mid j \leq n_i\}$.

1. Suppose $x$ is such that $\exp(x) \in \text{acl}(M_i) - M_i$. We want to extend the sequences so that $x, \exp(x) \in M_{i+1}$.

Since $X_i$ is closed, it is easy to see that $x \not\in \text{acl}(M_i)$.

We let $n_{i+1} = n_i + 1$ and let $x_{n_{i+1}} = x$. We let $m_{i+1} = m_i + 2$, $y_{m_{i+1}}^{1/\ell} = \exp((1/l) \cdot x)$ and define $y_{m_{i+1}}^{1/\ell}$ so that $y_{m_{i+1}}^{1/\ell} = x$ and for some $z$, $y_{m_{i+1}}^{1/\ell} = \exp((1/l) \cdot z)$.

Now let $N_0$ be as given by Theorem 4.5 for $(y_0^{1/\ell}, \ldots, y_{m_i+1}^{1/\ell})$ and $B$ (these determine $W^{1/l}$). Let $z$ be such that $(z_0^{1/N_0}, \ldots, z_{m_i}^{1/N_0}, z) \in W^{1/N_0}$. Let $z'$ be such that
exp(z') = z. As above z' is transcendental over B ∪ \{z_0^{1/N_0}, ..., z_{m_i}^{1/N_0}, z\}. Then we let z_{m_i+1}^{1/l} = \exp((N_0/l) \cdot z') and z_{m_i+1}^{1/l} be such that z_{m_i+1}^{1/l} = N_0 \cdot z' and (g) holds. It is easy to see that these are as wanted.

2. Suppose \( x \not\in X_i \) is such that \( \exp((1/l) \cdot x) = y_k^{1/l} \) for some \( k \leq m_i \) and all \( l \) (unless \( y_k^{1/l} = \exp(x_j) \) for some \( j \leq n_i \), such \( x \) exists by (g)). Again, we see that \( x \not\in acl(M_i) \) and we can proceed as before.

3. Suppose \( x \in M_i - X_i \) and we want to extend the sequences so that \( \exp(x) \in M_{i+1} \). Again, \( \exp(x) \) is transcendental over \( M_i \) and \( \exp(g_i(x)) \) is transcendental over \( g_i(M_i) \). Thus we can let \( n_{i+1} = n_i + 1, x_{n_{i+1}} = x, m_{i+1} = m_i + 1, y_{m_{i+1}}^{1/l} = \exp((1/l) \cdot x) \) and \( z_{m_{i+1}}^{1/l} = \exp((1/l) \cdot g_i(x)) \).

We can improve Lemma 4.3 (ii):

**Lemma 4.7.** For all \( a \) and \( B \) there is finite \( A \subseteq B \) such that \( a_{\mathcal{A}}^d B \).

**Proof.** Let \( \mathcal{B} = scl(B) \), \( C \) an \( R \)-witness for \( (a, B) \) and \( \mathfrak{A} = scl(C \cap B) \). Since \( \delta(C/C \cap acl(a \cup \mathfrak{A})) \geq 0 \), we may assume that \( C \subseteq acl(a \cup \mathfrak{A}) \). Clearly, it suffices to prove that \( a_{\mathcal{A}}^d \mathcal{B} \). For a contradiction, suppose that this is not the case. By Lemma 4.3 (i), this means that \( acl(a \cup \mathfrak{A}) \cap \mathcal{B} \neq \mathfrak{A} \).

Then we can find \( d_i \in scl(\mathfrak{A} \cup a), i \leq n, \) and \( p < k \leq l < n \) such that

(a) \( (d_i)_{i \leq n} \) is linearly independent,
(b) if \( D \) is the relevant set generated by \( \{d_0, ..., d_n\} \), then \( D \cap \mathfrak{A} \) is generated by \( \{d_0, ..., d_{k-1}\} \),
(c) if \( E \) is the relevant set generated by \( \{d_0, ..., d_{n-1}\} \), then \( E \cap \mathfrak{A} = E \cap \mathfrak{A} \),
(d) \( D \cap \mathfrak{A} \) is closed,
(e) \( \{d_0, ..., d_p, d_k, ..., d_l\} \) generate the relevant set \( C \) and for all \( l < i \leq n \), either \( d_i \) or \( \exp(d_i) \) is algebraic over \( \{d_0, ..., d_{i-1}, \exp(d_0), ..., \exp(d_{i-1})\} \),
(f) \( d_n, \exp(d_n) \in \mathcal{B} \).

Then

\[
\deg((d_k, ..., d_{n-1}, \exp(d_k), ..., \exp(d_{n-1}))/\mathcal{B}) < \\
\deg((d_k, ..., d_{n-1}, \exp(d_k), ..., \exp(d_{n-1}))/\mathfrak{A}) = n - k
\]

and \( \dim((d_k, ..., d_{n-1}))/\mathcal{B} = n - k \). This contradicts the choice of \( C \).

**Lemma 4.8.** Suppose \( A \subseteq B \subseteq \mathfrak{M} \). If \( a_{\mathcal{A}}^d B \), then \( t(a/B) \) divides over \( A \).

**Proof.** If \( scl(A \cup a) \not\subseteq scl(A) \), it is easy to see that \( t(a/B) \) divides over \( A \). So we assume that \( scl(A \cup a) \subseteq scl(A) \). Then the relevant set generated by \( scl(B) \cup scl(A \cup a) \) is not closed. Adapting notation from [BL], let \( b \) be a (possibly infinite) sequence such that \( B = rng(b) \). Let \( \lambda \) be a large enough cardinal, \( \mathfrak{A} = scl(A) \) and let \( b_i, i < \lambda \), be such that

(a) \( (b_i)_{i < \lambda} \) is \( \mathfrak{A} \)-indiscernible,
(b) \( b_0 = b \),
(c) for all \( i < \lambda \), \( b_i \upharpoonright \mathcal{A} \cup b_j \),
(d) for all \( i < \lambda \), the relevant set \( B_i^r \) generated by \( \cup_{j < i} scl(b_j) \) is closed.

By (the proof of) Lemma 4.3 (iii), homogeneity and Erdős-Rado theorem, such sequences \( b_i \) exist ((d) follows from (c)).

For a contradiction, suppose \( a^* \) is such that for all \( i < \lambda \), \( t(a^* \upharpoonright b_i) = t(a \upharpoonright b_i) \). Let \( \mathfrak{A}^* = scl(\mathfrak{A} \cup a^*), \mathfrak{B}_i = scl(b_i) \) and \( \mathfrak{B}_i^* = scl(b_i \cup a^*) \). Clearly we may assume that for all \( i < \lambda \),
(e) $\mathcal{B}_i \downarrow_\mathcal{A} \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$ (not $\downarrow_\mathcal{A}$!)

Also there is $n < \omega$ such that for all $i < \omega$, $0 \leq R(a^* / B_i^*) < n$. Thus the following claim implies the required contradiction.

**Claim 4.8.1.** Let $i < \omega$, and let $C$ be the relevant set generated by $\mathcal{B}_i \cup \mathcal{A}^*$ and $D$ be the relevant set generated by $\text{scl}(\cup_{j<i} \mathcal{B}_j^*) \cup C$. Then (i) and (ii) below hold.

(i) If $D$ is closed, then so is $C$ (i.e. $D$ is not closed).

(ii) If $D$ is not closed, then $R(a^* / B_{i+1}^*) < R(a^*/B_i^*)$.

**Proof.** (i): This can be proved essentially as the related claim was proved in Lemma 2.2. However, since there is a small difference, we repeat the proof for the reader's convenience.

Suppose $E \subseteq D$ is a relevant set and has finite dimension. Since $D$ is closed, it suffices to prove that $\delta(E/E \cap C) \geq 0$. Let $X = \{e_0, ..., e_n\}$ be a linear base for $E$ such that $X \cap C = \{e_k, ..., e_m\}$ is a base for $E \cap C$. Since we may increase $E \cap C$ without increasing $\delta(E/E \cap C)$ we may assume that for all $e \in E$, there is $c \in C \cap E$ and $b \in \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$ such that $e = c + b$. Thus we may assume that $X$ was chosen so that for all $m < k$, $e_m \in \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$. By (e) we may also assume that writing $E'$ for $E \cap \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$,

$$E' \cup \exp(E') \downarrow_{(E \cap \mathcal{A}^*) \cup \exp(E \cap \mathcal{A}^*)} C \cup \exp(C).$$

But then $\delta(E'/E' \cap \mathcal{A}^*) = \delta(E/E \cap C)$. Since $\mathcal{A}^*$ is closed, $\delta(E'/E' \cap \mathcal{A}^*) \geq 0$.

(ii): If $D$ is not closed, then there is a relevant set $E \subseteq \text{scl}(D) = \text{scl}(B_{i+1}^*)$ such that it has finite dimension and

$\delta(E/E \cap D) < 0$.

Notice that as before, we may increase $E$ by increasing the set $E \cap D$ and (*) remains true. Also as before, for every $N \subseteq \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$ with finite dimension there is $N^* \subseteq \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$ such that $N \cup (E \cap \text{scl}(\cup_{j<i} \mathcal{B}_j^*)) \subseteq N^*$ and $N^*$ is an $R$-witness for $R(a^*/B_i^*)$. So since $\text{scl}(\cup_{j<i} \mathcal{B}_j^*) \subseteq D$, we may assume that $E' = E \cap \text{scl}(\cup_{j<i} \mathcal{B}_j^*)$ is an $R$-witness for $R(a^*/B_i^*)$. Furthermore, since every elements in $D$ depends linearly from $H = \text{scl}(\cup_{j<i} \mathcal{B}_j^*) \cup \mathcal{B}_i$, we may assume that every element of $E \cap D$ depends linearly from $E \cap H$. Thus, letting $D'$ be the relevant set generated by $\text{scl}(B_i^*) \cup \mathcal{B}_i$, by (e), we may assume that

(1) $\delta(E \cap D / E \cap D') = R(a^*/B_i^*)$.

Since both $B_i^*$ and $B_{i+1}^*$ are closed, essentially repeating the argument above, we can see that

(2) $D'$ is closed.

Then by (1) and (**), $\delta(E/E \cap D') < R(a^*/B_i^*)$. By (2), $\delta(E \cap \text{scl}(B_{i+1}^*) / E \cap D') \geq 0$. Thus $\delta(E/E \cap \text{scl}(B_{i+1}^*)) \leq \delta(E/E \cap D') < R(a^*/B_i^*)$. □

□

**Lemma 4.9.** Suppose $A \subseteq B \subseteq \mathcal{M}$ and $B$ is countable. If $\downarrow_A^B B$, then $t(a/B)$ does not divide over $A$.

**Proof.** Again, let $b$ be a (possibly infinite) sequence such that $B = \text{rng}(b)$. We need to prove that $t(a/b)$ does not divide over $A$. For this let $(b_i)_{i < \alpha}$ be $A$-indiscernible sequence such that $t(b_0 / A) = t(b/A)$ and $\alpha$ is an infinite ordinal. Notice that it is enough to find $a^*$ and $b_i'$, $i < \alpha$, such that
(i) there is a \((L)\)-elementary \(F : \cup_{i < \alpha} b'_i \rightarrow b_i\) such that \(F \upharpoonright A = \text{id}\) and for all \(i < \alpha, F(b'_i) = b_i\).

(ii) for all \(i < \alpha, t(a^* \smallsetminus b'_i) = t(a \smallsetminus b_0)/\).

By homogeneity we may assume that \(b_0 = b\). Let \(\mathfrak{A} = \text{scl}(A)\). Again by homogeneity and the usual Erdős-Rado-argument, we may assume that there are sets \(\mathfrak{B}_i, i < \alpha,\) such that

(a) \(\text{scl}(\mathfrak{B}_i) = \mathfrak{B}_i,\)

(b) \(\text{scl}(b) = \mathfrak{B}_0,\)

(c) for all \(i < \alpha,\) there is an isomorphism \(f_i : \mathfrak{B}_0 \rightarrow \mathfrak{B}_i\) such that \(f_i \upharpoonright \mathfrak{A} = \text{id}\) and \(f_i(b_0) = b_i,\)

(d) \((\mathfrak{B}_i)_{i < \alpha}\) is \(\mathfrak{A}\)-indiscernible.

It is enough to find \(a^*\) such that for all \(i < \alpha, t(a^*/\mathfrak{B}_i) = f_i(t(a/\mathfrak{B}_0)).\)

Let \((I, <)\) be an ordering such that it extends \((\alpha, <),\) the order type of \(I - \alpha\) is \(\omega_1\) and for all \(x \in I - \alpha\) and \(y \in \alpha, x < y.\) Choose (using homogeneity) \(\mathfrak{B}_i, i \in I - \alpha,\) so that \((\mathfrak{B}_i)_{i \in I}\) is \(\mathfrak{A}\)-indiscernible. Let \(\mathfrak{C} = \text{scl}(\cup_{i \in I - \alpha} \mathfrak{B}_i)\) and for \(i < \alpha, \mathfrak{C}_i = \text{scl}(\mathfrak{B}_i \cup \mathfrak{C}).\) By Lemma 4.3 (iii), (iv) and (v), for all \(i < \alpha\) and finite subsequences \(b' \) of \(b_i,\)

\[b' \upharpoonright \mathfrak{C}_i \cup \mathfrak{C}_j \]

Thus by Lemma 4.3 (v) and the definition of \(\downarrow^d,\) for all \(i < \alpha,\)

\[\mathfrak{C}_i \upharpoonright \mathfrak{C}_j \]

By Lemma 4.3 (iii) and (iv), we may assume that \(a \upharpoonright \mathfrak{C}_i \mathfrak{C}_j.\) So, and this was the purpose the considerations above, we may assume that for all \(i < \alpha,\)

\[\mathfrak{B}_i \upharpoonright \mathfrak{B}_j \cup \mathfrak{C}_i \mathfrak{C}_j.\]

Also we may assume that \(a \upharpoonright \mathfrak{B}_i \cup \mathfrak{B}_j.\)

For all \(i < \alpha, \) let \(\mathfrak{D}_i = \text{scl}(a \cup \mathfrak{B}_i).\) Then

(e) for all \(i < j < \alpha, \mathfrak{D}_i \cap \mathfrak{D}_j = \text{scl}(a \cup \mathfrak{A}),\)

(f) for all \(i < \alpha,\) there is an isomorphism \(g_i : \mathfrak{D}_0 \upharpoonright (\rho - \{\log\}) \rightarrow \mathfrak{D}_i \upharpoonright (\rho - \{\log\})\) such that \(f_i \subseteq g_i\) and \(g_i(a) = a\) (by Lemma 4.6).

Since \(\alpha\) is arbitrary, by the pigeon-hole principle (and homogeneity) we may assume that for all \(i < \alpha, g_i \upharpoonright \text{scl}(a \cup \mathfrak{A}) = \text{id}.\) By \(\mathfrak{A}^*\) we denote \(\text{scl}(A \cup a).\) By \(\mathfrak{C}\) we denote the model \(\text{scl}(\cup_{i < \alpha} \mathfrak{D}_i)^* \in \mathfrak{R}^s.\) Also we denote \(\mathfrak{B}^+ = \text{scl}(\cup_{i < \alpha} \mathfrak{B}_i) \subseteq \mathfrak{C}.\) By Lemmas 3.4 and 3.8, the existence of \(a^*\) follows if there is \(\mathfrak{C}^* \subseteq \mathfrak{R}\) (i.e. there is an interpretation \(\log^*\) for \(\log\) on \(\mathfrak{C}\)) such that \((\mathfrak{C}^*)^s = \mathfrak{C},\)

\[\text{ for all } x, y \in \mathfrak{B}^+, \mathfrak{C}^* \models \log(x) = y \text{ iff } \mathfrak{M} \models \log(x) = y\]

and

\[\text{ for all } x, y \in \mathfrak{D}_0 \text{ and } i < \alpha, \mathfrak{D}_0 \models \log(x) = y \text{ iff } \mathfrak{C}^* \models \log(g_i(x)) = g_i(y).\]

We prove this.

By the construction, it is easy to see that for all \(i < \alpha, \mathfrak{D}_i \cap \mathfrak{B}^+ = \mathfrak{B}_i\) and that \(\mathfrak{D}_i \cap (\cup_{j \in (a - \{i\})} \mathfrak{D}_j) = \mathfrak{A}^*.\) Thus for all \(i < \alpha, \mathfrak{D}_i \cap (\mathfrak{B}^+ \cup \cup_{j \in (a - \{i\})} \mathfrak{D}_j) \subseteq \mathfrak{A}^* \cup \mathfrak{B}^+.\)

So one can define \(\log^* \upharpoonright C\) so that (g) and (h) hold, where \(C = (\mathfrak{B}^+ \cup \cup_{j < \alpha} \mathfrak{D}_j).\)

We are left to define \(\log^* \upharpoonright (\mathfrak{C} - C)\) so that \(\log^*\) satisfies the requirements (5) and (7) from the definition of \(\mathfrak{R}\) (notice that \(\text{exp}(C) = C).\)

Choose a linear base \(X = \{x_i \mid i < a^*\}\) for \(\mathfrak{C}\) so that \(x_0 = \pi.\) For finite \(Y \subseteq X - \{x_0\}, \mathfrak{C}_Y\) is defined as in the proof of Lemma 2.3. By the proof of Lemma 2.3, it is enough to define \(\log^* \upharpoonright (\mathfrak{C} - C)\) so that ((5) holds and) (7) holds for \(\mathfrak{C}_Y\) for all finite \(Y \subseteq X - \{x_0\}.\) Since \(C\) is a union of relevant sets,
(i) for all \( a \in \mathcal{E}_Y \), \( a \in C \) iff for some \( q \in \mathbb{Q} - \{0\} \) and \( r \in \mathbb{Q} \), \( q \cdot a + r \cdot \pi \in C \) iff for all \( q \in \mathbb{Q} \) and \( r \in \mathbb{Q} \), \( q \cdot a + r \cdot \pi \in C \).

With this, a simple diagonalization argument, as in the proof of Lemma 2.3, gives \( \log^+ (\mathcal{E} - C) \) such that (5) holds and (7) holds for \( \mathcal{E}_Y - C \) for all finite \( Y \subseteq X - \{x_0\} \). But then, by (i) and the pigeon-hole principle, (7) holds for \( \mathcal{E}_Y \) if there is finite \( J \subseteq \alpha \) such that \( \mathcal{E}_Y \cap C \subseteq \mathcal{B}^+ \cup \bigcup_{j \in J} \mathcal{D}_j \). This is clear by (*) above (i.e. \( (\mathcal{D}_i)_{i < \alpha} \) is linearly independent over \( \mathfrak{A}^* \)) together with finite dimension of \( E_Y \).

So we have proved:

**Proposition 4.10.** \( a \downarrow^A \beta \) iff there is finite \( A' \subseteq A \) such that for all finite \( B' \subseteq B \), \( t(a/A' \cup B') \) does not divide over \( A' \).

In particular, we have proved that \( \mathfrak{M} \) (or more precisely, the finite diagram of \( \mathfrak{M} \)) is supersimple in the sense of [BL].

### 5. On \( a \)-prime models

This section is an adaptation of S. Shelah’s theory of \( a \)-prime models from [S5] to our context.

**Definition 5.1.**

(i) We write \( t^e(a/A) = t^e(b/A) \) if there is an isomorphism \( f : \text{scl}(a \cup A) \upharpoonright (\rho - \{\log\}) \rightarrow \text{scl}(b \cup A) \upharpoonright (\rho - \{\log\}) \) such that \( f \upharpoonright \text{scl}(A) = \text{id} \) and \( f(a) = b \).

(ii) We write \( t^u(a/A) = t^u(b/A) \) if for all countable \( B \subseteq A \), \( t^e(a/B) = t^e(b/B) \).

(iii) For \( B \subseteq A \), we write \( t^u(a/B) = t^u(a/A) \) if for all \( b \), \( t^u(b/B) = t^u(a/B) \) implies \( t^u(b/A) = t^u(a/A) \).

(iv) We write \( t(a/A) \in F^e_\kappa(B) \) and say that \( t(a/A) \) is \( ae \)-\( \kappa \)-isolated if \( B \subseteq A \) has power \( \kappa \) and for all \( b \), if \( t^u(b/B) = t^u(a/B) \), then \( b \downarrow^B A \).

(v) By \( \kappa(\mathfrak{M}) \) we denote the least \( \kappa \) such that for all \( a \) and \( A \) there is \( B \subseteq A \) of power \( \kappa \) such that \( a \downarrow^B A \).

(vi) We say that \( t(a/A) \) is \( ae \)-isolated if it is \( ae \)-\( \kappa(\mathfrak{M}) \)-isolated and write \( F^e_\kappa \) in place of \( F^e_{\kappa(\mathfrak{M})} \).

Notice that by Lemma 4.7, \( \kappa(\mathfrak{M}) = \omega \).

**Lemma 5.2.** For all \( B \subseteq A \), \( a \) and \( \omega_1 \geq \kappa \geq |B| \), the following are equivalent:

(i) If there is an isomorphism \( f : \text{scl}(a \cup B) \upharpoonright (\rho - \{\log\}) \rightarrow \text{scl}(b \cup B) \upharpoonright (\rho - \{\log\}) \) such that \( f \upharpoonright \text{scl}(B) = \text{id} \) and \( f(a) = b \), then for all countable \( B' \subseteq A \), there is an isomorphism \( g : \text{scl}(a \cup B') \upharpoonright (\rho - \{\log\}) \rightarrow \text{scl}(b \cup B') \upharpoonright (\rho - \{\log\}) \) such that \( f(a) = b \) and \( g \upharpoonright \text{scl}(A) = \text{id} \).

(ii) \( t^u(a/B) = t^u(a/A) \).

(iii) \( t(a/A) \in F^e_\kappa(B) \).

**Proof.** (i) implies (ii) trivially and the implication (ii)\( \Rightarrow \) (iii) follows immediately from Lemma 4.3. The implication (iii)\( \Rightarrow \) (i) follows from Lemma 4.6.

**Lemma 5.3.** Suppose \( f : \text{scl}(A \cup a) \upharpoonright (\rho - \{\log\}) \rightarrow \text{scl}(B \cup b) \upharpoonright (\rho - \{\log\}) \) is an isomorphism such that \( f[A] = B \) and \( f(a) = b \). Then for all \( C \subseteq A \), \( t(a/A) \in F^e_\kappa(C) \) iff \( t(b/B) \in F^e_\kappa(f[C]) \).
PROOF. By symmetry, it suffices to prove the implication from left to right. Suppose \( t(b/B) \notin F_{\kappa}^{ae}(f[C]) \). Let \( c \) witness this. By Lemma 2.3, there is \( \mathcal{C} \in \mathfrak{A} \) such that there is an isomorphism \( g : \mathcal{C} \upharpoonright (\rho - \{ \log \}) \to \text{scl}(B \cup c) \upharpoonright (\rho - \{ \log \}) \) and for all \( d, d' \in \text{scl}(B) \), \( \mathcal{C} \models \log(g^{-1}(d)) = g^{-1}(d') \) iff \( \log(f^{-1}(d)) = f^{-1}(d') \). By the properties of \( \mathfrak{M} \), there are \( c' \in \mathfrak{M} \) and an isomorphism \( h : \mathcal{C} \to \text{scl}(A \cup c') \) such that for all \( d \in \text{scl}(B) \), \( h(g^{-1}(d)) = f^{-1}(d) \) and \( h(g^{-1}(c)) = c' \). Then \( c' \) witnesses that \( t(a/A) \notin F_{\kappa}^{ae}(C) \). \( \square \)

**Definition 5.4.**

(i) We say that \( \mathfrak{A} \subseteq \mathfrak{M} \) is \( ae-\kappa \)-saturated if for \( A \subseteq \mathfrak{A} \) and \( a \) the following holds: if \( t(a/A) \) is \( ae-\kappa \)-isolated, then there is \( b \in \mathfrak{A} \) such that \( t^u(b/A) = t^u(a/A) \).

(ii) We say that \( A \) is \( ae-\kappa \)-constructible over \( B \subseteq A \), if there are \( a_i \) and \( A_i \), \( i < \alpha \), such that \( A = B \cup \{ a_i \mid i < \alpha \} \) and for all \( i < \alpha \), \( t(a_i/B \cup \{ a_j \mid j < i \}) \in F_{\kappa}^{ae}(A_i) \).

Then we also say that \( (B, (a_i, A_i)_{i<\alpha}) \) is an \( F_{\kappa}^{ae} \)-construction.

(iii) We say that \( \mathfrak{A} \) is \( ae-\kappa \)-primary if it is \( ae-\kappa \)-constructible and \( ae-\kappa \)-saturated.

(iv) We write \( ae \)-isolated for \( ae-\kappa(\mathfrak{M}) \)-isolated. The notions \( ae \)-saturated, \( ae \)-constructible and \( ae \)-primary are defined similarly.

**Lemma 5.5.** \( \mathfrak{A} \) is \( ae \)-saturated if for all \( A \subseteq \mathfrak{A} \) of power \( < \omega \) and \( a \) there is \( b \in \mathfrak{A} \) such that \( t^u(b/A) = t^u(a/A) \).

**Proof.** Immediate by Lemma 5.2. \( \square \)

**Lemma 5.6.** Suppose \( \mathfrak{A} \) is \( ae \)-primary over \( A \).

(i) For all \( a \in \mathfrak{A} \), there is \( B \subseteq A \) such that \( t(a/A) \in F_{\kappa}^{ae}(B) \).

(ii) \( \mathfrak{A} \) is \( ae \)-primary over any \( B \) such that \( A \subseteq B \subseteq \text{scl}(A) \).

**Proof.** (i): Suppose not. Let sequences \( a_i \) and sets \( A_i \), \( i < \alpha \), witness the fact that \( \mathfrak{A} \) is \( ae \)-constructible over \( A \). Then we can find (essentially as in the proofs above, see e.g. the proof of Lemma 5.3) \( I \subseteq \alpha, B \subseteq A \) (both of power \( < \kappa(\mathfrak{M}) \)) and \( b_i, i \in I \), such that

(a) there is an isomorphism \( f : \text{scl}(B \cup \{ a_i \mid i \in I \}) \upharpoonright (\rho - \{ \log \}) \to \text{scl}(B \cup \{ b_i \mid i \in I \}) \upharpoonright (\rho - \{ \log \}) \) such that \( f \upharpoonright \text{scl}(B) = \text{id} \) and for all \( i \in I \), \( f(a_i) = b_i \),

(b) for all \( i \in I \), \( t(a_i/A \cup \{ a_j \mid j < i, j \in I \}) \in F_{\kappa}^{ae}(B \cup \{ a_j \mid j < i, j \in I \}) \),

(c) \( \{ b_i \mid i \in I \} \not\in B \).

By choosing these so that \( I \) is minimal, we may assume that there is \( i^* \in I \) such that \( I \subseteq i^* + 1 \) and letting \( I^* = I \setminus \{ i^* \} \),

(d) \( \{ b_i \mid i \in I^* \} \not\in B \).

Then there is countable \( A' \subseteq A \) such that \( B \subseteq A' \) and

(e) \( \{ b_i \mid i \in I \} \not\in B \).

By (d) and Lemma 4.6, there is an isomorphism \( g : \text{scl}(A' \cup \{ a_i \mid i \in I^* \}) \upharpoonright (\rho - \{ \log \}) \to \text{scl}(A' \cup \{ b_i \mid i \in I^* \}) \upharpoonright (\rho - \{ \log \}) \) such that \( g \upharpoonright \text{scl}(A') = \text{id} \).

Also by the homogeneity properties of \( \mathfrak{M} \upharpoonright (\rho - \{ \log \}) \), we can find \( c \) such that there is an isomorphism \( h : \text{scl}(A' \cup \{ a_i \mid i \in I^* \}) \upharpoontright (\rho - \{ \log \}) \to \text{scl}(A' \cup \{ c \}) \) such that \( g \upharpoonright \text{scl}(A') = \text{id} \). Then \( b_i \)-witnesses that \( t(c/A \cup \{ b_i \mid i \in I^* \}) \notin F_{\kappa}^{ae}(B \cup \{ b_i \mid i \in I^* \}) \). This contradicts Lemma 5.3.

(ii): Let sequences \( a_i \) and sets \( A_i \), \( i < \alpha \), witness the fact that \( \mathfrak{A} \) is \( ae \)-constructible over \( A \). It suffices to prove that for all \( i < \alpha \), \( t(a_i/B \cup \{ a_j \mid j < i \}) \) is \( ae \)-isolated. This is clear since for all \( c \) and \( C \subseteq D, c \not\in_C D \) iff \( c \not\in_C \text{scl}(D) \). \( \square \)
Lemma 5.7. Suppose $\mathfrak{A}$ is a $\text{ae}$-saturated model, $A \subseteq \mathfrak{A}$ has power $\leq \omega_1$ and $B$ is $\text{ae}$-constructible over $A$ witnessed by $a_i$, $A_i$, $i < \omega_1$. Let $C \subseteq \text{scl}(A)$ be countable and suppose that there are, sequences $b_i \in \text{scl}(A)$, $i < \omega_1$, such that $A \subseteq \text{scl}(C \cup \bigcup_{i<\omega_1} b_i)$ and

(*) for all $i < \omega_1$, if $g$ is an automorphism of $\text{scl}(C \cup \bigcup_{j<i} b_i) \upharpoonright (\rho - \{\log\})$ such that $g \upharpoonright C \cup \bigcup_{j<i} b_i = \text{id}$, then there is an automorphism $g'$ of $\text{scl}(C \cup \bigcup_{j<i} b_i) \upharpoonright (\rho - \{\log\})$ such that $g \subseteq g'$ and $g(b_i) = b_i$.

Then there is $f \in \text{Aut}(\mathfrak{M} \upharpoonright (\rho - \{\log\}))$ such that

(i) $B \subseteq f(A)$,
(ii) $f \upharpoonright \text{scl}(C) = \text{id}$,
(iii) $f(\text{scl}(A)) = \text{scl}(A)$.

Proof. Clearly we may assume that for all $i < \omega_1$, $A_i \subseteq \text{scl}(C \cup \bigcup_{j<i} b_j \cup \bigcup_{j<i} a_j)$ (add void $a_i$ to the construction of $B$). By induction on $i < \omega_1$, we choose $c_i \in \mathfrak{A}$ so that for all $i < \omega_1$, there is an isomorphism $f_i : \text{scl}(C \cup \bigcup_{j<i} b_j \cup \bigcup_{j<i} c_j) \upharpoonright (\rho - \{\log\}) \rightarrow \text{scl}(C \cup \bigcup_{j<i} b_j \cup \bigcup_{j<i} c_j) \upharpoonright (\rho - \{\log\})$ such that

(a) for all $j < i$, $f_i(b_j) = b_j$ and $f_i(c_j) = a_j$,
(b) $f_i \upharpoonright \text{scl}(C) = \text{id}$,
(c) for $j < i$, $f_j \subseteq f_i$.

Clearly this suffices.

We let $f_0 = \text{id}_{\text{scl}(C)}$ and for limit $i$, $f_i = \bigcup_{j<i} f_j$. Clearly these are as required.

So suppose we have chosen $f_i$, we need to find $c_i$ and $f_{i+1}$. There is an automorphism $F$ of $\mathfrak{M} \upharpoonright (\rho - \{\log\})$ such that $f_i \subseteq F$. Let $b' = F(b_i)$. By (*) there is an automorphism $f'$ of $\text{scl}(C \cup \bigcup_{j<i} b_j \cup b') \upharpoonright (\rho - \{\log\})$ such that $f_i^{-1} \upharpoonright \text{scl}(C \cup \bigcup_{j<i} b_j) \subseteq f'$ and $f'(b') = b'$. Thus $t^e(b'/\text{scl}(C \cup \bigcup_{j<i} b_j)) = t^e(b_i/\text{scl}(C \cup \bigcup_{j<i} b_j))$. Since $b' \downarrow_{\text{scl}(C \cup \bigcup_{j<i} b_j)} \bigcup_{j<i} b_j$ and $b_i \downarrow_{\text{scl}(C \cup \bigcup_{j<i} b_j)} \bigcup_{j<i} a_j$, by Lemma 4.6, there is an isomorphism $f'' : \text{scl}(C \cup \bigcup_{j<i} b_j \cup b' \cup \bigcup_{j<i} a_j) \upharpoonright (\rho - \{\log\}) \rightarrow \text{scl}(C \cup \bigcup_{j<i} b_j \cup b_i \cup \bigcup_{j<i} a_j) \upharpoonright (\rho - \{\log\})$ such that $f'' \upharpoonright \text{scl}(C \cup \bigcup_{j<i} b_j \cup \bigcup_{j<i} a_j) = \text{id}$ and $f''(b') = b_i$. Let $F'$ be an automorphism of $\mathfrak{M} \upharpoonright (\rho - \{\log\})$ such that $f'' \subseteq F'$. Let $G = F' \circ F$. Since $G(\mathfrak{A})$ is $\text{ae}$-saturated, there is $c' \in G(\mathfrak{A})$ such that $t^e(c'/\text{scl}(C \cup \bigcup_{j<i} b_j \cup b_i \cup \bigcup_{j<i} a_j)) = t^e(a_i/\text{scl}(C \cup \bigcup_{j<i} b_j \cup b_i \cup \bigcup_{j<i} a_j))$. Then we let $c_i = G^{-1}(c')$ and $f_{i+1}$ be such that $f_{i+1} \upharpoonright \text{scl}(C \cup \bigcup_{j<i} b_j \cup b_i \cup \bigcup_{j<i} a_j) = G \upharpoonright \text{scl}(C \cup \bigcup_{j<i} b_j \cup b_i \cup \bigcup_{j<i} a_j)$ and $f_{i+1}(c_i) = a_i$. Clearly these are as wanted.

Lemma 5.8. For all $A$ there exists a model $\mathfrak{A}$ such that it is $\text{ae}$-primary over $A$.

Proof. Immediate by Lemmas 4.7 and 5.5.

Lemma 5.9. Suppose $B, C \subseteq A$, $t(a/A \cup \{b\}) \in F^{\text{ae}}(B)$ and $t(b/A) \in F^{\text{ae}}(C)$. Then $t(b/A \cup \{a\}) \in F^{\text{ae}}(C)$.

Proof. By the definition of $F^{\text{ae}}$, we may assume that $A$ is countable. Suppose $t^e(c/C) = t^e(b/C)$. For a contradiction, suppose $t^e(c/A \cup \{a\}) \neq t^e(b/A \cup \{a\})$ (i.e. $c \not\equiv_A a$). As before, we can find $a' \in \mathfrak{M}$ such that $t^e((a',b)/A) = t^e((a,c)/A)$. Then $t^e(a'/B) = t^e(a/B)$ but $t^e(a'/A \cup \{b\}) \neq t^e(a/A \cup \{b\})$, a contradiction.
LEMMA 5.10. Let \((A, (a_i, B_i)_{i<\alpha})\) be an ae-construction and \(s: \beta \to \alpha\) be one-one and onto. If for all \(i < \beta\), \(B_{s(i)} \subseteq A \cup \bigcup_{j<i} a_{s(j)}\), then \((A, (a_{s(i)}, B_{s(i)})_{i<\beta})\) is an ae-construction.

PROOF. Let \(i < \beta\). For all \(j \leq \alpha\), we write \(D_j = A \cup B_{s(i)} \cup \bigcup\{a_{s(k)}\mid k < i, s(k) < j\}\). By induction on \(j \leq \alpha\), we show that \(t(a_{s(i)}, D_j) \in F^{ae}(B_{s(i)})\). This is enough, since \(D_{\alpha} = A \cup \bigcup_{k<i} a_{s(k)}\).

If \(j \leq s(i) + 1\), then \(D_j \subseteq A \cup \bigcup_{k<s(i)} a_k\) and so the claim clear. If \(j\) is limit, then the claim follows from the induction assumption (using the definition of ae-isolation, not the equivalent forms from Lemma 5.2). So assume \(j = k + 1\) and \(k > s(i)\). We may also assume that \(D_j = D_k \cup \{a_k\}\), since otherwise there is nothing to prove. Then there is \(m < i\) such that \(s(m) = k\). By the assumption on \(s, B_k \subseteq D_k\). Then \(t(a_k, D_k \cup a_{s(i)}) \in F^{ae}(B_k)\). By the induction assumption and Lemma 5.9, the claim follows.

LEMMA 5.11. Assume \(\mathfrak{A}\) is ae-primary over \(A\) and \((a_i)_{i<\alpha} \subseteq \mathfrak{A}\) and countable \(B \subseteq A\) are such that

(a) for \(j < i < \alpha\), \(a_j \neq a_i\) and \(t^e(a_j/B) = t^e(a_i/B)\),
(b) for all \(i < \alpha\), \(A \upharpoonright \{a_j\mid j < i\} \subseteq \mathfrak{A}\),
(c) for all finite \(I \subseteq \alpha\), \(t(\bigcup_{i \in I} a_i/A) \in F^{ae}(B')\) for some \(B' \subseteq B\).

Suppose for all sets \(D \subseteq scl(A)\) of power \(\leq \omega_1\) and countable \(C' \subseteq scl(A)\), there are a countable \(C \subseteq scl(A)\) and \(b_i \in scl(A)\), \(i < \omega_1\) such that

(i) \(D \subseteq scl(C \cup \bigcup_{i<\omega_1} a_i)\) and \(C' \subseteq C'\),
(ii) for all \(i < \omega_1\), if \(g\) is an automorphism of \(scl(C \cup \bigcup_{j<i} b_j) \upharpoonright (\rho - \{log\})\) such that \(g \upharpoonright C \cup \bigcup_{j<i} b_j = id\), then there is an automorphism \(g'\) of \(scl(C \cup \bigcup_{j<i} b_j) \upharpoonright (\rho - \{log\})\) such that \(g \subseteq g'\) and \(g(b_i) = b_i\).

Then \(\alpha \leq \omega_1\).

PROOF. For a contradiction, suppose \(\alpha = \omega_1\). Then we may also assume that \(|A| = \omega_1\). Let \(\mathfrak{B}\) be a model such that it is ae-primary over \(A \cup \{a_i\mid i < \omega\}\).

By Lemma 5.10, we can find an ae-constructible set \(A'\) over \(A\) such that for all \(i < \omega_1, a_i \in A'\) and the length of the construction of \(A'\) is \(\omega_1\).

So by Lemma 5.7, there is an automorphism \(f\) of \(\mathfrak{M} \upharpoonright (\rho - \{log\})\) such that \(f(A') \subseteq \mathfrak{B}\) and \(f \upharpoonright scl(B) = id\) and \(f(scl(A)) = scl(A)\).

CLAIM 5.11.1. For all \(i < \omega_1\), \(f(a_i) \upharpoonright \mathfrak{A} \{a_j\mid j < \omega\}\).

PROOF. By Lemma 5.6 (i), there is \(C \subseteq A \cup \{a_j\mid j < \omega\}\) such that

(*) \(t(f(a_i)/A \cup \{a_j\mid j < \omega\}) \in F^{ae}(C)\).

Let \(j < \omega\) be such that for all \(k < \omega_1\), if \(a_k \in C\), then \(k < j\). If \(f(a_i) \upharpoonright \mathfrak{A} \{a_k\mid k < \omega\}\), then for all \(j < k < \omega\), \(t^e(f(a_i)/C) = t^e(a_k/C)\). Clearly this contradicts (*).

By Claim 5.11.1, Lemma 4.3 and the pigeon-hole principle, we can find finite \(I \subseteq \omega\) and \(J \subseteq \omega_1\) of power \(\omega_1\) such that for all \(j \in J\), \(f(a_j) \upharpoonright \mathfrak{A} \upharpoonright \{a_i\mid i \in I\}\). Since for all \(j \in J\), \(f(a_j) \upharpoonright \mathfrak{A} \{a_k\mid k < j, k \in J\}\), this contradicts the basic properties of \(\mathfrak{M}\) and the fact that \(\kappa(\mathfrak{M}) < \omega_1\).

LEMMA 5.12. Suppose \(A\) is ae-saturated and \(B\) is ae-primary over \(A \cup \{a\}\). If \(b \upharpoonright a\), then \(b \upharpoonright A B\).

PROOF. By Lemma 5.6, this can be proved essentially as Lemma 5.3. □
6. Non-structure of $\mathcal{R}^e$

In this section we prove non-structure theorems for the class $\mathcal{R}^e = \{ A \upharpoonright \rho - \{ \log \} \upharpoonright A \in \mathcal{R} \}$.

**Definition 6.1.** We say that $t(a/B)$ is orthogonal to $A \subseteq B$ if for all $C \supseteq B$ and $b$ the following holds: If $a \upharpoonright_B C$ and $b \upharpoonright_A C$, then $a \upharpoonright_C b$.

**Lemma 6.2.** $\mathcal{R}^e$ has 'dop' (see e.g. [S5]) i.e. there are $A_i \subseteq M$, $i < 4$, and $a \in M$ such that

(i) for all $i < 4$, $A_i = scl(A_i)$,

(ii) $A_0 = A_1 \cap A_2$, $A_1 \upharpoonright_{A_0} A_2$ and $A_3 = scl(A_1 \cup A_2)$,

(iii) $a \not\in A_3$ and $t(a/A_3)$ is orthogonal to both $A_1$ and $A_2$.

Furthermore, these can be chosen so that $A_3$ is countable and there are $a_i \in A_i$, $i \in \{1,2\}$ such that if $f$ is an automorphism of $A_3 \upharpoonright (\rho - \{ \log \})$, $f \upharpoonright A_0 = id$ and $f(a_i) = a_i$ for $i = 1,2$, then there is an automorphism $g$ of $scl(a \cup A_3) \upharpoonright (\rho - \{ \log \})$ such that $f \subseteq g$ and $g(a) = a$.

**Proof.** Choose $A_i$, $i < 4$, so that (i) and (ii) hold and that there are elements $a_i \in A_i - A_0$ for $i \in \{1,2\}$ such that $R(a_i/A_0) = 2$ (if wanted one can let $A_0$ be countable and $A_i = scl(A_0 \cup a_i)$ for $i = 1,2$ but this is not needed until in the proof of the furthermore part).

**Claim 6.2.1.** There is a $a \not\in A_3$ such that $exp(a) = a \cdot exp(a_1) \cdot exp(a_2)$.

**Proof.** Choose $a$ so that $a \not\in A_3$ and choose for rationals $q > 0$, $b_q$ so that $b_1 = a \cdot exp(a_1) \cdot exp(a_2)$ and for all integers $z > 0$, $(b_q)^z = b_{q^z}$. For negative rationals $q$, let $b_q = (b_{-q})^{-1}$. Let $A$ be the relevant set generated by $A_3 \cup \{ a \}$ and define $f : A \to M$ so that for all $b \in A_3$ and rationals $q \neq 0$, $f(b + q \cdot a) = exp(b) \cdot b_q$. It suffices to prove that $f$ is a homomorphism from $(A, +)$ to $(M - \{ 0 \}, \cdot)$ and that the kernel of $f$ is $\pi Z$.

It is routine to check that $f$ is a homomorphism. So we only need to show that $Ker(f) = \pi Z$. For this, let $b \in A_3$ and $q \in Q$ be such that $f(b + q \cdot a) = 1$. It suffices to show that $q = 0$. Suppose not. Then $exp(b) \cdot b_q \in A_3$. Thus $b_q \in A_3$. Since $A_3$ is algebraically closed, $a \in A_3$, a contradiction.

**Claim 6.2.2.** Let $a$ be as in Claim 6.2.1. Then $t(a/A_3)$ is orthogonal to both $A_1$ and $A_2$.

**Proof.** Let $A \supseteq A_3$ be such that $scl(A) = A$ and $a \upharpoonright_{A_3} A$. Then $R(a/A) = 1$. Let $b$ be such that $b \upharpoonright_{A_3} A$. We need to prove that $a \upharpoonright_A b$.

Let $B = scl(A \cup b)$. We show first that $a, exp(a) \not\in B$. For this it suffices to prove the following subclaim.

**Subclaim 6.2.2.1.**

(i) Let $B$ be the (closed) relevant set generated by $A \cup scl(A_1 \cup b)$. Then $a$ and $exp(a)$ are transcendental over $B \cup exp[B]$.

(ii) Suppose $C \supseteq B$ is a closed relevant set and $a$ and $exp(a)$ are transcendental over $C \cup exp[C]$. Let $c$ be such that $exp(c)$ is algebraic over $C \cup exp[C]$ and $D$ the relevant set generated by $C \cup \{ c \}$. Then $a$ and $exp(a)$ are transcendental over $D \cup exp[D]$.

(iii) Suppose $C \supseteq B$ is a closed relevant set and $a$ and $exp(a)$ are transcendental over $C \cup exp[C]$. Let $c$ be such that it is algebraic over $C \cup exp[C]$ and $D$ the relevant set generated by $C \cup \{ c \}$. Then $a$ and $exp(a)$ are transcendental over $D \cup exp[D]$. 

Proof. (i): We show first that $a \not\in B$. For a contradiction, suppose $a = c + d$ for some $c \in \text{scl}(\mathfrak{A}_1 \cup \{ b \})$ and $d \in \mathfrak{A}$. Then

\[ \exp(c) \cdot \exp(d) = a \cdot \exp(a_1) \cdot \exp(a_2) = c \cdot \exp(a_1) \cdot \exp(a_2) + d \cdot \exp(a_1) \cdot \exp(a_2). \]

So $c$ and $\exp(c)$ realize a linear equation over $\mathfrak{A}$. Since $\text{scl}(\mathfrak{A}_1 \cup \{ b \}) \subseteq \mathfrak{A}_1$, $c$ and $\exp(c)$ realize a linear equation over $\mathfrak{A}_1$ i.e. there are $p, r \in \mathfrak{A}_1$ such that $\exp(c) = p \cdot c + r$. Then (1) gives

\[ c \cdot (\exp(a_1) \cdot \exp(a_2 - d) - p) + d \cdot \exp(a_1) \cdot \exp(a_2 - d) - r = 0. \]

Since $c \not\in \mathfrak{A}$, (2) implies that

\[ \exp(a_1) \cdot \exp(a_2 - d) - p = 0. \]

This means that $\exp(a_2 - d) \in \mathfrak{A}_1$ i.e. $a_2 - d \in \mathfrak{A}_1$. So we may assume that $d = a_2$. Then

\[ c = a - a_2 \quad \text{and (by (1)), } \exp(c) = a \cdot \exp(a_1). \]

Clearly these do not realize any linear equation over $\mathfrak{A}_1$, a contradiction.

But then $a$ is transcendental over $B \cup \exp[B]$ since otherwise

\[ \text{deg}((a, \exp(a))/B \cup \exp[B]) = 0 \]

and $\dim(a/B) = 1$ contradicting the fact that $B$ is closed. Similarly, $\exp(a)$ is transcendental over $B \cup \exp[B]$.

(ii): Again we show first that $a \not\in D$. For a contradiction, suppose there are $d \in C$ and $\eta \in \mathbb{Q}$ such that $a = d + \eta c$. Then $'\exp(a) = \exp(d) \cdot \exp(c)'$. This is impossible since by our assumptions, $'\exp(d) \cdot \exp(c)'$ is algebraic over $C \cup \exp[C]$. Since $D$ is closed, as one can easily see, the claim follows as in (i).

(iii): The proof goes essentially as in (ii) (but is easier). □

Now we continue as in the proof of Lemma 4.7: By Lemma 4.3, if $a \not\in \mathfrak{B}$, then $\text{scl}(\mathfrak{A} \cup \{ a \}) \cap \mathfrak{B} \not\subseteq \mathfrak{A}$. For a contradiction, suppose that this is the case. Then we can find $d_i \in \text{scl}(\mathfrak{A} \cup \{ a \})$, $i \leq n$, and $k < n$ such that

(a) $(d_i)_{i \leq n}$ is linearly independent,

(b) if $D$ is the relevant set generated by $\{ d_0, \ldots, d_n \}$, then $D \cap \mathfrak{A}$ is generated by $\{ d_0, \ldots, d_{k-1} \}$,

(c) if $C$ is the relevant set generated by $\{ d_0, \ldots, d_{n-1} \}$, then $C \cap \mathfrak{B} = C \cap \mathfrak{A}$,

(d) $D \cap \mathfrak{A}$ is closed,

(e) $d_k = a$ and for all $k < i \leq n$, either $d_i$ or $\exp(d_i)$ is algebraic over $\{ d_0, \ldots, d_{i-1}, \exp(d_0), \ldots, \exp(d_{i-1}) \}$,

(f) $d_n, \exp(d_n) \in \mathfrak{B}$.

Then

\[ \text{deg}((d_k, \ldots, d_{n-1}, \exp(d_k), \ldots, \exp(d_{n-1}))/\mathfrak{B}) < \]

\[ \text{deg}((d_k, \ldots, d_{n-1}, \exp(d_k), \ldots, \exp(d_{n-1}))/\mathfrak{A}) = n - k \]

and $\dim((d_k, \ldots, d_{n-1}))/\mathfrak{B}) = n - k$. This contradicts the fact that $\mathfrak{B}$ is closed. □

We are left to prove the furthermore part. For this it suffices to show that if $b$ is transcendental over $\mathfrak{A}_3$ and $\exp(b) = b \cdot \exp(a_1) \cdot \exp(a_2)$, then $t^e(b/\mathfrak{A}_3) = t^e(a/\mathfrak{A}_3)$. Since the proof of this is exactly as the proof of Lemma 4.6 (notice that the relevant set generated by $\{ a_1, a_2, a \}$ is closed and $R$-witness for $(a, \mathfrak{A}_3)$, $a \in M(B \cup \{ \exp(a) \})$ and $\exp(a)$ is transcendental over $\mathfrak{A}_3$), we skip the proof. □

We can now prove an easy non-structure theorem.

Proposition 6.3. For all regular uncountable cardinals $\kappa$, there are $2^\kappa$ pairwise non-isomorphic models of power $\kappa$ in $\mathcal{R}^\circ$. 

□
PROOF. Let $I = (I, \prec)$ be a linear ordering and choose $\mathcal{A}_i$, $i < 4$, $a_i \in \mathcal{A}_i - \mathcal{A}_0$, $i \in \{1, 2\}$ and $a$ as in (the proof of) Lemma 6.2. Clearly we may assume in addition that $\mathcal{A}_0 = \mathcal{A}_\text{pr}$ and for $i \in \{1, 2\}$, $\mathcal{A}_i = \text{scl}(\mathcal{A}_0 \cup \{a_i\}) (= \text{scl}(\{a_i\}))$. For all $i \in I$, choose $b_i$ and $c_i$ so that for all $i \in I$,

(i) $(b_i, c_i) \vdash_{\mathcal{A}_0} \{b_j, c_j \mid j \prec i\}$,
(ii) there is an automorphism $F$ of $\mathcal{M}$ such that $F \upharpoonright \mathcal{A}_0 = \text{id}$, $F(b_i) = a_1$ and $F(c_i) = a_2$.

By homogeneity, we may choose these so that
(iii) $((b_i, c_i))_{i \in I}$ is order indiscernible over $\mathcal{A}_0$.

By Lemma 4.6 and (i) above, for all $i, j \in I$, $i \neq j$,
(iv) there is an isomorphism

$$f : \text{scl}(\mathcal{A}_0 \cup \{b_i, c_j\}) \upharpoonright (\rho - \{\text{log}\}) \to \text{scl}(\{a_1, a_2\}) \upharpoonright (\rho - \{\text{log}\})$$

such that $f \upharpoonright \mathcal{A}_0 = \text{id}$, $f(b_i) = a_1$ and $f(c_j) = a_2$.

So for all $i, j \in I$, $i < j$, we can find $d_{ij}$ so that $d_{ij} \not\in \text{scl}(\{b_i, c_j\})$ and $\text{exp}(d_{ij}) = d_{ij} \cdot \text{exp}(b_i) \cdot \text{exp}(c_j)$. Clearly these can be chosen so that for all $i < j$,

$$d_{ij} \vdash_{\text{scl}(\{b_i, c_j\})} \{b_k, c_k \mid k \in I\} \cup \{d_{kl} \mid k < l, (i, j) \neq (k, l) \in I^2\}$$

(in fact we can not choose the elements in any other way). Again by homogeneity, we can find these so that in addition,

(v) for all partial isomorphism $g : I \to I$ there is an automorphism $f$ such that $f \upharpoonright \mathcal{A}_0 = \text{id}$, for all $i \in \text{dom}(g)$, $f(b_i) = b_{g(i)}$ and $f(c_i) = c_{g(i)}$ and for all $i, j \in I$, $i < j$, $f(a_{ij}) = a_{g(i)g(j)}$.

We let $\mathcal{A}_I$ be $\text{scl}(\{b_i, c_i \mid i \in I\} \cup \{a_{ij} \mid i, j \in I, i < j\})$.

CLAIM 6.3.1. For all $i, j \in I$, $i < j$ iff there is $d \in \mathcal{A}_I$ such that $d \not\in \text{scl}(\{b_i, c_j\})$ and $\text{exp}(d) = d \cdot \text{exp}(b_i) \cdot \text{exp}(c_j)$.

PROOF. If $i < j$, then clearly $d$ exists. So assume that $j \leq i$. Let $d (\in \mathcal{M})$ be such that $d \not\in \text{scl}(\{b_i, c_j\})$ and $\text{exp}(d) = d \cdot \text{exp}(b_i) \cdot \text{exp}(c_j)$. We show that $d \not\in \mathcal{A}_I$.

Let $A = \langle b_k, c_l \mid k, l \in I, k \neq i, l \neq j \rangle \cup \{d_{kl} \mid k, l \in I, k < l, k \neq i, l \neq j\}$. Then $A \vdash_{\mathcal{A}_0} \{b_i, c_j\}$. Since $t(d/\text{scl}(\{b_i, c_j\}))$ is orthogonal to $\mathcal{A}_0$, $d \not\in \text{scl}(\{b_i, c_j\} \cup A)$. Denote $A_b = \text{scl}(\{b_i\} \cup A)$ and $A_c = \text{scl}(\{c_j\} \cup A)$. By the basic properties of $\vdash_d$, $A_b \vdash_{\text{scl}(A)} A_c$. Thus, by the proof of Lemma 6.2, $t(d/\text{scl}(\{b_i, c_j\} \cup A))$ is orthogonal to both $A_b$ and $A_c$.

Let $B = \{d_{ik} \mid i < k\}$ and $C = \{d_{kj} \mid k < j\}$. A standard application of the basic properties of our independence notion gives that

(a) $B \vdash_{A_b} \text{scl}(\{b_i, c_j\} \cup A)$,
(b) $C \vdash_{A_c} \text{scl}(\{b_i, c_j\} \cup A \cup B)$.

By (a) and Lemma 6.2, $d \not\in \text{scl}(B \cup A \cup \{b_i, c_j\})$ and $t(d/\text{scl}(B \cup A \cup \{b_i, c_j\}))$ is orthogonal to $A_c$. So by (b), $d \not\in \text{scl}(A \cup B \cup C \cup \{b_i, c_j\}) = \mathcal{A}_I$.

Let $\phi(x, y)$ be the $L_{\omega_1, \omega}$-formula in the similarity type $\rho - \{\text{log}\}$, which says that there is $z$ such that $z \not\in \text{scl}(\{x, y\})$ and $\text{exp}(z) = z \cdot \text{exp}(x) \cdot \text{exp}(y)$. Let $\psi(x_0, x_1, y_0, y_1) = \phi(x_0, y_1) \land \neg\phi(x_1, y_0)$. By Claim 6.3.1 and (v) above, it is easy to see that if $I$ is a dense linear ordering, then $(b_i, c_i)_{i \in I}$ is weakly $(\omega, \psi)$-skeleton like in $\mathcal{A}_I$ (see [S6]). Thus the claim in our proposition follows immediately from [S6] (see e.g [H2] where this kind of argument is done in details).
Since the models in Proposition 6.3 are not necessarily existentially closed in the class $\mathfrak{R}$, the proposition is not completely satisfactory. So we try to do better.

**Lemma 6.4.** For all linear orderings $\eta = (\eta, \prec)$, there is an $\text{ae}$-saturated model $\mathfrak{A} = \mathfrak{A}_\eta$ for which the following holds: There are elements $b_i, c_i \in \mathfrak{A} - \text{scl}()$, $i \in \eta$, such that

(i) $b_i \downarrow^d \{b_j, c_j \mid j < i\}$, $c_i \downarrow^d \{b_j, c_j \mid j < i\} \cup \{b_i\}$ and $R(b_i/\text{scl}()) = R(c_i/\text{scl}()) = 2$,

(ii) there are sets $I_{ji} = \{a_{ji}^i \mid k < \omega_1\}$, $i, j \in \eta$ and $j < i$, such that

(a) $\mathfrak{A}$ is $\text{ae}$-primary over $\{b_i, c_i \mid i \in \eta\} \cup \{I_{ji} \mid j, i \in \eta, j < i\}$,

(b) if $g$ is a partial order-preserving function from $\eta$ to $\eta$, then there is an automorphism $f$ of $\mathfrak{M}$ such that for all $j < i$ from $\text{dom}(g)$ and $k < \omega_1$, $f(b_j) = b_{g(j)}$, $f(c_j) = c_{g(j)}$ and $f(a_{ji}^i) = a_{g(j)g(i)}^i$ and if $g \uparrow J = \text{id}$, $J \subseteq \text{dom}(g)$, then $f \uparrow \text{scl}() \{b_i, c_i, a_{ji}^i \mid i, j \in J, k < \omega_1\} = \text{id}$,

(c) for all $j < i$, $I_{ji} \downarrow^d \{b_k, c_k \mid k \in \eta\} \cup \{I_{kl} \mid k < l, (k, l) \neq (j, i)\}$,

(iii) $j < i$ iff $(*)_{ji}$ below holds:

$(*)_{ji} = (\forall)_{c, j, b_i}^\mathfrak{A}$: there are $a_k \in \mathfrak{A}$, $k < \omega_1$, such that

(a) $a_k \not\in \text{scl}() \{c_j, b_i\}$,

(b) $a_k \downarrow^d \text{scl}() \{c_j, b_i\}$ \{ap \mid p < k\},

(c) $\exp(a_k) = a_k \cdot \exp(c_j) \cdot \exp(b_i)$,

(d) for $k < l < \omega_1$, $t^e(a_k/\text{scl}() \{c_j, b_i\}) = t^e(a_l/\text{scl}() \{c_j, b_i\})$.

**Proof.** First, we want to choose elements $b_i, c_i \in \mathfrak{M} - \text{scl}()$, $i \in \eta$, so that (i) holds and for all $j < i$, we want to choose choose $I_{ji} = \{a_{ji}^i \mid k < c^+\}$ so that (a)-(d) from (iii) hold for these elements. In addition, we want to choose these so that (ii) (c) holds. If $\eta$ is an ordinal, it is easy to see that such elements can be found. But then by Erdős-Rado-theorem (and homogeneity of $\mathfrak{M}$), we can find these for any linear ordering $\eta$ and in addition so that (ii) (b) holds. Letting $\mathfrak{A}$ be an $\text{ae}$-primary model over $\{b_i, c_i \mid i < \kappa\} \cup \{I_{ji} \mid j < i < k\}$, (i) and (ii) hold. We show that also (iii) holds.

If $j < i$, then clearly $(*)_{ji}$ holds. So suppose $i \leq j$ and for a contradiction, assume that $(*)_{ji}$ holds. Let $a_k, k < \omega_1$, witness the assumption that $(*)_{ji}$ holds. Let $A = \{c_k \mid k \in \eta, k \neq i\} \cup \{b_k \mid k \in \eta, k \neq i\} \cup \{I_{kl} \mid k < l, k \neq j, l \neq i\}$, $A_c = \text{scl}(A \cup \{c_j\})$ and $A_b = \text{scl}(A \cup \{b_i\})$. Then it is easy to see that $A_c \downarrow^d \text{scl}(A) A_b$. By also (the proof of) Lemma 6.2, for all $k < \omega_1$, $t(a_k/\text{scl}() \{c_j, b_i\})$ is orthogonal to $\mathfrak{A}_0$ and since $A \downarrow^d_\mathfrak{A} \{c_j, b_i\}$, it is easy to see that

(iv) \{ak \mid k < \omega_1\} \downarrow^d \text{scl}(A \cup \{c_j, b_i\}) \text{scl}(A \cup \{c_j, b_i\})).

Thus by applying the proof of Lemma 6.2 again, we see that (v) for all $k < \omega_1$, $t(a_k/\text{scl}(A \cup \{c_j, b_i\})$ is orthogonal to both $A_c$ and $A_b$. Let $C = \cup I_{jl} \mid j < l\}$ and $B = \cup I_{kl} \mid k < i\}$. Then $C \downarrow^d A_c \text{scl}(A_c \cup A_b)$ and $B \downarrow^d A_b \text{scl}(C \cup A_c \cup A_b)$. From this and (v) a routine argument gives

(vi) \{ak \mid k < \omega_1\} \downarrow^d \text{scl}(A_c \cup A_b \cup C \cup B).

So by (iv) and (vi), for all $k < \omega_1$, ak \downarrow^d \text{scl}(A_c \cup A_b \cup C \cup B) \{ap \mid p < k\}. Also the proofs of (iv) and (vi) imply that for all finite $J \subseteq \omega_1$, $t(\cup k \in J ak/\text{scl}(A_c \cup A_b \cup C \cup B)) \in F^e(\{c_j, b_i\})$. So letting $B = \text{scl}() \{c_j, b_i\}$, the requirements (a)-(c) from Lemma 5.11 hold.
Let $D, C' \subseteq \text{scl}\{\{b_i, c_i, a_k^{ji} \mid i \in \eta, k < \omega_1, j \in \eta, j < i\}\}$ be such that $D$ is of power $\leq \omega_1$ and $C'$ is countable. For a contradiction, it suffices to find $C$ and $d_i$, $i < \omega_1$, so that (i) and (ii) from Lemma 5.11 hold.

There is $J \subseteq \eta$ of power $\omega_1$ such that $D, C' \subseteq \text{scl}\{\{b_i, c_i, a_k^{ji} \mid i \in J, k < \omega_1, j \in J, j < i\}\}$. We let $d_i$, $i < \omega_1$, be such that

(a) $\{d_i \mid i < \omega_1\} = \{b_i, c_i, a_k^{ji} \mid i \in J, k < \omega_1, j \in J, j < i\}$,

(b) if $d_i = a_k^{ji}$, then there are $i', i'' < i$ such that $d_{i'} = c_j$ and $d_{i''} = b_j$.

Then we let $C = \text{scl}(\cup_{i<i^*} d_i)$, where $i^* < \omega_1$ is large enough so that $C' \subseteq C$ (notice that $\text{scl}(C \subseteq \text{scl}(C)$ always). Then clearly (i) from Lemma 5.11 hold. Let $i < \omega_1$ and $g$ an automorphism of $\mathfrak{B} = \text{scl}(C \cup \bigcup_{j<i} d_j) \upharpoonright (\rho - \{\log\})$ such that $g \upharpoonright \text{id}$ and $g(d_j) = d_j$ for all $j < i$. We need to find $g'$ as in (ii) from Lemma 5.11.

We assume that $d_i = a_k^{ji'}$ for some $j, j' < J$ and $k < \omega_1$. The other case is immediate by Lemma 4.6. Let $G$ be an automorphism of $\mathfrak{M} \upharpoonright (\rho - \{\log\})$ such that $g \subseteq G$. Let $\mathfrak{C} = \text{scl}(\{c_j, b_{j'}\}) \subseteq \mathfrak{B}$. Then $g \upharpoonright \mathfrak{C}$ is an automorphism of $\mathfrak{C} \upharpoonright (\rho - \{\log\})$, $g \upharpoonright \text{scl}(\{c_j, b_{j'}\}) = i_d, g(c_j) = c_j$ and $g(b_{j'}) = b_{j'}$. Thus by Lemma 6.2, there is an automorphism $F$ of $\mathfrak{M} \upharpoonright (\rho - \{\log\})$ such that $F \upharpoonright \mathfrak{C} = g$ and $F(d_i) = d_i$. Let $d_i' = G(d_i)$. Then $F^{-1} \circ G$ show that $t^*(d_i'/\mathfrak{C}) = t^*(d_i/\mathfrak{C})$. Since $d_i \upharpoonright \mathfrak{B} \upharpoonright \mathfrak{C}$ and $d_i \upharpoonright \mathfrak{B}$, By Lemma 4.6, there is an automorphism $G'$ of $\mathfrak{M} \upharpoonright (\rho - \{\log\})$ such that $G' \upharpoonright \text{id}$ and $G'(d_i) = d_i'$. Then $g' = (G' \circ G) \upharpoonright \text{scl}(\mathfrak{B} \cup d_i)$ is as required in (ii) from Lemma 5.11.

**Lemma 6.5.** There is a formula $\Phi(x, y)$ in the similarity type $\rho - \{\log\}$ such that for all submodels $\mathfrak{A} = \text{scl}(\mathfrak{A})$ of $\mathfrak{M}$ and $c, b \in \mathfrak{A}$, $\mathfrak{A} \models \Phi(c, b)$ iff $\{c, b\}$ is closed and $(*)^{c, b}_{c, b}$ holds. Furthermore for all submodels $\mathfrak{B} \subseteq \mathfrak{C}$ of $\mathfrak{M}$, if $\text{scl}(\mathfrak{B}) = \mathfrak{B}$ and $\mathfrak{B} \models \Phi(c, b)$, then $\mathfrak{C} \models \Phi(c, b)$.

**Proof.** Clearly 'is $\{x, y\}$ a closed set?' can be decided in $\mathfrak{M} \upharpoonright (\rho - \{\log\})$ (and it is expressible in $\text{L}_{\omega_1\omega}$). Also it is easy to see that if $\{c, b\}$ is closed and $\text{scl}(\mathfrak{A}) = \mathfrak{A}$ then $(*)^{c, b}_{c, b}$ is expressible with an $\text{L}_{\omega_1\omega}$-formula in $\mathfrak{A}$ and in the similarity type $\rho - \{\log\}$ (notice that by Lemma 4.3, if $\{c, b\}$ is closed, then for $a_k \upharpoonright \text{scl}(\{c, b\}) \{a_p \mid p < k\}$ it suffices to say that if $z \in \text{scl}(\{c, b\} \cup \{a_p \mid p < k\}) \cap \text{scl}(\{c, b, a_k\})$ then $z \in \text{scl}(\{c, b\})$ and that for $X = \{c, b\}$ or $X = \{c, b, a_k\}$ or $X = \{c, b\} \cup \{a_p \mid p < k\}$, $\text{scl}(X)$ is the least $Y \subseteq \mathfrak{A}$ such that $X \subseteq Y$, $Y$ is closed under exp and $\exp^{-1}$ and $Y$ is algebraically closed subset of $F_{\mathfrak{A}}$). The furthermore part is immediate by the definition of $(*)^{c, b}_{c, b}$.

**Theorem 6.6.** Suppose $\kappa = \kappa^\omega > \omega_2$ is a regular cardinal. Then there are $\text{ae}$-saturated models $\mathfrak{A}_i \in \mathfrak{A}$, $i < 2^\kappa$, of power $\kappa$ such that for all $i < j < 2^\kappa$, $\mathfrak{A}_i \upharpoonright (\rho - \{\log\}) \not\equiv \mathfrak{A}_j \upharpoonright (\rho - \{\log\})$.

**Proof.** Let $\eta$ be a linear ordering and $\mathfrak{A}_\eta$ as in Lemma 6.4. Now the claim follows from [S6] (again, see e.g. [H2]), if we can show that that assuming $\eta$ is $\omega_1$-dense, then $\{(b_i, c_i) \mid i \in \eta\}$ is weakly $\omega_1$-$\text{ae}$-constructible over $a \cup S$, where $S = \{b_i, c_i, a_k^{ji} \mid i \in \eta, k < \omega_1, j \in \eta, j < i\}$.

(i) $\mathfrak{A}_\eta$ is $\text{ae}$-constructible over $a \cup S$, where $S = \{b_i, c_i, a_k^{ji} \mid i \in \eta, k < \omega_1, j \in \eta, j < i\}$.

(ii) $b, c \in a$,
(iii) letting $S' = \{ b_i, c_i, a^j_k \mid i \in J, k \in J', j \in J, j < i \}$, then $t(a/S) \in F^{ae}(S')$.

We claim that $J$ is as required in the definition of weakly skeleton like.

For this suppose that $n, m \in \eta - J$ are such that for all $i \in J$, $i < n$ iff $i < m$ and $\Phi(c_n, b)$ holds. We need to show that $\Phi(c_m, b)$ holds (the other cases are similar).

By Lemma 6.4 and the assumption that $\eta$ is $\omega_1$-dense, we can find $J^* \subseteq \eta$ and $D$ such that

(iv) $J^*$ has power $\omega_1$, $J \subseteq J^*$ and $m \in J^*$,

(v) $D$ is ae-constructible over $a \cup S^*$ and the length of the construction is $\omega_1$, where $S^* = \{ b_i, c_i, a^j_k \mid i \in J^*, k < \omega_1, j \in J^*, j < i \}$,

(vi) $\Phi(c_m, b)$ holds in $\text{scl}(D)$.

Since $A_{\eta}$ is $ae$-saturated, by Lemma 5.7, it suffices to prove that there are countable $C \subseteq \text{scl}(a \cup S^*)$ and $d_i \in \text{scl}(a \cup S^*)$, $i < \omega_1$, such that (*) from Lemma 5.7 holds for these, $\text{scl}\{\{b, c_m\}\} \subseteq C$ and $a \cup S^* \subseteq \text{scl}\{C \cup \bigcup_{i<\omega_1} d_i\}$.

For this, we let $C = \text{scl}\{\{a, c_m\} \cup S^*\}$ and choose $d_i$ so that

(a) $\{d_i \mid i < \omega_1\} = S^*$,

(b) if $d_i = a^j_k$, then there are $i', i'' < i$ such that $d_{i'} = c_j$ and $d_{i''} = b_j$.

The proof that these are as required is the same as the related proof inside the proof of Lemma 6.4.

\[\square\]

**Remark 6.7.** The ideas behind the construction in the proof of Theorem 6.6 can be used to get non-structure theorems for $\mathcal{R}^e$ that are much stronger than Theorem 6.6, see [HT] or [S3] and e.g. [FR] and [HR].

**References**


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Definability, semidefinability, and asymptotic structure in analysis

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Abstract. We study the notions of definability and semidefinability of types (in the sense of S. Shelah) in a non-first-order context which includes classes of functional-analytic structures. We also present some applications; we exhibit connections between these concepts and the theory of asymptotic structure, and prove a dichotomy which, among other things, sharpens results of G. Androulakis, E. Odell, Th. Schlumprecht, and N. Tomczak-Jaegermann concerning the structure of spreading models of Banach spaces.

0. Introduction

In this paper we study the model theoretic concepts of definability and semidefinition in a non-first-order context, namely, the class of normed space structures introduced by C. W. Henson and the author in [HI02]. Examples of such structures include Banach spaces, C*-algebras, and operator spaces, among many others. (The definition, introduced in [HI02], is quoted in Section 1.) The language used is that of positive bounded formulas and approximate satisfaction, which has been shown to be the "correct" model theoretic language in this context. (See [Iov01].)

We show that, when interpreted in the model theory of normed space structures, concepts such as type definability, semidefinition, and averages, which were introduced by S. Shelah within "pure" model theory [She90] correspond naturally to concepts that have been devised independently in analysis for different purposes. Notably, Shelah's concept of average corresponds precisely to the concept of spreading model, the classical tool for the study of asymptotic structure in Banach space theory.

The last section of the paper is devoted to applications. There, we focus on the Banach space language, and prove the following dichotomy for asymptotic structure: every basic sequence in a Banach space has a block base \((e_n)\) such that \((e_n)\) has either a spreading model isometric to one of the classical sequence spaces \(\ell_p\) and \(c_0\), or has the largest possible number on nonisomorphic spreading models. This follows from a dichotomy for averages proved in earlier sections in combination with previous work by the author on stable types, and sharpens the following result, obtained recently by G. Androulakis, E. Odell, Th. Schlumprecht, and N. Tomczak-Jaegermann [AOSTJ]: if a given Banach space has only one spreading model, then this spreading model must be isomorphic to \(\ell_p\) or \(c_0\). (This answers positively a question of S. Argyros.)
The rest of the paper is organized as follows. In Section 2 we introduce some notational conventions which help us to treat the quantifier-free fragment and the full language simultaneously. In Section 3 we introduce the concept of definability for normed space structures and prove an analog of Beth’s Definability Theorem. In Section 4 we introduce the counterpart in this context of Shelah’s concept of semidefinability. In Section 5 we state a version, for real-valued terms, of the classical Ramsey Theorem. In Section 6 we define the concept of spreading model for normed space structures, highlighting the connection between spreading models and semidefinability. In Section 7 we introduce the concept of heir of a type. The material included in Section 8 is preparatory for Section 9, where two adaptations of Shelah’s 2-rank are introduced and compared. The properties of these ranks are used in Section 10 to prove a dichotomy for spreading models (Corollary 10.4). As mentioned previously, the last section, Section 11, is devoted to applications to Banach space theory.

The exposition should be accessible to logicians as well as analysts (and mathematicians) who have had some exposure to basic model theory or nonstandard analysis. We use the model theoretic framework of [HI02], but familiarity with this apparatus is not strictly required; the concepts and results from [HI02] that will be used are recalled in Section 1, with pointers to the parts of [HI02] where they are discussed in more detail.

Normed space structures provide a concrete and mathematically rich setting in which to work. However, the arguments presented here can be carried out, with relatively minor adjustments, in wider settings, e.g., multi-sorted structures where each sort is a metric space. It should be noted that in these contexts, the approach of positive bounded formulas is equivalent to that of compact abstract theories developed recently by I. Ben-Yaacov [BY03].

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1. Preliminaries

**Normed space structures.** The class of structures studied in this paper is the class of normed space structures introduced by C. W. Henson and the author in [HI02]. The text of Definition 1.1 and the subsequent list of examples (namely, 1.2) is a direct quotation from [HI02, Chapter 2].

1.1. **Definition.** A normed space structure \( \mathcal{M} \) consists of the following items:

1. A family \( \{ M^{(s)} \mid s \in S \} \) of normed spaces.
2. A collection of functions of the form

\[ F: M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)}, \]

each of which is uniformly continuous on every bounded subset of its domain.

---

1The connections between both approaches will be discussed elsewhere.

2We thank Henson for allowing us to reprint the text.
The normed spaces $M^{(s)}$ are called the sorts of $\mathcal{M}$, and the structure $\mathcal{M}$ is said to be based on $(M^{(s)} \mid s \in S)$. If every sort of $\mathcal{M}$ is a Banach space, we say that $\mathcal{M}$ is a Banach space structure.

The functions of $\mathcal{M}$ of arity 0 correspond naturally to elements of the sorts of $\mathcal{M}$, and are called the constants of $\mathcal{M}$.

We require that every normed space structure include a special sort and certain special functions. Namely, if $\mathcal{M}$ is a normed space structure based on $(M^{(s)} \mid s \in S)$,

- The field $\mathbb{R}$ must occur as a distinguished sort of $\mathcal{M}$, i.e., the sort index set $S$ contains a distinguished element $s = s_{\mathbb{R}}$ for which $M^{(s)}$ is $\mathbb{R}$.
- For every $s \in S$, the vector space operations and the norm of $M^{(s)}$ must occur as functions of $\mathcal{M}$, and the additive identity $0_{M^{(s)}}$ of $M^{(s)}$ must occur as a constant of $\mathcal{M}$.

If $\mathcal{M}$ is a normed space structure, an element of $\mathcal{M}$ is an element of a sort of $\mathcal{M}$.

1.2. Examples.

1) Normed spaces $X$ over $\mathbb{R}$: the sorts are $X$ and $\mathbb{R}$, and the functions are the vector space operations, the additive identity $0_X$ and the norm of $X$, as well as the field operations, the additive identity 0 and the absolute value function on $\mathbb{R}$.

2) Normed spaces $X$ over $\mathbb{C}$: these can be regarded as normed space structures in several ways. For example we may add $\mathbb{C}$ as a sort together with its field structure and absolute value, and the scalar multiplication operation as a map from $\mathbb{C} \times X$ into $X$, as well as the inclusion map from $\mathbb{R}$ into $\mathbb{C}$. Alternatively, we may simply include a unary function from $X$ into itself, corresponding to scalar multiplication by $\sqrt{-1}$, in addition to the usual operations that come from regarding $X$ as a normed space over $\mathbb{R}$.

3) Normed vector lattices $(X, \vee, \wedge)$: this is the result of expanding the normed space structure corresponding to $X$ (see above) by adding the lattice operations $\vee$ and $\wedge$ on $X$ and the functions max and min on $\mathbb{R}$.

4) Normed algebras: multiplication is included as an operation; if the algebra has a multiplicative identity, it may be included as a constant.

5) $C^*$-algebras: multiplication and the *-map are included as operations.

6) Hilbert spaces with inner product, in which the pairing is included as a function.

7) Dual pairs $(X, X')$, where $X$ is a Banach space, $X'$ is the dual of $X$, and the pairing between $X$ and $X'$ is included as a function.

8) Triples $(X, X', X''')$, where $X'$ and $X'''$ are the dual and the double dual of $X$ and the pairing between $X$ and $X'$, the pairing between $X'$ and $X'''$, and the embedding $X \rightarrow X'''$ are included as functions.

9) Operator spaces, which include for each $n \geq 1$ a real-valued function of $n^2$ arguments mapping each $n \times n$ matrix $(a_{i,j})$ of elements of the underlying Banach space to its operator norm.

10) If $\mathcal{M}$ is a normed space structure, and $T$ is a bounded linear operator between sorts of $\mathcal{M}$, then the expansion $(\mathcal{M}, T)$ is a normed space structure, in which $T$ is a distinguished function.
If $\mathcal{M}$ is a normed space structure, $M^{(s)}$ is a sort of $\mathcal{M}$, and $A$ is a given subset of $M^{(s)}$, then $\mathcal{M}$ can be expanded by adding the real-valued function $x \mapsto \text{dist}(x, A)$, where $x$ ranges over $M^{(s)}$ and dist denotes the distance function with respect to the norm on $M^{(s)}$. The same can be done with subsets of finite cartesian products of sorts.

The concepts of substructure of a normed space structure, isomorphism between substructures, embedding of structure in another, and automorphism of a structure are defined as in standard (multi-sorted) model theory. The detailed definitions can be found in [HI02, Definitions 3.4 and 3.5]. We write $\mathcal{M} \cong \mathcal{N}$ if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, and $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure of $\mathcal{N}$. If $\mathcal{M}$ is a normed space structure based on $(M^{(s)} \mid s \in S)$ and $(A^{(s)} \mid s \in S)$ is family of sets such that $A^{(s)} \subseteq M^{(s)}$ for every $s \in S$, the substructure of $\mathcal{M}$ generated by $(A^{(s)} \mid s \in S)$ is the $L$-substructure of $\mathcal{M}$ that results from closing $(A^{(s)} \mid s \in S)$ and $\mathbb{R} = M^{(s_\emptyset)}$ under all the functions of $\mathcal{M}$.

**Signatures and real-valued terms.** A signature $L$ for a normed space structure consists of

- A sort index set $S$.
- An element $s_\mathbb{R} \in S$ such that $M^{(s_\emptyset)} = \mathbb{R}$.
- For each function $F : M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)}$

of $\mathcal{M}$, a triple of the form $(f, (s_1, \ldots, s_n), s_0)$, where $f$ is a syntactic symbol called a function symbol for $F$. In this case, we write $F = f^\mathcal{M}$ and call $F$ the interpretation of $f$ in $\mathcal{M}$. We say that $t$ is a term of range sort $s_0$. If $t$ is of range sort $s_\mathbb{R}$, we say that $t$ is a real-valued term. If $m = 0$, $f$ is called a constant symbol.

We express the fact that $L$ is a signature for $\mathcal{M}$ by saying that $\mathcal{M}$ is an $L$-structure.

If $L$ is a signature, a normed space $L$-structure $\mathcal{M}$ is an $L$-structure in the sense of standard (multi-sorted) model theory; therefore, the syntactic concepts of $L$-term and $L$-formula are defined, as are the concepts of evaluation of a term of and satisfaction ($\models$). The reader is referred to [HI02, Definition 5.3] for the detailed definition of term and evaluation of an $L$-term in an $L$-structure. The evaluation of an $L$-term $t$ in an $L$-structure $\mathcal{M}$ is denoted $t^\mathcal{M}$.

**Positive bounded formulas and approximations.** A smooth model theory of normed space structures is obtained by focusing on a particular subclass of the class of all formulas, called the class of positive bounded formulas, and a weakening of the satisfaction relation $\models \mathcal{A}$ which is denoted $\models_{\mathcal{A}}$ and called approximate satisfaction. Below, we outline the definitions; more details can be found in [HI02].

Let $L$ be a signature for normed space structures. The set of positive bounded $L$-formulas is defined recursively as follows.

- If $t$ is a real-valued $L$-term (see page 1) and $r$ is a fixed rational number, then the expressions $r \leq t$, $t \leq r$
are positive bounded $L$-formulas.\footnote{Since signature $L$ includes a symbol for the absolute value function on $\mathbb{R}$, we can assume that it also includes a symbol for the relation $\leq$; for instance, we may regard the expressions $t \leq r$ and $t \geq r$ as abbreviations of $|r - t| = t - r$ and $|r - t| = r - t$, respectively.}

- If $\varphi_1$ and $\varphi_2$ are positive bounded formulas, then the expressions
  $$(\varphi_1 \land \varphi_2), \quad (\varphi_1 \lor \varphi_2)$$
  are positive bounded formulas.

- If $\varphi$ is a positive bounded formula, $x$ is a variable, and $r$ is a positive rational number, then the expressions
  $$\exists x(\|x\| \leq r \land \varphi), \quad \forall x(\|x\| \leq r \rightarrow \varphi)$$
  are positive bounded formulas.

We write $\exists_r x \varphi$ and $\forall_r x \varphi$ as abbreviations of $\exists x(\|x\| \leq r \land \varphi)$ and $\forall x(\|x\| \leq r \rightarrow \varphi)$, respectively.

The 	extit{weak negation} of a positive bounded formula $\varphi$, denoted $\text{neg}(\varphi)$, is defined inductively, as indicated by the following table.

<table>
<thead>
<tr>
<th>If $\varphi$ is:</th>
<th>neg$(\varphi)$ is:</th>
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<tbody>
<tr>
<td>$t \leq r$</td>
<td>$t \geq r$</td>
</tr>
<tr>
<td>$t \geq r$</td>
<td>$t \leq r$</td>
</tr>
<tr>
<td>$(\psi_1 \land \psi_2)$</td>
<td>$(\text{neg}(\psi_1) \lor \text{neg}(\psi_2))$</td>
</tr>
<tr>
<td>$(\psi_1 \lor \psi_2)$</td>
<td>$(\text{neg}(\psi_1) \land \text{neg}(\psi_2))$</td>
</tr>
<tr>
<td>$\exists_r x \psi$</td>
<td>$\forall_r x \text{neg}(\psi)$</td>
</tr>
<tr>
<td>$\forall_r x \psi$</td>
<td>$\exists_r x \text{neg}(\psi)$</td>
</tr>
</tbody>
</table>

We have:

(1) $\text{neg}(\text{neg}(\varphi))$ is $\varphi$;
(2) If $M \not\models \varphi[a_1, \ldots, a_n]$, then $M \models \text{neg}(\varphi)[a_1, \ldots, a_n]$.

A fundamental concept in the model theory of normed space structures is that of 	extit{approximation} of a positive bounded formula. Intuitively, an approximation of a positive bounded formula $\varphi$ is a formula $\varphi'$ which results from relaxing all the norm estimates in $\varphi$ by replacing all the norm bounds that occur in $\varphi$ by weaker bounds. We write $\varphi < \varphi'$ to indicate that $\varphi'$ is an approximation of $\varphi$. The precise definition of approximation can be found in \cite[page 25]{HI02}. However, the definition will not be strictly needed here; most arguments involving approximations, and all the arguments in this paper, can be justified by one of the properties included in the following list (items (1)–(5) should be intuitively clear from our informal definition; item (5) is discussed in \cite[Propositions 5.15 and 9.1]{HI02}):

(1) Every approximation of $\varphi$ has the same quantifier complexity as $\varphi$;
(2) Whenever $\varphi < \psi$ there exists $\varphi'$ of such that $\varphi < \varphi' < \psi$;
(3) $\varphi < \varphi'$ if and only if $\text{neg}(\varphi') < \text{neg}(\varphi)$;
(4) If $M \models \varphi[a_1, \ldots, a_n]$, then $M \models \varphi'[a_1, \ldots, a_n]$ for every $\varphi' > \varphi$;
(5) $\mathcal{M} \not\models \varphi[a_1, \ldots, a_n]$ if and only there exists $\varphi' > \varphi$ such that $\mathcal{M} \models \text{neg}(\varphi'[a_1, \ldots, a_n])$;

(6) The Perturbation Lemma. See below.

1.3. PROPOSITION (Perturbation Lemma). Suppose that $\mathcal{M}$ is an $L$-structure and $\varphi(x_1, \ldots, x_n)$ is a positive bounded $L$-formula. Then for every $r > 0$ and every approximation $\varphi'$ of $\varphi$ there exists $\delta > 0$ such that whenever $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are elements of suitable sorts of $\mathcal{M}$, we have that

$$\mathcal{M} \models \bigwedge_{1 \leq i \leq n} \|a_i\| \leq r \wedge \bigwedge_{1 \leq i \leq n} \|a_i - b_i\| \leq \delta \wedge \varphi[a_1, \ldots, a_n]$$

implies

$$\mathcal{M} \models \varphi'[b_1, \ldots, b_n].$$

Approximate satisfaction. Suppose that $\mathcal{M}$ is a normed space $L$-structure based on $(M(s) | s \in S)$. Let $\varphi(x_1, \ldots, x_n)$ be a positive bounded $L$-formula, where $x_i$ is a variable of sort $s_i$, for $i = 1, \ldots, n$, and let $a_1, \ldots, a_n$ be such that $a_i \in M(s_i)$ for each $i$. We say that $\mathcal{M}$ approximately satisfies $\varphi(x_1, \ldots, x_n)$ at $a_1, \ldots, a_n$, and write

$$\mathcal{M} \models^A \varphi[a_1, \ldots, a_n],$$

if $\mathcal{M} \models \varphi'[a_1, \ldots, a_n]$ for every approximation $\varphi'$ of $\varphi$.

Suppose that $\mathcal{M}$ and $\mathcal{N}$ are normed space $L$-structures based on $(M(s) | s \in S)$ and $(N(s) | s \in S)$, respectively.

We say that $\mathcal{M}$ and $\mathcal{N}$ are approximately elementarily equivalent, and write

$$\mathcal{M} \equiv^A \mathcal{N},$$

if $\mathcal{M}$ and $\mathcal{N}$ approximately satisfy exactly the same positive bounded $L$-sentences.

If $\mathcal{M} \subseteq \mathcal{N}$ we say that $\mathcal{M}$ is an approximate elementary substructure of $\mathcal{N}$, and write

$$\mathcal{M} \prec^A \mathcal{N},$$

if the following condition holds: whenever $\varphi(x_1, \ldots, x_n)$ is a positive bounded $L$-formula and $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$, we have

$$\mathcal{M} \models^A \varphi[a_1, \ldots, a_n] \text{ if and only if } \mathcal{N} \models^A \varphi[a_1, \ldots, a_n].$$

In this case, we also say that $\mathcal{N}$ is an approximate elementary extension of $\mathcal{M}$.

1.4. PROPOSITION. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are normed space $L$-structures with $\mathcal{M} \subseteq \mathcal{N}$. Then $\mathcal{M} \prec^A \mathcal{N}$ if and only if for every positive bounded $L$-formula $\varphi(x_1, \ldots, x_n, y)$ and every approximation $\varphi'$ of $\varphi$, the following condition holds: if $a_1, \ldots, a_n$ are elements of suitable sorts of $\mathcal{M}$ and $b$ is in an element of $\mathcal{N}$ such that

$$\mathcal{N} \models \varphi[a_1, \ldots, a_n, b],$$

then there exists an element $c$ of $\mathcal{M}$ such that

$$\mathcal{N} \models \varphi'[a_1, \ldots, a_n, c].$$

PROOF. See [HI02, Proposition 6.6]. \qed
Ultraproducts and ultrapowers of normed space structures. In [HI02], concepts of ultraproduct and ultrapower for normed space structures were defined. These are different from the algebraic concepts of ultraproduct an ultrapower studied in first-order model theory (an ordinary ultrapower of even the simplest type of normed space structure — a normed space — is generally not a normed space structure). However, in the model theory of normed space structures, they play a role analogous to that played by their algebraic counterparts in first-order model theory. In particular,

1. Every ultrapower of a normed space structure $\mathcal{M}$ is an approximate elementary extension of $\mathcal{M}$;

2. Two $L$-structures are approximately elementary equivalent if and only if they have isomorphic ultrapowers. See Theorem 1.12 below, and the remarks following its statement.

Unlike in first-order model theory, if $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a family of normed space $L$-structures and $\mathcal{U}$ is an ultrafilter on $\Lambda$, in this context, the $\mathcal{U}$-ultraproduct of $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is not always defined; it is defined only when $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is a uniform family of structures. The definition of uniform family of structures will not be needed in this paper. (See [HI02, Chapter 8].) What we will use about uniform families of structures is that if there exists a normed space structure $\mathcal{M}$ such that $\mathcal{M} = \mathcal{M}_\xi$ for every $\xi \in \Lambda$, then the family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is uniform, and therefore the $\mathcal{U}$-ultraproduct of $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ is defined. This ultrapower, by definition, is the $\mathcal{U}$-ultrapower of $\mathcal{M}$.

The $\mathcal{U}$-ultraproduct of a family $(\mathcal{M}_\xi \mid \xi \in \Lambda)$ of normed space structures (when it is defined) is denoted

$$\prod_{\xi \in \Lambda} \mathcal{M}_\xi \mathcal{U}.$$ 

The $\mathcal{U}$-ultrapower of a normed space structure $\mathcal{M}$ is denoted $(\mathcal{M})_{\mathcal{U}}$.

The precise definition of ultrapower for normed space structures will not be needed here. (See [HI02, Chapter 4].) However, the following fact, which in fact follows directly from the definition, will be invoked in Section 3:

If $\mathcal{M}$ is an $L$-structure based on $(M^{(s)} \mid s \in S)$ and $\mathcal{U}$ is an ultrafilter on a set $\Lambda$, then $(\mathcal{M})_{\mathcal{U}}$ is based on $((M)^{(s)}_{\mathcal{U}} \mid s \in S)$ where, for each $s \in S$, $(M)^{(s)}_{\mathcal{U}}$ is a normed space whose elements are equivalence classes on $\ell_\infty(M^{(s)}, \Lambda)$, the set of bounded families in $M^{(s)}$ of the form $(a_\xi)_{\xi \in \Lambda}$. The equivalence class of a family $(a_\xi)_{\xi \in \Lambda} \in \ell_\infty(M^{(s)}, \Lambda)$ is denoted $((a_\xi)_{\xi \in \Lambda})_{\mathcal{U}}$. If

$$F: M^{(s_1)} \times \cdots \times M^{(s_n)} \to \mathbb{R}$$

is uniformly continuous on every subset of its domain, then $F$ has an extension

$$(F)_{\mathcal{U}}: (M^{(s_1)})_{\mathcal{U}} \times \cdots \times (M^{(s_n)})_{\mathcal{U}} \to \mathbb{R}$$

which is also uniformly continuous and bounded on every subset of its domain, and furthermore,

$$(\mathcal{M}, F) \prec_\mathcal{A} ((\mathcal{M})_{\mathcal{U}}, (F)_{\mathcal{U}}).$$

The function $(F)_{\mathcal{U}}$ is defined as follows: if $(a_\xi^i)_{\xi \in \Lambda}$ is a bounded family in $M^{(s_i)}$, for $i = 1, \ldots, n$,

$$(F)_{\mathcal{U}}(((a_\xi^1)_{\xi \in \Lambda})_{\mathcal{U}}, \ldots, ((a_\xi^n)_{\xi \in \Lambda})_{\mathcal{U}}) = \lim_{\mathcal{U}} F(a_\xi^1, \ldots, a_\xi^n).$$
The uniform continuity assumption on \( F \) ensures that the function \( (F)_U \) is well-defined. See [HI02, Definition 4.1].

**Fragments.**

1.5. Definition. If \( L \) is a signature, a fragment of \( L \) is a set of positive bounded \( L \)-formulas which contains all the quantifier-free positive bounded \( L \)-formulas and is closed under \( \land, \lor, \neg \), and approximations.

1.6. Notation. Suppose that \( L \) is a signature and \( \Phi \) is a fragment of \( L \). If \( \mathcal{M} \) is a normed space \( L \)-structure based on \( (M(s) \mid s \in S) \) and \( A = (A(s) \mid s \in S) \) is a family of sets such that \( A(s) \subseteq M(s) \) for every \( s \in S \), we denote by \( L(A) \) the signature that results from adding to \( L \), for each \( s \in S \) and each element \( a \in A(s) \), a new constant symbol \( c_a \) of range sort \( s \). We denote by \( (\mathcal{M}, A) \) the natural expansion of \( \mathcal{M} \) to an \( L(A) \)-structure, where the interpretation of \( c_a \) is \( a \).

To simplify the notation, we often identify \( a \) with \( c_a \).

If \( \Phi \) is a fragment of \( L \), a \( \Phi(A) \)-formula is an \( L(A) \)-formula of the form

\[
\varphi(x_1, \ldots, x_m, a_1, \ldots, a_n)
\]

where \( \varphi(x_1, \ldots, x_m, y_1, \ldots, y_n) \) is a positive bounded \( L \)-formula in \( \Phi \) and \( a_1, \ldots, a_n \) are constants in \( L(A) \) such that the range sort of \( a_i \) equals the sort \( y_i \), for \( i = 1, \ldots, n \).

**Types.**

1.7. Definition. Suppose that \( L \) is a signature and \( \Phi \) is a fragment of \( L \), \( \mathcal{M} \) is a normed space \( L \)-structure based on \( (M(s) \mid s \in S) \), and \( A = (A(s) \mid s \in S) \) is a family of sets such that \( A(s) \subseteq M(s) \) for every \( s \in S \). If \( c = c_1, \ldots, c_n \) is a tuple of elements of \( \mathcal{M} \), the \( \Phi \)-type of \( c \) over \( A \), denoted

\[
\text{tp}_\Phi(c/A),
\]

is the set of all positive bounded \( L(A) \)-formulas \( \varphi(x_1, \ldots, x_n) \) such that

\[
(\mathcal{M}, A) \models A \varphi[c_1, \ldots, c_n].
\]

If \( A(s) = M(s) \) for every \( s \in S \), \( \text{tp}_\Phi(c/A) \) is also denoted \( \text{tp}_\Phi(c/M) \). A \( \Phi \)-type over \( A \) is a set \( t(\bar{x}) \) of positive bounded \( \Phi(A) \)-formulas, where \( \bar{x} = x_1, \ldots, x_n \), such that there exists a structure \( \mathcal{N} \) with \( \mathcal{M} \prec_A \mathcal{N} \) and a tuple \( \bar{c} = c_1, \ldots, c_n \) of elements of \( \mathcal{N} \) satisfying

\[
t(\bar{x}) = \text{tp}_\Phi(\bar{c}/A).
\]

In this case, we say that \( \bar{c} \) realizes \( t(\bar{x}) \) in \( \mathcal{N} \).

If \( \Phi \) is the set of all positive bounded \( L \)-formulas, \( \text{tp}_\Phi(c/A) \) is denoted \( \text{tp}(c/A) \) and \( \text{tp}_\Phi(c/M) \) is denoted \( \text{tp}(c/M) \); if \( \Phi \) is the set of all quantifier-free positive bounded \( L \)-formulas, \( \text{tp}_\Phi(c/A) \) is denoted \( \text{tp}_{\text{QF}}(c/A) \) and \( \text{tp}_\Phi(c/M) \) is denoted \( \text{tp}_{\text{QF}}(c/M) \).

1.8. Remarks. Suppose that \( L \) is a signature, \( \Phi \) is a fragment of \( L \), and \( \mathcal{M} \) is a normed space \( L \)-structure based on \( (M(s) \mid s \in S) \).

(1) By the Perturbation Lemma (Proposition 1.3), if

\[
A = (A(s) \mid s \in S), \quad B = (B(s) \mid s \in S)
\]

are such that \( A(s) \subseteq B(s) \subseteq M(s) \) and \( A(s) \) is a dense subset of \( B(s) \) for every \( S \) (with respect to the norm of \( M(s) \)), then \( \text{tp}_\Phi(\bar{c}/B) \) is completely determined by \( \text{tp}_\Phi(\bar{c}/A) \).
(2) If $\mathcal{M}_0$ is the substructure of $\mathcal{M}$ generated by $A$, then $\text{tp}(\bar{c}/\mathcal{M}_0)$ is completely determined by $\text{tp}(\bar{c}/A)$.

Let $\Sigma(x_1, \ldots, x_n)$ be a set of $L$-formulas and let $\mathcal{M}$ be a normed space $L$-structure. We say that $\Sigma(x_1, \ldots, x_n)$ is \textit{satisfiable in} $\mathcal{M}$ if there exist elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$ such that $\mathcal{M} \models \Sigma[a_1, \ldots, a_n]$. We say that $\Sigma(x_1, \ldots, x_n)$ is \textit{approximately satisfiable in} $\mathcal{M}$ if there exist elements $a_1, \ldots, a_n$ of suitable sorts of $\mathcal{M}$ such that $\mathcal{M} \models_A \Sigma[a_1, \ldots, a_n]$.

1.9. PROPOSITION. Suppose that $L$ is a signature and $\Phi$ is a fragment of $L$, $\mathcal{M}$ is a normed space $L$-structure based on $(M^s \mid s \in S)$, and $A = (A^s \mid s \in S)$ is a family of sets such that $A^s \subseteq M^s$ for every $s \in S$.

If $t(x_1, \ldots, x_n)$ is a set of $L(A)$-formulas, then $t(x_1, \ldots, x_n)$ is a $\Phi$-type over $A$ if and only if the following conditions hold:

1. There exists a rational number $r$ such that the formula $\bigwedge_{i=1}^n \|x_i\| \leq r$ is in $t(x_1, \ldots, x_n)$;

2. Every finite subset of $t$ is approximately satisfiable in $(M, A)$;

3. Whenever $\varphi(x_1, \ldots, x_n)$ is a positive bounded $L(A)$-formula and $\varphi'$ is an approximation of $\varphi$, we have either $\varphi \in t$ or $\text{neg}(\varphi') \in t$.

Big models.

1.10. DEFINITION. Let $\mathcal{M}$ be a normed space $L$-structure and let $\kappa$ be an infinite cardinal. We say that $\mathcal{M}$ is \textit{strongly $\kappa$-homogeneous} if the following condition holds: if

$$A = (A^s \mid s \in S), \quad B = (B^s \mid s \in S)$$

are such that $A^s, B^s \subseteq M^s$ and

$$\sum_{s \in S} \text{density}(A^s) < \kappa, \quad \sum_{s \in S} \text{density}(B^s) < \kappa,$$

then

$$(\mathcal{M}, A) \equiv_A (\mathcal{M}, B)$$

implies

$$(\mathcal{M}, A) \cong (\mathcal{M}, B).$$

1.11. DEFINITION. Let $\mathcal{M}$ be a normed space $L$-structure and let $\kappa$ be an infinite cardinal. We say that $\mathcal{M}$ is \textit{$\kappa$-saturated} if the following condition holds: whenever

$$A = (A^s \mid s \in S)$$

is such that $A^s \subseteq M^s$ and

$$\sum_{s \in S} \text{density}(A^s) < \kappa$$

and $\Sigma(x_1, \ldots, x_n)$ is a set of positive bounded $L(A)$-formulas containing the formulas $\|x_i\| \leq r$ for $i = 1, \ldots, n$ (where $r$ is a rational number), if every finite subset of $\Sigma$ is approximately satisfiable in $(\mathcal{M}, A)$, then the entire set $\Sigma$ is approximately satisfiable in $(\mathcal{M}, A)$.

The following theorem is one of the central results of [HI02]. The proof involves intricate combinatorial arguments. See [HI02, Chapter 10].
1.12. **Theorem.** Let \( \kappa \) be an infinite cardinal and let \( \lambda = 2^\kappa \). Then there exists an ultrafilter \( U \) on \( \lambda \) satisfying the following properties: if \( L \) is a signature with \( \text{card}(L) \leq \lambda \) and \( (M_\xi \mid \xi < \lambda) \) and \( (N_\xi \mid \xi < \lambda) \) are families of normed space \( L \)-structures such that the density character of each sort of \( M_\xi \) and \( N_\xi \) is at most \( \kappa \) for all \( \xi < \lambda \), then

\[
\left( \prod_{\xi < \lambda} M_\xi \right)_U \equiv_A \left( \prod_{\xi < \lambda} N_\xi \right)_U
\]

implies

\[
\left( \prod_{\xi < \lambda} M_\xi \right) U \cong \left( \prod_{\xi < \lambda} N_\xi \right) U.
\]

Moreover, in each such situation, the ultrapower \( \left( \prod_{\xi < \lambda} M_\xi \right) U \) is \( \kappa^+ \)-saturated and strongly \( \lambda^+ \)-homogeneous.

Throughout the paper we will have a fixed signature \( L \) in the background, and we will work within a normed space \( L \)-structure \( C \) which is \( \kappa \)-saturated and strongly \( \kappa \)-homogeneous, where \( \kappa \) is an infinite cardinal with \( \kappa > \text{card}(L) \), and \( \kappa \) is large enough so that if \( M \) is any other normed space structure mentioned in the paper such that \( M \equiv_A C \) and \( M \) is based on \( (M^{(s)} \mid s \in S) \), then

\[
\sum_{s \in S} \text{density}(M^{(s)}) < \kappa.
\]

We will express the occurrence of this inequality by saying that \( M \) is *not too large*. The existence of such a structure \( C \) is guaranteed by Theorem 1.12.

The saturation of \( C \) implies that if \( M \equiv_A C \) and \( M \) is not too large, then \( M \) has an isomorphic copy that is an approximate elementary substructure of \( C \). Thus, without loss of generality, we can assume that whenever \( M \equiv_A C \) we have \( M \triangleleft_A C \); furthermore, we can assume that every type over \( M \) is realized in \( C \).

The homogeneity assumption on \( C \) implies that the following condition holds. If \( M \subseteq C \), \( M \) is not too large, and \( M \) is based on \( (M^{(s)} \mid s \in S) \), then whenever \( A = (A^{(s)} \mid s \in S) \) is such that \( A^{(s)} \subseteq M^{(s)} \) for every \( s \in S \) and \( \bar{a} = a_1, \ldots, a_n \) and \( \bar{b} = b_1, \ldots, b_n \) are tuples of elements of \( C \) such that

\[
\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A).
\]

there exists an automorphism \( f \) of \( C \) such that

\[
\cdot \quad f(\bar{a}) = \bar{b}, \text{ i.e., } f(a_i) = b_i \text{ for } i = 1, \ldots, n,
\]

\[
\cdot \quad f \text{ is the identity on } A^{(s)}, \text{ for every } s \in S.
\]

Following the tradition established by Shelah in model theory, we will refer to \( C \) as the **monster model**. The letter \( T \) will denote the complete positive bounded theory of \( C \), i.e., the set of all positive bounded \( L \)-sentences \( \varphi \) such that \( C \models_A \varphi \).

The term "model" will mean "model of \( T \)". By the preceding lines, we may assume that all models are approximate elementary submodels of the monster model. Unless specified otherwise, all models mentioned, except for the monster model, will be assumed to be not too large (see above). Similarly, the word "structure" will be reserved to refer to substructures of the monster model. For notational simplicity, we will identify syntactic terms with their evaluation in the monster model.

Whenever we refer to a finite tuple \( \bar{a} = a_1, \ldots, a_n \), it is assumed that \( a_1, \ldots, a_n \) are elements of the monster model. The **norm of \( \bar{a} \)**, denoted \( \| \bar{a} \| \), is \( \max_{1 \leq i \leq n} \| a_i \| \).

If \( t(\bar{x}) \) is a type, the **norm of \( t \)**, denoted \( \| t \| \), is defined as \( \| \bar{a} \| \), where \( \bar{a} \) is any realization of \( t \).
The logic topology. Let $\Phi$ be a fragment of $L$. If $\varphi(\bar{x})$ is a $\Phi$-formula, we let

$$[\varphi]_\Phi = \{ t(\bar{x}) \mid t \text{ is a } \Phi\text{-type and } \varphi \in t \}.$$ 

The logic topology on $\Phi$-types is defined as follows. If $t$ is a $\Phi$-type, the basic neighborhoods of $t$ are the sets of the form $[\varphi]_\Phi$, where $\varphi$ is an approximation of a $\Phi$-formula in $t$.

Note that the logic topology is Hausdorff. Indeed, if $\Phi$ is a fragment of $L$ and $t, t'$ are distinct $\Phi$-types, there exist a formula $\varphi$ and an approximation $\varphi'$ of $\varphi$ such that $t \in [\varphi]_\Phi$ and $t \in [\text{neg}(\varphi')]_\Phi$, while $[\varphi]_\Phi$ and $[\text{neg}(\varphi')]_\Phi$ are disjoint neighborhoods.

1.13. PROPOSITION. Suppose that $\Phi$ is a fragment of $L$, $M$ is an $L$-structure based on $(M(s) \mid s \in S)$ that is not too large, and $A = (A(s) \mid s \in S)$ is such that $A(s) \subseteq M(s)$ for every $s \in S$. Then, for any list of variables $\bar{x} = x_1, \ldots, x_n$ and any real number $r > 0$, the set of $\Phi$-types $t(\bar{x})$ over $A$ such that $\|t\| \leq r$ is compact with respect to the logic topology.

PROOF. This is a restatement of the saturation property of the monster model.

Proposition 1.13 will be used many times in this paper, in the following form: if $(t_i(\bar{x}))_{i \in I}$ is a a family of $\Phi$-types over $A$ of uniformly bounded norm and $\mathcal{U}$ is an ultrafilter on $I$, then

$$\lim_{i, \mathcal{U}} t_i(\bar{x})$$

is a $\Phi$-type over $A$.

2. Some Notational Conventions

Of particular interest in this paper will be two fragments of $L$, namely, the fragment of all quantifier-free positive bounded $L$-formulas and the fragment of all positive bounded formulas. Hereafter, the letter $\Phi$ will stand for one of these two fragments. Several of the results proved will involve $\Phi$ as a parameter, and the information provided by such a result will generally depend on which of the two interpretations the letter $\Phi$ is given.

In this section we introduce some notation which will make some arguments serve this dual purpose in a smooth manner.

2.1. NOTATION. Suppose that $M$ is a model (i.e., an approximate elementary substructure of the monster model) and that $N$ is an $L$-structure (i.e., a substructure of the monster model). We introduce the notation

$$M \prec^\Phi_A N.$$ 

The meaning of this notation will be as follows:

1. If $\Phi$ is the fragment of all positive bounded formulas, then $M \prec^\Phi_A N$ will mean $M \prec_A N$;

2. If $\Phi$ is the fragment of all quantifier-free positive bounded formulas, then $M \prec^\Phi_A N$ will mean $M \subseteq N$.

2.2. PROPOSITION. Suppose that $M$ is a model and $N$ is an $L$-structure. Then $M \prec^\Phi_A N$ if and only if whenever $b_1, \ldots, b_n$ are elements of $N$, the type

$$tp_\Phi(b_1, \ldots, b_n/M)$$
is in the closure of
\[ \{ \text{tp}_\Phi(a_1, \ldots, a_n/M) \mid a_1, \ldots, a_n \text{ are elements of } M \} \].

PROOF. By Proposition 1.4 and the fact that \( M \) is an approximate elementary substructure of the monster model. \( \square \)

2.3. NOTATION. Suppose that \( \bar{a}, \bar{b} \) are tuples of elements of the monster model, \( M \) is an \( L \)-structure, and \( A = (A(s) \mid s \in S) \) is such that \( A(s) \subseteq M(s) \) for every \( s \in S \).

(1) If \( \Phi \) is the fragment of all positive bounded formulas, we will say that \( f \) is a \( \Phi(A) \)-isomorphism if \( f \) is an automorphism of the monster model which fixes \( A(s) \) pointwise, for every \( s \in S \).

(2) If \( \Phi \) is the fragment of all quantifier-free positive bounded formulas, we will say that \( f \) is a \( \Phi(A) \)-isomorphism if \( f \) is an isomorphism between two substructures of the monster model which fixes \( A(s) \) pointwise, for every \( s \in S \). If \( f \) is a \( \Phi(A) \)-isomorphism and \( \bar{a} \) and \( \bar{b} \) are tuples of elements of the monster model, we will say that that \( f(\bar{a}) = \bar{b} \) to imply that: (i) the domain of \( f \) contains \( A \), (ii) the range of \( f \) contains \( \bar{b} \), and (iii) \( f(\bar{a}) = \bar{b} \).

A \( \Phi(M) \)-isomorphism is a \( \Phi(A) \)-isomorphism, where \( A \) is as above and If \( A(s) = M(s) \) for every \( s \in S \).

The motivation for the preceding notation is the following proposition.

2.4. PROPOSITION. Suppose that \( \bar{a}, \bar{b} \) are tuples of elements of the monster model, \( M \) is an \( L \)-structure, and \( A = (A(s) \mid s \in S) \) is such that \( A(s) \subseteq M(s) \) for every \( s \in S \). Then \( \text{tp}_\Phi(\bar{a}/A) = \text{tp}_\Phi(\bar{b}/A) \) if and only if there is an \( \Phi(A) \)-isomorphism \( f \) such that \( f(\bar{a}) = \bar{b} \).

PROOF. If \( \Phi \) is the fragment of all positive bounded formulas, the proposition follows from the homogeneity of the monster model. In the case when \( \Phi \) is the fragment of all quantifier-free positive bounded formulas, the assertion is trivially true. \( \square \)

3. Definable Real-Valued Functions

Suppose that \( M \) is an \( L \)-structure based on \( (M(s) \mid s \in S) \) and that
\[ F : M^{(s_1)} \times \cdots \times M^{(s_n)} \to \mathbb{R} \]
is uniformly continuous on every subset of its domain. We claim that if \( N \) is an \( L \)-structure based on \( (N(s) \mid s \in S) \) such that \( M \prec_N N \), then \( F \) has an extension
\[ G : N^{(s_1)} \times \cdots \times N^{(s_n)} \to \mathbb{R} \]
such that \( (M, F) \prec_N (N, G) \). Indeed, without loss of generality, we can assume that \( N \) is an ultrapower of \( M \), say, \( (M)_{\mathcal{U}} \), in which case, \( G \) can be defined as \( (F)_{\mathcal{U}} \) (see page 7 for the definition of \( (F)_{\mathcal{U}} \)).

3.1. DEFINITION. Suppose that \( M \) is a normed space \( L \)-structure based on \( (M(s) \mid s \in S) \), and let
\[ F : M^{(s_1)} \times \cdots \times M^{(s_n)} \to \mathbb{R} \]
be a real-valued function. We will say that \( F \) is \( \Phi \)-definable if the following condition holds. For every choice of \( K, \epsilon > 0 \) and every interval \( I \) there exist
a \Phi(\mathcal{M})\text{-formula } \theta(x_1, \ldots, x_n), \text{ where } x_i \text{ is a variable of sort } s_i \text{ for } i = 1, \ldots, n, \text{ and}

\text{an approximation } \theta' \text{ of } \theta

such that whenever } a_i \in M^{(s_i)} \text{ and } \|a_i\| \leq K \text{ for } i = 1, \ldots, n, \text{ we have}

(i) \quad F(a_1, \ldots, a_n) \in I \text{ implies } \theta(a_1, \ldots, a_n);
(ii) \quad \theta'(a_1, \ldots, a_n) \text{ implies } F(a_1, \ldots, a_n) \in I + [-\epsilon, \epsilon].

3.2. Proposition. Suppose that \mathcal{M} \text{ is a normed space } L\text{-structure based on } (M^{(s)} | s \in S) \text{ and that }

F : M^{(s_1)} \times \cdots \times M^{(s_n)} \rightarrow \mathbb{R}

is uniformly continuous on every bounded subset of its domain. Then, the following conditions are equivalent:

(1) \quad F \text{ is } \Phi\text{-definable;}
(2) \quad Whenever } \mathcal{N} \text{ is an } L\text{-structure based on } (N^{(s)} | s \in S) \text{ such that } \mathcal{M} \prec^\Phi \mathcal{N} \text{ there exists a unique function }

G : N^{(s_1)} \times \cdots \times N^{(s_n)} \rightarrow \mathbb{R}

such that } (\mathcal{M}, F) \prec^\Phi (\mathcal{N}, G).

Proof. (1) \Rightarrow (2): \text{ Fix an } L\text{-structure } \mathcal{N} \text{ based on } (N^{(s)} | s \in S) \text{ and a function }

G : N^{(s_1)} \times \cdots \times N^{(s_n)} \rightarrow \mathbb{R}

such that } (\mathcal{M}, F) \prec^\Phi (\mathcal{N}, G). \text{ Take now positive numbers } K, K', \delta, \epsilon, \epsilon' \text{ such that }

0 < K < K', \quad 0 < \delta < \epsilon < \epsilon'

and a real number } r. \text{ Then use Definition 3.1 to find a } \Phi(\mathcal{M})\text{-formula } \theta(\bar{x}), \text{ where } \bar{x} = x_1, \ldots, x_n \text{ and } x_i \text{ is a variable of sort } s_i \text{ for } i = 1, \ldots, n, \text{ and an approximation } \theta' \text{ of } \theta \text{ such that whenever } \|\bar{a}\| \leq K', \text{ where } \bar{a} = a_1, \ldots, a_n \text{ and } a_i \in M^{(s_i)} \text{, we have }

(i_M) \quad F(\bar{a}) \in [r - \delta, r + \delta] \text{ implies } \theta(\bar{a});
(ii_M) \quad \theta'(\bar{a}) \text{ implies } F(\bar{a}) \in [r - \epsilon, r + \epsilon].

Take now } \Phi(\mathcal{M})\text{-formulas } \sigma, \sigma' \text{ such that }

\theta < \sigma < \sigma' < \theta'.

We claim that whenever } \|\bar{b}\| \leq K, \text{ where } \bar{b} = b_1, \ldots, b_n \text{ and } b_i \in N^{(s_i)}, \text{ we have }

(i)_N \quad G(\bar{b}) = r \text{ implies } \sigma(\bar{b});
(ii)_N \quad \sigma'(\bar{b}) \text{ implies } G(\bar{b}) \in [r - \epsilon', r + \epsilon'].

Indeed, since } (\mathcal{M}, F) \prec^\Phi (\mathcal{N}, G), \text{ by Proposition 2.2, any counterexample to } (i)_N \text{ would yield a counterexample to } (i)_M, \text{ and any counterexample to } (ii)_N \text{ would yield a counterexample to } (ii)_M.

Thus, if } \bar{b} = b_1, \ldots, b_n, \text{ where } b_i \in N^{(s_i)}, \text{ the value of } G(\bar{b}) \text{ is completely determined by } \text{tp}_\Phi(\bar{b}/\mathcal{M}), \text{ so the uniqueness of } G \text{ is given by Proposition 2.2.}

(2) \Rightarrow (1): \text{ Assume that } F \text{ is not definable, and choose } K, \epsilon > 0 \text{ and an interval } I \text{ such that there does not exist a } \Phi(\mathcal{M})\text{-formula } \theta(x_1, \ldots, x_n) \text{ and an approximation } \theta' \text{ of } \theta \text{ satisfying the conditions of Definition 3.1.}

Let } \Sigma(\bar{x}) \text{ be the set of all } \Phi(\mathcal{M})\text{-formulas } \theta(\bar{x}), \text{ where } \bar{x} = x_1, \ldots, x_n \text{ and } x_i \text{ is a variable of sort } s_i \text{ for } i = 1, \ldots, n, \text{ such that the following holds: whenever } \|\bar{a}\| \leq K, \text{ where } \bar{a} = a_1, \ldots, a_n \text{ and } a_i \in M^{(s_i)}, \text{ we have }


Notice that $\Sigma$ is closed under finite conjunctions. By our assumption, for every 
\( \theta(\bar{x}) \in \Sigma \) and every approximation $\theta'$ of $\theta$ there exists a tuple $\bar{a}$ of elements of $\mathcal{M}$ with $\|\bar{a}\| \leq K$ such that $\theta'(\bar{a})$ and $F(\bar{a}) \notin I + [-\epsilon, \epsilon]$. Thus, by the saturation of the monster model, there exist a structure $\mathcal{N}$, an extension $G$ of $F$ to $\mathcal{N}$, and a tuple $\bar{a}$ of elements of $\mathcal{N}$ such that

(i) $(\mathcal{M}, F) \prec_{\Phi} (\mathcal{N}, G)$,
(ii) $\theta(\bar{a})$, for every $\Phi(\mathcal{M})$-formula $\theta \in \Sigma$,
(iii) $G(\bar{a}) \notin I + (-\epsilon, \epsilon)$.

Now notice that if $\theta(\bar{x})$ is a $\Phi(\mathcal{M})$-formula such that $\theta(\bar{a})$, then for every approximation $\theta'$ of $\theta$ there exists a tuple $\bar{b}$ of elements of $\mathcal{M}$ with $\|\bar{b}\| \leq K$ such that $F(\bar{b}) \in I$ and $\theta'(\bar{b})$; otherwise, we would have $\text{neg}(\theta') \in \Sigma$, contradicting the satisfiability of $\Sigma$ (which is given by (ii), above). Therefore, by the saturation of the monster model, there exist a structure $\mathcal{N'}$, an extension $G'$ of $F$ to $\mathcal{N}'$, and a tuple $\bar{a}'$ of elements of $\mathcal{N}$ such that

(iv) $(\mathcal{M}, F) \prec_{\Phi} (\mathcal{N}', G')$,
(v) $\theta(\bar{a'})$, for every $\Phi(\mathcal{M})$-formula such that $\theta(\bar{a})$,
(vi) $G'(\bar{a'}) \in I$.

Without loss of generality, we can assume that both $\mathcal{N}$ and $\mathcal{N}'$ equal the monster model. By (v) above and the homogeneity of the monster model, we can assume $\bar{a} = \bar{a}'$ (we have $\text{tp}_{\Phi}(\bar{a}/\mathcal{M}) = \text{tp}_{\Phi}(\bar{a'}/\mathcal{M})$, so by Proposition 2.4, there is a $\Phi(\mathcal{M})$-isomorphism $f$ such that $f(\bar{a}) = \bar{a}'$). Thus, Condition (2) does not hold.

\[
F(\bar{a}) \in I \quad \text{implies} \quad \theta(\bar{a}).
\]

3.3. DEFINITION. Suppose that $\mathcal{M}$ is a normed space $L$-structure based on $(M^{(s)} | s \in S)$, $\mathbf{A}$ is a set of elements of $\mathcal{M}$, and $t(\bar{x})$ is a $\Phi$-type over $\mathbf{A}$. We will say that $t$ is $\Phi$-definable if the following condition holds. For every $\Phi$-formula $\varphi(\bar{x}, \bar{y})$, where $\bar{y} = y_1, \ldots, y_n$ and $y_i$ is a variable of sort $s_i$ for $i = 1, \ldots, n$, every approximation $\varphi'$ of $\varphi$, and every $K > 0$ there exist

- a $\Phi(\mathbf{A})$-formula $\theta(\bar{y})$, and
- an approximation $\theta'$ of $\theta$

such that whenever $\|\bar{a}\| \leq K$, where $\bar{a} = a_1, \ldots, a_n$ and $a_i \in M^{(s_i)}$, we have

(i) $\varphi(\bar{x}, \bar{a}) \in t$ implies $\theta(\bar{a})$;
(ii) $\theta'(\bar{a})$ implies $\varphi'(\bar{x}, \bar{a}) \in t$.

3.4. PROPOSITION. Suppose that $\mathcal{M}$ is a normed space $L$-structure based on $(M^{(s)} | s \in S)$ and $t(\bar{x})$ is a $\Phi$-type over $\mathcal{M}$. Then for every $\Phi$-formula $\varphi(\bar{x}, \bar{y})$, where $\bar{y} = y_1, \ldots, y_n$ and $y_i$ is a variable of sort $s_i$ for $i = 1, \ldots, n$, there exists a function

$$F_{\varphi} : M^{(s_1)} \times \cdots \times M^{(s_n)} \to \mathbb{R}$$

such that

(1) $F_{\varphi}$ is uniformly continuous on every bounded subset of its domain;
(2) $t$ is definable if and only if $F_{\varphi}$ is definable for every $\varphi$.

PROOF. Fix $t(\bar{x})$ and $\varphi(\bar{y})$ as in the statement of the proposition. Choose a family

$$\left( \varphi_r | r \in \mathbb{Q} \cap (0, 1) \right)$$
of approximations of $\varphi$ such that for every $r \in \mathbb{Q} \cap (0, 1)$:

- $\varphi < \varphi_r < \varphi_s$, iff $r < s$,
- whenever $\varphi < \psi < \psi'$ there exist $r, s \in \mathbb{Q} \cap (0, 1)$ such that
  $$\varphi < \varphi_r < \psi < \varphi_s < \psi'.$$

Define

$$F_{\varphi} : M^{(a_1)} \times \ldots \times M^{(a_n)} \to [0, 1]$$

as follows. If $\bar{a} = a_1, \ldots, a_n$, where $a_i \in M^{(a_i)}$ for $i = 1, \ldots, n$,

$$F_{\varphi}(\bar{a}) = \begin{cases} 
\inf\{ r \in \mathbb{Q} \cap (0, 1) \mid \varphi_r(x, \bar{a}) \in t \}, & \text{if } \{ r \in \mathbb{Q} \cap (0, 1) \mid \varphi_r(x, \bar{a}) \in t \} \neq \emptyset \\
1, & \text{otherwise.}
\end{cases}$$

The Perturbation Lemma (Proposition 1.3) applied to $\varphi$ says that $F_{\varphi}$ is uniformly lower semicontinuous on every bounded subset of its domain; applied to $\text{neg}(\varphi)$, it says that $F_{\varphi}$ is uniformly upper semicontinuous on every bounded subset of its domain. Thus, $F_{\varphi}$ is uniformly continuous on every bounded subset of its domain and (1) of the proposition holds. Clearly, $t$ is definable if and only if $F_{\varphi}$ is definable for every $\varphi$, so (2) holds too.

\[ \square \]

4. Semidefinition and Limits

In order to simplify the notation, at this point we will explicitly deal with structures that have only one sort (which, as usual, is called the universe of the structure), in addition to the special sort $\mathbb{R}$. It will be clear that the arguments can be adjusted, with only minor notational modifications, to the general multi-sorted context. The Roman letters $A, B$, etc. will be used to denote subsets of the universe of structures that are no too large (see page 10).

4.1. DEFINITION. Suppose that $A \subseteq B$ and $t(\bar{x})$ is a $\Phi$-type over $B$. We will say that $t$ is semidefinitional over $A$ if for every formula $\varphi(\bar{x}) \in t$ and every approximation $\varphi'$ of $\varphi$ there exists $\bar{a} \in A$ such that $\varphi'(\bar{a})$.

4.2. REMARK. A $\Phi$-type $t$ over $B$ is semidefinitional over $A$ if there exists a family $(\bar{a}_i)_{i \in I}$ in $A$ and an ultrafilter $\mathcal{U}$ on $I$ such that

$$t = \lim_{i, \mathcal{U}} \text{tp}_\Phi(\bar{a}_i/B),$$

where the limit is taken in the logic topology (see page 11).

4.3. PROPOSITION. Suppose that $A \subseteq B$ and that $t(\bar{x})$ is a $\Phi$-type over $B$ which is semidefinitional over $A$. Then, for every superset $C$ of $A$ there exists a $\Phi$-type $\hat{t}(\bar{x})$ over $C$ such that $\hat{t}$ extends $t$ and is semidefinitional over $A$; furthermore, if $(\bar{a}_i)_{i \in I}$ is a family in $A$ and $\mathcal{U}$ is an ultrafilter on $I$ such that

$$t = \lim_{i, \mathcal{U}} \text{tp}_\Phi(\bar{a}_i/B),$$

then $\hat{t}$ can be chosen so that

$$\hat{t} = \lim_{i, \mathcal{U}} \text{tp}_\Phi(\bar{a}_i/C).$$
PROOF. We claim that if $\psi(x) \in t$, $\psi'$ is an approximation of $\psi$, and $\varphi(x, y)$ is a $\Phi$-formula such that

\[(*) \quad \{ i \in I \mid \varphi(a_i, c_i) \} \notin U,\]

where $c \in C$, then

\[\{ i \in I \mid \psi'(a_i) \land \neg(\varphi(a_i, c_i)) \} \in U.\]

Indeed, we have

\[\{ i \in I \mid \neg(\varphi(a_i, c_i)) \} \in U,\]

and by the hypothesis that $t = \lim_{i, U} \text{tp}_\Phi(a_i/B)$ we also have

\[\{ i \in I \mid \psi'(a_i) \} \in U,\]

so the claim follows. Let

\[\Sigma(x) = \{ \neg(\varphi(x, c)) \mid c \in C \text{ and } \{ i \in I \mid \varphi(a_i, c) \} \notin U \}.\]

By the claim, every finite subset of $t \cup \Sigma(x)$ approximately is satisfiable (in the monster model). For every $\Phi$-formula of the form $\varphi(x, c)$, where $c \in C$, we have either $\varphi \in t \cup \Sigma$ or $\neg(\varphi) \in t \cup \Sigma$, so $t \cup \Sigma$ is a $\Phi$-type over $C$ (see Proposition 1.9). Furthermore, if $\varphi(x, c) \in t \cup \Sigma$ and $\varphi' > \varphi$, we must have $\{ i \in I \mid \varphi'(a_i, c) \} \in U$. Thus, $\lim_{i, U} \text{tp}_\Phi(a_i/C) = t \cup \Sigma$.

\[\square\]

5. Ramsey’s Theorem

Recall that a term $t$ is said to be real-valued if its range sort of is the special sort $s_\mathbb{R}$ (see page 1). By convention, we are identifying every $L$-term with its evaluation in the monster model. Thus, every $k$-ary real-valued $L$-term $t(x_1, \ldots, x_k)$ is being regarded as a $k$-ary function with range $\mathbb{R}$; furthermore as such, $t$ is uniformly continuous, and hence bounded, on every subset of its domain (see Definition 1.1). Hence, if $t(x_1, \ldots, x_k)$ is a real-valued $L$-term, where $x_j$ is a variable of sort $s_j$, and $(a_{i,j})_{i \in I}, \ldots, (a_{i,k})_{i \in I}$ are bounded families in the monster model such that $(a_{i,j})_{i \in I}$ is in the sort of the monster model indexed by $s_j$, for $j = 1, \ldots, k$, then the elements of the form

\[t(a_{i,1}, \ldots, a_{i,k})\]

are contained in a compact subset $\Gamma$ of $\mathbb{R}$. Given any ultrafilter $U$ on $I$, we can compute

\[\lim_{t_{i_1, U} \ldots t_{i_k, U}} t(a_{i,1}, \ldots, a_{i,k})\]

within $\Gamma$. Furthermore, by the saturation of the monster model, there exists elements $c_1, \ldots, c_k$ of the monster model such that

\[t(c_1, \ldots c_k) = \lim_{t_{i_1, U} \ldots t_{i_k, U}} t(a_{i,1}, \ldots, a_{i,k}).\]

The map

\[(a_{i,1})_{i \in I}, \ldots, (a_{i,k})_{i \in I} \mapsto \lim_{t_{i_1, U} \ldots t_{i_k, U}} t(a_{i,1}, \ldots, a_{i,k})\]

provides a "coloring" on the $k$-tuples of the form $((a_{i,1})_{i \in I}, \ldots, (a_{i,k})_{i \in I})$. The set of colors is $\Gamma$, which is not finite, but is compact.

The proofs of Propositions 5.1 and 5.2 below are an analog for the context of positive bounded formulas and approximate satisfaction of the model-theoretic proof of the classic Ramsey Theorem (see, for example, [Hod93, Theorem 11.1.3]). Proposition 5.2 will be used in subsequent sections to simplify iterated limits.
5.1. Proposition. Suppose that

\[(a_i)_{i \in I}, \quad (b_i)_{i \in I}\]

are bounded families and \(\mathcal{U}\) is an ultrafilter on \(I\). Then, whenever \(t(x, y)\) is a real-valued \(L\)-term and \(c, d\) are such that

\[t(c, d) = \lim_{i, \mathcal{U}} \lim_{j, \mathcal{U}} t(a_i, a_j),\]

there exist \(i(0), i(1), \ldots\) in \(I\) such that

\[t(c, d) = \lim_{m < n} t(a_{i(m)}, a_{i(n)}).\]

Proof. By hypothesis, for every \(\epsilon > 0\) there exists \(U_\epsilon \in \mathcal{U}\) such that

\[i \in U_\epsilon \quad \text{implies} \quad |\lim_{j, \mathcal{U}} t(a_i, b_j) - t(c, d)| \leq \epsilon.\]

Also, for every \(i \in I\) and every \(\epsilon > 0\) there exists \(V_\epsilon^i \in \mathcal{U}\) such that

\[j \in V_\epsilon^i \quad \text{implies} \quad |t(a_i, b_j) - \lim_{j, \mathcal{U}} t(a_i, b_j)| \leq \epsilon.\]

Choose \(i(0), i(1), \ldots\) such that,

\[i(0) \in U_{1/2},\]
\[i(n + 1) \in U_{1/2^{n+1}} \cap V_{1/2}^{i(0)} \cap \cdots \cap V_{1/2^{n+1}}^{i(n)}.

Then,

\[m < n \quad \text{implies} \quad |t(a_{i(m)}, b_{i(n)}) - t(c, d)| \leq 1/2^m.\]

\[\square\]

5.2. Proposition. Suppose that

\[(a_{0, i})_{i \in I}, \ldots, (a_{k, i})_{i \in I}\]

are bounded families and \(\mathcal{U}\) is an ultrafilter on \(I\). Then, whenever \(t(x_1, \ldots, x_k)\) is a real-valued \(L\)-term and \(c_0, \ldots, c_k\) are such that

\[t(c_0, \ldots, c_k) = \lim_{i_0, \mathcal{U}} \lim_{i_k, \mathcal{U}} t(a_{i_0}, \ldots, a_{i_k}),\]

there exist \(i(0), i(1), \ldots\) in \(I\) such that

\[t(c_0, \ldots, c_k) = \lim_{n_0 < \cdots < n_k} t(a_{i(n_0)}, \ldots, a_{i(n_k)}).\]

Proof. Using Proposition 5.1 and induction on \(k\).

\[\square\]

5.3. Remark. The continuity of \(t\) on bounded sets did not play a role in the proof of Propositions 5.2 and 5.2. The fundamental requirement in these propositions is that the limits mentioned exist. To guarantee this, it suffices to assume that \(t\) is a function such that the image under \(t\) of a bounded set is a compact Hausdorff space. (See [Iov99] for a somewhat more general version of Proposition 5.2.)
6. Spreading Models

The construction of spreading model was introduced in analysis by A. Brunel and L. Sucheston in the early 1970’s [Bru74, BS74, BS76], and since then has had fundamental applications in Banach space geometry. See [Ode02] for a survey and pointers to the extensive literature. See also the remarks in Section 11.) In this section we use model theoretic language to generalize this construction to arbitrary normed space structures.

Suppose that \((\bar{a}_i)_{i \in I}\) is a bounded family in a set \(A\) and that \(\mathcal{U}\) is an ultrafilter on \(I\). By Proposition 4.3, for any ordinal \(\gamma\) we can construct a sequence

\[(\bar{c}_\alpha \mid \alpha < \gamma)\]

such that for every \(\beta < \gamma\),

\[\text{tp}_\Phi(\bar{c}_\beta / A \cup \{\bar{c}_\alpha \mid \alpha < \beta\}) = \lim_{i, \mathcal{U}} \text{tp}_\Phi(\bar{a}_i / A \cup \{\bar{c}_\alpha \mid \alpha < \beta\}).\]

Obviously, then

1. For every \(\beta < \gamma\), \(\text{tp}_\Phi(\bar{c}_\beta / A \cup \{\bar{c}_\alpha \mid \alpha < \beta\})\) is semidefinable over \(A\);
2. Whenever \(\beta < \beta' < \gamma\),

\[\text{tp}_\Phi(\bar{c}_\beta / A \cup \{\bar{c}_\alpha \mid \alpha < \beta\}) = \text{tp}_\Phi(\bar{c}_{\beta'} / A \cup \{\bar{c}_\alpha \mid \alpha < \beta\}).\]

6.1. Definition. Let \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\) be as above. The substructure of the monster model generated by the sequence \((\bar{c}_\alpha \mid \alpha < \gamma)\) and \(A\) is called a spreading model. More specifically, we call this structure a \((\Phi, \gamma)\)-spreading model over \(A\), generated by \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\). The sequence \((\bar{c}_\alpha \mid \alpha < \gamma)\) is called a fundamental sequence for this spreading model.

6.2. Definition. Let \(A\) be a set and let \(\gamma\) be an ordinal. A sequence \((\bar{c}_\alpha \mid \alpha < \gamma)\) is said to be \(\Phi\)-indiscernible over \(A\) if

\[\text{tp}_\Phi(\bar{c}_{\alpha(0)}, \ldots, \bar{c}_{\alpha(n)}/A) = \text{tp}_\Phi(\bar{c}_0, \ldots, \bar{c}_n/A),\]

whenever \(\alpha(0) < \cdots < \alpha(n) < \gamma\).

6.3. Proposition. If \((\bar{c}_\alpha \mid \alpha < \gamma)\) is a fundamental sequence for a \((\Phi, \gamma)\)-spreading model over \(A\), then \((\bar{c}_\alpha \mid \alpha < \gamma)\) is \(\Phi\)-indiscernible over \(A\).

Proof. Suppose that \((\bar{c}_\alpha \mid \alpha < \gamma)\) is a fundamental sequence for the \((\Phi, \gamma)\)-spreading model over \(A\) generated by the family \((\bar{a}_i)_{i \in I}\) in \(A\) and the ultrafilter \(\mathcal{U}\). We prove by induction of \(n\) that if \(\alpha(0) < \cdots < \alpha(n) < \gamma\),

\[\text{tp}_\Phi(\bar{c}_{\alpha(n)}/A \cup \{\bar{c}_{\alpha(0)}, \ldots, \bar{c}_{\alpha(n-1)}\}) = \text{tp}_\Phi(\bar{c}_n/A \cup \{\bar{c}_0, \ldots, \bar{c}_{n-1}\}).\]

For \(n = 0\), the assertion is immediate. Assume, inductively, that the equation holds for \(n-1\)-tuples. Then we have the following equalities (the second one follows from the induction assumption):

\[\text{tp}_\Phi(\bar{c}_{\alpha(n)}/A \cup \{\bar{c}_{\alpha(0)}, \ldots, \bar{c}_{\alpha(n-1)}\}) =\]
\[\lim_{i, \mathcal{U}} \text{tp}_\Phi(\bar{a}_i/A \cup \{\bar{c}_{\alpha(0)}, \ldots, \bar{c}_{\alpha(n-1)}\})\]
\[=\]
\[\lim_{i, \mathcal{U}} \text{tp}_\Phi(\bar{a}_i/A \cup \{\bar{c}_0, \ldots, \bar{c}_{n-1}\})\]
\[= \text{tp}_\Phi(\bar{c}_n/A \cup \{\bar{c}_0, \ldots, \bar{c}_{n-1}\}).\]

\(\square\)
6.4. COROLLARY. Suppose that \((c_\alpha \mid \alpha < \gamma)\) is a fundamental sequence for a \((\Phi, \gamma)\)-spreading model over \(A\) generated by \((a_i)_{i \in I}\) and \(U\). Then, if \(t(x_1, \ldots, x_k, \bar{y})\) is a real-valued \(L\)-term, for arbitrary \(\alpha(0) < \cdots < \alpha(k-1) < \gamma\) and \(\bar{a} \in A\), we have
\[
\lim_{i_k, \bar{u}_0, \bar{u}}^{i_k, \bar{u}} t(a_{i_0}, \ldots, a_{i_k}, \bar{a}).
\]

PROOF. By Proposition 6.3. \(\square\)

6.5. COROLLARY. Suppose that \((a_i)_{i \in I}\) is a bounded family in a set \(A\), \(U\) is an ultrafilter on \(I\), and \((c_\alpha \mid \alpha < \gamma)\) is a fundamental sequence for a \((\Phi, \gamma)\)-spreading model over \(A\) generated by \((a_i)_{i \in I}\) and \(U\). Then, if \(t(x_1, \ldots, x_k, \bar{y})\) is a real-valued \(L\)-term, for arbitrary \(\alpha(0) < \cdots < \alpha(k-1) < \gamma\) and \(\bar{a} \in A\) there exists a sequence \((i(n))_{n \in \mathbb{N}}\) in \(I\) such that
\[
\lim_{n_k < \cdots < n_0}^{n_k, \bar{u}} t(a_{i(n_0)}, \ldots, a_{i(n_k)}, \bar{a}).
\]
Furthermore, if \(A\) is separable, the sequence \((i(n))_{n \in \mathbb{N}}\) can be taken to be independent of \(\bar{a}\).

PROOF. The first part of the proposition follows directly from Corollary 6.4 and 5.2; the “furthermore” statement follows by a standard diagonalization argument. \(\square\)

7. Heirs and Limits

7.1. DEFINITION. Suppose that \(A \subseteq B\), \(t(\bar{x})\) is a \(\Phi\)-type over \(A\), and \(\hat{t}(\bar{x})\) is an extension of \(t\) over \(B\). We will say that \(t\) is an heir of \(\hat{t}\) if whenever \(\varphi(\bar{x}) \in \hat{t}\) and \(\varphi'\) is an approximation of \(\varphi\), one has \(\varphi'(\bar{x}) \in t\).

The following observation, and the refinement of it given by Proposition 7.6 below, will underlie key arguments in the paper.

7.2. REMARK. If \(A\) is a set and \(\bar{a}, \bar{b}\) are tuples of elements (in the monster model), then \(tp_\Phi(\bar{a}/A \cup \{\bar{b}\})\) is an heir of \(tp_\Phi(\bar{a}/A)\) if and only if \(tp_\Phi(\bar{b}/A \cup \{\bar{a}\})\) is semidefinable over \(A\).

7.3. PROPOSITION. Suppose that \(t(\bar{x})\) is a \(\Phi\)-type over a model \(M\). Then \(t\) is \(\Phi\)-definable if and only if for every structure \(N\) with \(M \prec_\Phi N\) there exists a unique \(\Phi\)-type \(\hat{t}(\bar{x})\) over \(N\) such that \(\hat{t}\) is an heir of \(t\).

PROOF. By Propositions 3.2 and 3.4. \(\square\)

7.4. DEFINITION. Suppose that \(A \subseteq B\) and that \(t(\bar{x})\) is a type over \(B\) which is semidefinable over \(A\). If \((\bar{a}_i)_{i \in I}\) is a bounded family in \(A\) and \(U\) is an ultrafilter on \(I\) such that
\[
t = \lim_{i, \bar{u}}^{i, \bar{u}} \text{tp}_\Phi(\bar{a}_i/B)
\]
we will say that \(\text{tp}_\Phi(\bar{c}/A \cup \{\bar{b}\})\) is semidefinable over \(A\) through \((\bar{a}_i)_{i \in I}\) and \(U\).

7.5. DEFINITION. Suppose that \(\text{tp}_\Phi(\bar{c}/A \cup \{\bar{b}\})\) is an heir of \(\text{tp}_\Phi(\bar{c}/A)\). If \((\bar{a}_i)_{i \in I}\) is a bounded family in \(A\) and \(U\) is an ultrafilter on \(I\) such that
\[
\text{tp}_\Phi(\bar{c}/A \cup \{\bar{b}\}) = \lim_{i, \bar{u}}^{i, \bar{u}} \text{tp}_\Phi(\bar{c}/A \cup \{\bar{a}_i\}),
\]
we will say that \(\text{tp}_\Phi(\bar{c}/A \cup \{\bar{b}\})\) inherits from \(\text{tp}_\Phi(\bar{c}/A)\) through \((\bar{a}_i)_{i \in I}\) and \(U\).
Notice that the homogeneity of the monster model (and specifically, Proposition 2.4) ensures that this notion is well defined, i.e., whether \( \text{tp}_\Phi(c/A \cup \{\bar{b}\}) \) inherits from \( \text{tp}_\Phi(\bar{c}/A) \) through \((\bar{a})_{i \in I} \) and \( \mathcal{U} \) depends on \( \text{tp}_\Phi(\bar{c}/A \cup \{\bar{b}\}) \) (and \((\bar{a})_{i \in I} \) and \( \mathcal{U} \)), not directly on \( \bar{c} \).

7.6. **Proposition.** Let \((\bar{a})_{i \in I} \) be a bounded family in \( A \) and let \( \mathcal{U} \) be an ultrafilter on \( I \). Then the following conditions are equivalent:

1. \( \text{tp}_\Phi(\bar{b}/A \cup \{\bar{c}\}) \) is semidefinable over \( A \) through \((\bar{a})_{i \in I} \) and \( \mathcal{U} \);
2. \( \text{tp}_\Phi(\bar{c}/A \cup \{\bar{b}\}) \) inherits from \( A \) through \((\bar{a})_{i \in I} \) and \( \mathcal{U} \).

**Proof.** Immediate from the definitions. □

8. **Separation of Types**

8.1. **Definition.** Suppose that \( t_1(\bar{x}) \) and \( t_2(\bar{x}) \) are types over \( A \) and that \( \psi_1(\bar{x}, \bar{y}) \) and \( \psi_2(\bar{x}, \bar{y}) \) are \( L \)-formulas. We will say that \( (\psi_1, \psi_2) \) separates \((t_1, t_2)\) (or that \((t_1, t_2) \) is separated by \( (\psi_1, \psi_2)\)) if

- \{\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y})\} is not satisfiable, and
- there exists \( \bar{a} \in A \) such that \( \psi_1(\bar{x}, \bar{a}) \in t_1(\bar{x}) \) and \( \psi_2(\bar{x}, \bar{a}) \in t_2(\bar{x}) \).

In this case we say that \( (\psi_1, \psi_2) \) separates \((t_1, t_2)\) at \( \bar{a} \).

8.2. **Remark.** Suppose that \( t_1(\bar{x}), t_2(\bar{x}) \) are \( \Phi \)-types over \( A \). If \( t_1 \) and \( t_2 \) are distinct, there exist a \( \Phi \)-formula \( \varphi(\bar{x}, \bar{y}) \), an approximation \( \varphi' \) of \( \varphi \), and a tuple \( \bar{a} \in A \), such that

\( \varphi(\bar{x}, \bar{a}) \in t_1(\bar{x}), \quad \neg(\varphi'(\bar{x}, \bar{a})) \in t_2(\bar{x}). \)

Hence, two \( \Phi \)-types \( t_1(\bar{x}) \) and \( t_2(\bar{x}) \) are distinct if and only if \((t_1, t_2)\) is separated by a pair of \( \Phi \)-formulas.

8.3. **Proposition.** Suppose that \( t(\bar{x}) \) is a \( \Phi \)-type over \( A \), and that \( t_1, t_2 \) are extension of \( t \) such that

- \( t_1 \) and \( t_2 \) are heirs of \( t \),
- \((t_1, t_2) \) is separated by \( (\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y})) \) at \( \bar{b} \),
- \( t_1, t_2 \) inherit from \( t \) through \((\bar{b}_i)_{i \in I} \) and \( \mathcal{U} \), and
- \( t_1, t_2 \) are semidefinable over \( A \) through \((\bar{c}_j)_{j \in J} \) and \( \mathcal{V} \).

Then, if \( \hat{t} \) is an heir of \( t \), there exist extensions \( \hat{t}_1, \hat{t}_2 \) of \( \hat{t} \) such that

1. \( \hat{t}_1 \) and \( \hat{t}_2 \) are heirs of \( \hat{t} \);
2. \( \hat{t}_1, \hat{t}_2 \) is separated by \( (\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y})) \) at a tuple \( \bar{b}' \) with \( \text{tp}_\Phi(\bar{b}'/A) = \text{tp}_\Phi(\bar{b}/A) \);
3. \( \hat{t}_1, \hat{t}_2 \) inherit from \( \hat{t} \) through \((\bar{b}_i)_{i \in I} \) and \( \mathcal{U} \);
4. \( \hat{t}_1, \hat{t}_2 \) are semidefinable over \( A \) through \((\bar{c}_j)_{j \in J} \) and \( \mathcal{V} \).

**Proof.** Suppose that \( t_1 \) and \( t_2 \) are heirs of \( t \) over \( A \cup \{\bar{b}\} \), and that \((t_1, t_2)\) is separated by \( (\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y})) \) at \( \bar{b} \), i.e., \( \psi_1(\bar{x}, \bar{b}) \in t_1 \) and \( \psi_2(\bar{x}, \bar{b}) \in t_2 \). Fix a realization \( \bar{c}_1 \) of \( t_1 \) and a realization \( \bar{c}_2 \) of \( t_2 \).

Suppose that \( A \subseteq D \) and \( \hat{t}(\bar{x}) \) is an heir of \( t(\bar{x}) \) over \( D \). Let \( \rho(\bar{x}, \bar{a}, \bar{d}) \) be an arbitrary \( \Phi(D) \)-formula in \( \hat{t} \), where \( \bar{a} \) is a list of the parameters in \( A \) that occur in \( \rho \) and \( \bar{d} \) is a list of all the parameters of \( D \setminus A \) that occur in \( \rho \). Since \( \hat{t} \) is an heir of \( t \), for every approximation \( \rho' \) of \( \rho \) there exists a tuple \( \bar{d}' \) in \( A \) such that \( \rho'(\bar{x}, \bar{a}, \bar{d}') \) is in \( t \). Fix such \( \rho' \) and \( \bar{d}' \). Then,

\( \rho'(\bar{c}_1, \bar{a}, \bar{d}') \land \psi_1(\bar{c}_1, \bar{b}), \)
\[ \rho'(\tilde{c}^2, \tilde{a}, \tilde{d}') \land \psi_2(\tilde{c}^2, \tilde{b}). \]

We claim that if \( \sigma(\tilde{x}, \tilde{b}, \tilde{a}, \tilde{d}) \) and \( \tau(\tilde{x}, \tilde{b}, \tilde{a}, \tilde{d}) \) are \( \Phi(D) \)-formulas (where \( \tilde{a} \) and \( \tilde{d} \) are as above and all the parameters of \( \sigma \) and \( \tau \) are exhibited) and \( \sigma', \tau' \) are approximations of \( \sigma, \tau \), respectively, such that

\[ \{ i \in I \mid \sigma'(\tilde{x}, \tilde{b}_i, \tilde{a}, \tilde{d}') \} \notin \mathbb{U} \]

and

\[ \{ j \in J \mid \tau'(\tilde{c}_j, \tilde{b}, \tilde{a}, \tilde{d}') \} \notin \mathbb{V}, \]

then we have

\[ (\dagger) \quad \rho'(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}') \land \psi_1(\tilde{c}^2, \tilde{b}) \land \text{neg}((\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}')) \land \text{neg}(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}'). \]

\[ (\ddagger) \quad \rho'(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}') \land \psi_2(\tilde{c}^2, \tilde{b}) \land \text{neg}(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}'). \]

Indeed, if \( \text{neg}(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}') \) were not true, we would have \( \sigma(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}'); \) but then, by the hypothesis on \( (\tilde{b}_i)_{i \in I} \) and \( \mathbb{U} \), for every formula \( \sigma^* \) with \( \sigma < \sigma^* < \sigma' \) we would have

\[ \{ i \in I \mid \sigma^*(\tilde{x}, \tilde{b}_i, \tilde{a}, \tilde{d}') \} \in \mathbb{U}, \]

which contradicts the choice of \( \sigma \). Similarly, if \( \text{neg}(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}') \) were not true, we would have \( \tau(\tilde{c}^2, \tilde{b}, \tilde{a}, \tilde{d}'); \) and then, by the hypothesis on \( (\tilde{c}_j)_{j \in J} \) and \( \mathbb{V} \), if \( \tau^* \) is such that \( \tau < \tau^* < \tau' \) we would have

\[ \{ j \in J \mid \tau^*(\tilde{c}_j, \tilde{b}, \tilde{a}, \tilde{d}') \} \in \mathbb{V}, \]

contradicting the choice of \( \tau \). This proves \((\dagger)\) of the claim. The proof of \((\ddagger)\) is analogous.

Since \( \rho, \sigma, \) and \( \tau \) are arbitrary, by the saturation and homogeneity of the monster model, there exists a \( \Phi(A) \)-isomorphism \( f \) such that

(i) \( \rho(\tilde{c}^1, \tilde{a}, f(\tilde{d})), \) whenever \( \rho(\tilde{x}, \tilde{a}, \tilde{d}) \in \tilde{t}, \)

(ii) \( \rho(\tilde{c}^2, \tilde{a}, f(\tilde{d})), \) whenever \( \rho(\tilde{x}, \tilde{a}, \tilde{d}) \in \tilde{t}, \)

(iii) \( \psi_1(\tilde{c}^2, \tilde{b}), \)

(iv) \( \psi_2(\tilde{c}^1, \tilde{b}), \)

(v) \( \{ i \in I \mid \sigma'(\tilde{x}, \tilde{b}_i, \tilde{a}, f(\tilde{d})) \} \in \mathbb{U}, \) whenever \( \sigma(\tilde{c}^1, \tilde{b}, \tilde{a}, f(\tilde{d})); \) and \( \sigma < \sigma' \),

(vi) \( \{ i \in I \mid \sigma'(\tilde{x}, \tilde{b}_i, \tilde{a}, f(\tilde{d})) \} \in \mathbb{U}, \) whenever \( \sigma(\tilde{c}^2, \tilde{b}, \tilde{a}, f(\tilde{d})); \) and \( \sigma < \sigma' \),

(vii) \( \{ j \in J \mid \tau'(\tilde{c}_j, \tilde{b}, \tilde{a}, f(\tilde{d})) \} \in \mathbb{V}, \) whenever \( \tau(\tilde{c}^2, \tilde{b}, \tilde{a}, f(\tilde{d})); \) and \( \tau < \tau' \),

(viii) \( \{ j \in J \mid \tau'(\tilde{c}_j, \tilde{b}, \tilde{a}, f(\tilde{d})) \} \in \mathbb{V}, \) whenever \( \tau(\tilde{c}^2, \tilde{b}, \tilde{a}, f(\tilde{d})); \) and \( \tau < \tau' \).

Let

\[
\tilde{b}' = f^{-1}(\tilde{b}), \]

\[
\tilde{t}_1 = f^{-1}\left( \text{tp}_\Phi(\tilde{c}^3/f(D) \cup \{\tilde{b}\}) \right) = \text{tp}_\Phi(f^{-1}(\tilde{c}^1)/D \cup \{\tilde{b}'\}), \]

\[
\tilde{t}_2 = f^{-1}\left( \text{tp}_\Phi(\tilde{c}^2/f(D) \cup \{\tilde{b}\}) \right) = \text{tp}_\Phi(f^{-1}(\tilde{c}^2)/D \cup \{\tilde{b}'\}). \]

Then (i)–(viii) can be rewritten as:

(i)' \( \rho(\tilde{x}, \tilde{a}, \tilde{d}) \in \tilde{t}_1, \) whenever \( \rho(\tilde{x}, \tilde{a}, \tilde{d}) \in \tilde{t}, \)

(ii)' \( \rho(\tilde{x}, \tilde{a}, \tilde{d}) \in \tilde{t}_2, \) whenever \( \rho(\tilde{x}, \tilde{a}, \tilde{d}) \in \tilde{t}, \)

(iii)' \( \psi_1(\tilde{x}, \tilde{b}') \in \tilde{t}_1, \)

(iv)' \( \psi_2(\tilde{x}, \tilde{b}') \in \tilde{t}_2, \)

(v)' \( \{ i \in I \mid \sigma'(\tilde{x}, \tilde{b}_i, \tilde{a}, \tilde{d}) \} \in \mathbb{U}, \) whenever \( \sigma(\tilde{x}, \tilde{b}', \tilde{a}, \tilde{d}) \in \tilde{t}_1 \) and \( \sigma < \sigma' \),

(vi)' \( \{ i \in I \mid \sigma'(\tilde{x}, \tilde{b}_i, \tilde{a}, \tilde{d}) \} \in \mathbb{U}, \) whenever \( \sigma(\tilde{x}, \tilde{b}', \tilde{a}, \tilde{d}) \in \tilde{t}_2 \) and \( \sigma < \sigma' \),

(vii)' \( \{ j \in J \mid \tau'(\tilde{c}_j, \tilde{b}, \tilde{a}, \tilde{d}) \} \in \mathbb{V}, \) whenever \( \tau(\tilde{x}, \tilde{b}', \tilde{a}, \tilde{d}) \in \tilde{t}_1 \) and \( \tau < \tau' \).
(viii)' \{ j \in J \mid \tau'(\bar{c}_i, \bar{b}, \bar{a}, \bar{d}) \} \in \mathcal{V}, \text{ whenever } \tau(\bar{x}, \bar{b}', \bar{a}, \bar{d}) \in \hat{t}_2 \text{ and } \tau < \tau'.

By (i)' and (ii)', \hat{t}_1 \text{ and } \hat{t}_2 \text{ extend } \hat{t} \text{ over } D \cup \{ \bar{d} \}; \text{ by (iii)' and (iv)', } (\hat{t}_1, \hat{t}_2) \text{ is separated by } (\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y})) \text{ at } \bar{b}'; \text{ by (v)' and (vi)', } \hat{t}_1 \text{ and } \hat{t}_2 \text{ inherit from } \hat{t} \text{ through } (\bar{b}_i)_{i \in I} \text{ and } \mathcal{U}; \text{ by (vii)' and (viii)', } \hat{t}_1 \text{ and } \hat{t}_2 \text{ are semidefinable over } A \text{ through } (\bar{c}_i)_{j \in J} \text{ and } \mathcal{V}.

\square

8.4. Proposition. Let \((\bar{a}_i)_{i \in I}\), be a bounded family in A, and let \(\mathcal{U}\) be an ultrafilter on I. Let

\[
\begin{align*}
t(\bar{x}) &= t_p(\bar{b}/A), \\
u(\bar{y}) &= t_p(\bar{c}/A).
\end{align*}
\]

Then the following conditions are equivalent:

(1) \(t(\bar{x})\) has two extensions \(t_1(\bar{x}), t_2(\bar{x})\) over \(A \cup \{\bar{c}\}\) such that
(a) \(t_1(\bar{x})\) and \(t_2(\bar{x})\) are semidefinable over \(A\) through \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\),
(b) \((t_1, t_2)\) is separated by \((\psi_1(\bar{x}, \bar{y}), \psi_2(\bar{x}, \bar{y}))\) at \(\bar{c}\);

(2) \(u(\bar{y})\) has two extensions \(u_1(\bar{y}), u_2(\bar{y})\) over \(A \cup \{\bar{b}\}\) such that
(a) \(u_1(\bar{x})\) and \(u_2(\bar{x})\) inherit from \(A\) through \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\),
(b) \((u_1, u_2)\) is separated by \((\psi_1(\bar{y}, \bar{x}), \psi_2(\bar{y}, \bar{x}))\) at \(\bar{b}\).

Proof. We prove (1) \(\Rightarrow\) (2); the proof of (2) \(\Rightarrow\) (1) is analogous.

Fix \(t_1(\bar{x})\) and \(t_2(\bar{x})\) as given by (1), and let \(\bar{b}_1\) and \(\bar{b}_2\) be realizations of \(t_1\) and \(t_2\), respectively. Then

\[
\begin{align*}
t_p(\bar{b}_1/A \cup \{\bar{c}\}) = t_p(\bar{b}_2/A \cup \{\bar{c}\}),
\end{align*}
\]

are semidefinable over \(A\) through \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\), and hence, by Proposition 7.6,

\[
\begin{align*}
t_p(\bar{c}/A \cup \{\bar{b}_1\}) = t_p(\bar{c}/A \cup \{\bar{b}_2\})
\end{align*}
\]

inherit from \(A\) through \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\). Since \(t_p(\bar{b}_1/A) = t_p(\bar{b}_2/A) = t\), there exists a \(\Phi(A)\)-isomorphism \(f\) such that \(f(\bar{b}_1) = \bar{b}_2\). Then,

\[
\begin{align*}
\phi_p(f(\bar{c})/A \cup \{\bar{b}_2\}) = f(\phi_p(\bar{c}/M \cup \{\bar{b}_1\})),
\end{align*}
\]

so \(\phi_p(f(\bar{c})/A \cup \{\bar{b}_2\})\) inherits from \(A\) through \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\). We have \(\psi_1(\bar{b}_2, f(\bar{c}))\), since \(\psi_1(\bar{b}_1, \bar{c})\). We are also given \(\psi_2(\bar{b}_2, \bar{c})\). Hence, the \(\Phi\)-types

\[
\begin{align*}
\hat{u}_1(\bar{y}) &= \phi_p(f(\bar{c})/A \cup \{\bar{b}_2\}), \\
\hat{u}_2(\bar{y}) &= \phi_p(\bar{c}/A \cup \{\bar{b}_2\})
\end{align*}
\]

are extensions of \(\phi_p(\bar{c}/A) = u(\bar{y})\) which inherit from \(A\) through \((\bar{a}_i)_{i \in I}\) and \(\mathcal{U}\). Furthermore, \((\hat{u}_1, \hat{u}_2)\) is separated by \((\psi_1(\bar{y}, \bar{x}), \psi_2(\bar{y}, \bar{x}))\) at \(\bar{b}_2\). Thus, condition (2) of the proposition follows by taking an \(\Phi(A)\)-isomorphism \(g\) such that \(g(\bar{b}_2) = b\), and defining

\[
\begin{align*}
u_1(\bar{y}) &= g(\hat{u}_1(\bar{y})) = \phi_p(g(f(\bar{c}))/A \cup \{\bar{b}\})), \\
u_2(\bar{y}) &= g(\hat{u}_1(\bar{y})) = \phi_p(g(\bar{c}))/A \cup \{\bar{b}\}).
\end{align*}
\]

\square
8.5. Proposition. Suppose that $M$ is a model such that every $\Phi$-type over $M$ is definable. Then for every $\Phi$-type $t$ over $M$ and every structure $N$ with $M \prec_{\Phi} N$ there exists a unique $\Phi$-type $t(\bar{x})$ over $N$ such that $t$ extends $t$ and is semidefinable over the universe of $M$.

Proof. By Propositions 7.3 and 8.4.

9. Two Ranks

Ordinal ranks are a fundamental tool in contemporary model theory, and have played an important role in Banach space theory. For an excellent survey of applications of ordinal ranks in Banach space theory, we refer the reader to [Ode].

9.1. Definition. Suppose that
\begin{itemize}
  \item $a = (\bar{a}_i)_{i \in I}$ is a bounded family in $A$ and $U$ is an ultrafilter on $I$,
  \item $b = (\bar{b}_j)_{j \in J}$ is a bounded family in $A$ and $V$ is an ultrafilter on $J$,
  \item $t(\bar{x})$ and $u(\bar{y})$ are $\Phi$-types over $A$ such that $t$ is semidefinable over $A$ through $(\bar{b}_j)_{j \in J}$ and $V$,
  \item $\varphi(\bar{x}, \bar{y})$ is a $\Phi$-formula and $\varphi'$ is an approximation of $\varphi$.
\end{itemize}

We define the heir rank

$$R_h[t, u, \varphi, \varphi', a, b, U, V].$$

The heir rank $R_h[t, u, \varphi, \varphi', a, b, U, V]$ is either an ordinal or $\infty$. The relation

$$R_h[t, u, \varphi, \varphi', a, b, U, V], \geq \alpha,$$

where $\alpha$ is an ordinal, is defined by induction on $\alpha$ as follows.

1. $R_h[t, u, \varphi, \varphi', a, b, U, V] \geq 0$ in every case;
2. $R_h[t, u, \varphi, \varphi', a, b, U, V] \geq \delta$ when $\delta$ is a limit ordinal if
   $$R_h[t, u, \varphi, \varphi', a, b, U, V], \geq \beta, \quad \text{for every } \beta < \delta;$$
3. $R_h[t, u, \varphi, \varphi', a, b, U, V] \geq \alpha + 1$ if there exist sets $A_1, A_2$, tuples $\bar{c}_1, \bar{c}_2, \bar{b}$, and $\Phi$-types $t_1(\bar{x}), t_2(\bar{x}), u_1(\bar{y}), u_2(\bar{y})$ such that
   (a) $A \subseteq A_1$, and $\bar{c}_1, \bar{b}$ are in $A_1$,
   (b) $A \subseteq A_2$, and $\bar{c}_2, \bar{b}$ are in $A_2$,
   (c) $t_1, u_1$ are over $A_1$,
   (d) $t_2, u_2$ are over $A_2$,
   (e) $t = t_1(\bar{x}) \cup \{\bar{b}: \in t_1$ and $t \subseteq t_1(\bar{x}) \cup \{\bar{b}\} \subseteq t_2$,
   (f) $u = t_2(\bar{b}/A) \subseteq u_2$ and $u \subseteq t_1(\bar{b}/A) \subseteq u_2$,
   (g) $t_1$ and $t_2$ inherit from $A$ through $(\bar{a}_i)_{i \in I}$ and $U$,
   (h) $t_1$ and $t_2$ are semidefinable over $A$ through $(\bar{b}_j)_{j \in J}$ and $V$,
   (i) $\varphi(\bar{x}, \bar{b}) \in t_1$ and neg$(\varphi')(\bar{x}, \bar{b}) \in t_2$,
   (j) $R_h[t_1, u_1, \varphi, \varphi', a, b, U, V], \geq \alpha$ and $R_h[t_2, u_2, \varphi, \varphi', a, b, U, V], \geq \alpha$.

We write

$$R_h[t, u, \varphi, \varphi', a, b, U, V] = \alpha$$

if

$$R_h[t, u, \varphi, \varphi', a, b, U, V] \geq \alpha \quad \text{but} \quad R_h[t, u, \varphi, \varphi', a, b, U, V] \not\geq \alpha + 1.$$

If there is no such ordinal $\alpha$, i.e., if $R_h[t, u, \varphi, \varphi', a, b, U, V] \geq \alpha$ for every $\alpha$, we say that $R_h[t, u, \varphi, \varphi', a, b, U, V]$ is unbounded, and write

$$R_h[t, u, \varphi, \varphi', a, b, U, V] = \infty.$$
9.2. **Proposition.** Suppose that \( t(\bar{x}) \) is a \( \Phi \)-type over a model \( \mathcal{M} \) and \( t \) is semidefinable over \( \mathcal{M} \) through \( \bar{b} = (b_j)_{j \in J} \) and \( \mathcal{V} \). Then, one and only one of the following conditions holds:

1. \( t \) is \( \Phi \)-definable, in which case \( R_h[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] = 0 \) for every \( \Phi \)-type \( u(\bar{y}) \), every bounded family \( \bar{a} = (\bar{a}_i)_{i \in I} \), any ultrafilter \( \mathcal{U} \) on \( I \), any \( \Phi \)-formula \( \varphi(\bar{x},\bar{y}) \), and any approximation \( \varphi' \) of \( \varphi \);
2. There exist a \( \Phi \)-type \( u(\bar{x}) \), a bounded family \( \bar{a} = (\bar{a}_i)_{i \in I} \), an ultrafilter \( \mathcal{U} \) on \( I \), a \( \Phi \)-formula \( \varphi(\bar{x},\bar{y}) \), and an approximation \( \varphi' \) of \( \varphi \) such that
   \[
   R_h[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] = \infty.
   \]

**Proof.** If \( t \) is definable, then, by Proposition 7.3, \( R_h[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \neq 1 \) for any \( \Phi \)-type \( u(\bar{y}) \), any bounded family \( \bar{a} = (\bar{a}_i)_{i \in I} \), every ultrafilter \( \mathcal{U} \) on \( I \), every \( \Phi \)-formula \( \varphi(\bar{x},\bar{y}) \), and every approximation \( \varphi' \) of \( \varphi \). If \( t \) is not definable, Proposition 7.3 provides \( u, \varphi, \varphi', a, a \), and \( \mathcal{U} \) such that \( R_h[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq 1 \). Then, one shows, by induction on \( \alpha \), that \( R_h[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \alpha \) for every ordinal \( \alpha \): when \( \alpha \) is limit, the induction is trivial; at the successor stage, one invokes Proposition 8.3. \( \Box \)

9.3. **Definition.** Suppose that

- \( \bar{a} = (\bar{a}_i)_{i \in I} \) is a bounded family in \( A \) and \( \mathcal{U} \) is an ultrafilter on \( I \),
- \( \bar{b} = (b_j)_{j \in J} \) is a bounded family in \( A \) and \( \mathcal{V} \) is an ultrafilter on \( J \),
- \( t(\bar{x}) \) and \( u(\bar{y}) \) are \( \Phi \)-types over \( A \) such that \( u \) is semidefinable over \( A \) through \( (\bar{b})_{j \in J} \) and \( \mathcal{V} \),
- \( \varphi(\bar{x},\bar{y}) \) is a \( \Phi \)-formula and \( \varphi' \) is an approximation of \( \varphi \).

We define the **coheir rank**

\[
R_c[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}].
\]

The coheir rank \( R_c[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \) is either an ordinal or \( \infty \). The relation

\[
R_c[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \alpha,
\]

where \( \alpha \) is an ordinal, is defined by induction on \( \alpha \) as follows.

1. \( R_c[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq 0 \) in every case;
2. \( R_c[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \delta \) when \( \delta \) is a limit ordinal if
   \[
   R_c[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \beta, \text{ for every } \beta < \delta;
   \]
3. \( R_h[t,u,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \alpha + 1 \) if there exist sets \( A_1, A_2 \), tuples \( \bar{c}_1, \bar{c}_2, \bar{b} \), and \( \Phi \)-types \( t_1(\bar{x}), t_2(\bar{x}), u_1(\bar{y}), u_2(\bar{y}) \) such that
   - \( A \subseteq A_1 \), and \( \bar{c}_1, \bar{b} \) are in \( A_1 \),
   - \( A \subseteq A_2 \), and \( \bar{c}_2, \bar{b} \) are in \( A_2 \),
   - \( t_1, u_1 \) are over \( A_1 \),
   - \( t_2, u_2 \) are over \( A_2 \),
   - \( t \subseteq \text{tp}_\Phi(\bar{c}_1/A \cup \{\bar{b}\}) \subseteq t_1 \) and \( t \subseteq \text{tp}_\Phi(\bar{c}_2/A \cup \{\bar{b}\}) \subseteq t_2 \),
   - \( u \subseteq \text{tp}_\Phi(\bar{b}/A) \subseteq u_1 \) and \( u \subseteq \text{tp}_\Phi(\bar{b}/A) \subseteq u_2 \),
   - \( t_1 \) and \( t_2 \) are semidefinable over \( A \) through \( (\bar{a}_i)_{i \in I} \) and \( \mathcal{U} \),
   - \( u_1 \) and \( u_2 \) are semidefinable over \( A \) through \( (b_j)_{j \in J} \) and \( \mathcal{V} \),
   - \( \varphi(\bar{x},\bar{b}) \in t_1 \) and \( \neg \varphi'(\bar{x},\bar{b}) \in t_2 \),
   - \( R_c[t_1,u_1,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \alpha \) and \( R_c[t_2,u_2,\varphi,\varphi',a,b,\mathcal{U},\mathcal{V}] \geq \alpha \).
We write
\[ R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] = \alpha \]
if
\[ R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq \alpha \quad \text{but} \quad R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \not\geq \alpha + 1. \]
If there is no such ordinal \( \alpha \), i.e., if \( R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq \alpha \) for every \( \alpha \), we say that \( R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \) is unbounded, and write
\[ R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] = \infty. \]
Recall that if \( \varphi \) is a positive bounded formula and \( \varphi' \) is an approximation of \( \varphi \), then \( \neg(\varphi) \) is an approximation of \( \neg(\varphi') \).

9.4. **Proposition.** Suppose that
- \( a = (a_i)_{i \in I} \) is a bounded family in \( A \) and \( \mathcal{U} \) is an ultrafilter on \( I \),
- \( b = (b_j)_{j \in J} \) is a bounded family in \( A \) and \( \mathcal{V} \) is an ultrafilter on \( J \),
- \( t(\bar{x}) \) and \( u(\bar{y}) \) are \( \Phi \)-types over \( A \) such that \( u \) is semidefinable over \( A \) through \((\bar{b})_{j \in J} \) and \( \mathcal{V} \),
- \( \varphi(\bar{x}, \bar{y}) \) is a \( \Phi \)-formula and \( \varphi' \) is an approximation of \( \varphi \).

Then, if \( \psi(\bar{y}, \bar{x}) \) is defined as \( \varphi(\bar{x}, \bar{y}) \), we have
\[ R_h[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] = R_c[u, t, \neg(\varphi'), \neg(\psi), a, b, \mathcal{U}, \mathcal{V}]. \]

**Proof.** One proves the proposition by showing that the following implications hold for every ordinal \( \alpha \):
\[ R_h[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq \alpha \quad \text{implies} \quad R_c[u, t, \neg(\varphi'), \neg(\psi), a, b, \mathcal{U}, \mathcal{V}] \geq \alpha \]
and
\[ R_c[u, t, \neg(\varphi'), \neg(\psi), a, b, \mathcal{U}, \mathcal{V}] \geq \alpha \quad \text{implies} \quad R_h[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq \alpha. \]
Both implications are proved by induction on \( \alpha \); the cases when \( \alpha = 0 \) or \( \alpha \) is a limit ordinal are immediate, and the successor case is given by Proposition 8.4.

9.5. **Corollary.** Suppose that \( t(\bar{x}) \) is a \( \Phi \)-type over a model \( \mathcal{M} \) and \( t \) is semidefinable over \( \mathcal{M} \) through \( b = (b_j)_{j \in J} \) and \( \mathcal{V} \). Then, one and only one of the following conditions holds:

1. \( t \) is \( \Phi \)-definable, in which case \( R_c[u, t, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] = 0 \) for every \( \Phi \)-type \( u(\bar{y}) \), every bounded family \( a = (a_i)_{i \in I} \), every ultrafilter \( \mathcal{U} \) on \( I \), every \( \Phi \)-formula \( \varphi(\bar{x}, \bar{y}) \), and every approximation \( \varphi' \) of \( \varphi \);
2. There exist a \( \Phi \)-type \( u(\bar{x}) \), a bounded family \( a = (a_i)_{i \in I} \), an ultrafilter \( \mathcal{U} \) on \( I \), a \( \Phi \)-formula \( \varphi(\bar{x}, \bar{y}) \), and an approximation \( \varphi' \) of \( \varphi \) such that
\[ R_c[u, t, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] = \infty. \]

**Proof.** From Propositions 9.2 and 9.4.

10. **Ranks and Spreading Models**

10.1. **Lemma.** Suppose that

- \( a = (a_i)_{i \in I} \) is a bounded family in \( A \) and \( \mathcal{U} \) is an ultrafilter on \( I \),
- \( b = (b_j)_{j \in J} \) is a bounded family in \( A \) and \( \mathcal{V} \) is an ultrafilter on \( J \),
- \( t(\bar{x}) \) and \( u(\bar{y}) \) are \( \Phi \)-types over \( A \) such that \( u \) is semidefinable over \( A \) through \((\bar{b})_{j \in J} \) and \( \mathcal{V} \),
\[ \varphi(x, y) \text{ is a } \Phi\text{-formula and } \varphi' \text{ is an approximation of } \varphi. \]

Then, if
\[ R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq \omega, \]
for every ordinal \( \gamma \), there exists a family of sets
\[ (A_\eta \mid \eta \in 2^{<\gamma}) \]
and families of tuples
\[ (\bar{c}_{\xi, \alpha} \mid \xi \in 2^\gamma, \alpha < \gamma), \quad (\bar{b}_\eta \mid \eta \in 2^{<\gamma}) \]
such that:

1. \( \bar{c}_{\xi, \alpha} \) realizes \( t(\bar{x}) \), for \( \xi \in 2^\alpha \) and \( \alpha < \gamma \);
2. \( \bar{b}_\eta \) realizes \( u(\bar{y}) \), for \( \eta \in 2^{<\alpha} \);
3. If \( \xi \in 2^\gamma, \alpha < \gamma, \) and \( \eta \in 2^\alpha \) is such that \( \eta = \xi \upharpoonright \alpha \), then
   \[ \cdot \varphi(\bar{c}_{\xi, \alpha}, \bar{b}_\eta), \text{ if } \xi(\alpha) = 0, \]
   \[ \cdot \neg \varphi(\varphi')(\bar{c}_{\xi, \alpha}, \bar{b}_\eta), \text{ if } \xi(\alpha) = 1; \]
4. \( A_0 = A \cup \{\bar{b}_0\} \);
5. If \( \xi \in 2^\gamma, \alpha < \alpha + 1 < \gamma, \) and \( \eta \in 2^\alpha \) is such that \( \eta = \xi \upharpoonright \alpha \), then
   \[ \cdot A_{\eta \uparrow 0} = A_{\eta} \cup \{\bar{c}_{\xi, \alpha}, \bar{b}_{\eta \uparrow 0}\}, \text{ if } \xi(\alpha) = 0, \]
   \[ \cdot A_{\eta \uparrow 1} = A_{\eta} \cup \{\bar{c}_{\xi, \alpha}, \bar{b}_{\eta \uparrow 1}\}, \text{ if } \xi(\alpha) = 1; \]
6. If \( \xi \in 2^\gamma, \alpha < \alpha' < \gamma, \) and \( \eta \in 2^\alpha \), \( \eta' \in 2^{\alpha'} \) are such that \( \eta' = \xi \upharpoonright \alpha' \) and \( \eta = \eta' \upharpoonright \alpha \), then
   \[ \text{tp}_\Phi(\bar{c}_{\xi, \alpha}/A_\alpha) \subseteq \text{tp}_\Phi(\bar{c}_{\xi, \alpha'}/A_{\alpha'}); \]
7. If \( \xi \in 2^\gamma, \alpha < \gamma, \) and \( \eta \in 2^\alpha \), then \( \text{tp}_\Phi(\bar{c}_{\xi, \alpha}/A_\alpha) \) is semidefinite over \( A \)
   through \( (\bar{a}_i)_{i \in I} \) and \( \mathcal{U} \);
8. If \( \delta < \gamma, \delta \) is a limit ordinal, and \( \eta \in 2^\delta \), then
   \[ A_\eta = \bigcup_{\alpha < \delta} A_{\eta \uparrow \alpha}. \]

PROOF. By the saturation of the monster model, it suffices to prove that for every \( n < \omega \) there exists a family of sets
\[ (A_\eta \mid \eta \in 2^{<n}) \]
and families of tuples
\[ (\bar{c}_{\xi, m} \mid \xi \in 2^n, m < n), \quad (\bar{b}_\eta \mid \eta \in 2^{<n}) \]
such that (1)–(8) hold with \( n \) in place of \( \gamma \).

When \( n = 0 \) the implication is immediate. Assume that (1)–(8) hold with \( n \) in place of \( \gamma \). Since
\[ R_c[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq n + 1, \]
there exist sets \( A^0, A^1 \), tuples \( \bar{c}^0, \bar{c}^1, \bar{b} \), and \( \Phi\)-types \( t_0(\bar{x}), t_1(\bar{x}), u_0(\bar{y}), u_1(\bar{y}) \) such that

1. \( A \subseteq A^0 \), and \( \bar{c}^0, \bar{b} \) are in \( A^0 \),
2. \( A \subseteq A^1 \), and \( \bar{c}^1, \bar{b} \) are in \( A^1 \),
3. \( t_0, u_0 \) are over \( A^0 \),
4. \( t, u_1 \) are over \( A^1 \),
5. \( t \subseteq \text{tp}_\Phi(\bar{c}^0/A \cup \{\bar{b}\}) \subseteq t_0 \) and \( t \subseteq \text{tp}_\Phi(\bar{c}^1/A \cup \{\bar{b}\}) \subseteq t_1 \),
6. \( u \subseteq \text{tp}_\Phi(\bar{b}/A) \subseteq u_0 \) and \( t \subseteq \text{tp}_\Phi(\bar{b}/A) \subseteq u_1 \)
7. \( t_0 \) and \( t_1 \) are semidefinite over \( A \) through \((\bar{a}_i)_{i \in I}\) and \( \mathcal{U} \),
(8) \( u_0 \) and \( u_1 \) are semidefinable over \( A \) through \( (\bar{b}_j)_{j \in J} \) and \( V \),
(9) \( \varphi(\bar{x}, \bar{b}) \in t_0 \) and \( \neg \varphi(\varphi')(\bar{x}, \bar{b}) \in t_1 \),
(10) \( R_e[t_0, u_0, \varphi, \varphi', a, b, u, \nu, \nu] \geq n \) and \( R_e[t_1, u_1, \varphi, \varphi', a, b, u, \nu] \geq n \).

By (10) and the induction hypothesis, for \( i \in \{0, 1\} \) we can fix families of sets
\[ (\bar{A}_{\eta}^{i} \mid \eta \in 2^{<n}) \]
and families of tuples
\[ (\bar{c}_{\xi, m}^{i} \mid \xi \in 2^n, m < n), \quad (\bar{b}_{\eta}^{i} \mid \eta \in 2^{<n}) \]
such that:

1. \( \bar{c}_{\xi, m}^{i} \) realizes \( t_i(\bar{x}) \), for \( \xi \in 2^m \) and \( m < n \);
2. \( \bar{b}_{\eta}^{i} \) realizes \( u(\bar{y}) \), for \( \eta \in 2^{<m} \);
3. If \( \xi \in 2^n, m < n \), and \( \eta \in 2^m \) is such that \( \eta = \xi \restriction m \), then
   \[ \varphi(\bar{c}_{\xi, m}^{i}, \bar{b}_{\eta}^{i}), \text{ if } \xi(m) = 0, \]
   \[ \neg \varphi(\varphi')(\bar{c}_{\xi, m}^{i}, \bar{b}_{\eta}^{i}), \text{ if } \xi(m) = 1; \]
4. \( A_{\emptyset}^{i} = A \cup \{\bar{c}^{i}, \bar{b}_{\emptyset}^{i}\} \);
5. If \( \xi \in 2^n, m < m' < n \), and \( \eta \in 2^m \) is such that \( \eta = \xi \restriction m \), then
   \[ A_{\eta \restriction 0}^{i} = A_{\emptyset}^{i} \cup \{\bar{c}_{\xi, m}^{i}, \bar{b}_{\eta \restriction 0}^{i}\}, \text{ if } \xi(m) = 0, \]
   \[ A_{\eta \restriction 1}^{i} = A_{\emptyset}^{i} \cup \{\bar{c}_{\xi, m}^{i}, \bar{b}_{\eta \restriction 1}^{i}\}, \text{ if } \xi(m) = 1; \]
6. If \( \xi \in 2^n, m < m' < n \), and \( \eta \in 2^m, \eta' \in 2^{m'} \) are such that \( \eta' = \xi \restriction m' \) and \( \eta = \eta' \restriction m \), then
   \[ \text{tp}_\Phi(\bar{c}_{\xi, m}^{i}/A_{\eta}^{i}) \subseteq \text{tp}_\Phi(\bar{c}_{\xi, m'}^{i}/A_{\eta}^{i}); \]
7. If \( \xi \in 2^n, m < \gamma, \) and \( \eta \in 2^m \), then \( \text{tp}_\Phi(\bar{c}_{\xi, m}^{i}/A_{\eta}^{i}) \) is semidefinable over \( A \) through \( (\bar{a}_i)_{i \in I} \) and \( \nu \).

Let
\[ \bar{b}_{\emptyset} = \bar{b}, \quad A_{\emptyset} = A \cup \{\bar{b}\}. \]

Then the lemma follows by defining, for \( \eta \in 2^{<n} \) and \( i \in \{0, 1\} \),
\[ \bar{b}_{i \restriction \eta} = \bar{b}_{\eta}^{i}, \quad A_{i \restriction \eta} = A_{\eta}^{i} \cup \{\bar{b}\}, \]
and for \( \xi \in 2^n, \)
\[ \bar{c}_{i \restriction \xi, 0}^{i} = \bar{c}^{i}, \]
\[ \bar{c}_{i \restriction \xi, k+1}^{i} = \bar{c}_{\xi, k}^{i}. \]

If
\[ c = (c_j \mid j \in J), \]
\[ c' = (c'_j \mid j \in J) \]
are families of elements of the monster model and \( A \) is a set, we write
\[ \text{tp}_\Phi(c/A) = \text{tp}_\Phi(c'/A) \]
if whenever \( j_1, \ldots, j_n \in J, \)
\[ \text{tp}_\Phi(c_{j_1}, \ldots, c_{j_n}/A) = \text{tp}_\Phi(c'_{j_1}, \ldots, c'_{j_n}/A). \]

If \( \Phi \) is the set of all quantifier-free \( L \)-formulas, we may write
\[ \text{tpqf}(c/A). \]
instead of $tp_{\Phi}(c/A)$.

We can extend Definition 8.1 to this concept of type naturally as follows. If $\psi_1(\bar{x}, \bar{y})$ and $\psi_2(\bar{x}, \bar{y})$ are $L$-formulas, we will say that $(\psi_1, \psi_2)$ separates

$$(tp_{\Phi}(c/A), tp_{\Phi}(c'/A))$$

if there exist $j_1, \ldots, j_n \in J$ such that $(\psi_1, \psi_2)$ separates

$$(tp_{\Phi}(c_1, \ldots, c_n/A), tp_{\Phi}(c'_1, \ldots, c'_n/A))$$

in the sense of Definition 8.1.

10.2.Lemma. Suppose that

- $a = (\bar{a}_i)_{i \in I}$ is a bounded family in $A$ and $\mathcal{U}$ is an ultrafilter on $I$,
- $b = (\bar{b}_j)_{j \in J}$ is a bounded family in $A$ and $\mathcal{V}$ is an ultrafilter on $J$,
- $t(\bar{x})$ and $u(\bar{y})$ are $\Phi$-types over $A$ such that $u$ is semidefinable over $A$ through $(\bar{b}_j)_{j \in J}$ and $\mathcal{V}$,
- $\varphi(\bar{x}, \bar{y})$ is a $\Phi$-formula and $\varphi'$ is an approximation of $\varphi$,

and

$$R_{\gamma}[t, u, \varphi, \varphi', a, b, \mathcal{U}, \mathcal{V}] \geq \omega,$$

Then for every ordinal $\gamma$ there exists a family

$$(\bar{c}_{\xi, \alpha} \mid \xi \in 2^\gamma, \alpha < \gamma)$$

such that for each $\xi \in 2^\gamma$,

$$c_\xi = (\bar{c}_{\xi, \alpha} \mid \alpha < \gamma)$$

is a fundamental sequence for a $(\Phi, \gamma)$-spreading model over $A$ generated by $a$ and $\mathcal{U}$, and

$$tp_{\Phi}(c_{\xi}/A) \neq tp_{\Phi}(c_{\xi'}/A), \text{ for } \xi \neq \xi'.$$

Furthermore, if $\psi$ and $\psi'$ are $\Phi$-formulas such that

$$\varphi < \psi < \psi' < \varphi',$$

then every pair of the form

$$(tp_{\Phi}(c_{\xi}/A), tp_{\Phi}(c_{\xi'}/A)),$$

where $\xi < \xi'$ in the lexicographic order, is separated by $(\psi, neg(\psi')).$

PROOF. Let

$$(A_\eta \mid \eta \in 2^{<\gamma}), \quad (\bar{c}_{\xi, \alpha} \mid \xi \in 2^\gamma, \alpha < \gamma), \quad (\bar{b}_\eta \mid \eta \in 2^{<\gamma})$$

be as given by Lemma 10.1, and for $\xi \in 2^\gamma$ let $c_\xi = (\bar{c}_{\xi, \alpha} \mid \alpha < \gamma)$.

Fix $\xi, \xi' \in 2^\gamma$ such that $\xi < \xi'$ in the lexicographic and let $\alpha$ be the smallest ordinal such that $\xi \upharpoonright \alpha = \xi' \upharpoonright \alpha$, but $\xi(\alpha) = 0$ and $\xi'(\alpha) = 1$. Then,

$$c_\xi \upharpoonright \alpha = c_{\xi'} \upharpoonright \alpha.$$

However, as we now show,

$$tp_{\Phi}(c_{\xi} \upharpoonright \alpha + 1 / A) \neq tp_{\Phi}(c_{\xi'} \upharpoonright \alpha + 1 / A).$$

Indeed, by (3) of Lemma 10.1,

- $\varphi(\bar{c}_{\xi, \alpha}, \bar{b}_{\xi_1\alpha}),$
- $\neg(\varphi')(\bar{c}_{\xi, \alpha}, \bar{b}_{\xi_1\alpha}),$
so, since
\[ \tp_{\Phi}(\bar{b}_{\xi|\alpha} / A) = u = \lim_{j,y} \tp_{\Phi}(\bar{b}_j / A), \]
there exists \( j \in J \) such that
\[ \cdot \psi(\bar{c}_{\xi,\alpha}, \bar{b}_j), \]
\[ \cdot \neg(\psi')(\bar{c}_{\xi,\alpha}, \bar{b}_j). \]
Thus,
\[ ( \tp_{\Phi}(c_\xi \upharpoonright \alpha + 1/A), \tp_{\Phi}(c_{\xi'} \upharpoonright \alpha + 1/A) ) \]
is separated by \((\psi, \neg(\psi'))\).

\[ \square \]

10.3. Definition. Let \((S_\xi | \xi \in \Xi)\) be a family of \((\Phi, \gamma)\)-spreading models over \(A\), and suppose that
\[ c = (\bar{c}_{\xi,\alpha} | \alpha < \gamma) \]
is a fundamental sequence for \(S_\xi\). We will say that \((S_\xi | \xi \in \Xi)\) is a family of uniformly inequivalent spreading models if there exist a \(\Phi\)-formula \(\varphi\) and an approximation \(\varphi'\) of \(\varphi\) such that whenever \(\xi \neq \xi'\), the pair \((\tp_{\Phi}(c_\xi / A), \tp_{\Phi}(c_{\xi'} / A))\) is separated by either \((\varphi, \neg(\varphi'))\) or \((\neg(\varphi'), \varphi)\).

10.4. Corollary. For every \(\Phi\)-type \(t\) over \(\mathcal{M}\), one and only one of the following conditions holds:

1. \(t\) is \(\Phi\)-definable;
2. There exist a bounded family \(a = (\bar{a}_i)_{i \in I}\) in \(\mathcal{M}\) and an ultrafilter \(\mathcal{U}\) on \(I\) such that the following holds. For every ordinal \(\gamma\) there exists a family \((S_\xi | \xi \in 2^\gamma)\) of uniformly inequivalent \((\Phi, \gamma)\)-spreading models over \(\mathcal{M}\) such that for each \(\xi \in 2^\gamma\), \(S_\xi\) is a \((\Phi, \gamma)\)-spreading model over \(\mathcal{M}\) generated by \(a\) and \(\mathcal{U}\).

11. Applications to Banach spaces

In this section we consider applications of Corollary 10.4 to normed spaces over \(\mathbb{R}\).

Every normed space \(X\) over \(\mathbb{R}\) is naturally a normed space structure: the sorts are \(X\) and \(\mathbb{R}\), and the functions are the vector space operations, the additive identity \(0_X\), and the norm on \(X\), as well as the field operations, the additive identity 0, and the absolute value function on \(\mathbb{R}\).

The corresponding signature will be denoted \(L_0\) (using the convention followed in [HI02]). For notational convenience, we often identify a Banach space \(L_0\)-structure with its universe.

11.1. Definition. Let \(\varphi\) be a positive bounded \(L_0\)-formula. For every rational number \(\lambda > 1\) we define an approximation \(\varphi_\lambda\) of \(\varphi\). The definition is by induction on the complexity of \(\varphi\), as given by the following table:
If $\varphi$ is: 
\begin{align*}
    \|t\| & \leq r & \text{then } \varphi_\lambda \text{ is: } \|t\| & \leq \lambda r \\
    \|t\| & \geq r & \text{then } \varphi_\lambda \text{ is: } \|t\| & \geq \frac{r}{\lambda} \\
    (\psi \land \theta) & & (\psi_\lambda \land \theta_\lambda) \\
    (\psi \lor \theta) & & (\psi_\lambda \lor \theta_\lambda) \\
    \exists_r x \psi & & \exists_{\lambda r} x \psi_\lambda \\
    \forall_r x \psi & & \forall_{\lambda r} x \psi_\lambda
\end{align*}

11.2. REMARKS. Let $\varphi$ be a positive bounded $L_0$-formula. Then:

1. For each $\lambda > 1$, $\varphi_\lambda$ is an approximation of $\varphi$;
2. For every approximation $\varphi'$ of $\varphi$ there exists a rational number $\lambda > 1$ such that $\varphi < \varphi_\lambda < \varphi'$;
3. If $\lambda, \mu > 1$, then $(\varphi_\lambda)_\mu$ is $\varphi_{\lambda \mu}$;
4. If $\lambda < \mu$, then $(\neg(\varphi_\mu))_\lambda$ is $\neg(\varphi_{\mu / \lambda})$.

If $X$ and $Y$ are Banach spaces and $C$ is a real number with $C \geq 1$, we say that $X$ and $Y$ are $C$-isomorphic if there is a linear isomorphism $f : X \to Y$ such that

$$
\frac{1}{C} \|x\| \leq \|f(x)\| \leq C \|x\|,
$$

that is, $\|f\|, \|f^{-1}\| \leq C$. In this case we say that the function $f$ is said to be a $C$-isomorphism.

Notice that $f : X \to Y$ is an isomorphism if and only if for every quantifier-free formula $\varphi(x_1, \ldots, x_n)$ and every rational number $\lambda$ with $\lambda > C$,

$$
X \models \varphi[a_1, \ldots, a_n] \quad \text{implies} \quad Y \models \varphi_\lambda[f(a_1), \ldots, f(a_n)].
$$

11.3. DEFINITION. If $X, Y$ are Banach spaces and $C$ is a real number with $C \geq 1$, we write $X \equiv_A^C Y$ if for every positive bounded $L_0$-sentence $\varphi$,

$$
X \models \varphi \quad \text{implies} \quad Y \models \varphi_\lambda, \quad \text{for every } \lambda > C.
$$

The following result was proved in [HI02, Theorem 13.26].

11.4. THEOREM. For every infinite cardinal $\kappa$ there exists an ultrafilter $U$ on $D$ on $2^\kappa$ satisfying the following property: if $C$ is a real number with $C \geq 1$ and $(X_\xi | \xi \in 2^\kappa)$ and $(Y_\xi | \xi < 2^\kappa)$ are families of Banach spaces such that $\text{density}(X_\xi)$ and $\text{density}(Y_\xi)$ are uniformly bounded by $\kappa$ and

$$
\left( \prod_{\xi \in 2^\kappa} X_\xi \right)_U \equiv_A^C \left( \prod_{\xi \in 2^\kappa} Y_\xi \right)_U,
$$

then $(\prod_{\xi \in 2^\kappa} X)_U$ and $(\prod_{\xi \in 2^\kappa} Y)_U$ are $C$-isomorphic.

An immediate consequence is the following characterization of $X \equiv_A^C Y$ in terms of ultrapowers; the separable case was first proved in [HH86]:

11.5. THEOREM. Let $X$ and $Y$ be Banach spaces and let $C$ be a real number with $C \geq 1$. The following conditions are equivalent:

1. $X \equiv_A^C Y$;
(2) There exist Banach space ultrapowers $\hat{X}$ and $\hat{Y}$ of $X$ and $Y$, respectively, such that $\hat{X}$ and $\hat{Y}$ are $C$-isomorphic.

11.6. DEFINITION. Let $\Phi$ be a fragment of $L_0$. Suppose that $(S_\xi \mid \xi \in \Xi)$ is a family of $(\Phi, \gamma)$-spreading models over $A$,

$$c_\xi = (\bar{c}_{\xi, \alpha} \mid \alpha < \gamma)$$

is a fundamental sequence for $S_\xi$, and $C$ is a real number with $C > 1$. We will say that the $(S_\xi \mid \xi \in \Xi)$ is a family of uniformly $C$-inequivalent spreading models if there exist a $\Phi$-formula $\varphi(x, y)$ and $\lambda > C$ such that whenever $\xi \neq \xi'$, the pair $(\text{tp}_{\Phi}(c_\xi / A), \text{tp}_{\Phi}(c_{\xi'} / A))$ is separated by either $(\varphi, \text{neg}(\varphi, \lambda))$ or $(\text{neg}(\varphi, \lambda), \varphi)$.

In this context, Corollary 10.4 can be restated as follows.

11.7. COROLLARY. Let $\Phi$ be a fragment of $L_0$. Suppose that $T$ is $\Phi$-model-complete and that $X$ is a Banach space. Then, for every $\Phi$-type $t$ over $X$, one and only one of the following conditions holds:

1. $t$ is $\Phi$-definable;
2. There exist a bounded family $a = (a_i)_{i \in I}$ in $X$, an ultrafilter $U$ on $I$, and $C > 1$ such that the following holds. For every ordinal $\gamma$ there exists a family $(S_\xi \mid \xi \in 2^\gamma)$ of uniformly $C$-inequivalent $(\Phi, \gamma)$-spreading models over $X$ such that for each $\xi \in 2^\gamma$, $S_\xi$ is a $(\Phi, \gamma)$-spreading model over $X$ generated by $a$ and $U$.

We now focus our attention to the quantifier-free case, i.e., the case when $\Phi$ consists of all the quantifier-free $L_0$-formulas. But first we recall some basic facts from Banach space theory.

If $X$ is a Banach space, a sequence $(x_n)$ is said to be basic if $(x_n)$ is a basis for its closed linear span $\text{span}(\{x_n \mid n \in \mathbb{N}\})$. Alternatively, $(x_n)$ is basic if and only if $\lim_{n \to \infty} x_n = 0$ and there exists a nonnegative constant $K$ such that whenever $m < n$ and $r_0, \ldots, r_n$ are scalars $\|\sum_{i=0}^{m} r_i x_i\| \leq K \|\sum_{i=0}^{n} r_i x_i\|$. The smallest such $K$ is called the basis constant of $(x_n)$, and $(x_n)$ is said to be $K$-basic. (The proof can be found in a Banach space theory textbook, e.g., [FHH+01].)

If $(x_n)$ is a basic sequence in a Banach space, a sequence $(y_n)$ is a block base of $(x_n)$ if there exist finite subsets $F_0, F_1, \ldots$ of $\mathbb{N}$ such that $\max F_n < \min F_{n+1}$ for every $n$ and $y_n \in \text{span}\{x_k \mid k \in F_n\}$ for every $n \in \mathbb{N}$. A block space is a space generated by a block base.

A basic sequence $(x_n)$ is said to be unconditional if each $x_n \neq 0$ for every $n$ and there exists a nonnegative constant $K$ such that whenever $m < n$ and $r_0, \ldots, r_n$ are scalars $\|\sum_{i=0}^{n} \pm r_i x_i\| \leq K \|\sum_{i=0}^{n} r_i x_i\|$.

The concept of spreading model in analysis was introduced by A. Brunel and L. Sucheston in the 1970’s [Bru74, BS74, BS76], and since then has become the fundamental tool to study asymptotic geometry of sequences in Banach space geometry. (Alternative notions of asymptotic structure have been proposed recently; see [Ode02] for pointers to the extensive literature.)

Brunel and Sucheston used Ramsey’s Theorem to prove that every normalized basic sequence $(x_n)$ in a Banach space $X$ has a subsequence $(y_n)$ such that the limits
\[
\lim_{n_0 < \ldots < n_k} \left\| \sum_{i=0}^{k} r_i y_i \right\|
\]
exist for all scalars \(r_0, \ldots, r_k\) in \([-1, 1]\). Given such a subsequence \((y_n)\), one can define a norm on \(c_{00}\) (the vector space of all finitely supported sequences of real numbers) as follows: if \((e_n)\) is the standard unit basis of \(c_{0,0}\) \(r_0, \ldots, r_k\) are scalars in \([-1, 1]\), the norm
\[
\left\| \sum_{i=0}^{k} r_i e_i \right\| = \lim_{n_0 < \ldots < n_k} \left\| \sum_{i=0}^{k} r_i y_i \right\|.
\]
The completion of \(c_{00}\) with this norm is called a spreading model of \((x_n)\) or of \(X\). The sequence \((e_n)\) is called the fundamental sequence of the spreading model.

If the initial sequence \((x_n)\) is \(K\)-basic, so is \((e_n)\). Every spreading model of \(X\) is finitely representable in \(X\), i.e., if \(E\) is a finite-dimensional block space of \(X\) and \(\epsilon > 0\), there exists a finite-dimensional subspace \(F\) of \(X\) such that \(E\) and \(F\) are \((1 + \epsilon)\)-isomorphic.

If \(Y\) is a subspace of \(X\), the sequence \((y_n)\) above can be chosen so that the limit
\[
\lim_{n_0 < \ldots < n_k} \left\| \sum_{i=0}^{k} r_i y_i + x \right\|
\]
eexists for every choice of scalars \(r_0, \ldots, r_k\) in \([-1, 1]\) and \(x \in Y\). Thus, the norm on the spreading model can be defined so that
\[
\left\| \sum_{i=0}^{k} r_i e_i + x \right\| = \lim_{n_0 < \ldots < n_k} \left\| \sum_{i=0}^{k} r_i y_i + x \right\|, \quad \text{for all } x \in Y.
\]
The completion of \(c_{00} \cup Y\) with this norm is called a spreading model of \((x_n)\) over \(Y\).

If \((x_n)\) has a spreading model with fundamental sequence \((e_n)\) information about the finite-dimensional subspaces of \(\text{span}\{(e_n \mid n \in \mathbb{N})\}\) can be recovered (approximately) in \(\text{span}\{(x_n \mid n \in \mathbb{N})\}\); however, for infinite-dimensional spaces, this is not the case. For example, H. Rosenthal [Ros83] proved that \((e_n)\) always has a block base which is 1-unconditional. On the other hand, by the celebrated negative solution by W. T. Gowers and B. Maurey to the unconditional basic sequence problem [GM93], not every Banach space has an unconditional sequence.

The following question was motivated by Tsirelson’s famous space (the first space exhibited which does not contain \(\ell_p\) or \(c_0\)):

**Question 1:** Does every Banach space have a spreading model isomorphic to \(\ell_p\) for some \(1 \leq p < \infty\) or \(c_0\)?

A counterexample was exhibited by E. Odell and Th. Schlumprecht [OS95]. This raised the following two questions (Question 2 was posed by V. D. Milman and Question 3 by S. Argyros).

**Question 2:** Does every Banach space admit a spreading model which is either reflexive or isomorphic to \(\ell_1\) or \(c_0\)?

In order to state the third question, we need to recall the following well-known concept:
Let \( X \) be a Banach space, let \( E \) be a subspace of \( X \), and let \( C \) be real number with \( C \geq 1 \). If \( 1 \leq p < \infty \), a sequence \( (x_n) \) is said to be \( C \)-equivalent over \( E \) to the standard unit basis of \( \ell_p \) if whenever \( x \in E \) and \( \lambda_0, \ldots, \lambda_n \) are scalars,

\[
C^{-1} \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| \leq \left\| x + \left( \sum_{i=0}^{n} |\lambda_i|^{p} \right)^{1/p} x_0 \right\| \leq C \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| .
\]

The sequence \( (x_n) \) is \( C \)-equivalent over \( E \) to the standard unit basis of \( c_0 \) if whenever \( x \in E \) and \( \lambda_0, \ldots, \lambda_n \) are scalars,

\[
C^{-1} \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| \leq \left\| x + \left( \max_{0 \leq i \leq n} |\lambda_i| \right) x_0 \right\| \leq C \left\| x + \sum_{i=0}^{n} \lambda_i x_i \right\| .
\]

When \( E \) is the trivial subspace \( \{0\} \), the clause "over \( E \)" is omitted. We say that \( (x_n) \) is equivalent (or \( C \)-equivalent over \( E \)) to the standard unit basis of \( \ell_p \) if there exists \( C \geq 1 \) such that \( (x_n) \) is \( C \)-equivalent (respectively, \( C \)-equivalent over \( E \)) to the standard unit basis of \( \ell_p \). Similarly for \( c_0 \).

Notice that if \( (x_n) \) is a basic sequence and \( (x_n) \) is \( C \)-equivalent over \( E \) to the standard unit basis of \( \ell_p \) (or \( c_0 \)), then the same is true about any block base of \( (x_n) \).

**Question 3:** If \( X \) has a unique spreading model up to equivalence, must the fundamental sequence be equivalent to the standard unit basis of \( \ell_p \) or \( c_0 \), for some \( p \) with \( 1 \leq p < \infty \)?

Questions 2 and 3 have been answered recently, in the negative, by G. Androulakis, E. Odell, Th. Schlumprecht, and N. Tomczak-Jaegermann. See [AOSTJ]. Below (Theorem 11.9) we present a dichotomy for spreading models which strengthens the answer of Androulakis-Odell-Schlumprecht-Tomczak-Jaegermann to Question 3.

The classical notion of spreading model corresponds precisely to the notion of \((\Phi, \omega)\)-spreading model, where \( \Phi \) is the fragment of quantifier-free \( L_0 \)-formulas. In what follows, the term "spreading model" will have this specific interpretation. Also, "quantifier-free type" stands for \( \Phi \)-type, where \( \Phi \) denotes the fragment all quantifier-free formulas in the Banach space language \( L_0 \), and "quantifier-free-definable" stands for \( \Phi \)-definable, for the same \( \Phi \).

The following result was proved by the author in [Iov].

**11.8. Theorem.** Let \( (x_n) \) be a normalized basic sequence in a Banach space \( X \). Then the following conditions are equivalent:

1. There exists a normalized block base \( (y_n) \) of \( (x_n) \) such for every finite-dimensional subspace \( E \) of \( X \), every type over \( \text{span}(E \cup \{y_n \mid n \in \mathbb{N}\}) \) is quantifier-free definable;

2. One of the following two conditions holds:
   a. There exists \( 1 \leq p < \infty \) such that for every \( \epsilon > 0 \) and every finite dimensional subspace \( E \) of \( X \) there exists a normalized block base of \( (x_n) \) which is \((1 + \epsilon)\)-equivalent over \( E \) to the standard unit basis of \( \ell_p \);
   b. For every \( \epsilon > 0 \) and every finite dimensional subspace \( E \) of \( X \) there exists a normalized block base of \( (x_n) \) which is \((1 + \epsilon)\)-equivalent over \( E \) to the standard unit basis of \( c_0 \).

Combining this result with Corollary 11.7, we obtain the following dichotomy.
11.9. **Theorem.** Every basic sequence in a Banach space $X$ has a normalized block base $(x_n)$ such that one and only one of the following conditions is true:

1. $(x_n)$ has a spreading model whose fundamental sequence is $1$-equivalent over $X$ to the standard unit basis of either $\ell_p$, for some $p$, or $c_0$;
2. For every normalized block base $(y_n)$ of $(x_n)$ there exists $C > 1$ and a family $(S_\xi \mid \xi \in 2^\omega)$ of uniformly $C$-inequivalent spreading models such that for each $\xi \in 2^\omega$, $S_\xi$ is a spreading model of $(y_n)$.

**Proof.** Conditions (1) and (2) are mutually exclusive, because if $(x_n)$ has a spreading model whose fundamental sequence is $1$-equivalent the standard unit basis of either $\ell_p$, (or $c_0$), the same is true about every normalized block base of $(x_n)$, so (2) cannot hold.

Without loss of generality we can start with a normalized basic sequence $(x_n)$ in $X$ such that the limit

$$\lim_{n_0 < \cdots < n_k} \left\| \sum_{i=0}^k r_i x_i + x \right\|$$

exists for every choice of scalars $r_0, \ldots, r_k$ in $[-1, 1]$ and $x \in X$. Let us consider two cases:

**Case 1.** The sequence $(x_n)$ has a normalized block base $(y_n)$ of $(x_n)$ such for every finite-dimensional subspace $E$ of $X$, every type over $\text{span}(E \cup \{y_n \mid n \in \mathbb{N}\})$ is quantifier-free definable. Let $(E_n \mid n \in \mathbb{N})$ be a family of finite-dimensional subspaces of $X$ such that $\bigcup_n E_n$ is dense in $X$, and take positive numbers $\epsilon_0 > \epsilon_1 > \ldots$ such that $\lim_n \epsilon_n = 0$. By (1)$\Rightarrow$(2) of Theorem 11.8, we can construct sequences $(y_n^k)$ for $k \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$,

- $(y_n^0)$ is $(y_n)$,
- $(y_n^{k+1})$ is a normalized block base of $(y_n^k)$,
- $\max(\text{support of } y_n^k) < \min(\text{support of } y_n^{k+1})$,
- $(y_n^k)$ is $(1 + 1/n)$-equivalent over $E_n$ to the standard unit basis of $\ell_p$, or $c_0$, as given by (1)$\Rightarrow$(2) of Theorem 11.8.

Then the diagonal sequence $(z_n)$ defined by $z_n = y_n^n$ has a spreading model whose fundamental sequence is $1$-equivalent over $X$ to the standard unit basis of either $\ell_p$, or $c_0$.

**Case 2.** Case 1 does not hold. Then (2) follows from Corollary 11.7.

\[\square\]

**References**


[FHH+01] Marián Fabian, Petr Habala, Petr Hájek, Vicente Montesinos Santalucía, Jan Pelant, and Václav Zizler. *Functional analysis and infinite-dimensional geometry*. CMS Books


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DEPENDENCE RELATIONS IN NON-ELEMENTARY CLASSES

ALEXEI S. KOLESNIKOV

ABSTRACT. The goal of this paper is to identify the properties of dependence relations in certain non-elementary classes that, firstly, characterize the model-theoretic properties of those classes; and secondly, allow to uniquely describe an abstract dependence itself in a very concrete way. We investigate totally transcendental atomic models and finite diagrams, stable finite diagrams, and a subclass of simple homogeneous models from this point of view.

INTRODUCTION

In the last 25 years, significant effort was made to develop classification theory for non-elementary classes. While for the general case (the abstract elementary classes) existence of a satisfactory dependence relation is a major open question, good dependence relations were defined and used in several non-first order frameworks. In this paper we study dependence relations in the following non-elementary classes:

1) totally transcendental classes of atomic models and finite diagrams. The known dependence relation in atomic models was developed by S. Shelah in [Sh 48, Sh 87ab], it is called a stable amalgamation. For finite diagrams, it was introduced by O. Lessmann in [Le1] via an appropriate 2-rank.

2) stable finite diagrams. The dependence relation is strong splitting, introduced and studied by S. Shelah in [Sh 3], with extensions in [GrLe, HySh, HyLe] to name a few.

3) simple homogeneous models. The dependence relation is dividing, due to S. Buechler and O. Lessmann in [BuLe].

The goal is to characterize dependence relations for these classes in the following two ways.

First, we identify the properties of dependence relations that allow us to conclude from existence of a dependence relation on a non-elementary class


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that the class has certain model-theoretic properties (e.g., totally transcendental, stable, etc.). For the first order case, the work in this direction was started in 1974 by J. Baldwin and A. Blass. In [BB] they deal with axiomatization of rank function, and with the question of what do the properties of the rank imply about the theory. In 1978 S. Shelah introduced axiomatizations of various isolation notions in his book [Sh:a]. The axiomatization of $F^f$ is an implicit axiomatization of forking for stable theories. Axiomatic treatment of forking in stable theories appeared in [HH]. Abstract dependence relations were systematically studied in the book [Ba book] by J. Baldwin that appeared in 1988. In 1996, B. Kim and A. Pillay showed in [KP] that, for simple theories, forking satisfies almost all the properties it has for stable theories. Moreover, they showed that a first order theory must be simple if it has an (abstract) dependence relation with certain properties of forking. To prove the last fact, it was shown that any abstract dependence relation with certain properties must actually coincide with forking.

This brings us to the second aspect of our study: determine whether or not the specific dependence relation used in analysis of a non-elementary class is the unique "nice" dependence relation for the class. We isolate the properties that allow us to uniquely describe any abstract dependence relation with those properties in a concrete way. For stable first order theories, such a characterization of forking was derived from [La] by J. Baldwin in [Ba book]. For simple first order theories, the characterization of forking was obtained by B. Kim and A. Pillay in [KP]. Their analysis was useful in particular as a tool to establish that a certain theory is simple, see for example [ChP]. On the non-first order front, a characterization of dependence was obtained by T. Hyttinen and O. Lessmann in [HyLe] for finite diagrams that are both simple and stable.

The abstract approach to dependence relations goes back to the works of Van der Waerden. In model theory, the abstract treatment of dependence was introduced in [Ba] by J. Baldwin, with many extensions in [Ba book]. This paper was inspired by [GrSh], some results from which were presented by R. Grossberg in a model theory course at Carnegie Mellon.

The paper is organized as follows. In Section 1 we describe the general context in which we define the notion of an abstract dependence relation and identify the properties of abstract dependence that allow us to characterize totally transcendental, stable, and simple classes. As we show later in the paper, the abstract dependence has to coincide with the specific dependence relations introduced for the classes, i.e., is unique in certain sense.

Section 2 deals with totally transcendental classes of atomic models. We present motivation, basic definitions, and dependence relation for this case. The dependence relation is not defined for all sets, it is restricted to good Tarski-Vaught pairs of sets. We discuss the reasons for such restrictions. We then prove that a class of atomic models with an abstract dependence relation must be totally transcendental. Moreover, we prove that Shelah's
stable amalgamation relation must be the only “reasonable” dependence relation in atomic models.

In Section 3 we discuss a similar case of totally transcendental finite diagrams. We find the situation there is analogous to the atomic case. The major differences between the contexts are that finite diagrams have a monster model that is a member of the class (while atomic models do not), but the types in finite diagrams are not necessarily isolated, as they are in atomic case.

In Section 4 we prove that a finite diagram is stable if and only if it has a “stable” dependence relation. Moreover, we show that, over models, any stable dependence relation must coincide with (non) strong splitting. As a byproduct of our study, we conclude that the strong splitting relation is optimal in the sense that it has the smallest local character possible for a stable dependence relation.

Section 5 is devoted to analysis of dependence relations in a simple homogeneous model with type amalgamation over all small sets. We prove an analogous result to the characterization of forking and simplicity obtained for the first order case by B. Kim and A. Pillay; the main difficulty is getting around failure of compactness theorem, that was heavily used in the first order case.

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1. Abstract dependence relations

We first describe the general context for the notion of an abstract dependence relation. The context generalizes the cases of atomic models and finite diagrams that we study in this paper. Background and motivation remarks for the classes of atomic models and finite diagrams are postponed to the sections in which those classes are studied.

1.1. Preliminary definitions. Fix a first order theory $T$, let $\mathcal{C}$ be a monster model of $T$.

**Definition 1.1.** For a set $A \subset \mathcal{C}$, the set of types $D(A) := \{tp(\bar{a}/\emptyset) \mid \bar{a} \in A\}$ is called the diagram of $A$. The diagram of $T$ is $D(T) := D(\mathcal{C})$, where $\mathcal{C}$ is the monster model of the first order theory $T$.

For a fixed $D \subset D(T)$, we call $A$ a $D$-set if $D(A) \subset D$. If $M \models T$ and $D(M) \subset D$, we call $M$ a $D$-model.

The object of our study is essentially the class of $D$-submodels of $\mathcal{C}$ for a fixed diagram $D$, with some extra assumptions either on the diagram $D$ (e.g., $D$ is atomic) or on the class of $D$-models. We restrict ourselves to those subsets of $\mathcal{C}$ because even though the underlying theory $T$ may be
too complex from the classification theory point of view, the collection of
$D$-models could well have nice model-theoretic properties.

**Definition 1.2.** We denote by $S^n_D(A)$ the collection of all complete types
in $n$ variables such that for all $\bar{c} \models p$ the set $A \cup \bar{c}$ is a $D$-set. Accordingly,
$S^*_D(A) := \bigcup_{n<\omega} S^n_D(A)$.

A $D$-model $M$ is called $(D, \lambda)$-homogeneous if $M$ realizes all the types
$\{ p \in S_D(A) \mid A \subseteq M, |A| < \lambda \}$.

Compactness theorem no longer holds in our context, so the first-order intuition,
and many of the methods, do not work. In particular, it is not clear if it is possible to realize the $D$-types over sets in some $D$-model containing the set without any additional assumptions on the class of all the $D$-structures. Two particular cases of $D$-structures studied are (1) atomic models, when $D$ is the collection of atomic types; and (2) finite diagrams, where the extra assumption is existence of a monster $D$-model, i.e., $(D, \chi)$-homogeneous model for some very large $\chi$.

1.2. Abstract dependence relation. It is natural to define the dependence relation only on the “relevant” sets and models in our context. Let $A$, $B$, and $C$ be $D$-sets such that $A \cup B \cup C$ is a $D$-set as well. The expression
$A \downarrow_C B$ reads “$A$ is independent from $B$ over $C$,” we use the superscript $(A)$
to distinguish an abstract dependence relation from a concrete dependence relation in each context.

**Definition 1.3.** We call a relation $\downarrow_C$ totally transcendental if it satisfies
the following conditions:

(a) **Invariance:** If $f \in \text{Aut}(C)$, then

$$A \downarrow_C B \quad \text{if and only if} \quad f(A) \downarrow f(C)$$

(b) **Monotonicity:** Suppose $A \downarrow_C B$. For any $B', C'$ such that $C \subseteq C' \subseteq B' \subseteq B$ we have $A \downarrow_{C'} B'$.

(c) **Finite Character:** If $A \not\subseteq_C B$, then there are finite tuples $\bar{a} \in A,$

$\bar{b} \in B$ such that $\bar{a} \not\subseteq_C \bar{b}$.

(d) **Stationarity over finite subsets of models:** Suppose that $M$ is
a $(D, \aleph_0)$-homogeneous model, and $\bar{a}M$ is a $D$-set. There is a finite
tuple $\bar{c} \in M$ such that $\bar{a} \downarrow_\bar{c} M$ and for any $D$-set $B$ containing $\bar{c}$ the
type \( \text{tp}(\vec{a}/\vec{c}) \) can be uniquely extended to a \( \sqsubseteq \)-independent \( D \)-type over \( B \).

We use the name "totally transcendental" for such a dependence relation because we prove in the subsequent sections that existence of such a relation implies that a class of \( D \)-structures is totally transcendental.

The properties (1)--(4) imply other properties of dependence (such as Extension, Symmetry, and Transitivity). We say more about this after Definition 1.6.

In the totally transcendental case, the stationarity property holds over \((D, \aleph_0)\)-homogeneous models; when we go to the stable case, stationarity can be guaranteed for a smaller class of \( D \)-structures.

**Definition 1.4.** Let \( A \) be a \( D \)-set, suppose \( \vec{a}, \vec{b} \) are finite tuples such that \( A\vec{a}\vec{b} \) is a \( D \)-set as well. We say that \( \vec{a} \) and \( \vec{b} \) have the same Lascar strong type over \( A \) and write \( \text{lstp}(\vec{a}/A) = \text{lstp}(\vec{b}/A) \) if \( \vec{a}E\vec{b} \) for every \( A \)-invariant equivalence relation \( E \) with fewer than \( |\mathcal{C}| \) equivalence classes.

A \( D \)-model is Lascar \((D, \lambda)\)-homogeneous if it realizes all the Lascar strong types over its subsets of size less than \( \lambda \).

**Remark 1.5.** In [HySh], the term "\( a \)-saturated" is used for Lascar \((D, \kappa)\)-homogeneous, for certain \( \kappa \). We want the cardinal explicitly mentioned in the property.

**Definition 1.6.** Let \( A, B, C \) be as above. A relation \( \sqsubseteq \) is stable if it has the invariance, monotonicity, and finite character properties and in addition it satisfies:

**Local Character:** There is a cardinal \( \kappa \) such that for \( A = \vec{a} \), there is \( C \subset B \), \( |C| < \kappa \), with \( \vec{a} \sqsubseteq C \).

**Stationarity over models:** Let \( \kappa \) be as above and suppose that \( M \) is a Lascar \((D, \kappa)\)-homogeneous model, and \( \vec{a}M \) is a \( D \)-set. Then for every \( D \)-set \( B \supset M \) there is a unique \( p \in S_D(B) \) such that for all \( \vec{a}' \models p \) we have \( \vec{a}' \sqsubseteq M \).

If \( \kappa = \aleph_0 \), we call the above relation superstable.

The symmetry and transitivity properties for stable and totally transcendental dependence relations do hold. We show this in a rather indirect way. We prove in Subsections 2.5, 3.2 and 4.2 that abstract dependence has to coincide with the relations of strong splitting (for the stable case) and splitting (for totally transcendental). For those specific relations symmetry and transitivity hold, hence we can conclude that abstract dependence relations must have them as well. For the first order case, such approach was used by J. Baldwin (see [Ba book], Chapter 7). Beyond the stable context, symmetry can no longer be derived from other properties.
In the case of simple $D$-structures, we work in the context described by S. Buechler and O. Lessmann in [BuLe]. They require extension property over all the $D$-sets. In totally transcendental and stable case extension is a part of stationarity, and is guaranteed to hold over certain models only (or only for certain pairs of sets, see Fact 2.13(10)).

**Definition 1.7.** Let $A$, $B$, $C$ be as above. A relation $\downarrow$ is *simple* if it has the invariance, monotonicity, finite and $\kappa$-local character properties and in addition it satisfies:

**Extension:** If $\overset{(A)}{\bar{a} \downarrow_B A}$, $B \subset A$, then for all $C$ there is $\overset{(A)}{\bar{a}' \models tp(\bar{a}/A)}$ such that $\overset{(A)}{\bar{a}' \downarrow_B C}$.

**Symmetry:**

\[
\overset{(A)}{A \downarrow_C B} \text{ if and only if } \overset{(A)}{B \downarrow_C A}.
\]

**Transitivity:** If $B \subset C \subset D$, then

\[
\overset{(A)}{A \downarrow_B C} \text{ and } \overset{(A)}{A \downarrow_C D} \text{ if and only if } \overset{(A)}{A \downarrow_B D}.
\]

**Type amalgamation:** Suppose $\overset{(A)}{\bar{a}_1}$, $\overset{(A)}{\bar{a}_2}$ are tuples of length less than $\kappa$; $\overset{(A)}{\bar{b}_1}$, $\overset{(A)}{\bar{b}_2}$ are tuples of arbitrary size. If $\overset{(A)}{lstp(\bar{a}_1/C)} = \overset{(A)}{lstp(\bar{a}_2/C)}$, $\overset{(A)}{\bar{b}_1 \downarrow_C \bar{b}_2}$, and $\overset{(A)}{\bar{a}_i \downarrow_C \bar{b}_i}$, $i = 1, 2$, then there is $\overset{(A)}{\bar{a} \models lstp(\bar{a}_1/C\bar{b}_1) \cup lstp(\bar{a}_2/C\bar{b}_2)}$ such that $\overset{(A)}{\bar{a} \downarrow_C \bar{b}_1 \bar{b}_2}$.

If $\kappa = \aleph_0$, we call the above relation *supersimple*.

## 2. Atomic models

In this section we give some background information about totally transcendental classes of atomic models, in particular, we present the dependence relation for the classes introduced by Shelah in [Sh 87ab]. We then prove that existence of an (abstract) totally transcendental dependence relation on a class of atomic models implies that the class is totally transcendental and that the abstract relation coincides with that defined by Shelah.

### 2.1. Motivation

A major motivating question for developing classification theory in non first order situation is Shelah's categoricity conjecture for $L_{\omega_1,\omega}$: if a sentence $\psi \in L_{\omega_1,\omega}$ is categorical in some cardinality above $\beth_{\omega_1}$, then it is categorical in every cardinality above $\beth_{\omega_1}$.

One of the tools to deal with this context was introduced by Shelah in [Sh 48]. There he suggests to expand the language with predicates that isolate complete types in the sense of $L_{\omega_1,\omega}$, and deal with atomic sets,
types, and models of the appropriate first order theory. Of course, the class of atomic models is closely tied to the original class of models as described in the theorem below. Atomicity allows to make sure that atomic types over certain sets are realized in some atomic model.

In [Sh 48] Shelah showed the following result (Lemmas 2.5 and 3.1); we state it in the form closer to what he has in [Sh 87ab]. For a class of models $\mathcal{K}$, let $I(\lambda, \mathcal{K})$ denote the number of non-isomorphic models in $\mathcal{K}$ of cardinality $\lambda$.

**Theorem 2.1** (Shelah). Let $\psi$ be an $L_{\omega_1, \omega}$ sentence, and suppose that in some uncountable $M^* \models \psi$ only countably many $L_{\omega_1, \omega}$ types are realized. Then there is a first order theory $T$ in an expanded language $L(T)$ such that, letting $\mathcal{K}$ be the class of atomic models of $T$,

1. every formula in $L(T)$ is equivalent to an atomic formula modulo $T$;
2. $\mathcal{K}$ has an uncountable model, and if $\text{Mod}(\psi)$ has arbitrarily large models, then so does $\mathcal{K}$;
3. every model in $\mathcal{K}$ can be made into a model of $\psi$, so $I(\lambda, \mathcal{K}) \leq I(\lambda, \text{Mod}(\psi))$ for all $\lambda$. More precisely, $I(\lambda, \mathcal{K}) = I(\lambda, \{M \models \psi \mid M \equiv_{\omega_1, \omega} M^*\})$.

The assumption on countably many $L_{\omega_1, \omega}$ types in an uncountable model is not too restricting. Shelah proved that this holds if $\psi$ has countably many non-isomorphic models in $\aleph_1$, or if it has less than $2^{\aleph_1}$ non-isomorphic models (the latter requires a mild set-theoretic assumption). It is easy to see that if $\psi$ has arbitrarily large models, the assumption also holds no matter how many models there are in $\aleph_1$.

Recall that for a first order theory $T$ and a set of types $\Gamma$ in the language of $T$, an $EC(T, \Gamma)$ class is a class of models of $T$ that omit the types in $\Gamma$. For an $EC(T, \Gamma)$ class, the above theorem works when the class has arbitrarily large models; or it can be axiomatized by an $L_{\omega_1, \omega}$ sentence with properties implying existence of an uncountable model with countably many $L_{\omega_1, \omega}$ types.

**Assumptions 2.2.** For the rest of this section, we fix a first order theory $T$ in a relational language such that every formula is equivalent to an atomic formula modulo $T$. Let $\mathfrak{C}$ be the monster model of $T$, and let $\mathcal{K}$ be the class of atomic elementary submodels of $\mathfrak{C}$. By default, $\models \varphi$ means satisfaction in $\mathfrak{C}$, and all the sets and elements are in $\mathfrak{C}$. By a “model” we mean a model in $\mathcal{K}$.

We further assume that $\mathcal{K}$ has an $\aleph_0$-amalgamation property, that is for all countable models $M \prec M_0, M_1$ there is a countable model $N$ and elementary embeddings $f_i : M_i \rightarrow N$, $i = 0, 1$ that coincide on $M$. The amalgamation property holds for instance, if $I(\aleph_1, \mathcal{K}) < 2^{\aleph_1}$ and $2^{\aleph_0} < 2^{\aleph_1}$ by [Sh 87ab] for $\aleph_0$-categorical $\mathcal{K}$.

### 2.2. Preliminary results and definitions.

We introduce now some important definitions and basic results for the context. All of the definitions
and almost all the results are due to Shelah [Sh 87ab]. We will use them extensively in this section.

**Definition 2.3.** (1) A set $B$ is **constructible over** $A$ if $B = A \cup \{b_i \mid i < \alpha\}$, where for all $i < \alpha$ the type $tp(b_i/A \cup \{b_j \mid j < i\}$ is isolated. A constructible model $M$ over $A$ is called **primary over** $A$.

(2) A model $M$ is **universal over** $A$ if $A \subseteq M$, $\|M\| = |A|$, and every $N \supset A$ of the same cardinality can be elementarily embedded into $M$ over $A$.

(3) An atomic set $A \subseteq \mathcal{C}$ is **good** if for each $\bar{a} \in A$ if $\models \exists \bar{x} \varphi(\bar{x}, \bar{a})$, then $\varphi(\bar{x}, \bar{a})$ belongs to a type $p \in S_D(A)$ (i.e., to an isolated type over $A$).

In [Sh 87ab], the set of types $S_D(A)$ is denoted $D_A$.

**Remark 2.4.** If $T$ is an $\aleph_0$-stable countable first order theory, then every set $A \subseteq \mathcal{C}$ is good. In our context, it is possible to have a situation when the class $\mathcal{K}$ is $\aleph_0$-stable (i.e., $|S_D(M)| = \aleph_0$ for all countable atomic $M$), but the underlying theory $T$ is not. Example in [HaSh] shows in particular that not every set is good in general for an $\aleph_0$-stable class $\mathcal{K}$.

**Definition 2.5.** The pair $(A, B)$, $A \subseteq B$, satisfies the **Tarski-Vaught condition** if for every $\bar{b} \in B$ and $\bar{a} \in A$ if $\models \varphi[\bar{b}, \bar{a}]$, then there is $\bar{b}' \in A$ such that $\models \varphi[\bar{b}', \bar{a}]$.

We call such pair $(A, B)$ a **Tarski-Vaught pair** and write $A \subset_{TV} B$.

We list some properties of Tarski-Vaught pairs that we will use later.

**Proposition 2.6.** If $M \in \mathcal{K}$ and $M \cup B$ is atomic, then $(M, M \cup B)$ satisfies the **Tarski-Vaught condition**.

**Proof.** Suppose $\bar{a} \in |M|$, $\bar{b} \in B$ and $\models \varphi[\bar{a}, \bar{b}]$. Since $M \cup B$ is atomic, there is $\psi(\bar{x}, \bar{y})$ isolating the type $tp(\bar{a}\bar{b}/\emptyset)$. Since $\models \exists \bar{y}\psi(\bar{a}, \bar{y})$, there is $\bar{b}' \in |M|$ such that $M \models \psi[\bar{a}, \bar{b}]$. Clearly, $M \models \psi[\bar{a}, \bar{b}']$.

**Claim 2.7.** Suppose $B \subset_{TV} C$, $\bar{d} \subseteq C$ is atomic, and assume $tp(\bar{d}/B)$ is isolated by $\varphi(\bar{x}, \bar{b})$ for some $\bar{b} \in B$. Then $\varphi(\bar{x}, \bar{b})$ isolates the type $tp(\bar{d}/C)$.

**Proof.** Let $\bar{d} \models \psi(\bar{x}, \bar{c})$ for some $\bar{c} \in C$. Since $B \subset_{TV} C$, there is $\bar{c}' \in B$ such that $tp(\bar{c}'/\bar{b}) = tp(\bar{c}/\bar{b})$. Since $\models \forall \bar{x} \varphi(\bar{x}, \bar{b}) \rightarrow \psi(\bar{x}, \bar{c}')$, conjugating $\bar{c}'$ to $\bar{c}$ over $\bar{b}$ we get $\models \forall \bar{x} \varphi(\bar{x}, \bar{b}) \rightarrow \psi(\bar{x}, \bar{c})$.

**Remark 2.8.** The dependence relation for the class $\mathcal{K}$ that we describe in the next subsection does not have the extension property in general. We can show that extension holds for pairs that satisfy the Tarski-Vaught condition, and get a nice description of the relation for this case. Shelah defined the dependence relation for pairs that satisfy the Tarski-Vaught condition in [Sh 87ab]. Getting the extension property is probably the main reason for introducing the concept in this context.
2.3. Rank and dependence relation in atomic models. Let $\mathcal{K}$ be a class of atomic models of a first order theory $T$ such that Assumptions 2.2 hold. We give the definition of a rank function (it is due to Shelah [Sh 48], though we present it in a slightly different form) and describe the resulting dependence relation for the class.

**Definition 2.9.** Let $M$ be a model, let $p$ be a finite type over (finite) $B \subset |M|$. 

1. $R_M[p] \geq 0$ if $p$ is realized in $M$.
2. For $\alpha$ limit ordinal, $R_M[p] \geq \alpha$ if $R_M[p] \geq \beta$ for all $\beta < \alpha$.
3. $R_M[p] \geq \alpha + 1$ if
   a. there are $\varphi(\bar{x}, \bar{y})$ and $\bar{a} \in M$ such that $R_M[p \cup \varphi(\bar{x}, \bar{a})] \geq \alpha$ and $R_M[p \cup \neg \varphi(\bar{x}, \bar{a})] \geq \alpha$;
   b. for every $\bar{b} \in |M|$ there is a complete formula $\psi(\bar{x}, \bar{y})$ such that $R_M[p \cup \psi(\bar{x}, \bar{b})] \geq \alpha$.

As usual, we say $R_M[p] = -1$ if $R_M[p] \not\geq 0$; $R_M[p] = \alpha$ if $R_M[p] \geq \alpha$ and $R_M[p] \not\geq \alpha + 1$; $R_M[p] = \infty$ if $R_M[p] \geq \alpha$ for all $\alpha \in \text{On}$; if $q$ is a type over a subset of $M$ is not necessarily finite, we let $R_M[q] := \text{Min}\{R_M[p] \mid p \subseteq q, p \text{ finite}\}$.

If the model $M \in \mathcal{K}$ is clear from the context, we omit the subscript $M$ in the notation for the rank.

The following properties of the rank appeared in [Sh 48].

**Fact 2.10 (Properties of the rank).** 

1. Invariance: if $f \in \text{Aut}(\mathcal{C})$, then $R_M[p] = R_M[f(p)]$;
2. Monotonicity: If $p \vdash q$, then $R_M[p] \leq R_M[q]$;
3. If $R_M[p] \geq \omega_1$, then $R_M[p] = \infty$;
4. Stationarity: Suppose $R_M[\bar{x} = \bar{x}] < \infty$. Let $N \in \mathcal{K}$ be a submodel of $M$ and let $p$ be a complete type over $N$. Then there exist $\bar{b} \in N$ and a formula $\varphi$ such that $R_M[p] = R_M[\varphi(\bar{x}, \bar{b})]$. Moreover, if $A \subset M$, $\bar{b} \in A$, then there is a unique type $p_A$ containing $\varphi(\bar{x}, \bar{b})$ with the same rank.

Certainly, the rank can be unbounded in general. To define a dependence relation, we need to assume that the rank is bounded. Boundedness of the rank can be obtained if, for instance, the class comes from an $L_{\omega_1, \omega}$ sentence that has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ under the assumption $2^{\aleph_0} < 2^{\aleph_1}$. The class $\mathcal{K}$ is called **totally transcendental** if $R$ is bounded. For the rest of this subsection, we assume that the class $\mathcal{K}$ is totally transcendental.
Facts 2.11. (1) In this context, if $M$ is a primary model over $A$, then it is unique over $A$ and is prime over $A$ (i.e., can be elementarily embedded in any $N \supset A$ over $A$).

(2) Let $A$ be a countable set. $A$ is good if and only if there is a countable primary model over $A$.

(3) If $A$ is countable, then $A$ is not good if and only if $|S_D(A)| = 2^{\aleph_0}$.

(4) Let $A$ be countable good set. There is a countable model $N$ that is $(D_A, \aleph_0)$-homogeneous. The model is unique and universal over $A$. Moreover, a countable $A$ is good if and only if there is a countable universal model over it.

The usual way to define a dependence relation from a rank is by saying $A$ is independent of $B$ over $C$ if for all $\bar{a} \in A$ the ranks of $\text{tp}(\bar{a}/B)$ and $\text{tp}(\bar{a}/C)$ coincide. The limitation is of course that the rank is computed inside a model in $\mathcal{K}$. In addition, one can get the extension property for the rank only for types over models.

The following dependence relation for $\mathcal{K}$ was suggested by Shelah (see [Sh 87ab]). It is defined only for good sets, essentially because these are the sets of interest here and since one cannot expect a good dependence relation for all atomic sets. We discuss this further in the next section.

Definition 2.12. Suppose $A \cup B \cup C$ is atomic and $C$ is good. Then $A$ is independent of $B$ over $C$ (we write $A \perp C$) if for each $\bar{a} \in A$, $\text{tp}(\bar{a}/B)$ does not split over some finite subset of $C$.

We summarize below the properties that this dependence relation has. Many of the properties were defined in Section 1, we restate some of them for this situation to avoid any ambiguities.

Fact 2.13 (Properties of $\perp$). The relation $\perp$ has the following properties:

(1) Invariance;

(2) Monotonicity;

(3) Local Character: For all $\bar{a}$ and $B$ such that $\bar{a} \cup B$ is atomic and $B$ is good, there is a finite $\bar{b} \in B$ such that $\bar{a} \perp B$.

(4) Existence: For all $A, C$ such that $A \cup C$ is atomic and $C$ is good we have $A \perp C$.

(5) Extension: If $\bar{a} \perp B$ and $C$ is a good set such that $B \subset C$ and $A \subset_{TV} C$, then there is $\bar{a}' \models \text{tp}(\bar{a}/B)$ such that $\bar{a}' \perp C$. In particular, there is always an independent extension of a type over a model.

(6) Finite Character;

(7) Symmetry over models;

(8) Transitivity over Tarski-Vaught pairs: Suppose $B \subset_{TV} C \subset_{TV} D$, and $B, C$ are good. If $\bar{a} \perp C$ and $\bar{a} \perp D$, then $\bar{a} \perp D$. 


(9) Stationarity over finite subsets of models: Suppose $M \in \mathcal{K}$, $\bar{a} \cup M \cup B$ is atomic. There is a finite $\bar{c} \in M$ such that $\bar{a} \perp \bar{c}$. Moreover, if $B$ is atomic, $B \supset M$, then there is $\bar{a}' \models \text{tp}(\bar{a}/M)$ such that $\bar{a}' \perp B$ and such an extension is unique over $\bar{c}$. We say that $\text{tp}(\bar{a}/\bar{c})$ is stationary, and the extension to $B$ is a stationarization of $\text{tp}(\bar{a}/\bar{c})$ in this situation.

(10) Weak stationarity over good sets: Suppose $A$ is good and $\bar{a}A$ is atomic. There is finite $\bar{c} \in A$ such that $\bar{a} \perp A$ and for any atomic set $B$, $A \subset_{TV} B$, the type $\text{tp}(\bar{a}/\bar{c})$ can be uniquely extended to an independent atomic type over $B$. We say that $\text{tp}(\bar{a}/\bar{c})$ is weakly stationary over $\bar{c}$, and the unique extension is the weak stationarization in this case.

(1) and (2) are obvious from the definition. (3) is proved in Lemma 2.2(1), last sentence, in [Sh 87ab] and (4) follows at once from (3). (5) is proved in Lemma 2.10(1) in [Sh 87ab]. (6) is immediate by the definition. (7) is proved in Theorem 1.4.1(c) in [Sh 87ab]. (8) is in Lemma 2.10(1), second paragraph, in [Sh 87ab]. (9) follows from Theorem 1.4.1(b) in [Sh 87ab]; (10) from local character and Lemma 2.10(1) in Shelah.

2.4. Some negative results. The purpose of this subsection is to complement the next section, where we prove a uniqueness result assuming existence of a dependence relation with certain properties. Here we present various results showing that requiring less from an abstract dependence relation is unreasonable.

We start by pointing out that we do need to restrict the dependence relation to good sets. Suppose $A$ is countable, not good. Then $D_A = \{\text{tp}(\bar{a}/A) \mid A\bar{a} is atomic\}$ is uncountable by Facts 2.11 (in fact, it has size continuum). So we get an instability phenomenon in this situation. The reason is that the theory $T$ need not be $\aleph_0$-stable (or stable at all). Our intention, however, is to investigate the totally transcendental atomic part of $T$, so we need to choose carefully which sets to deal with.

Next we show that one can define two dependence relations that coincide on Tarski-Vaught pairs, but differ on other (countable) sets.

Claim 2.14. Let $B, C$ be good countable with $B \subset_{TV} C$. Suppose further that $\bar{a}C$ is atomic. Let $M_B, M_C$ be primary models over $B$ and $C$ respectively. Suppose $\bar{a} \perp_{M_B} M_C$. Then $\bar{a} \perp_{B} C$.

Proof. By Theorem 1.6(1) of [Sh 87ab], the primary model $M_B$ is also prime over $B$, so we may assume that $M_B \prec M_C$. Let $\tilde{d} \in M_B$ be such that $\text{tp}(\bar{a}/\tilde{d})$ is stationary. Since $\tilde{d} \in M_B$ and $M_B$ is primary over $B$, the type of $\tilde{d}$ over $B$ is isolated. Let $\bar{b} \in B$ and $\varphi$ be such that $\varphi(\bar{x}, \bar{b}) \models \text{tp}(\tilde{d}/B)$. Since $B \subset_{TV} C$ and certainly $\tilde{d}C$ is atomic, by Claim 2.7 we get that $\varphi(\bar{x}, \bar{b})$ isolates $\text{tp}(\tilde{d}/C)$.
Now by a standard argument we conclude that since $tp(\bar{a}/M_C)$ is stationary over $\bar{d}$, and $\varphi(\bar{x},\bar{b})$ isolates $tp(\bar{d}/C)$, we have $tp(\bar{a}/C)$ does not split over $\bar{b}$.

So we can define another dependence relation on good atomic subsets.

**Definition 2.15.** For $B, C$ good, $\bar{a}C$ atomic $\bar{a}$ is independent from $C$ over $B$ if there are primary models $M_B \prec M_C$ over $B$ and $C$ respectively and $\bar{a} \downarrow M_C$ ($\downarrow$ is in the standard sense); if there is no primary model over $M_B$ one of the sets (they are not guaranteed to exist for uncountable good sets), then we leave the relation as was defined before.

The new dependence relation coincides with the standard one for Tarski-Vaught pairs by the claim above. However, for a countable good $B$ and all $\bar{a}$ such that $\bar{a}B$ is atomic we have $\bar{a}$ is independent from $M_B$ over $B$ according to Definition 2.15. If $B$ is not a model and $M_B$ contains at least two realizations $\bar{a}, \bar{b}$ of an isolated type over $B$, then certainly $tp(\bar{a}/M_B)$ splits over $B$ and hence, it splits over every finite subset of $B$. We illustrate this on the following simple example.

**Example 2.16.** Let $\varphi$ be a Scott sentence for an algebraically closed field of characteristic zero of infinite transcendence degree. Clearly, $\varphi$ has models in every infinite cardinality and is totally categorical. Let $\mathcal{K}$ be the class of atomic models in the expanded language constructed as described in Theorem 1.1. The situation in class $\mathcal{K}$ is actually very close to that in first order algebraically closed fields. Letting $\bar{a} := i, B := \mathbb{Q}$, and $M_B$ an algebraic closure of $\mathbb{Q}$ of infinite transcendence degree, we see that $\bar{a}$ is independent from $M_B$ over $B$ according to Definition 2.15 of dependence. It is also clear that $tp(\bar{a}/M_B)$ splits over $B$.

The example shows that, without restriction to the Tarski-Vaught pairs, one cannot hope to uniquely characterize the dependence relations in atomic models.

We also can look at this example from another angle. Every type over a good set can be split into a stationary and isolated parts (see fact below). As we show in the next section, the dependence relation for stationary types can be uniquely characterized (so the notion of stationarity is invariant in certain sense). Our example also shows that dependence for isolated types can be decided positively or negatively without affecting the stationary part.

**Fact 2.17** (Shelah). **Suppose $A$ is good. A type $p$ is atomic over $A$ if and only if there are $\bar{a}$ and $\bar{d}$ such that $p = tp(\bar{a}/A)$, the type $tp(\bar{a}/\bar{d})$ is stationary, $tp(\bar{a}/Ad)$ is the non-splitting extension of $tp(\bar{a}/\bar{d})$, and $tp(\bar{d}/A)$ is isolated.**

2.5. **Abstract dependence characterization.** In this subsection, we prove that if $\mathcal{K}$ has an abstract totally transcendental dependence relation in the sense of the Definition 1.3, then the rank $R$ has to be bounded for $\mathcal{K}$, i.e.,
$\mathcal{K}$ is totally transcendental. We also show that the relation $\downarrow^A$ coincides with the dependence relation introduced above over models. We show that a stronger stationarity assumption for $\downarrow^A$ implies that it coincides with $\downarrow$ over all good sets.

**Theorem 2.18.** Suppose $\mathcal{K}$ has an abstract totally transcendental dependence relation. Then the rank function $R$ is bounded on $\mathcal{K}$.

**Proof.** If the rank $R_M[p]$ is unbounded, then by Lemma 4.2 in [Sh 48] there is a countable model $M$ such that $|S_D(M)| \geq \aleph_1$. By stationarity, each of those types is $\downarrow^A$-independent over a finite subset of $M$. By the pigeonhole principle, there are at least $\aleph_1$ types in $S_D(M)$ that are independent over the same subset of $M$. Since there are only countably many $D$-types over a finite set, by pigeonhole principle again we conclude that there are $\aleph_1$ independent extensions of the same type over the stationary base. Contradiction to the stationarity over models.

The following definition will facilitate the proofs characterizing the abstract dependence relation in terms of $\downarrow$ over models and over good sets in general.

**Definition 2.19.** Let $C$ be a good set and $\bar{a}$ be such that $C\bar{a}$ is atomic. We say that $\text{tp}(\bar{a}/C)$ is $\downarrow^A$-weakly stationary if there is $\bar{c} \in C$ such that $\bar{a} \downarrow C^\bar{c}$ and for all atomic $B$ containing $\bar{c}$, $C \subset TV B \cup C$, the type $\text{tp}(\bar{a}/\bar{c})$ can be uniquely extended to a $\downarrow^A$-independent atomic type over $B$.

**Remarks 2.20.**

1. If $\mathcal{K}$ has an abstract totally transcendental dependence relation, and $C$ is the universe of a model, then every type over $C$ is weakly stationary for all atomic $B \supset C$. That is simply because of the stationary over models property.

2. If $\mathcal{K}$ is totally transcendental, then types over good sets are $\downarrow^A$-weakly stationary.

We first prove that the relations are the same for $\downarrow^A$-weakly stationary types. This will imply that stationarity in the sense of $\downarrow^A$ and $\downarrow$ is the same thing. So dependence relation is unique for types over models.

**Theorem 2.21.** Let $\downarrow^A$ be an abstract totally transcendental dependence relation on $\mathcal{K}$. Suppose $\bar{a}$ is a finite atomic tuple, $C$ is a good set, and $B$ is atomic with $C \subset TV B$. Suppose that $\text{tp}(\bar{a}/C)$ is $\downarrow^A$-weakly stationary. Then

$\bar{a} \downarrow C B$ if and only if $\bar{a} \downarrow C B$.
Proof. \(\Rightarrow\) Suppose \(\bar{a} \downarrow B\). Let \(\bar{c}\) be as in the definition of weak stationarity.

Then we have \(\bar{a} \downarrow B\) by uniqueness of independent extension. We prove that \(\text{tp}(\bar{a}/B)\) does not split over \(\bar{c}\). Suppose not; let \(\bar{b}_1, \bar{b}_2 \in B\) be such that \(\text{tp}(\bar{b}_1/\bar{c}) = \text{tp}(\bar{b}_2/\bar{c})\), but the types \(\text{tp}(\bar{a}\bar{b}_1/\bar{c})\) and \(\text{tp}(\bar{a}\bar{b}_2/\bar{c})\) are different. By monotonicity, we have

\[
\bar{a} \downarrow \bar{c}\bar{b}_1 \quad \text{and} \quad \bar{a} \downarrow \bar{c}\bar{b}_2.
\]

Let \(f \in \text{Aut}_\bar{c}(\mathcal{C})\) be such that \(f(\bar{b}_1) = \bar{b}_2\). Let \(\bar{a}_1 := f(\bar{a})\). Then by invariance \(\bar{a}_1 \downarrow \bar{c}\bar{b}_2\). Thus we get two distinct \(\downarrow\)-independent extensions of \(\text{tp}(\bar{a}/\bar{c})\) to \(\bar{c}\bar{b}_2\). Since \(C \subset_{TV} C \cup \bar{b}_2\), we get a contradiction to weak stationarity.

\(\Leftarrow\) Suppose \(\text{tp}(\bar{a}/B)\) does not split over a finite subset \(\bar{c}_1\) of \(C\). Suppose for contradiction that \(\bar{a} \not\in B\). Let \(\bar{a}' \models p\), \(p = \text{tp}(\bar{a}/C)\) be such that \(\bar{a}' \downarrow B\) (possible to find by extension requirement). By the first part of the proof, \(\text{tp}(\bar{a}'/B)\) does not split over a finite subset \(\bar{c}_2 \in C\). By invariance, \(\text{tp}(\bar{a}/B) \neq \text{tp}(\bar{a}'/B)\), so there are \(\varphi(\bar{x}, \bar{y})\) and \(\bar{d} \in B\) such that \(\varphi(\bar{x}, \bar{d}) \in \text{tp}(\bar{a}/B)\) and \(\neg \varphi(\bar{x}, \bar{d}) \in \text{tp}(\bar{a}'/B)\). Since \(C \subset_{TV} B\), we can find \(\bar{e} \in C\) such that \(\text{tp}(\bar{d}/\bar{c}_1 \cup \bar{c}_2) = \text{tp}(\bar{e}/\bar{c}_1 \cup \bar{c}_2)\). Since \(\varphi(\bar{x}, \bar{d}) \in \text{tp}(\bar{a}/B)\), necessarily \(\neg \varphi(\bar{x}, \bar{e}) \notin p\) (otherwise, \(\text{tp}(\bar{a}/B)\) splits over \(\bar{c}_1\)). Similarly, \(\varphi(\bar{x}, \bar{e}) \notin p\) by \(\neg \varphi(\bar{x}, \bar{d}) \in \text{tp}(\bar{a}'/B)\). So we get a contradiction to completeness of the type \(p\). \(\neg\)

Now from remarks above and Theorem 2.21 we conclude:

**Corollary 2.22.** If \(\mathcal{K}\) has a totally transcendental dependence relation \(\downarrow\), \(M \in \mathcal{K}\) is a model, \(B \supset M\), and \(\bar{a}B\) is atomic, then

\[
\bar{a} \downarrow M B \quad \text{if and only if} \quad \bar{a} \downarrow M B.
\]

Note that we did not require the symmetry property for the abstract dependence relation. The following is a (rather indirect) proof that symmetry property over models follows.

**Corollary 2.23** (Symmetry property). Suppose \(\mathcal{K}\) has an abstract dependence relation \(\downarrow\) satisfying (1)-(4). For \(M \in \mathcal{K}\), \(\bar{a}, \bar{b}\) such that \(M\bar{a}\bar{b}\) is atomic we have

\[
\bar{a} \downarrow M \bar{b} \iff \bar{b} \downarrow M \bar{a}.
\]

**Proof.** By Theorem 2.18 the rank \(R\) is bounded for the class \(\mathcal{K}\). By Lemmas 4.2 and 6.4 in [Sh 48], the dependence relation \(\downarrow\) has the symmetry
property. By Corollary 2.22, \( \downarrow \) has the symmetry over models property as well.

Suppose now that \( \downarrow \) satisfies the following additional property:

Uniqueness of extension:

Assume that \( C \) is good, \( \bar{a}C \) is atomic. There is finite \( \bar{c} \in C \) such that \( \bar{a} \not\downarrow_{\bar{c}} C \) and for any atomic set \( B, C \subset_{TV} B \cup C \), the type \( \text{tp}(\bar{a}/\bar{c}) \) can be uniquely extended to a \( \downarrow \)-independent atomic type over \( B \).

Remarks 2.24. (1) Certainly, the Uniqueness of extension property implies Stationarity over models. We have seen above that having Stationarity over models property already allows to draw many conclusion about the class \( \mathcal{K} \). Perhaps, in the non-first order situation, types over models are the right objects to look at.

(2) Assuming Uniqueness property, we characterize the dependence not only for stationary types, but also for Tarski-Vaught good pairs. That is because with Uniqueness, the types are \( \downarrow \)-weakly stationary.

Corollary 2.25. If \( \mathcal{K} \) has a totally transcendental dependence relation satisfying the Uniqueness property, \( C \) is good and \( C \subset_{TV} B \). If \( \bar{a}B \) is atomic, then

\[
\bar{a} \downarrow_{C} B \text{ if and only if } \bar{a} \downarrow_{C} B.
\]

3. Totally Transcendental Finite Diagrams

The goal of this section is to establish results parallel to those of Section 2 in the context of finite diagrams. The subject here is a class of models of a first order theory \( T \) such that each model omits a set of types \( \Gamma \). Such classes are denoted \( EC(T, \Gamma) \). A finite diagram is a class \( EC(T, \Gamma) \) with one extra assumption: everything happens inside a big homogeneous model that also belongs to the class. In other words we assume existence of a monster model that omits all the types in \( \Gamma \). Another way to look at the class \( EC(T, \Gamma) \) is to treat it as a class of \( D \)-models, where \( D = D(T) \setminus \Gamma \). So the extra assumptions translates into existence of a large homogeneous \( D \)-model. As the result of this assumption, we can, for instance, realize unions of \( D \)-types over sets of cardinality less than the cardinality of the monster model. For the remainder of the section, we agree to use the symbol \( \mathcal{C} \) for the monster \( D \)-model.

Finite diagrams in a stable context were introduced by S. Shelah in [Sh 3], with recent extensions due to R. Grossberg and O. Lessmann [GrLe] and T. Hyttinen and S. Shelah [HySh, HySh2]. Totally transcendental case was studied by O. Lessmann in [Le1].
Although there is a similarity in methods in the contexts of atomic models and finite diagrams, it is not true that either one case is a subcase of the other, so we cannot directly apply the results of the previous section.

### 3.1. Rank and dependence relation

A dependence relation for $\aleph_0$-stable finite diagrams was introduced by Olivier Lessmann in [Le1]. The dependence relation is defined via the rank function. The challenge is to make sure that unbounded rank gives uncountably many types over a countable set, and that the types are realized in the monster model. This is achieved by adding an extra condition to the definition of the 2-rank (due to Lessmann) that we give now. Note the similarity between this rank and the rank in Subsection 2.3

**Definition 3.1.** Let $p$ be a type over a finite $B \subset |\mathcal{C}|$.

1. $R[p] \geq 0$ if $p$ is realized in $\mathcal{C}$.
2. for $\alpha$ limit ordinal, $R[p] \geq \alpha$ if $R[p] \geq \beta$ for all $\beta < \alpha$.
3. $R[p] \geq \alpha + 1$ if
   - there are $\varphi(\bar{x}, \bar{y})$ and $\bar{a} \in \mathcal{C}$ such that $R[p \cup \varphi(\bar{x}, \bar{a})] \geq \alpha$ and $R[p \cup \neg \varphi(\bar{x}, \bar{a})] \geq \alpha$;
   - for every $\bar{b} \in |\mathcal{C}|$ there is a complete type $q(\bar{x}, \bar{y}) \in D$ such that $R[p \cup q(\bar{x}, \bar{b})] \geq \alpha$.

As usual, $R[p] = -1$ if $R[p] \nleq 0$;

- $R[p] = \alpha$ if $R[p] \geq \alpha$ and $R[p] \nleq \alpha + 1$;
- $R[p] = \infty$ if $R[p] \geq \alpha$ for all $\alpha \in \text{On}$;

if $q$ is a type over a subset of $\mathcal{C}$ which is not necessarily finite, we let $R[q] := \text{Min}\{R[p] \mid p \subseteq q, \text{dom}(p) \text{ finite}\}$.

The rank function has similar properties to the one defined in the previous section. For the rest of this subsection we assume that the diagram is totally transcendental, that is, the rank is bounded. Thus we can define the dependence relation by equality of the ranks:

**Definition 3.2.** Suppose $A$, $B$, $C$ are subsets of $\mathcal{C}$ such that $B \subset A$. Then $A \perp\limits_B C$ if and only if for all $\bar{a} \in A \ R[\text{tp}(\bar{a}/B)] = R[\text{tp}(\bar{a}/C)]$.

One of the key notions is stationarity. We give a definition formally different from the one suggested by Olivier Lessmann although the concepts we define are the same.

**Definition 3.3.** A $D$-type $p$ is stationary over $B \subseteq \text{dom}(p)$ if it has a unique $\perp$-independent extension to any superset of $B$.

For finite diagrams, stationary types are the most natural candidates for studying dependence relations. As well as in the atomic case, every type "splits" into stationary and isolated parts, for the right notion of isolation.
We give now some definitions and facts, all of them due to Olivier Lessmann [Le1].

**Definition 3.4.** A type \( p \in S_D(A) \) is \( D^s_\lambda \)-isolated if there is \( B \subset A, |B| < \lambda \), such that for any \( q \in S_D(A) \) extending \( p \upharpoonright B \) we have \( p = q \).

**Facts 3.5.** (1) Let \( p \in S_D(A) \) realized by \( \bar{a} \). There is \( \bar{d} \) such that the type \( tp(\bar{a}/\bar{d}) \) is stationary, \( tp(\bar{a}/A\bar{d}) \) is the stationarization of \( tp(\bar{a}/\bar{d}) \), and \( tp(\bar{d}/A) \) is \( D^s_{\aleph_0} \)-isolated.

(2) If \( M \) is a \((D,\aleph_0)\)-homogeneous model and \( p \in S_D(M) \), then \( p \) is stationary over a finite subset of \( M \).

(3) If \( tp(\bar{a}/B) \) is stationary, then \( \bar{a} \downarrow \bar{C} \) if and only if \( tp(\bar{a}/C) \) does not split over a finite subset of \( B \).

### 3.2. Abstract dependence characterization.

**Fact 3.6.** The dependence relation defined in [Le1] satisfies the properties of a totally transcendental abstract dependence relation.

Same theorems we proved in the previous section are true here as well. The proofs are almost the same, so we just state the results.

**Theorem 3.7.** Suppose \( \mathcal{C} \) has a totally transcendental dependence relation. Then the rank function \( R \) is bounded on \( \mathcal{C} \).

**Theorem 3.8.** If \( \mathcal{C} \) has a totally transcendental dependence relation, \( M \) is a \((D,\aleph_0)\)-homogeneous model, and \( B \supset M \), then

\[
\bar{a}^{(A)} \downarrow_M \bar{B} \quad \text{if and only if} \quad \bar{a} \downarrow_M \bar{B}.
\]

Note that the stationary bases may be different in different dependence relations. As before, we also get symmetry property for types over models.

**Corollary 3.9 (Symmetry property).** Suppose \( \downarrow \) is a totally transcendental dependence relation on \( \mathcal{C} \). If \( M \) is a \((D,\aleph_0)\)-homogeneous model, then

\[
\bar{a}^{(A)} \downarrow_M \bar{M}\bar{b} \quad \iff \quad \bar{b}^{(A)} \downarrow_M \bar{M}\bar{a}.
\]

We can extend our results to more sets, similar to what was done in the first section. We need the notion of a Tarski-Vaught pair. In finite diagrams, it translates to a relative saturation requirement; but we keep the Tarski-Vaught name.

**Definition 3.10.** We say that a pair of sets \((A,B), A \subset B\), satisfies the **\( D \)-Tarski-Vaught condition** if for every \( \bar{b} \in B, \bar{a} \in A, \) and \( q(\bar{x},\bar{y}) \in D \) if \( \bar{b} \models q(\bar{x},\bar{a}) \), then there is \( \bar{b}' \in A \) such that \( \bar{b}' \models q(\bar{x},\bar{a}) \). We write \( A \subset_{TV} B \).

Now if we replace Stationarity over finite subsets of models in the definition of a totally transcendental dependence relation by a stronger condition
Uniqueness of extension:
Assume in addition that \( A =: \vec{a} \) is finite. There is finite \( \vec{c} \in C \) such that \( \text{tp}(\vec{a}/C) \) is weakly stationary over \( \vec{c} \). That is, for any \( B, C \subset_{TV} B \), there is \( \vec{a}' \models \text{tp}(\vec{a}/C) \) such that \( \vec{a}' \upharpoonright \vec{c} \), and such an extension is unique over \( \vec{c} \).

then we can get a stronger result:

\[ \text{(A)} \]

**Theorem 3.11.** Suppose \( \downarrow \) is a totally transcendental dependence relation on \( \mathcal{C} \). Suppose \( \vec{a} \) is a finite tuple, \( C \) is a set, and \( B \) is such that \( C \subset_{TV} B \). Then

\[
\text{(A)} \quad \vec{a} \downharpoonright C \text{ if and only if } \vec{a} \downharpoonright C.
\]

4. Stable finite diagrams

In this section, we prove that a finite diagram is stable if and only if it has a stable dependence relation (see Definition 1.6). Moreover, we show that, over models, any stable dependence relation must coincide with (non) strong splitting.

4.1. Preliminary results. Fix a diagram \( D \), and let \( \mathcal{C} \) be a monster \( D \)-model. The following definitions are due to Shelah.

**Definition 4.1.**

1. The diagram \( D \) is stable in \( \lambda \) if for every \( A \subset \mathcal{C} \) of cardinality at most \( \lambda \) we have \( |S_D(A)| \leq \lambda \).
2. The diagram \( D \) is stable if it is stable in some \( \lambda \); \( D \) is superstable if there is \( \lambda \) such that \( D \) is stable in \( \mu \) for all \( \mu \geq \lambda \).
3. A type \( \text{tp}(\vec{c}/A) \) splits strongly over \( B \subset A \) if there is an indiscernible sequence \( \{a_i \mid i < \omega \} \) over \( B \) such that \( \vec{a}_0 \in A \) and for some \( \varphi(\vec{x}, \vec{y}) \) we have \( \vec{c} \models \varphi(\vec{x}, \vec{a}_0) \land \lnot \varphi(\vec{x}, \vec{a}_1) \).
4. We write \( A \downharpoonright B \) if for every finite \( \vec{a} \in A \), \( \text{tp}(\vec{a}/B) \) does not split strongly over \( C \).

Much is known about the structure/non-structure theory of stable finite diagrams (see [Sh 3, GrLe, HySh2]) as well as dependence relation of strong splitting ([HySh]). It is worth pointing out that the dependence relation in [HySh] is slightly different from the one we define. In [HySh] extension property is a part of the definition. Therefore, the Existence property \( \text{(A)} \)

and the more so Local Character hold only over extension bases. For the strong splitting relation, both properties hold over arbitrary \( D \)-sets in a stable finite diagram.

**Fact 4.2.** If \( D \) is a stable finite diagram, then the relation \( \downarrow \) is a stable dependence relation. In addition, if \( D \) is superstable, then the local character of strong splitting is \( \aleph_0 \).
The invariance, monotonicity, and finite character properties are clear; local character is established in a convenient for us form in [GrLe], Theorem 4.11 (see also [Sh 3, HySh]). The local character of non strong splitting $\kappa_s$ is less than or equal to the least stability cardinal. Stationarity is established in [HySh], Lemma 3.4 (remember that $\alpha$-saturated is Lascar $(D, \kappa)$-homogeneous in our terminology).

Under the assumption of stability with local character $\kappa = \kappa_s$, there are many Lascar $(D, \kappa)$-homogeneous models. Namely, the following holds.

**Facts 4.3.** (1) (Stability Spectrum theorem) If $D$ is stable, $\lambda_D$ is the least stability cardinal, and $\mu \geq \lambda_D$, then $D$ is stable in $\mu$ if and only if $\mu^{<\kappa_s} = \mu$.

(2) If $D$ is stable in $\lambda$ and $\lambda^{<\kappa} = \lambda$ for a regular $\kappa$, then every set of cardinality at most $\lambda$ is contained in a Lascar $(D, \kappa)$-homogeneous model of cardinality $\lambda$.

(1) is presented in [Sh 3, GrLe]; (2) is essentially Lemma 1.9(ii) in [HySh].

In the next subsection, we will need to use stationarity over Lascar $(D, \kappa)$-homogeneous models, for $\kappa$ the local character of $\downarrow$, without the stability assumption (our goal is to deduce stability from existence of a stable dependence relation).

Accordingly, we need to know that such models exist. First we state a few facts.

**Facts 4.4 ([BuLe]).** (1) If $I$ is an indiscernible sequence over $A$, then $\lstp(\bar{a}/A) = \lstp(\bar{b}/A)$ for all $\bar{a}, \bar{b} \in I$.

(2) There are fewer than $\beth_{(2^{|A|})}^+$ distinct Lascar strong types (in finitely many variables) over a set $A$.

(3) Equality of Lascar strong types over a set $A$ is the finest bounded $A$-invariant equivalence relation over $A$.

**Lemma 4.5.** Let $\kappa$ be a fixed cardinal. For every set $A$, there is a Lascar $(D, \kappa)$-homogeneous model $M$ containing $A$. If in addition $|A| \geq \beth_{(2^\kappa)}^+$ and $|A|^{<\kappa} = |A|$, then $M$ could be chosen of the same cardinality as $A$.

**Proof.** Let $\lambda := \max\{\beth_{(2^\kappa)}^+, |A|\}$. Construct $\{M_i \mid i < \kappa^+\}$ such that

1. $M_0 := A$, $|M_i| \leq \lambda$;
2. $M_{i+1}$ is a model that contains $M_i$ and realizes Lascar strong types in finitely many variables over all the subsets of $M_i$ of size less than $\kappa$;
3. if $i$ is a limit ordinal, $M_i := \bigcup_{j<i} M_j$.

To carry out the construction at the successor step, observe that there are at most $\lambda$ many subsets of $M_i$ of size less than $\kappa$ since the cofinality of $\lambda$ is at least $\kappa$. Over each such subset, there are at most $\lambda$ many Lascar strong types. So the set $A_{i+1}$ of representatives of each Lascar strong types over all subsets of $M_i$ of size less than $\kappa$ has cardinality at most $\lambda$. We let $M_{i+1}$ be a model containing $A_{i+1}$, $|M_{i+1}| \leq \lambda$. 
Let \( M := \bigcup_{i < \kappa^+} M_i \). Clearly, \( M \) is as needed: if \( B \subset M \), \( |B| < \kappa \), then \( B \subset M_i \) for some \( i < \kappa \), and so all Lascar strong types over \( B \) are realized in \( M \).

Note that if \( \kappa \) is a regular cardinal, then it is enough to construct \( M_i \) for \( i < \kappa \).

4.2. Abstract dependence characterization.

**Proposition 4.6.** If the finite diagram \( D \) has a stable dependence relation \( \downarrow \), then \( D \) is stable.

**(A)**

**Proof.** Let \( \kappa \) be the local character of \( \downarrow \) and let \( \lambda := \beth_{2^\kappa} \). We prove that \( D \) is stable in \( \mu := 2^\lambda \). Suppose for contradiction that there is \( A \) is such that \( |A| = \mu \), and \( |S_D(A)| \geq \mu^+ \). By Lemma 4.5 we may assume that \( A \) is the universe of a Lascar \((D, \kappa)\)-homogeneous model.

Let \( \{a_i \mid i < \mu^+\} \) be realizations of distinct types over \( A \). By local character, there are \( \{B_i \mid i < \mu^+\} \) such that \( |B_i| < \kappa \) and \( B_i \upharpoonright A \) for all \( i < \mu^+ \). Since there are \( \mu \) many subsets of \( A \) of size less than \( \kappa \), by pigeonhole principle we may assume that for some \( B \subset A \), \( |B| < \kappa \), \( B_i \upharpoonright A \) for all \( i < \mu^+ \). Let \( M \) be a Lascar \((D, \kappa)\)-homogeneous model containing \( B \). By Lemma 4.5 \( M \) could be chosen so that \( |M| \leq \lambda \), and \( M \subset A \). Monotonicity now gives \( a_i \upharpoonright M \) for \( i < \mu^+ \). Since there are at most \( \mu \) different types over \( M \), by pigeonhole principle we may assume that all \( \{a_i \mid i < \mu^+\} \) realize the same type \( p \in S_D(M) \). This contradicts the stationarity property.

We now prove that the stable dependence relation \( \downarrow \) is exactly that of non strong splitting over Lascar \((D, \kappa)\)-homogeneous models.

**Lemma 4.7.** Let \( \downarrow \) be a stable dependence relation. Suppose \( A, B \) are \( D \)-sets and a model \( M \) is Lascar \((D, \kappa)\)-homogeneous. If \( A \downarrow M B \), then \( A \downarrow M B \).

**(A)**

**Proof.** By finite character, we may assume \( A = M\bar{a}, B = M\bar{b} \) for finite \( \bar{a}, \bar{b} \). Assume \( \bar{a} \downarrow M\bar{b} \), but \( \text{tp}(\bar{a}/M\bar{b}) \) strongly splits over \( M \). Let \( \{\bar{b}_i \mid i < \omega\} \) witness strong splitting, with \( \bar{b}_0 = \bar{b} \). By extension, there is \( \bar{a}' \models \text{tp}(\bar{a}/M\bar{b}) \) such that \( \bar{a}' \downarrow M\{\bar{b}_i \mid i < \omega\} \). Let \( f \in \text{Aut}_M(\mathcal{C}) \) be such that \( f(\bar{b}_0) = \bar{b}_1 \).
By monotonicity, we have

\[ \bar{a}' \Downarrow_M M\bar{b}_0 \quad \text{and} \quad \bar{a}' \Downarrow_M M\bar{b}_1. \]

Let \( \bar{a}_1 := f(\bar{a}') \). Then by invariance \( \bar{a}_1 \Downarrow_M M\bar{b}_1 \). Thus we get two distinct \( \Downarrow_M \)-independent extensions of \( \text{tp}(\bar{a}/M) \) to \( M\bar{b}_1 \). Contradiction to stationarity over models.

To prove the converse, we need to establish the connection between the local character \( \kappa \) of \( \Downarrow_M \) and the local character \( \kappa_s \) of non strong splitting. Without loss of generality, we may assume that \( \kappa \) is a regular cardinal (clearly, \( \kappa \)-local character implies \( \kappa^+ \)-local character).

**Lemma 4.8.** Suppose \( D \) has a stable dependence relation \( \Downarrow_M \). Let \( \kappa \) be the (regular) local character cardinal of \( \Downarrow_M \). Then \( \kappa_s \leq \kappa \).

**Proof.** Suppose for contradiction that \( \kappa < \kappa_s \). By Proposition 4.6, the finite diagram \( D \) is stable. Let \( \lambda_D \) be the least stability cardinal. We know that \( \kappa_s < \lambda_D \), so \( \kappa < \lambda_D \). Let \( \mu > \lambda_D \) be a cardinal such that \( \mu^{<\kappa} = \mu \) and \( \mu^{<\kappa_s} > \mu \) (for example, the \( \kappa \)th successor of \( \lambda_D \) will work, here we use regularity of \( \kappa \)). By Stability Spectrum theorem, \( D \) is unstable in \( \mu \), so let \( A \) be a set of cardinality \( \mu \) such that \( |S_D(A)| \geq \mu^+ \).

**Claim 4.9.** We may assume that \( A \) has the following property. For every \( B \subset A \), \( |B| < \kappa \), there is a Lascar \( (D, \kappa) \)-homogeneous model \( M \subset A \), \( |M| = \lambda_D \) containing \( B \).

**Proof.** Construct a sequence \( \{A_i \mid i < \kappa\} \) such that

1. \( A_0 := A \), \( |A_i| = \mu \);
2. for every \( B \subset A_i \) of size less than \( \kappa \), \( A_{i+1} \) contains a Lascar \( (D, \kappa) \)-homogeneous model \( M \supset B \).
3. if \( i \) is a limit ordinal, \( A_i := \bigcup_{j < i} A_j \).

For the successor step, let \( \{B_{\alpha} \mid \alpha < \mu\} \) be an enumeration of all the subsets of \( A_i \) of size less than \( \kappa \) (there are \( \mu \) many of them since \( \mu^{<\kappa} = \mu \). By stability in \( \lambda_D \) we have \( \lambda_D^{<\kappa} \leq \lambda_D^{\kappa_s} = \lambda_D \), so the conditions of Fact 4.3(2) are satisfied. Therefore, for every \( \alpha < \mu \), there is a Lascar \( (D, \kappa) \)-homogeneous model \( M_\alpha \) containing \( B_\alpha \) such that \( |M_\alpha| = \lambda_D \). Let \( A_{i+1} := \bigcup_{\alpha < \mu} M_\alpha \). Clearly, \( A_{i+1} \) is as needed.

The set \( A_\kappa \) has the property required in the claim: if \( B \subset A_\kappa \), \( |B| < \kappa \), then \( B \in A_i \) for some \( i < \kappa \) (here we use regularity of \( \kappa \) again). So \( A_{i+1} \) (and therefore \( A_\kappa \)) contain the needed model. It is clear that \( |A_\kappa| = \mu \), and since \( A \subset A_\kappa \), there are at least \( \mu^+ \) many types over \( A_\kappa \). So we may take \( A_\kappa \) in place of \( A \).
Let \( \{a_i \mid i < \mu^+\} \) be an enumeration of distinct types over \( A \). By local character, there are \( \{B_i \subset A \mid i < \mu^+\} \), \( |B_i| < \kappa \), such that \( \bigwedge_{B_i} \langle a_i \rangle \mid B_i \). By pigeonhole principle, we may assume that for all \( i < \mu^+ \), \( \bigwedge_{B_i} \langle a_i \rangle \mid B_i \) for some \( B \subset A \). Let \( M \subset A \) be a Lascar \( (D, \kappa) \)-homogeneous model of cardinality \( \lambda_D \) containing \( B \). By stability in \( \lambda_D \), there are at most \( \lambda_D \) types over \( M \). By pigeonhole principle we may assume that \( \{\bigwedge_{B_i} \langle a_i \rangle \mid B_i \} \) realize the same type over \( M \). By monotonicity, \( \bigwedge_{B_i} \langle a_i \rangle \mid M \) for each \( i < \mu^+ \), so we get a contradiction to stationarity over models.

\[ \neg \]

**Theorem 4.10.** Suppose \( D \) has a stable dependence relation \( \downarrow \). Suppose \( A, B \) are \( D \)-sets and a model \( M \) is Lascar \( (D, \kappa) \)-homogeneous. Then \( A \downarrow B \)

if and only if \( A \downarrow M \).

**Proof.** One direction is established in Lemma 4.7.

We may assume \( A = M\bar{a} \) for finite \( \bar{a} \). Suppose \( \bar{a} \downarrow M \) but \( \bar{a} \not\in B \). Since \( \bar{a} \downarrow M \), by extension there is \( \bar{a}' \models \text{tp}(\bar{a}/M) \) such that \( \bar{a}' \downarrow B \). Since \( \bar{a} \not\in B \), by invariance, \( \text{tp}(\bar{a}'/B) \neq \text{tp}(\bar{a}/B) \). Now Lemma 4.7 gives \( \bar{a}' \downarrow B \). Since \( \kappa_s \leq \kappa \), \( M \) is also Lascar \( (D, \kappa_s) \)-homogeneous. So we get a contradiction to stationarity over models of non strong splitting.

\[ \neg \]

Since strong splitting has symmetry and transitivity properties over Lascar \( (D, \kappa_s) \)-homogeneous models, we also get the properties for \( \downarrow \).

**Corollary 4.11.** The relation \( \downarrow \) is symmetric and transitive over Lascar \( (D, \kappa) \)-homogeneous models.

If we let \( \kappa := \aleph_0 \), we get the following

**Corollary 4.12.** A finite diagram \( D \) is superstable if and only if it has a superstable dependence relation \( \downarrow \). Moreover, the relation must coincide with that of non strong splitting over Lascar \( (D, \aleph_0) \)-homogeneous models.

In particular, \( \downarrow \) must be symmetric and transitive over those models.

5. Simple homogeneous models

5.1. Preliminaries. We work in the context described in the paper [BuLe] by Steven Buechler and Olivier Lessmann. The class of structures we are
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dealing with here is larger than in the finite diagrams framework. Namely,
we study logical structures \((M, \mathcal{R})\), where \(M\) is a structure in a first order
language and \(\mathcal{R}\) is a collection of finitary relations on \(M\) closed under some
reasonable operations.

We further assume that \((M, \mathcal{R})\) is strongly \(\lambda\)-homogeneous for a large \(\lambda\),
see Definition 1.3 in [BuLe]. By large we mean the following: if we are
interested in types over set of size at most \(\pi\) in less than \(\pi\) many variables,
then \(\lambda\) should be at least \(\beth_{\left(2^\pi\right)^+}\).

For the rest of the section, we fix \(\pi\) and agree to consider the types in less
than \(\pi\) many variables over sets of cardinality at most \(\pi\). Furthermore, we
identify the sets of size less than \(\pi\) with some enumeration of those sets.

The following is an important property of strongly homogeneous structures (it appears in [BuLe] in Lemmas 1.3 and 1.4, see also assumption \(\Pi\)
in Section 2 there).

**Fact 5.1.** Let \((M, \mathcal{R})\) be a strongly \(\lambda\)-homogeneous structure. There is a
 cardinal \(\pi' \leq \lambda\) such that for every type \(p(\bar{x})\) over a \(A\) set of cardinality
less than \(\pi\) in less than \(\pi\) many variables, if \(\{\bar{a}_i \mid i \in X\}\) is a sequence of
realizations of \(p\) indexed by a linear order \(X\), \(|X| \geq \pi'\), then for every linear
order \(Y\), \(|Y| \leq \lambda\) there is an indiscernible over \(A\) sequence \(\{\bar{b}_i \mid i \in Y\}\) with
typing \((\bar{b}_{i_0}, \ldots, \bar{b}_{i_n}/A)\) realized by some increasing sequence \(\{\bar{a}_{j_0}, \ldots, \bar{a}_{j_n}\}\) for all
\(n < \omega\).

The following concept is a substitute for algebraic types in our context.

**Definition 5.2.** A type \(p\) is small, if the set of realizations of \(p\) has cardinality
less than \(\pi'\). A type \(p\) is called large otherwise.

The dependence relation in this situation is given by dividing.

**Definition 5.3.** (1) A type \(p(\bar{x}, \bar{b})\) divides over \(A\), if there is an infinite indiscernible sequence \(\{b_i \mid i \in X\}\) such that \(\bar{b} = \bar{b}_i\) for some \(i \in X\) and the
type \(\bigcup_{i \in X} p(\bar{x}, \bar{b}_i)\) is inconsistent.

(2) We say that \(A\) is free from \(B\) over \(C\) \((A \downarrow B)\) if for all finite tuples
\(\bar{a} \in A\) and \(\bar{b} \in B \cup C\) we have \(\text{tp}(\bar{a}, \bar{b})\) does not divide over \(C\).

It is clear that if \(\text{tp}(\bar{b}/A)\) is small, then \(p(\bar{x}, \bar{b})\) does not divide over \(A\).

In [BuLe], it is established that in a \(\kappa\)-simple homogeneous model dividing
has the all properties of a simple dependence relation, except the type
amalgamation, which does hold in the following “local” form.

**Fact 5.4** (Buechler, Lessmann). Suppose \(|C| < \kappa\) and \(\bar{a}_i, \bar{b}_i, i = 1, 2,\)
are tuples of length less than \(\kappa\). If \(\text{lstp}(\bar{a}_1/C) = \text{lstp}(\bar{a}_2/C)\), \(\bar{b}_1 \downarrow C \bar{b}_2\), and
\(\bar{a}_i \downarrow C \bar{b}_i, i = 1, 2,\) then there is \(\bar{a} \models \text{lstp}(\bar{a}_1/C\bar{b}_1) \cup \text{lstp}(\bar{a}_2/C\bar{b}_2)\) such that
\(\bar{a} \downarrow C \bar{b}_1 \bar{b}_2\).
Remark 5.5. In the definition of a simple dependence relation we require the type amalgamation to hold over any small $b_i, i = 1, 2$, and $C$ not necessarily of size less than $\kappa$. It is not clear whether this type amalgamation property would follow from the local version in general.

However, the local type amalgamation implies type amalgamation over all small sets for compact homogeneous models; and any counterexample would have to be quite exotic: it will be an example of a $\kappa$-simple homogeneous model which is not $\kappa'$-simple for some $\kappa' > \kappa$.

Definition 5.6. We say that a strongly $\lambda$-homogeneous structure $(M, R)$ is $\kappa$-simple with type amalgamation over all small sets if $(M, R)$ is $\kappa$-simple and the type amalgamation property holds for $b_1, b_2$, and $C$ of arbitrary small size.

Fact 5.7. If the homogeneous structure is $\kappa$-simple with type amalgamation over all small sets, then dividing is a simple dependence relation with local character $\kappa$.

Remark 5.8. In [BuLe], extension property for large types is required by the definition of a simple structure. For small types, extension for dividing holds trivially by Lemma 2.6 in [BuLe] and transitivity. For the purpose of characterizing the dependence relations, it is essential to formulate the extension for both large and small types.

Even for first order simple theories, one can define a dependence relation different from dividing that would satisfy all the properties of a simple relation, but not the extension for small (algebraic, in first order) types.

5.2. Abstract dependence characterization. The following result generalizes Kim-Pillay’s theorem for the simple case. Though the idea of the proof is similar, there are some added difficulties due to the failure of compactness and a different definition of a Lascar strong type.

Theorem 5.9. A strongly $\lambda$-homogeneous logical structure $(M, R)$ is simple with type amalgamation over all small sets if and only if it has a simple dependence relation. In addition, the abstract dependence relation coincides with the one defined by dividing.

Proof. One direction is given by Fact 5.7.

Suppose now that we have a simple dependence relation $\Downarrow^A$. First we prove that $\Downarrow^A$ relation coincides with the one defined by dividing. By Finite Character of $\Downarrow^A$, it is enough to show that for all finite $\bar{a}, \bar{b} \in M$, $A \subseteq M$

$$\bar{a} \Downarrow^A A \bar{b}$$

if and only if $\text{tp}(\bar{a}/A\bar{b})$ does not divide over $A$.

We split the proof into several lemmas.
Lemma 5.10 (Existence of $\downarrow$-Morley sequences). Let $\bar{a} \in M$, $B \subset M$, and $A \subset B$ be such that $\bar{a} \downarrow B$. Let $X$ be an infinite order. If $\text{tp}(\bar{a}/B)$ is large, then $M$ contains a $\downarrow$-Morley sequence $I = \{\bar{a}_i \mid i \in X\}$ for $\text{tp}(\bar{a}/B)$ over $A$.

Proof. Use the same argument as in Lemma 2.4 in [BuLe].

Lemma 5.11. If $\bar{a} \not\subseteq A\bar{b}$, then $\text{tp}(\bar{a}/A\bar{b})$ divides over $A$.

Proof. Denote $p(\bar{x}, \bar{b}) := \text{tp}(\bar{a}/A\bar{b})$. First we prove that $\bar{a} \not\subseteq A\bar{b}$ implies that $\text{tp}(\bar{b}/A)$ is large. Suppose $\text{tp}(\bar{b}/A)$ is small. Let $D$ be the set of all realizations of $\text{tp}(\bar{b}/A)$, let $\bar{b}' \models \text{tp}(\bar{b}/A)$ be such that $\bar{b}' \downarrow A \cup D$. Since $\bar{b}' \in D$, we get $\bar{b}' \downarrow A\bar{b}'$ by monotonicity, so invariance implies $\bar{b} \downarrow A\bar{b}$. By extension property, there is $\bar{b}'' \models \text{tp}(\bar{b}/A\bar{b})$ such that $\bar{b}'' \downarrow A\bar{b}$. Clearly, $\bar{b}'' = \bar{b}$, so we have $\bar{b} \downarrow A\bar{a}\bar{b}$, and by symmetry $\bar{a} \downarrow A\bar{b}$, contradiction.

Since $\text{tp}(\bar{b}/A)$ is large, we can find $I := \{\bar{b}_i \mid i < \kappa\}$ a $\downarrow$-Morley sequence for $\text{tp}(\bar{b}/A)$. We claim that $\bigcup_{i < \kappa} p(\bar{x}, \bar{b}_i)$ is inconsistent.

Suppose for contradiction that it is consistent and let $\bar{a}' \models \bigcup_{i < \kappa} p(\bar{x}, \bar{b}_i)$. By invariance, $\bar{a} \not\subseteq A\bar{b}$ implies $\bar{a}' \downarrow A\bar{b}_i$ for all $i < \kappa$. On the other hand, by local character $\bar{a}' \downarrow A \cup I$ for some $J \subset I$, $|J| < \kappa$. Let $i < \kappa$ be such that $J < i$, then by symmetry and transitivity of $\downarrow$ we have $\bar{a}' \downarrow A\bar{b}_i$, contradiction.

Lemma 5.12. If $\bar{a} \downarrow A\bar{b}$, then $\text{tp}(\bar{a}/A\bar{b})$ does not divide over $A$.

Proof. If $\text{tp}(\bar{b}/A)$ is small, then $\text{tp}(\bar{a}/A\bar{b})$ does not divide over $A$, so we are done.

Suppose now that $\text{tp}(\bar{b}/A)$ is large. Let $I = \{\bar{b}_i \mid i \in X\}$ be an indiscernible sequence in $\text{tp}(\bar{b}/A)$, with $\bar{b}_0 = \bar{b}$. We need to prove that $\bigcup_{i \in X} p(\bar{x}, \bar{b}_i)$ is consistent.

Take a long extension of the sequence $\{\bar{b}_i \mid i \in X\}$: let $\bar{X}$ be a linear order that extends $X$, we take it to be $\kappa^+$ copies of $X$, where $\kappa$ is the
local character of $\downarrow$, with an extra last element $i^*$. Accordingly, $\bar{I}$ is an indiscernible sequence that has $\kappa^+$ copies of $I$, with an extra element $\bar{b}_{i^*}$.

By the local character of $\downarrow$, there is a subsequence $I' \subset \bar{I}$, $|I'| < \kappa$, such that $\bar{b}_{i^*} \downarrow \bar{I}$. By regularity of $\kappa^+$, there is $\delta < \kappa^+$ such that $I' \subset \{\bar{I}_\alpha \mid \alpha < \delta\}$, where $\bar{I}_\alpha$ is the $\alpha$th copy of $I$ in $\bar{I}$. By monotonicity,

$$\bar{b}_{i^*} \downarrow_{\bar{I}_\delta^0} \bar{I}_\delta.$$  

Since $\bar{I}_\delta$ is a copy of $I$, we may assume that in fact $\bar{I}_\delta = \{\bar{b}_i \mid i \in X\}$, i.e., $\bar{I}_\delta = I$.

For a subset $Y \subset X$, and an index $i \in X$ we say $Y < i$ when $i$ is greater than any element in $Y$; the symbol $\bar{b}_Y$ stands for the sequence $\{\bar{b}_j \mid j \in Y\}$. With these notations, for any $Y \subset X$ and any $i > Y$, by indiscernibility of $\bar{I}$ over $A$, we have $\text{tp}(\bar{b}_{i^*}/AI'\bar{b}_Y) = \text{tp}(\bar{b}_{i}/AI'\bar{b}_Y)$. By invariance and $(\ast)$, from this we can conclude

$$\bar{b}_i \downarrow_{AI'} \bar{b}_Y$$

for any $Y \subset X$ and $i > Y$. Therefore $\{\bar{b}_i \mid i \in X\}$ is a $\downarrow$-Morley sequence over $AI'$. We will need two implications of this fact.

First, all $\bar{b}_i$ realize the same Lascar strong type over $AI'$. Second, by a standard argument, symmetry and transitivity imply that for any rearrangement of $I$ is a $\downarrow$-independent sequence over $AI'$ (of course, not necessarily indiscernible). Let $\lambda := |X|$. Rearranging the elements of $I$ in some order, we may assume that $I$ is a $\downarrow$-independent sequence over $AI'$ (not necessarily indiscernible), and that all the elements of $I$ have the same Lascar strong type over $AI'$.

For $i < \lambda$, let $f_i$ be a strong automorphism over $AI'$ such that $f_i(\bar{b}_0) = \bar{b}_i$. Let $\bar{a}_i := f_i(\bar{a})$. By invariance, we then have $\bar{a}_i \downarrow_{AI'} \bar{b}_i$.

Let $\bar{a}_0'$ be in a $\downarrow$-Morley sequence in $\text{tp}(\bar{a}_0/\bar{b}_0) AI'$. We can choose $\bar{a}_0'$ so that $\bar{a}_0' \downarrow_{AI'} I$. Then $\bar{a}_0 \downarrow_{AI'} \bar{b}_0 \bar{a}_0', \text{ lstp}(\bar{a}_0/\bar{b}_0) = \text{ lstp}(\bar{a}_0/\bar{b})$, and $\bar{a}_0' \bar{b}_0 \downarrow \{\bar{b}_j \mid 1 \leq j < \lambda\}$.

Let $q(\bar{x}, \bar{b}_0, \bar{a}_0') := \text{tp}(\bar{a}_0/\bar{b}_0 \bar{a}_0')$. By induction on $1 \leq \alpha \leq \lambda$ we construct types $q_\alpha(\bar{x})$ such that

1. $q_1(\bar{x}) := q(\bar{x}, \bar{b}_0, \bar{a}_0'); \text{ dom}(q_\alpha) = AI'\bar{a}_0' \{\bar{b}_i \mid i < \alpha\}$;
2. $q_\alpha \subset q_\beta$ for $\alpha < \beta < \lambda$, and $q_\alpha(\bar{x}) \supset \bigcup_{i < \alpha} p(\bar{x}, \bar{b}_i)$;
(3) if $\bar{c}_\alpha \models q_\alpha$, then $\text{lstp}(\bar{c}_\alpha/AI') = \text{lstp}(\bar{a}_0/AI')$ and

$$\left( A \right) \bar{c}_\alpha \downarrow \bar{a}_0'_{\{\bar{b}_i \mid i < \alpha \}};$$

(4) $\bar{c}_\alpha \downarrow \bar{a}_0'_{\{\bar{b}_i \mid i < \alpha \}}$.

The case $\alpha = 1$ is clear. Note that the Lascar strong type of $\bar{a}_0$ over $AI'$ became type definable when we added $\bar{a}_0'$. Hence, any independent realization of $q_1(\bar{x})$ will have the same Lascar strong type over $AI'$ as $\bar{a}_0$.

For the successor case, suppose we have a complete $q_\alpha(\bar{x})$ as in (1)–(4) above. By the choice of $\bar{a}_0'$, we have $\text{dom}(q_\alpha) \downarrow \bar{b}_\alpha$. Also $\text{lstp}(\bar{a}_0/AI') = \text{lstp}(\bar{c}_\alpha/AI') = \text{lstp}(\bar{c}_\alpha/AI')$. So we can apply type amalgamation (Property 7) and get $\bar{c}_{\alpha+1}$ that realizes $q_\alpha(\bar{x}) \cup p(\bar{x}, \bar{b}_\alpha)$ such that $\bar{c}_{\alpha+1} \downarrow \bar{a}_0'_{\{\bar{b}_i \mid i < \alpha + 1 \}}$. Let $q_{\alpha+1} := \text{tp}(\bar{c}_{\alpha+1}/AI' \bar{a}_0'_{\{\bar{b}_i \mid i < \alpha + 1 \}})$. Clearly, $q_{\alpha+1}$ is as needed.

For the limit case, we apply Lemma 1.2 in [BuLe] to the sequence of types $\{q_i(\bar{x}) \mid i < \alpha \}$ to conclude that the union $\bigcup_{i<\alpha} q_i(\bar{x})$ is consistent. By the choice of the type $q_1(\bar{x})$, the realization $\bar{c}_\alpha$ has the same Lascar strong type over $AI'$ as does $\bar{a}_0'$, which coincides with the Lascar strong type of $\bar{a}_0$ over $AI'$. So $\text{lstp}(\bar{c}_\alpha/AI') = \text{lstp}(\bar{a}_0/AI')$. Condition (4) in the construction holds for the union by finite character of $\downarrow$.

Finally, the construction gives that the union $\bigcup_{i \in \mathcal{X}} p(\bar{x}, \bar{b}_i)$ is consistent.

Thus, we have proved that $\downarrow$ relation coincides with the relation given by dividing. Therefore, dividing has the properties of a simple dependence relation, and hence the logical structure $(M, \mathcal{R})$ is simple.

REFERENCES


[HaSh] Bradd Hart and Saharon Shelah. Categoricity over $P$ for first order $T$ or categoricity for $\varphi \in L_{\omega_1, \omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$. *Isr. J. Math.*, 46, 219–235, 1990.


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AN INTRODUCTION TO EXCELLENT CLASSES

OLIVIER LESSMANN

ABSTRACT. In this paper, we present Shelah’s theory of excellence concluding with his categoricity theorem for the class of atomic models of a countable first order theory, under the assumption that there exists a large full model. This allows us to do the entire work within ZFC and without any assumption on the number of models of size $\aleph_1$, in contrast to Shelah’s original treatment.

INTRODUCTION

The problem of categoricity has had a major influence on the development of model theory. A class is categorical in some cardinal $\lambda$ if all the models of the class of size $\lambda$ are isomorphic; the problem of categoricity is whether categoricity in some cardinals implies categoricity in others. In the first order case, Morley’s solution to Łos conjecture [Mo], and Shelah’s generalisation to uncountable languages [Sh 70] constitute the beginning of classification theory culminating in Shelah’s Main Gap [Sh a], while Baldwin-Lachlan’s solution [BaLa] laid the ground for geometric stability theory.

Categoricity for classes which are not first order is a considerably more complicated problem. It is a very active area with many partial results (see [Ke], [Sh 3], [KoSh], [MaSh], [Sh 48], [Sh 87a], and [Sh 87b], [Sh 88], [Sh 394], [ShVi], [Sh 576], [Sh 600], [Sh 705] to name but a few). Shelah views it as the most important problem in model theory and lists it first in [Sh 666]. He conjectures that, if the class $\mathcal{K}$ of models of a sentence in $L_{\omega_1,\omega}$ is categorical in some large enough cardinal, then it must be categorical in all large enough cardinals.

Historically, excellence arose after this conjecture was verified under the assumption that the class $\mathcal{K}$ contains sufficiently homogeneous models [Ke], [Sh 3]. This marked the beginning of classification for homogeneous model theory, as this context is now known. In homogeneous model theory, we have good notions of stability [Sh 3], [Sh 54], [GrLe1], [Hy], superstability [HySh], [HyLe1], $\omega$-stability and total transcendence [Le1], and even simplicity [BuLe]. Moreover, we have a Baldwin-Lachlan [BaLa] style theorem [Le1], as well as the beginning of geometric model theory [HLS]. At the time of [Ke], Keisler asked whether the existence of sufficiently homogeneous models actually followed from categoricity. Shelah

answered negatively using an example of Markus [Ma], and introduced the theory of *excellence*. To do this, he first reduced the categoricity problem for the class of models of a sentence in $L_{\omega_1, \omega}$ to the categoricity of the class of atomic models of a countable first order theory [Sh 54]. This reduction is an important step (see also Baldwin [Ba] for more details); it shows that proving the categoricity conjecture for $L_{\omega_1, \omega}$ is equivalent to proving it for the apparently simpler context of the class of atomic models of a countable first order theory. Then, in [Sh 87a], [Sh 87b], Shelah showed:

**Theorem.** Assume $2^{n_1} < 2^{n_1+1}$ for each $n < \omega$. Let $\mathcal{K}$ be the class of atomic models of a countable first order theory and assume further that $\mathcal{K}$ has fewer than $2^{n_1}$ nonisomorphic models of size $n_1$.

1. If $\mathcal{K}$ is categorical in each $n_1$, for $n < \omega$, then $\mathcal{K}$ is excellent.
2. If $\mathcal{K}$ is excellent and categorical in some uncountable cardinal, then $\mathcal{K}$ is categorical in all uncountable cardinals.

Modulo some additional properties, (1) thus shows that categoricity implies excellence, while (2) is the parallel to Morley's theorem for excellent classes. Above, (1) has the flavour of a nonstructure result and (2) belongs to structure theory.

In this paper, we focus on the structure part. We present a proof of (2), under the assumption that there exists a large *full model* (see below for more details). The existence of full models follows from excellence, and, for uncountable models, we can give an equivalent definition which makes sense for any class of atomic models. The reason for using a full model is that it allows us to present the entire treatment within ZFC and to remove the assumption on the number of uncountable models; for example, we obtain all the properties of independence directly from $\omega$-stability.

Solving the categoricity problem for excellent classes marked the beginning of classification theory in this context; Grossberg and Hart developed orthogonality calculus, introduced regular types, and proved the Main Gap for excellent classes [GrHa] (see the related article [GrLe2] in this volume). We can also prove a Baldwin-Lachlan theorem, emphasising the role of quasiminimal types, and introduce a U-rank for types over models and obtain a picture very similar to the first order case [Le3]. Finally, quasiminimal types can be used to generalise Hrushovski's result [Hr] to the context of excellent classes [HLS], starting geometric stability theory proper. Excellence is the precursor to Shelah's work on good frames for the categoricity problem for abstract elementary classes [Sh 705].

Excellence appears naturally in several mathematical contexts. For example, it is the main dividing line when studying almost free algebras [MeSh] and it is also the key property in Zilber's work on complex exponentiation [Zi1] and [Zi2].
Let us now say a few words about fullness. Understanding which types are realised in the models of a class is a difficult problem as soon as the compactness theorem fails. For example, fix a homogeneous model $M$ of a first order theory $T$. There is no criterion to understand which types are realised inside $M$, except for complete types. We have weak compactness: A complete (first order) type $p \in S(A)$, with $A \subseteq M$ and $|A| < \|M\|$, is realised in $M$ if and only if $p \upharpoonright B$ is realised in $M$ for each finite subset $B$ of $A$. In full models, we have a similar condition but for complete types over models (at least when the full model is uncountable). An uncountable model $M$ of a countable first order theory $T$ is full, if $M$ realises every complete (first order) type $p \in S(N)$, where $N \prec M$ and $\|N\| < \|M\|$, provided that $p \upharpoonright B$ is realised in $M$ for each finite $B \subseteq N$. This gives us a way of dealing with complete types over models.

In this paper, we consider the class $\mathcal{K}$ of atomic models of a countable first order theory. We will assume that there is a sufficiently large model $\mathcal{C} \in \mathcal{K}$ which is full; we do not assume that every (small) model of $\mathcal{K}$ is isomorphic to an elementary submodel of $\mathcal{C}$ (this will be proved in the paper). We use $\mathcal{C}$ twice in the course of the paper. The first time is to show that the categoricity of $\mathcal{K}$ in some uncountable cardinal implies that the class is $\omega$-stable, which means here that $\mathcal{C}$ realises only countably many types over countable elementary submodels. The $\omega$-stability implies that $\mathcal{K}$ admits a bounded rank, which is then used to define an independence relation. We use $\mathcal{C}$ a second time to prove the symmetry of the independence relation. Provided we restrict our attention to types over models, we prove that the independence relation satisfies all the properties of nonforking: symmetry, extension, transitivity, stationarity. This allows us to define independent systems of models and excellence; i.e. $\mathcal{K}$ is excellent if there exists a primary model over any $n$-dimensional independent system of countable models, for each $n < \omega$. We then prove some of the basic results of excellence, namely the existence of primary models over other sets. Finally, we present Shelah’s categoricity theorem. The main difference with Shelah’s original approach is that Shelah uses the assumptions that $2^{\aleph_0} < 2^{\aleph_1}$ and that $\mathcal{K}$ has fewer than $2^{\aleph_1}$ many nonisomorphic models of size $\aleph_1$ to prove $\omega$-stability and to establish the symmetry property for nonsplitting; these are exactly the arguments which we replace by using $\mathcal{C}$. Once these properties have been established, excellence can be defined and the rest of Shelah’s treatment is in ZFC. We end the paper with a more detailed discussion of these issues.

This paper grew out of lecture notes for a class on excellence that I gave at Oxford University in 2002 during the Michaelmas term. It assumes only basic model theory, say, up to Morley’s theorem. For expositional purposes, a particular case of the general result is proved on two occasions, when the main idea is obscured by the additional technicality. We also complement the presentation with comparisons to the relevant theorems of homogeneous model theory to illustrate both the differences and the limitations of the theorems, but familiarity with homogeneous model theory is not essential.
0. TYPES AND $\omega$-STABILITY

Fix a complete first order theory $T$ in a countable language $L$. In this paper, we consider the class $\mathcal{K}$ of atomic models of $T$, i.e. $M \in \mathcal{K}$ if and only if $M \models T$ and for any finite sequence $c \in M$, there exists a formula $\phi(x) \in L$ such that $\phi(x) \vdash \text{tp}(c/\emptyset, M)$. We assume that $\mathcal{K} \neq \emptyset$.

As usual, we work in a large sufficiently saturated model $\bar{M}$ of $T$. The model $\bar{M}$ is not, in general, a member of $\mathcal{K}$ since $\bar{M}$ realises all types over the empty set and some may not be isolated (in fact $\bar{M} \in \mathcal{K}$ if and only if $T$ is $\aleph_0$-categorical, but this means that we are in the first order case). Satisfaction is defined with respect to $\bar{M}$. All sets and models are assumed to be inside $\bar{M}$ — so $\mathcal{K}$ is the class of atomic elementary submodels of $\bar{M}$. We use uppercase letters $A, B, C$ for sets, $M, N$ for models, and lowercase letters $a, b, c$ for finite sequences. We write $AB$ for the union of $A$ and $B$ and $Ac$ for the union of $A$ with the range of the sequence $c$.

We first make a few observations about the class $\mathcal{K}$. The next remark shows that $(\mathcal{K}, \prec)$, where $M \prec N$ if $M$ is an elementary submodel of $N$, is an abstract elementary class (see [GrLe2] in this volume for a definition). The proofs are left to the reader.

**Remark 0.1.** Let $\mathcal{K}$ be the class of atomic models of the countable theory $T$.

1. ($LS(\mathcal{K}) = \aleph_0$) If $A \subseteq M \in \mathcal{K}$, then there exists $N \prec M$, $A \subseteq N$, such that $|N| = |A| + \aleph_0$. Since $N \prec M$ and $M \in \mathcal{K}$, then $N \in \mathcal{K}$.
2. (Tarski-Vaught’s chain condition) If $(M_i : i < \alpha)$ is an increasing and continuous elementary chain of models such that $M_i \in \mathcal{K}$ for each $i < \alpha$, then $\bigcup_{i<\alpha} M_i \in \mathcal{K}$. Furthermore $M_0 \prec \bigcup_{i<\alpha} M_i$ and if $M_i \prec N \in \mathcal{K}$ for each $i < \alpha$ then also $\bigcup_{i<\alpha} M_i \prec N$.

Recall that $M$ is $\lambda$-homogeneous if for any elementary map $f : A \to M$, with $A \subseteq M$ of size less than $\lambda$, there exists an elementary map $g : B \to M$ extending $f$ such that $a \in B \subseteq M$. We say that $M$ is homogeneous if $M$ is $|M|$-homogeneous.

**Remark 0.2.**

1. Each model of $\mathcal{K}$ is $\omega$-homogeneous, and therefore embeds elementarily any countable atomic set.
2. There is a unique countable model in $\mathcal{K}$.

We now consider the problem of types. As usual, we denote by $S(A)$ the set of complete $L$-types over $A$ in finitely many variables. In the first order case, all types are realisable by models of a theory; this is an important consequence of the compactness theorem. In our context, the situation is a little more delicate; if $A \subseteq M \in \mathcal{K}$ and if $p \in S(A)$ is realised in $M$ by, say, $c$, then $A \cup c$ is an atomic set.
This gives us a \textit{necessary} condition for which types are realisable in the models of our class. We make the following definition:

\textbf{Definition 0.3.} We let $S_{\text{at}}(A)$ be the set of types $p \in S(A)$ such that for all $c \models p$, the set $Ac$ is atomic.

When $A$ is not atomic, $S_{\text{at}}(A)$ is clearly empty. Also, if $Ac$ is atomic for some $c \models p$, then $Ad$ is atomic for any $d \models p$. Furthermore, if $(A_i : i < \alpha)$ is an increasing and continuous sequence of atomic sets, and $(p_i : i < \alpha)$ is an increasing and continuous sequence of types, with $p_i \in S_{\text{at}}(A_i)$ for $i < \alpha$, then the type $p = \bigcup_{i<\alpha} p_i \in S_{\text{at}}(\bigcup_{i<\alpha} A_i)$.

However, given a (partial) type $p$ over an atomic set $A$ (indeed, even for a complete type $p \in S_{\text{at}}(B)$, where $B$ is a subset of $A$), there may not exist $q \in S_{\text{at}}(A)$ extending $p$. This may fail also for countable $A$.

Another problem is that, in general, there may be $A \subseteq M \in K$ and $p \in S_{\text{at}}(A)$ not realised in any $N \in K$. This only occurs for \textit{uncountable} $A$, though: If $c \in M$ realises $p \in S_{\text{at}}(A)$, then $Ac$ is atomic by definition. If $A$ is countable, then $Ac$ embeds elementarily inside the countable model $M_0$ of $K$. This embedding extends in $M$ to an elementary map whose range contains $M_0$. The image of $M_0$ under the inverse of this map produces an atomic model containing $Ac$. It follows that, for countable $A$, the set $S_{\text{at}}(A)$ corresponds exactly to the set of types over $A$ realised by models in $K$.

The fact that, for $A$ uncountable, some types in $S_{\text{at}}(A)$ will be omitted is unavoidable unless there exists an $(|A| + \aleph_0)$-homogeneous model in $K$ of size at least $|A| + \aleph_0$. But, as Shelah showed using Markus's example ([Sh 48], [Ma]), there are uncountably categorical atomic classes $K$ not containing any uncountable $\omega_1$-homogeneous model. So, outside of homogeneous model theory, types over general sets are intractable. In this paper, we will deal essentially with types over \textit{models}. We make the following hypothesis throughout the text.

\textbf{Hypothesis 0.4.} There exists a model $\mathcal{C} \in K$ of size at least $\bar{\kappa}$, for some suitably large cardinal $\bar{\kappa}$, with the property that if $p \in S_{\text{at}}(M)$ and $M \prec \mathcal{C}$ of size less than $\bar{\kappa}$, then $p$ is realised in $\mathcal{C}$.

In this paper, 'suitably large' means that $\bar{\kappa}$ is assumed to be at least the categoricity cardinal, and at least the \textit{Hanf number} for atomic classes ($= \beth_{\omega_1}$, see for example Chapter VII.5 of [Sh 1]). This latter condition ensures, in particular, that $K$ has arbitrarily large models. We will see in subsequent sections that $\mathcal{C}$ is \textit{full}. The existence of full models follows from excellence; by introducing them early, we can present the entire theory within ZFC in a very smooth way. The existence of full models does not imply the existence of homogeneous models (for a nice example, see [Zi 1]). Notice also that we do not assume that \textit{all} (small) models of $K$ embed in $\mathcal{C}$ (this will follow from excellence); for now, it is enough
that all countable models of $\mathcal{K}$ embed into $\mathcal{C}$, which is a consequence of the $\omega$-homogeneity of $\mathcal{C}$.

We now consider $\omega$-stability. There are several possible notions for $\omega$-stability, not all of which are equivalent to each other. We discuss this below.

**Definition 0.5.** The class $\mathcal{K}$ is $\lambda$-stable if $|S_{\text{at}}(M)| \leq \lambda$, for each $M \in \mathcal{K}$ of size $\lambda$.

**Proposition 0.6.** The following conditions are equivalent:

1. $\mathcal{K}$ is $\omega$-stable.
2. $\mathcal{C}$ realises only countably many types over countable subsets.
3. Each $M \in \mathcal{K}$ realises only countably many types over countable subsets.

**Proof.** (3) implies (2) is clear. To see that (2) implies (1), suppose that $S_{\text{at}}(M)$ is uncountable for some countable $M \in \mathcal{K}$. By $\omega$-homogeneity, we may assume that $M \prec \mathcal{C}$, and thus $\mathcal{C}$ is not $\omega$-stable since it realises each type in $S_{\text{at}}(M)$. For (1) implies (3), if $M \in \mathcal{K}$ realises uncountably many types over a countable subset, then it realises uncountably many types over a countable submodel $M_0 \prec M$. Thus $S_{\text{at}}(M_0)$ is uncountable, contradicting (1). \qed

$\mathcal{K}$ can be $\omega$-stable while $T$ is unstable: Consider the countable theory in the language $\{ N, +, \cdot, 0, 1 \}$, where $T$ has the first order theory of $\mathbb{N}$ in this language on the predicate $N$ and asserts that the complement of $N$ is infinite. $T$ is unstable since it has the strict order property. However, the class $\mathcal{K}$ of atomic models of $T$ has arbitrarily large homogeneous models (hence satisfies our hypothesis 0.4), and is $\omega$-stable.

Without additional assumptions, the $\omega$-stability of $\mathcal{K}$ does not even imply that $S_{\text{at}}(A)$ is countable for each countable atomic $A$: although each type $p \in S_{\text{at}}(A)$ is realisable inside a model, there may be no model realising jointly all types in $S_{\text{at}}(A)$. If we had an uncountable $\omega_1$-homogeneous model, we could do this (or amalgamation over sets, which is the same); it turns out to be equivalent, as is shown in the following fact [Le2]. The existence of countable sets $A$ with $S_{\text{at}}(A)$ uncountable is a core difference with the categoricity problem in the homogeneous case. It is the basic motivation behind excellence and will be revisited in Section 2.

**Fact 0.7** (Lessmann). Suppose that $S_{\text{at}}(A)$ is countable for each countable atomic set $A$. If $\mathcal{K}$ has an uncountable model, then $\mathcal{K}$ has arbitrarily large homogeneous models.

Throughout this paper, we will make occasional use of the following fact, often referred to as Morley's methods.

**Fact 0.8.** Suppose $(a_i : i < \omega_1) \subseteq M \in \mathcal{K}$. Let $L^*$ be an expansion of $L$ with Skolem functions and $T^*$ be the theory of $\mathcal{C}$ in this expanded language. Then there
exists an $L^*$-indiscernible sequence $(b_i : i < \omega)$ such that for each $n < \omega$ we can find $i_0 < \cdots < i_n < \bigvee \omega_1$ satisfying
\[ tp_{L^*}(b_0, \ldots, b_n/\emptyset) = tp_{L^*}(a_{i_0}, \ldots, a_{i_n}/\emptyset). \]
It follows that the reduct to $L$ of the Skolem Hull of $(b_i : i < \omega)$ is a model in $\mathcal{K}$.

Recall that $\mathcal{K}$ is $\lambda$-categorical for a cardinal $\lambda$ if all models of $\mathcal{K}$ of size $\lambda$ are isomorphic. $\mathcal{K}$ is always $\aleph_0$-categorical. We now connect categoricity with $\omega$-stability.

**Proposition 0.9.** If $\mathcal{K}$ is $\lambda$-categorical for some uncountable $\lambda$, then $\mathcal{K}$ is $\omega$-stable.

**Proof.** Suppose that $\mathcal{K}$ is not $\omega$-stable. Then $\mathfrak{C}$ realises uncountably many types over a countable subset. By the Downward Löwenheim theorem, we can find $M \prec \mathfrak{C}$ of size $\lambda$ realising uncountably many types over a countable subset. On the other hand, since $\mathcal{K}$ has arbitrarily large models, we can use Fact 0.8 to find an infinite $L^*$-indiscernible sequence $(b_i : i < \omega)$ inside some $N \in \mathcal{K}$. By compactness, we can extend $(b_i : i < \omega)$ to an $L^*$-indiscernible sequence $I$ ordered by the ordinal $\lambda$. By construction, the Skolem Hull of $I$ also omits all the nonisolated types of $T$ (a counterexample would otherwise provide one in $N$), and hence its reduct is a model in $\mathcal{K}$. It is easy to see that this reduct is $\omega$-stable of size $\lambda$, which contradicts $\lambda$-categoricity. \qed

Shelah’s example of $\omega_1$-categorical class with no $\omega_1$-homogeneous model shows that uncountable categoricity does not necessarily imply that $\mathfrak{S}_{\text{sat}}(A)$ is countable for all countable atomic $A$.

1. **Rank and Independence**

From now on, until the rest of the paper, we assume that $\mathcal{K}$ is $\omega$-stable (except in Proposition 1.6 for which it is part of the conclusion). In this section, we introduce a rank. The rank is bounded in the $\omega$-stable case, and equality of ranks provides an independence relation which we show to be well-behaved over models of $\mathcal{K}$.

**Definition 1.1.** For any formula $\phi(x)$ with parameters in $M \in \mathcal{K}$, we define the rank $R_M[\phi]$. The rank $R_M[\phi]$ will be an ordinal, $-1$, or $\infty$ and we have the usual ordering $-1 < \alpha < \infty$ for any ordinal $\alpha$. We define the relation $R_M[\phi] \geq \alpha$ by induction on $\alpha$.

1. $R_M[\phi] \geq 0$ if $\phi$ is realised in $M$;
2. $R_M[\phi] \geq \delta$, when $\delta$ is a limit ordinal, if $R_M[\phi] \geq \alpha$ for every $\alpha < \delta$;
3. $R_M[\phi] \geq \alpha + 1$ if the following two conditions hold:
   (a) There is $a \in M$ and a formula $\psi(x, y)$ such that
   \[ R_M[\phi(x) \land \psi(x, a)] \geq \alpha \quad \text{and} \quad R_M[\phi(x) \land \neg \psi(x, a)] \geq \alpha; \]
(b) For every $c \in M$ there is a formula $\chi(x, c)$ isolating a complete type over $c$ such that
\[
R_M[\phi(x) \land \chi(x, c)] \geq \alpha.
\]

We write:
\[
R_M[\phi] = -1 \text{ if } \phi \text{ is not realised in } M;
R_M[\phi] = \alpha \text{ if } R_M[\phi] \geq \alpha \text{ but it is not the case that } R_M[\phi] \geq \alpha + 1;
R_M[\phi] = \infty \text{ if } R_M[\phi] \geq \alpha \text{ for every ordinal } \alpha.
\]

For any set of formulas $p(x)$ over $A \subseteq M$, we let
\[
R_M[p] = \min\{ R_M[\phi] \mid \phi = \land q, q \subseteq p, \text{ } q \text{ finite } \}.
\]

Note that the condition $R[p] \geq 0$ does not imply that $p$ is realised in a model of $\mathcal{K}$.

We first write down a few properties of the rank. They are all basic and can be proved easily by induction or directly. (7) follows from (6) and (1) using the countability of the language $L$.

**Lemma 1.2.**

(1) $R_M[\phi(x, b)]$ depends on $\phi(x, y)$ and $\text{tp}(b/\emptyset)$ only.

(2) If $p$ is over $M$ and $N$, for $M, N \in \mathcal{K}$, then $R_M[p] = R_N[p]$.

(3) If $p$ is finite and $\phi$ is obtained by taking the conjunction of all the formulas in $p$, then $R[p] = R[\phi]$.

(4) (Finite Character) For each $p$ there is a finite subset $q$ of $p$ such that $R[p] = R[q]$.

(5) (Monotonicity) If $p \subseteq q$, then $R[p] \geq R[q]$.

(6) If $R[p] = \alpha$ and $\beta < \alpha$ there is $q$ such that $R[q] = \beta$.

(7) There exists an ordinal $\alpha_0 < \omega_1$, such that if $R[p] \geq \alpha_0$ then $R[p] = \infty$.

In view of (2), we have dropped the subscript $M$ from the rank in (3)–(7) and will drop it henceforth. We now show that $\omega$-stability implies that the rank is bounded (the converse is Proposition 1.6). The idea of the proof is essentially like the first order case: we construct a binary tree of formulas, whose branches give us continuum many types. There is one difference: to contradict $\omega$-stability, we need the types to be in $S_{\text{at}}(M)$ for some countable $M \in \mathcal{K}$. To achieve this, we simply choose isolating formulas along the way, and force the parameters to enumerate a model.

**Theorem 1.3.** $R[p] < \infty$ for every type $p$.

**Proof.** We prove the contrapositive. Suppose there is a type $p$ over some atomic model $M$ such that $R[p] = \infty$. We may assume that $p = \{ \phi(x, b) \}$ is a formula and also that $M$ is countable by the previous lemma. Let $\{a_i : i < \omega \}$ be an enumeration of $M$. 
We construct formulas $\phi_{\eta}(x, b_{\eta})$, for $\eta \in <^{<\omega}2$, such that:

1. $\phi_{\eta}(x, b_{\eta})$ isolates a complete type over $b_{\eta}$;
2. $\models \forall x[\phi_{\nu}(x, b_{\nu}) \rightarrow \phi_{\eta}(x, b_{\eta})]$ when $\eta \subseteq \nu$;
3. $\phi_{\eta|0}$ and $\phi_{\eta|1}$ are contradictory;
4. $R[\phi_{\eta}] = \infty$;
5. $a_{\ell(\eta)} \in b_{\eta}$.

This is possible: The construction is by induction on $\ell(\eta)$.

For $\langle \rangle$: Since $R[\phi] = \infty$, in particular $R[\phi] \geq \omega_{1} + 1$ so there exists $\phi_{\langle \rangle}(x, b_{\langle \rangle})$, where $b_{\langle \rangle} = ba_{0}$ isolating a complete type over $ba_{0}$ such that $R[\phi \land \phi_{\eta}] \geq \omega_{1}$. The formula $\phi_{\langle \rangle}$ is as required.

Assume that we have constructed $\phi_{\eta}(x, b_{\eta})$ with $\ell(\eta) < \omega$. Since $R[\phi_{\eta}] = \infty$, in particular $R[\phi_{\eta}] \geq (\omega_{1} + 1) + 1$. Hence, there is $c \in M$ and $\psi(x, y)$ such that

(*) $R[\phi_{\eta} \land \psi(x, c)] \geq \omega_{1} + 1$ and $R[\phi_{\eta} \land \neg\psi(x, c)] \geq \omega_{1} + 1$.

Let $b_{\eta|0} = b_{\eta|1} = cb_{\eta}a_{\ell(\eta)+1}$. By (*) and the definition of the rank ((3)(b)), there are $\phi_{\eta|\ell}(x, b_{\eta|\ell})$ isolating a complete type over $b_{\eta|\ell}$ for $\ell = 0, 1$, such that

$R[\phi_{\eta} \land \psi(x, c) \land \phi_{\eta|0}(x, b_{\eta|0})] \geq \omega_{1}$

and

$R[\phi_{\eta} \land \neg\psi(x, c) \land \phi_{\eta|1}(x, b_{\eta|1})] \geq \omega_{1}$

Then $\phi_{\eta|\ell}(x, b_{\eta|\ell})$ are as required, for $\ell = 0, 1$.

This is enough: For each $\eta \in ^{<\omega}2$, define $p_{\eta} := \bigcup_{n \in \omega} p_{\eta|n}$. Notice that each $p_{\eta}$ determines a complete type over $M$ with the property that $MC_{\eta}$ is atomic for any realisation $c_{\eta}$ of $p_{\eta}$. Hence, each $p_{\eta} \in S_{\text{at}}(M)$, so $S_{\text{at}}(M)$ has size continuum, which contradicts the $\omega$-stability of $\mathcal{K}$.

Recall that $p \in S_{\text{at}}(A)$ splits over $B \subseteq A$ if there exist $c, d \in A$ realising the same type over $B$ and $\phi(x, y)$ such that $\phi(x, c) \in p$ and $\neg\phi(x, d) \in p$. The next proposition examines the connection between the rank and nonsplitting. It also shows that we may have at most one same rank extension over a model.

**Proposition 1.4.** If $p \in S_{\text{at}}(M)$, $M \in \mathcal{K}$ and $\phi(x, b) \in p$ such that $R[p] = R[\phi(x, b)]$. Then $p$ does not split over $b$. Furthermore, $p$ is the only type in $S_{\text{at}}(M)$ extending $\phi(x, b)$ with the same rank.

**Proof.** Let $\alpha = R[p]$. Suppose that $p$ splits over $b$. Let $\psi(x, y)$ and $c, d \in M$ such that $tp(c/b) = tp(d/b)$ and $\psi(x, c) \in p$ and $\neg\psi(x, d) \in p$. Then $R[\phi(x, b) \land \psi(x, c)] \geq \alpha$, and $R[\phi(x, b) \land \neg\psi(x, d)] \geq \alpha$ by monotonicity. By Lemma 1.2, we also have $R[\phi(x, b) \land \psi(x, d)] \geq \alpha$, so (a) of the rank is satisfied at the successor stage. Now let $e \in M$. By monotonicity, $R[p \upharpoonright be] \geq \alpha$. There exists $\chi \in p \upharpoonright be$
isolating \( p \upharpoonright \subseteq \). By monotonicity, \( R[\phi \land \chi] \geq \alpha \), so (b) of the rank is satisfied at the successor stage. Hence \( R[\phi] \geq \alpha + 1 \), a contradiction.

For uniqueness, suppose that \( q \neq p \in S_{at}(M) \) extend \( \phi(x, b) \). Suppose \( \psi(x, c) \in q \) such that \( \neg \psi(x, c) \in p \). By monotonicity, we have \( R[\phi(x, b) \land \psi(x, c)] \geq \alpha \) and \( R[\phi(x, b) \land \neg \psi(x, c)] \geq \alpha \), so (a) of the rank is satisfied. It is easy to see that (b) of the rank is satisfied the same way as the previous paragraph.

The proposition shows that if \( p \in S_{at}(M) \) then \( p \) does not split over a finite set. This is not true for \( p \in S_{at}(A) \) in general.

**Proposition 1.5.** Let \( p \in S_{at}(M) \), \( M \in \mathcal{K} \). Let \( C \) be an atomic set containing \( M \), then there exists a unique \( q \in S_{at}(C) \) extending \( p \) such that \( R[p] = R[q] \).

**Proof.** Let \( \phi(x, b) \in p \) such that \( R[p] = R[\phi] \). By the previous proposition, \( p \) does not split over \( b \). Let \( q \) be the following set of formulas:

\[
\{ \psi(x, c) : \psi(x, c') \in p \text{ for some } c' \in M \text{ realising } \text{tp}(c/b), c \in C, \psi \in L \}.
\]

Since \( p \) does not split over \( b \), this is well-defined. Similarly, this determines a type \( q \in S_{at}(C) \). It is easy to check that \( q \) does not split over \( b \) and has the same rank as \( p \).

For uniqueness, suppose that \( R[q_\ell] = R[\phi] \), for \( \ell = 1, 2 \) both contain \( \phi \). Then \( R[q_\ell \upharpoonright M] = R[p] \) for \( \ell = 1, 2 \) by the previous proposition. Then \( q_1 = q_2 \) since neither \( q_1 \) nor \( q_2 \) splits over \( b \).

We now prove the converse to Theorem 1.3. As we pointed out, we do not assume that \( \mathcal{K} \) is \( \omega \)-stable. The theorem is interesting for two reasons; (1) \( \omega \)-stability is equivalent to boundedness of the rank, and (2) \( \omega \)-stability implies \( \lambda \)-stability for each infinite \( \lambda \) (this forms a particular case of the stability spectrum theorem).

**Proposition 1.6.** If \( R[p] < \infty \) for all types \( p \), then \( \mathcal{K} \) is \( \lambda \)-stable, for each infinite \( \lambda \).

**Proof.** Let \( M \in \mathcal{K} \). Let \( p \in S_{at}(M) \) and let \( \phi(x, b) \in p \) such that \( R[p] = R[\phi(x, b)] \). Since \( p \) is the only extension of \( \phi(x, b) \) with the same rank, the number of types in \( S_{at}(M) \) is at most the number of formulas over \( M \), which is \( \| M \| \).

In general, a type may fail to have an extension to a larger set, so in particular, it may fail to have a same rank extension. This is why we consider stationary types.

**Definition 1.7.** We say that \( p \in S_{at}(A) \) is stationary if there exists \( b \in A \), \( \phi(x, b) \in p \), and \( M \in \mathcal{K} \) containing \( A \) with \( q \in S_{at}(M) \), \( \phi(x, b) \in q \) and \( R[\phi(x, b)] = R[p] = R[q] \).
Clearly, complete types over models are stationary. Proposition 1.5 does not assert that the nonsplitting extension is actually realized in a model (it is, if we work inside $\mathcal{C}$). It only says that when $p \in S_{at}(A)$ is stationary and $C$ is atomic containing $A$, there is a unique extension in $S_{at}(C)$ of the same rank. We will denote this extension by $p \upharpoonright C$.

We now introduce fullness, a substitute for saturation. Full is called weakly full in [Sh 87a].

**Definition 1.8.** A model $M \in \mathcal{K}$ is $\lambda$-full over $A \subseteq M$ if for all stationary $p \in S_{at}(B)$ with $B \subseteq M$ finite, and for all $C \subseteq M$ of size less than $\lambda$, $M$ realizes $p \upharpoonright ABC$. We say that $M$ is $\lambda$-full if $M$ is $\lambda$-full over each subset of size less than $\lambda$. Finally, we say that $M$ is full if $M$ is $\lambda$-full for $\lambda = ||M||$.

Note that if $M$ is $\lambda$-full then $M$ realizes each $p \in S_{at}(N)$ for $N \prec M$ of size less than $\lambda$. Also, if $M$ is $\lambda$-full over $A$, then it is $\lambda$-full over $B$, for any $B \subseteq A$. We have the following easy proposition, which is left to the reader. It shows that $\mathcal{C}$ is $\kappa$-full:

**Proposition 1.9.** Let $M \in \mathcal{K}$ be uncountable. The following are equivalent:

1. $M$ is $\lambda$-full.
2. $M$ realizes each $p \in S_{at}(N)$, with $N \prec M$ of size less than $\lambda$.

We are going to prove Symmetry over models. For this, we will use the order property which was introduced by Shelah in [Sh 12]. In the first order case, compactness implies that if there exists a formula linearly ordering an infinite set, then that formula linearly orders sets of arbitrarily large size. Without compactness, this fails (see the example after Proposition 0.6), which is the reason why we need to consider longer orders. By using Morley's methods Fact 0.8, it is easy to see that the existence of a formula linearly ordering a set of size $\beth_\omega$ (the Hanf number) ensures that this formula linearly orders sets of arbitrarily large size. This justifies the next definition.

**Definition 1.10.** We say that $\mathcal{K}$ has the order property if there exist a model $M \in \mathcal{K}$, a formula $\phi(x,y) \in L$ and $(d_i : i < \beth_\omega) \subseteq M$ such that

$$M \models \phi(d_i, d_j) \quad \text{if and only if} \quad i < j.$$ 

We can now show that the order property contradicts ($\omega$-)stability.

**Proposition 1.11.** $\mathcal{K}$ does not have the order property.

**Proof.** Suppose that $\mathcal{K}$ has the order property. Let $(d_i : i < \beth_\omega) \subseteq M \in \mathcal{K}$ such that $M \models \phi(d_i, d_j)$ if and only if $i < j$. By Fact 0.8, there exists $(b_n : n < \omega)$ $L^*$-indiscernible such that $\models \phi(b_n, b_m)$ if and only if $n < m$, and furthermore, the reduct to $L$ of the Skolem Hull of $(b_n : n < \omega)$ is in $\mathcal{K}$. By compactness, we can
find \((b_i : i \in \mathbb{R})\) \(L^*\)-indiscernible such that any finite subsequence of it satisfies the same \(L^*\)-type as any finite subsequence of \((b_n : n < \omega)\) of the same length. Let \(N\) be the reduct to \(L\) of the Skolem Hull of \((b_i : i \in \mathbb{R})\). By construction \(N \in \mathcal{K}\) (it is a model of \(T\) and must be atomic since a counterexample can be used to provide a counterexample in \(M\)). Furthermore \(N \models \phi(b_i, b_j)\) if and only if \(i < j\). Letting \(B = \bigcup_{i \in \mathbb{Q}} b_i\) gives us a countable subset of \(N\) over which \(2^{\aleph_0}\) types are realised by density of \(\mathbb{Q}\). This contradicts the \(\omega\)-stability of \(\mathcal{K}\) (Proposition 0.6).

We can now prove the symmetry property of the rank.

**Proposition 1.12.** (Symmetry) Let \(a, c\) and \(M \in \mathcal{K}\) be such that \(Mac\) is atomic. Then \(R[\text{tp}(a/Mc)] = R[\text{tp}(a/M)]\) if and only if \(R[\text{tp}(c/Ma)] = R[\text{tp}(c/M)]\).

**Proof.** Notice that by Finite Character and Monotonicity, we may assume that \(M\) is countable, and so by \(\omega\)-homogeneity of \(\mathcal{C}\), we may assume that \(a, c\) and \(M\) are inside \(\mathcal{C}\). Suppose the conclusion of the proposition fails; we will contradict the \(\omega\)-stability of \(\mathcal{K}\) by showing that it has the order property. We can choose a formula \(\psi(x, y)\) over \(M\) such that

\[
R[\text{tp}(a/Mc)] = R[\psi(x, c)] = R[\text{tp}(a/M)]
\]

while

\[
R[\text{tp}(c/Ma)] = R[\psi(a, y)] < R[\text{tp}(c/M)].
\]

Define \((a_i, c_i : i < \bigcup \omega_1) \subseteq \mathcal{C}\) and \(B_i = M \cup \{a_j, c_j : j < i\}\) such that

1. \(c_i \in \mathcal{C}\) realises \(\text{tp}(c/M)\) and \(R[\text{tp}(c_i/B_i)] = R[\text{tp}(c/M)]\),
2. \(a_i \in \mathcal{C}\) realises \(\text{tp}(a/M)\) and \(R[\text{tp}(a_i/B_i c_i)] = R[\text{tp}(a/M)]\).

This is possible since both \(\text{tp}(a/M)\) and \(\text{tp}(c/M)\) are stationary and thus the realisations \(a_i\) and \(c_i\) in (1) and (2) of the unique nonsplitting extensions of these types exist by fullness of \(\mathcal{C}\).

This implies the order property: Suppose that \(i > j\). Then \(a_j \in B_i\). If \(\models \psi(a_j, c_i)\), then \(R[\text{tp}(c_i/B_i)] \leq R[\psi(a_j, y)] = R[\psi(a, y)] < R[\text{tp}(c_i/B_i)]\) contradicting (1) (we used the fact that \(\text{tp}(a_i/M) = \text{tp}(a/M)\) to see that the middle equality holds). Hence \(\models \neg \psi(a_j, c_i)\) if \(i > j\).

Now if \(i \leq j\), then we have \(\text{tp}(a_j/Mc) = \text{tp}(a/Mc)\), by uniqueness of same rank extensions, so \(\models \psi(a_j, c)\). Since \(\text{tp}(a_j/B_j c_j)\) does not split over \(B\) by (2) and \(\text{tp}(c/M) = \text{tp}(c_i/M)\), we must have \(\models \psi(a_j, c_i)\).

Thus, \(\phi(x_1, x_2, y_1, y_2) := \psi(x_1, y_2)\) and \(d_i := c_i a_i\), for \(i < \bigcup \omega_1\), witness the order property. \(\square\)
We can now define a natural independence relation using the rank: For $A, B, C$ such that $A \cup B \cup C$ is atomic, we write

$$A \downarrow C \quad \text{if} \quad B \quad R[\text{tp}(a/BC)] = R[\text{tp}(a/B)], \quad \text{for all} \quad a \in A.$$ 

We will say that $A$ is free from $C$ over $B$ if $A \downarrow C$. We now gather the properties we have established for this dependence relation. The reader used to the first order case may wonder whether (6) and (8) hold with sets instead of models. The answer is no in general.

**Proposition 1.13.** Assume $A, B, C, D$ are sets whose union is atomic.

1. **(Invariance)** If $f$ is an elementary map, then

   $$A \downarrow C \quad \text{if and only if} \quad \frac{A}{B} \downarrow \frac{f(A)}{f(B)}.$$

2. **(Monotonicity)** If $A' \subseteq A$ and $C' \subseteq C \cup B$ and $A \downarrow C$, then $A' \downarrow C'$.

3. **(Finite Character)**

   $$A \downarrow C \quad \text{if and only if} \quad A' \downarrow C', \quad \text{for each finite} \quad A' \subseteq A \quad \text{and finite} \quad C' \subseteq C.$$

4. **(Transitivity)** If $B \subseteq C \subseteq D$, then

   $$A \downarrow C \quad \text{and} \quad A \downarrow D \quad \text{if and only if} \quad A \downarrow D.$$

5. **(Local Character)** If $a \cup C$ is atomic, then there exists a finite $B \subseteq C$ such that $a \downarrow C$.

6. **(Extension over models)** Let $A \cup M$ be atomic, $M \in K$ and $C$ be atomic containing $A$. Then there exists $a'$ realising $\text{tp}(a/M)$ such that $a' \cup C$ is atomic and $a' \downarrow C$.

7. **(Stationarity over models)** Suppose that $a_\ell$ realises $\text{tp}(a/M)$, $a_\ell \cup C$ is atomic, and $a_\ell \downarrow C$ for $\ell = 1, 2$. Then $\text{tp}(a_1/C) = \text{tp}(a_2/M)$.

8. **(Symmetry over models)** If $A \cup C \cup M$ is atomic, then

   $$A \downarrow C \quad \text{if and only if} \quad C \downarrow A.$$

In the sequel, we will use these properties extensively; on occasions, we will simply say ‘by independence calculus’ when establishing the independence of certain sets from others by using a sequence of these properties.

**Remark 1.14.** Alexei Kolesnikov (see [Ko] in this volume) proved that, in an excellent class, the independence relation $\downarrow$ is uniquely determined when we work
over models; i.e. \( M_1 \downarrow M_2 \) if and only if for each \( a \in M_1 \) there is a finite subset \( C \subseteq M_0 \) such that \( \text{tp}(a/M_0 \cup M_2) \) does not split over \( C \).

2. **Good Sets, Primary, and Full Models**

Recall that \( \mathcal{K} \) is \( \omega \)-stable. In order to define excellence, we will also need **primary** models.

**Definition 2.1.** We say that \( M \in \mathcal{K} \) is **primary over** \( A \), if \( M = A \cup \{ a_i : i < \alpha \} \), and for each \( i < \alpha \) the type \( \text{tp}(a_i/A \cup \{ a_j : j < i \}) \) is isolated.

The sequence \( (a_i : i < \alpha) \) is referred to as a **construction** of \( M \) over \( A \). It is a standard fact that if \( M \in \mathcal{K} \) is primary over \( A \) then for each \( c \in M \), the type \( \text{tp}(c/A) \) is isolated. If \( M \) is primary over \( A \), then it is easy to see that it is prime over \( A \). Recall that a model \( M \in \mathcal{K} \) is **prime over** \( A \), if for each \( N \in \mathcal{K} \) containing \( A \), there is an elementary map \( f : M \to N \) which is the identity on \( A \).

The main tool for producing primary models over countable sets is the following corollary to Henkin's omitting type theorem:

**Fact 2.2.** Let \( T \) be a countable theory. Assume that for each consistent formula \( \phi(x) \) there exists a complete type over the empty set containing \( \phi(x) \) which is isolated. Then there exists a countable atomic model of \( T \).

This leads to the next definition, important mostly for countable sets.

**Definition 2.3.** An atomic set \( A \) is **good** if for each \( \phi(x,a) \) with \( a \in A \) and \( \models \exists x \phi(x,a) \), there is a complete type \( p \in S_{\text{at}}(A) \) containing \( \phi(x,a) \) which is isolated.

The next lemma is the motivation behind the definition of good sets.

**Lemma 2.4.** Let \( A \) be countable and atomic. If \( A \) is good, then there is a primary model over \( A \).

**Proof.** Form the theory \( T_A \) by expanding \( T \) with countably many constants for the elements of \( A \). The assumptions of the previous fact are satisfied for \( T_A \) since \( A \) is good, so there exists a countable atomic model \( M(A) \) for \( T_A \). It is easy to see that the reduct of \( M(A) \) to the original language is a primary model over \( A \).

We will find several equivalent properties for good sets in a few more lemmas culminating in Corollary 2.8.

**Lemma 2.5.** Let \( A \) be a countable atomic set. If \( S_{\text{at}}(A) \) is countable, then \( A \) is good.
Proof: Suppose $A$ is not good: Then there exists $\phi(x, a)$ with $a \in A$ and $\models \exists x \phi(x, a)$, but no isolated extension of $\phi(x, a)$ exists in $S_{at}(A)$. Thus, for each $\psi(x, b)$ with $b \in A$ with $\models \forall x (\psi(x, b) \rightarrow \phi(x, a))$, there is $b' \in A$ such that $\psi(x, b)$ has at least two extensions in $S_{at}(a b b')$. We will use this to contradict the countability of $S_{at}(A)$, in a similar way to the proof of boundedness of the rank.

Let $A = \{a_i : i < \omega\}$. We construct $\psi_\eta(x, b_\eta)$ for $\eta \in {}^{<\omega}2$ such that $\psi_{\eta_0}(x, b_{\eta_0}) = \phi(x, a)$, if $\eta < \nu$ then $\models \forall x (\psi_\eta(x, b_\eta) \rightarrow \psi_\nu(x, b_\nu))$, each $\psi_\eta(x, b_\eta)$ isolates a complete type over $b_\eta$, $b_\eta$ contains $a_i$ if $\ell(\eta) > i$, and $\psi_{\eta_0}(x, b_{\eta_0})$ and $\psi_{\eta_1}(x, b_{\eta_1})$ are contradictory. This is possible and implies that $S_{at}(A)$ has size continuum.

Lemma 2.6. If $M \in \mathcal{K}$ is countable and $M c$ is atomic, then $M c$ is good.

Proof: For each $tp(d/Mc) \in S_{at}(Mc)$, consider $tp/dc/M) \in S_{at}(M)$. It is easy to see that this induces an injection from $S_{at}(Mc)$ into $S_{at}(M)$. Hence $S_{at}(Mc)$ is countable, since $S_{at}(M)$ is countable by $\omega$-stability of $\mathcal{K}$. Hence, $Mc$ is good.

We now consider the dual notion to prime models. We say that $N$ is universal over $A$, if $A \subseteq N$, and for $M \in \mathcal{K}$ with $A \subseteq M$ and $\|M\| = \|N\|$, there exists an elementary map $f : M \rightarrow N$ which is the identity on $A$.

Lemma 2.7. If $M \in \mathcal{K}$ is countable, then there exists a countable universal model $N \in \mathcal{K}$ over $M$.

Proof: Let $(M_n : n < \omega)$ be an increasing sequence of countable models such that $M_0 = M$ and $M_{n+1}$ realises all types in $S_{at}(M_n)$. We could do this at once using $\mathcal{C}$, but $\mathcal{C}$ is not necessary here: Let $\{p_i : i < \omega\}$ be an enumeration of $S_{at}(M_n)$. Let $a_0$ realise $p_0$ (which exists since $M_n$ is countable) and let $M_0'$ be primary over $M_0 a_0$, which exists by the three previous lemmas. Since $p_1$ is stationary, there exists a unique free extension $q_1$ over $M_0'$. Let $a_1$ realise $q_1$. Let $M_1'$ be primary over $M_0' a_1$. Continue like this inductively. Let $M_{n+1} = \bigcup_{i < \omega} M_i'$.

Let $N = \bigcup_{n < \omega} M_n$. We claim that $N$ is universal over $M$. Let $M' \in \mathcal{K}$ be countable such that $M \prec M'$. Write $M' = \{a_i : i < \omega\}$. We construct an increasing sequence of elementary maps

$$f_i : M \cup \{a_0, \ldots, a_i\} \rightarrow N,$$

which is the identity on $M$ for $i < \omega$. This is enough as $\bigcup_{i < \omega} f_i$ is an elementary map sending $M'$ into $N$, which is the identity on $M$.

Let us now construct the $f_i$s. For $i = 0$, let $b_0$ realise $tp(a_0/M) \in S_{at}(M)$ which exists in $N$ by construction, and let $f_0$ be the partial elementary map from $M a_0$ which is the identity on $M$ and sends $a_0$ to $b_0$. Having constructed $f_i$, let $M^*$ be a primary model over $M \cup \{a_0, \ldots, a_i\}$ which exists, since $M \cup \{a_0, \ldots, a_i\}$ is good. There exists $k < \omega$ such that $a_0, \ldots, a_i \in M_k$. By definition, we can extend
$f_i$ to $f_i^*: M^* \to M_k$, which is the identity on $M$. Then the image of $\text{tp}(a_{i+1}/M^*)$ under $f_i^*$ can be extended to a type in $S_{\text{at}}(M_k)$ (by stationarity), which is then realised by some element $b_{i+1}$ of $M_{k+1}$. Let $f_{i+1}$ be the partial elementary map extending $f_i$ and sending $a_{i+1}$ to $b_{i+1}$. This finishes the construction. 

So, we finally have:

**Corollary 2.8.** Let $A$ be a countable atomic set. The following conditions are equivalent:

1. $A$ is good.
2. There is a primary model over $A$.
3. There is a countable universal model over $A$.
4. $S_{\text{at}}(A)$ is countable.

**Proof.** We showed (1) implies (2) and (4) implies (1). For (2) implies (3): Let $M$ be primary over $A$. By the previous lemma there exists $N$ universal over $M$. This implies immediately that $N$ is universal over $A$. (3) implies (4) is clear: Let $N$ be universal over $A$. Each $p \in S_{\text{at}}(A)$ is realised in some countable model $M_p$, which embeds in $N$ over $A$ by universality of $N$. Hence, each $p \in S_{\text{at}}(A)$ is realised in $N$ so $|S_{\text{at}}(A)| \leq \|N\|$. 

Unless we are in the homogeneous case, there are countable atomic sets $A$ such that $S_{\text{at}}(A)$ is uncountable, so some sets are good and others are not. The next remark follows by counting types.

**Remark 2.9.** Let $A$ be countable and good. Let $c$ realise $p \in S_{\text{at}}(A)$. Then $Ac$ is good.

We pointed out in a previous section that a type $p \in S_{\text{at}}(A)$ may fail to have an extension in $S_{\text{at}}(M)$ where $M$ contains $A$; this only happens when $A$ is not good as the next lemma indicates. In fact, a countable set $A$ is good if and only if each type in $S_{\text{at}}(A)$ extends to a type over $M$ for any $M$ containing $A$.

**Lemma 2.10.** Let $A$ be a good countable set. Let $M \in \mathcal{K}$ containing $A$ and let $p \in S_{\text{at}}(A)$. Then there is $q \in S_{\text{at}}(M)$ extending $p$.

**Proof.** Let $c \models p$. Then $Ac$ is countable and atomic. By the previous remark, $Ac$ is good since $A$ is countable and good. Let $M'$ be primary over $Ac$. By primeness, we may assume that $M' < M$, so $\text{tp}(c/M) \in S_{\text{at}}(M)$ and extends $p$. 

We finish this section with the problem of existence and uniqueness of countable full models over countable sets.

**Proposition 2.11.** Let $A$ be countable and atomic. Then there exists a countable $M \in \mathcal{K}$ which is full over $A$. 
Proof. Let $M_0$ be any countable model containing $A$. Let $M_{n+1}$ be countable realising each type in $S_{\text{at}}(M_n)$. Let $N = \bigcup_{n<\omega} M_n$. Then $N \in \mathcal{K}$ is countable and contains $A$. We claim that $N$ is full over $A$. Let $p \in S_{\text{at}}(Ac)$ be the unique free extension of a stationary type in $S_{\text{at}}(c)$. There is $n < \omega$ such that $c \in M_n$, so $Ac \subseteq M_n$. There is a unique free extension of $p$ in $S_{\text{at}}(M_n)$, and this extension is realised in $M_{n+1}$, hence in $N$. 

For uniqueness, $A$ needs to be good.

**Proposition 2.12.** Suppose that $A$ is good and countable. Suppose that $M$ and $N \in \mathcal{K}$ are countable and full over $A$. Then $M$ is isomorphic to $N$ over $A$.

Proof. Let $M = A \cup \{a_i : i < \omega\}$ and $N = A \cup \{b_i : i < \omega\}$. We construct an increasing and continuous sequence of partial elementary maps $f_i : M \to N$ such that $\text{dom}(f_i) = A, \text{ran}(f_i) = B_i, f_i \upharpsim A = id_A$ and $a_i \in A_{2i}$ and $b_i \in B_{2i+1}, A_iB_i \setminus A$ is finite.

This is clearly enough. Let us see that this is possible. We first construct $f_0$. Since $A$ is good, there is a primary model $M'$ over $A$. Without loss of generality $M' < M$. There is an elementary map $f : M' \to N$, which is the identity on $A$. Consider the stationary type $\text{tp}(a_0/M')$. It is the unique free extension of the stationary type $p = \text{tp}(a_0/Ac)$ for some $c \in M'$. The image of $p$ under $f$ is realised in $N$ by some $b$, by fullness of $N$, and $\text{tp}(a_0/A) = \text{tp}(b/A)$. Let $f_0$ be the map extending $id_A$ sending $a_0$ to $b$.

Let $M''$ be primary over $Ab$ ($M''$ exists since $A$ is good and $Ab$ is atomic). There exists $g : M'' \to M$ extending the inverse of $f_0$. The stationary type $\text{tp}(b_0/M'')$ is the unique extension of some $q \in S_{\text{at}}(Abd)$, with $d \in M''$. The image of $q$ under $g$ is realised in $M$ by fullness of $M$ over $A$, so the map $f_1$ extending $f_0$ sending a realisation of $g(q)$ to $b_0$ is elementary.

Now assume that $f_{2i+1}$ has been constructed. Then $A_{2i+1} \setminus A$ is finite, so $A_{2i+1}$ is a good set containing $A$. Hence, there is a primary model $M'$ containing $A_{2i+1}$. The type $\text{tp}(a_{i+1}/M')$ is stationary and is the unique free extension of a type $p = \text{tp}(a_{i+1}/A_{2i+1}c)$ for some $c \in M'$. There is $f : M' \to N$ extending $f_{2i+1}$. The image of $p$ under $f$ is realised in $N$, which allows us to find $f_{2i+2}$ with domain $A_{2i+1} \cup a_{i+1}$, extending $f_{2i+1}$. The construction of $f_{2i+3}$ is similar. \qed

3. INDEPENDENT SYSTEMS, EXCELLENCE, AND CATEGORICITY THEOREM

In this section, we assume that $\mathcal{K}$ is $\omega$-stable. To motivate the definition of excellence, let us consider the problem of existence and uniqueness of uncountable full models. For full models of size $\aleph_1$ over a countable (good) set, it is still manageable. For existence: simply iterate $\omega_1$-times the construction of Proposition 2.11 (or, under our assumption, to consider an appropriate submodel of $\mathbb{C}$). For
uniqueness: To show that $M, N \in \mathcal{K}$ of size $\aleph_1$, full over the good countable set $A$, are isomorphic, choose $(M_i : i < \omega_1)$, and $(N_i : i < \omega_1)$ increasing and continuous chain of countable models such that $\bigcup_{i<\omega_1} M_i = M$ and $\bigcup_{i<\omega_1} N_i = N$, with $M_0$ and $N_0$ primary over $A$, $M_{i+1}$ full over $M_i$ and $N_{i+1}$ full over $N_i$. The isomorphism between $M$ and $N$ is then defined inductively by using Proposition 2.12.

However, to prove the existence or uniqueness of uncountable full models over larger sets (for example over a model of size $\aleph_2$), or to prove the existence and uniqueness of full models of size at least $\aleph_2$ is more problematic. The key ingredient in both proofs is the existence of a primary model over $Ma$, where $M \in \mathcal{K}$ countable and $Ma$ is atomic. We have not proved this for $M \in \mathcal{K}$ uncountable. Here is a possible strategy to prove this for $M$ of size, say, $\aleph_1$:

Choose an increasing and continuous chain of countable models $(M_i : i < \omega_1)$ such that $\bigcup_{i<\omega_1} M_i = M$. There exists a primary model $N_0$ over $M_0 a$, since $M_0 a$ is good. Suppose $N_0 \cup M_1$ is atomic and good. Then we can find $N_1$ primary over $N_0 \cup M_1$. Inductively, if $N_i \cup M_{i+1}$ is atomic and good for each $i < \omega_1$, then we can continue this process, and by taking unions at limit, obtain an increasing and continuous chain of models $(N_i : i < \omega_1)$ such that $N_{i+1}$ is primary over $N_i \cup M_{i+1}$. The hope is then that, pasting together the constructions shows that $\bigcup_{i<\omega_1} N_i$ is primary over $Ma$. To help us carry out this construction, we will use independence. First choose $M_0$ countable such that, in addition, $a$ is free from $M$ over $M_0$; this, we will see, ensures that the primary model $N_0$ is free from $M$ over $M_0$ and $N_0 \cup M$ is atomic. Inductively, assume that $N_i$ is free from $M$ over $M_i$ and $N_i \cup M$ is atomic. Then, if $N_i \cup M_{i+1}$ is good, we can find $N_{i+1}$ primary over $N_i \cup M_{i+1}$ such that, in addition, $N_{i+1}$ is free from $M$ over $N_{i+1}$ and $N_{i+1} \cup M$ is atomic. Taking unions at limit allows us to construct $N = \bigcup_{i<\omega_1} N_i$ (provided $N_i \cup M_{i+1}$ is good at each stage), and now, it is easy to see that $N$ is primary over $Ma$. Thus, the problem of finding a primary model over $Ma$ is reduced to finding primary models over countable sets of the form $M_1 \cup M_2$, where $M_1$ is free from $M_2$ over $M_0$, and $M_1 \cup M_2$ is atomic. The gain is that the models involved are countable; the cost is that we have to consider 2-dimensional (independent) systems: $N_{i+1}$ completes a square whose vertices are $M_i$, $M_{i+1}$, and $N_i$ and edges given by the relation $\prec$.

Now consider $Ma$, with $M \in \mathcal{K}$ with $Ma$ atomic, but this time with $M$ of size $\aleph_2$. Using the same idea leads us to ask about the existence of primary models over the atomic set $M_1 \cup M_2$, where $M_0, M_1, M_2 \in \mathcal{K}$, $M_0 \prec M_1, M_2$, $M_1$ is free from $M_2$ over $M_0$, and $M_\ell$ has size $\aleph_1$ for $\ell = 0, 1, 2$, i.e. an independent 2-dimensional system of models of size $\aleph_1$. We can repeat the same procedure to analyse this 2-dimensional system. Fix increasing and continuous chains $(M^i_\ell : i < \omega_1)$ of countable models such that $\bigcup_{i<\omega_1} M^i_\ell = M_\ell$, for $\ell = 0, 1, 2$. We can try to construct the primary model over $M_1 \cup M_2$ as the union of an increasing and continuous chain of models $(N_i : i < \omega_1)$ such that $N_0$ is primary over $M^0_1 \cup M^0_2$, and $N_{i+1}$ primary over $M^i+1_1 \cup M^{i+1}_2 \cup N_i$.
the countable sets $M_1^{i+1} \cup M_2^{i+1} \cup N_i$ are atomic and good at each stage $i < \omega_1$; here again independence will play an important part. In all, the gain is that the models are now countable, but the cost is that we have to consider 3-dimensional systems: $N_{i+1}$ completes the cube whose 7 other vertices are $N_i$, $M_1^i$, $M_2^i$, $M_0^i$, and $M_1^{i+1}$, $M_2^{i+1}$, and $M_0^{i+1}$, again edges are given by $\prec$. We formalise these ideas next.

We consider the following partial order $\mathcal{P}^-(n) := \mathcal{P}(n) \setminus \{n\}$ with respect to inclusion. We write $s \subset t$, if $s$ is a strict subset of $t$, so $\mathcal{P}^-(n) = \{s : s \subset n\}$, where $n = \{0, 1, \ldots, n - 1\}$. Then, $\mathcal{P}^-(0)$ is empty, $\mathcal{P}^-(1) = \{0\}$ is a point, $\mathcal{P}^-(2) = \{0, \{0\}, \{1\}\}$ is a square without one of its vertices, $\mathcal{P}^-(3) = \{0, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$ is a cube without one of its vertices, and so forth.

**Definition 3.1.** An independent $(\lambda, n)$-system is a collection of models $(M_s : s \subset n)$ such that

1. Each $M_s \in \mathcal{K}$ has size $\lambda$.
2. $M_s \prec M_t$ if $s \subset t$.
3. The set $A_s = \bigcup_{t \subset s} M_t$ is atomic.
4. For each $s$, $M_s \downarrow B_s$, where $B_s = \bigcup_{t \subset s} M_t$.

We will omit the parameters when they are either obvious or not important.

The next definition is not formally made in [Sh 87b]. Also, in [Sh 87b], $(\lambda, n)$-existence refers to a different property.

**Definition 3.2.** We say that $\mathcal{K}$ has $(\lambda, n)$-existence if there exists a primary model over $\bigcup_{s \subset n} M_s$, for each $(\lambda, n)$-independent system $(M_s : s \subset n)$.

Thus, in this paper, $(\aleph_0, n)$-existence is equivalent to the requirement that $\bigcup_{s \subset n} M_s$ is good for any independent $(\aleph_0, n)$-system. The next definition is the main definition of this paper.

**Definition 3.3.** $\mathcal{K}$ is excellent if $\mathcal{K}$ has $(\aleph_0, n)$-existence, for each $n < \omega$.

**Examples 3.4.** The simplest possible example for an excellent atomic $\mathcal{K}$ is probably the class of models of any $\aleph_0$-categorical and $\aleph_0$-stable first order theory. The former condition ensures that all models are atomic, while the latter that the appropriate primary models exist.

Another natural source of examples comes from homogeneous model theory. If $\mathcal{K}$ is atomic and $\omega$-stable and has arbitrarily large homogeneous models, then there are primary models over all sets, so $\mathcal{K}$ is excellent. The class of infinitely generated free groups $F(X)$ with a predicate for the set of generators $X$ fits into this context but is not first order (see [Le3] for details).
As we pointed out, there are also examples which are not homogeneous. The work of Boris Zilber has shown that excellence is relevant to understand complex exponentiation. See [Zi1], [Zi2] for interesting examples of excellent non-homogeneous classes.

We will now show how the existence of primary models over some (countable) sets implies the existence of primary models over other sets. If we had primary models over all countable atomic sets, then we would have them over all atomic sets [Le2] (the next fact does not assume Hypothesis 0.4).

**Fact 3.5** (Lessmann). Assume that $\mathcal{K}$ is $\omega$-stable and has an uncountable model. The following are equivalent:

1. There is a primary model over each countable atomic set.
2. There is a primary model over each atomic set.
3. There are arbitrarily large homogeneous models.

In our case, the situation is a bit more delicate. We now prove a lemma, which we refer to as **dominance**.

**Lemma 3.6.** Suppose that $A \perp B$, where $M \in \mathcal{K}$ and $ABM$ is an atomic set. Then $M' \perp B$.

**Proof.** By finite character of independence, it is enough to show this for $A$ and $B$ finite. Write $A = a$ and $B = b$. It is also enough to show that if $\text{tp}(c/Ma)$ is isolated and $a \perp b$, then $c \perp ab$. But this is clear since $\text{tp}(c/Ma)$ isolates $\text{tp}(c/Mab)$ (as $M$ is a model). $\square$

We now formalise the proof discussed at the beginning of this section with two theorems. The first shows that the existence of primary models over sets of the form $Ma$ does indeed follow from the existence of primary models over independent 2-dimensional systems of models of smaller size.

**Theorem 3.7.** Suppose that $\mathcal{K}$ has $(\mu, 2)$-existence, for each $\mu < \lambda$. Then, if $M \in \mathcal{K}$ has size $\lambda$ and $Ma$ is atomic, there exists a primary model over $Ma$.

**Proof.** Let $B$ be a finite subset of $M$ such that $c \perp M$. Choose an increasing and continuous sequence of models $(M_i : i < \lambda)$, such that $M_i \in \mathcal{K}$, $B \subseteq M_i$, $\|M_i\| \leq |i| + \aleph_0$, and $\bigcup_{i < \lambda} M_i = M$.

Now construct an increasing and continuous sequence of models

$(N_i : i < \lambda)$
such that $N_i \perp M$, $N_i \cup M$ is atomic, $N_0$ is primary over $M_0$, and $N_{i+1}$ is primary over $N_i \cup M_{i+1}$.

For $i = 0$, a primary model $N_0$ over $M_0$ exists, since $M_0$ is countable (and so $M_0$ is good). By independence calculus, we may assume that $M \cup N_0$ is atomic and $N_0 \perp M$. Hence, $(M_0, N_0, M_1)$ form an independent $(\aleph_0, 2)$-system. Inductively, notice that $(M_i, N_i, M_{i+1})$ forms an independent $(|i| + \aleph_0, 2)$-system. By $(|i| + \aleph_0, 2)$-existence, there exists $N_{i+1}$ primary over $M_{i+1} \cup N_i$. By induction hypothesis and monotonicity, $N_i \perp M$, so $N_i \cup M_{i+1} \perp M$. Thus $N_{i+1} \perp M_{i+1}$ by dominance. At limits, take the union and notice that the independence and atomicity follows by finite character of these notions.

Let $N = \bigcup_{i<\lambda} N_i$. Then, by pasting the constructions together and using independence, it is not difficult to see that $N$ is primary over $M$. □

The next theorem states that the same principle extends to larger dimensions.

**Theorem 3.8.** Let $\lambda$ be an infinite cardinal and $n < \omega$. Suppose that $\mathcal{K}$ has $(\mu, n)$ and $(\mu, n+1)$-existence, for each $\mu < \lambda$. Then $\mathcal{K}$ has $(\lambda, n)$-existence.

**Proof.** We prove the particular case when $n = 2$. Suppose $M_0 \!<\! M_\ell$, for $\ell = 1, 2$ forms an independent $(\lambda, 2)$-system.

Choose an increasing and continuous sequences $(M^i_\ell : i < \lambda)$ of models of $\mathcal{K}$ such that $||M^i_\ell|| \leq |i| + \aleph_0$, for $\ell = 0, 1, 2$, with $\bigcup_{i<\lambda^+} M^i_\ell = M_\ell$ and

$$M^i_1 \perp M_2 \quad \text{and} \quad M^i_2 \perp M_1.$$  

This is done as follows: Enumerate $M_\ell = \{a^i_\ell : i < \lambda\}$, for $\ell = 0, 1, 2$. For $i$ a limit, define $M^i_\ell$ by continuity. For $i = 0$ or a successor, having chosen $M^i_\ell$ containing $a^i_\ell$, choose $M^i_0$ containing $a^i_0$ of size $|i| + \aleph_0$ such that $M^i_\ell \perp M_0$, for $\ell = 1, 2$. Then, we obtain (*) by transitivity, since $M_1 \perp M_2$.

Now construct an increasing and continuous sequence of models $(N_i : i < \lambda)$ such that:

1. $N_0$ is primary over $M^0_1 \cup M^0_2$,
2. $N_1 \cup M_1$ is atomic,
3. $N_{i+1}$ is primary over $M^i_{i+1} \cup M^{i+1}_2 \cup N_i$. 


(4) \( N_i = \bigcup_{j<i} N_j \), for \( i \) a limit.

This is possible: For \( i = 0 \), we use \( (\aleph_0, 2) \)-existence (atomicity is obtained by extension). At successor stage \( i \), we use \( (|i| + \aleph_0, 3) \)-existence after checking that \( (M^i_\ell, M^{i+1}_\ell, N_i : \ell = 0, 1, 2) \) forms an independent \( (|i| + \aleph_0, 3) \)-system (use independence calculus and dominance, just as the previous theorem). At limit stages, we define \( N_i \) by continuity (again, atomicity is preserved and so is independence).

This is enough, as \( \bigcup_{i<\lambda} N_i \) is primary over \( M_1 \cup M_2 \) (pasting the constructions and using independence).

We now show two theorems showing that excellence implies the existence of primary models also over uncountable sets (this can be further extended to other systems of models [GrHa]).

**Theorem 3.9.** Suppose that \( \mathcal{K} \) is excellent. Then, \( \mathcal{K} \) has \( (\lambda, n) \)-existence for all \( n < \omega \) and for all cardinals \( \lambda \).

**Proof.** We prove this by induction on \( \lambda \) for all \( n < \omega \). For \( \lambda = \aleph_0 \), this is the definition of excellence. Assume now that \( \lambda > \aleph_0 \), and that \( (\mu, n) \)-existence holds for each \( \mu < \lambda \) and for all \( n < \omega \). Then \( (\lambda, n) \)-existence follows from Theorem 3.8.

We can finally prove:

**Theorem 3.10.** Suppose that \( \mathcal{K} \) is excellent. Then for any \( M \in \mathcal{K} \) and \( a \) such that \( Ma \) is atomic, there exists a primary model over \( Ma \).

**Proof.** By Theorems 3.7 and 3.9.

The previous theorem was the key idea behind extending the proof of the existence of full models to higher cardinalities. Before doing this, we prove a lemma.

**Lemma 3.11.** Assume that \( \mathcal{K} \) is excellent. Suppose that \( (M_i : i < \lambda) \) is an increasing and continuous chain of models in \( \mathcal{K} \), \( p \in S(C) \) is stationary with \( C \subseteq M_0 \), and for each \( i < \lambda \), \( a_i \in M_{i+1} \setminus M_i \) and the type \( \text{tp}(a_i/M_i) \) is the unique free extension of \( p \). Let \( I = (a_i : i < \lambda) \). Then

1. \( I \) is indiscernible over \( M_0 \).
2. For each \( b \in M_\lambda \), there is a finite set \( J \subseteq I \) such that \( I \setminus J \) is indiscernible over \( b \).

**Proof.** Indiscernibility over \( M_0 \) is clear. Let us prove (2). Construct \( (N_i : i < \lambda) \) increasing and continuous such that \( N_i < M_i \) and \( N_0 \) is countable and atomic, and
$N_{i+1}$ is primary over $N_i \cup a_i$. This is possible by excellence. Let $N = \bigcup_{i<\lambda} N_i < M$. Then $N$ is primary over $N_0 \cup \{a_i : i < \lambda\}$.

Let $b \in M_\lambda$. Then there is $c \in N$ and $i_1 < \ldots < i_n < \lambda$ such that $\text{tp}(b/N)$ does not split over $c$. Now $\text{tp}((c/N_0\{a_i : i < \lambda\}))$ is isolated by a formula $\phi(x, d, a_{i_1}, \ldots, a_{i_n})$, where $d \in N_0$ and $i_1 < \ldots < i_n < \lambda$.

We claim that $I \setminus \{a_{i_1}, \ldots, a_{i_n}\}$ is indiscernible over $b$. Let $\bar{a}, \bar{a}'$ two finite subsequences of $I \setminus \{a_{i_1}, \ldots, a_{i_n}\}$ of the same length. Notice that

$$\text{tp}(\bar{a}/da_{i_1}, \ldots, a_{i_n}) = \text{tp}(\bar{a}'/da_{i_1}, \ldots, a_{i_n}),$$

since $d \in N_0$ and $I$ is indiscernible over $N_0$. Since $\text{tp}(c/N)$ is isolated over $da_{i_1}, \ldots, a_{i_n}$, then necessarily $\text{tp}(\bar{a}/c) = \text{tp}(\bar{a}'/c)$. By nonsplitting, it follows that $\text{tp}(\bar{a}b/0) = \text{tp}(\bar{a}'b/0)$. This shows that $I \setminus \{a_{i_1}, \ldots, a_{i_n}\}$ is indiscernible over $b$. 

In general, indiscernible sequences in an excellent class cannot be extended; however those obtained as above can. This gives us a way of extracting an extensible indiscernible sequence from any uncountable set (see the proof of categoricity for more details). We can now construct full models directly. The next theorem is much stronger than our Hypothesis 0.4. There we assumed the existence of some suitably large full model; here we show that every model extends to a full model.

**Theorem 3.12.** Suppose that $K$ is excellent. Let $M \in K$. Then there is a full model $N$ over $M$ of size $\lambda$ for any $\lambda \geq \|M\|$. 

**Proof.** Let $M \in K$ be given and let $\lambda \geq \|M\|$. We construct an increasing and continuous sequence of models $(M_i : i < \lambda)$ such that $M_0 = M$, $\|M_i\| = |i| + \aleph_0$, and $M_{i+1}$ realises all types in $S_{\text{at}}(M_i)$. This is done as in the countable case using excellence: Having constructed $M_i$ of size at most $\lambda$, by $\omega$-stability, $S_{\text{at}}(M_i) = \{p_j : j < \lambda\}$ (since $K$ is $\lambda$-stable by Proposition 1.6). Construct an increasing and continuous sequence of models $M'_j$, such that $M'_{j+i}$ is primary over $M'_{j}a_j$, and $a_j$ realises the unique free extension of $p_j$ over $M'_j$. This is possible by stationarity of each $p_j$ and the fact that a primary model exists over each set of the form $Ma$, where $a$ realises a type in $S_{\text{at}}(M)$.

Let $M = \bigcup_{i<\lambda} M_i$. We claim that $M$ is full. Let $p \in S(C)$ with $C \subseteq M$ of size less than $\lambda$, be stationary. Let $c \in C$, such that $p$ does not split over $c$. Without loss of generality, we may assume that $c \in M_0$. Let $I = \{a_i : i < \lambda\}$ be such that $a_i \in M_{i+1}$ realises $p \upharpoonright M_i$. The $a_i$s exist by construction. Now, if $a_i = a_j$ for $i < j$, then $p$ is realised by $a_i$. Otherwise, $I$ has size $\lambda$. By the previous fact, $I$ is an indiscernible sequence and there exists $J \subseteq I$ of size less than $\lambda$, such that $I \setminus J$ is indiscernible over $C$. Thus, each element of $I \setminus J$ realises $p$, so $p$ is realised in $M$. 

\[\square\]
We now consider the problem of uniqueness of full models. Similarly to the previous theorem, we could prove uniqueness of full models like Proposition 2.12 by using Theorem 3.10. Instead, we illustrate again this idea of decomposing a certain problem into a larger dimensional problem involving smaller models. For this, we define full independent systems. We could have used this idea to construct full models also.

**Definition 3.13.** A \((\lambda, n)\)-independent system is called a full \((\lambda, n)\)-independent system if \(M_s\) is full over \(A_s\) for each \(s \subset n\).

**Definition 3.14.** We say that \(\mathcal{K}\) satisfies \((\lambda, n)\)-uniqueness, if there is a unique full model over \(A_n\), for any full \((\lambda, n)\)-independent system \((M_s : s \subset n)\).

**Lemma 3.15.** If \(\mathcal{K}\) is excellent, then \(\mathcal{K}\) satisfies \((\aleph_0, n)\)-uniqueness, for all \(n < \omega\).

**Proof.** By excellence, the set \(\bigcup_{s \subset n} M_s\) is good, for any \((\aleph_0, n)\)-independent system \((M_s : s \subset n)\). There is a full model over it, and by Proposition 2.12, the full model over it is unique.

**Proposition 3.16.** Assume that \(\mathcal{K}\) is excellent. If \(\mathcal{K}\) has \((\mu, n)\)- and \((\mu, n + 1)\)-uniqueness, for each \(\mu < \lambda\), then \(\mathcal{K}\) has \((\lambda, n)\)-uniqueness.

**Proof:** We prove this for \(n = 1\). Suppose that \(M_\ell\) is full over \(M_0\) for \(\ell = 1, 2\) and \(M_\ell \in \mathcal{K}\) have size \(\lambda\), and \(M_0\) is full. We must show that \(M_1\) is isomorphic to \(M_2\) over \(M_0\).

We construct three increasing and continuous chains of models

\[
(M_\ell^i : i < \lambda),
\]

with \(|M_\ell^i| \leq |i| + \aleph_0\), such that \(M_\ell = \bigcup_{i < \lambda} M_\ell^i\), for \(\ell = 0, 1, 2\) and the following conditions hold:

1. \(M_0^{i+1}\) is full over \(M_0^i\);
2. \(M_\ell^i\) is full over \(M_0^i\) for \(\ell = 1, 2\);
3. \(M_\ell^{i+1}\) is full over \(M_\ell^i \cup M_0^{i+1}\);
4. \(M_\ell^{i+1} \cup M_0^{i+1}\) is atomic, for \(\ell = 1, 2\);
5. \(M_\ell^i \downarrow M_0^{i+1}\), for \(\ell = 1, 2\).

This is possible: For \(i = 0\), to reconcile (1) and (2) with (5), we construct \(M_\ell^0\) for \(\ell = 0, 1, 2\) by taking \(\omega\)-chains of models. Choosing first the approximation to \(M_{\ell, n}^0 < M_\ell\) which are full over \(M_0^{0,n-1}\) (for \(\ell = 1, 2\), and then \(M_0^{0,n+1} < M_0\) so \(M_{\ell, n}^0 \downarrow M_0\) and \(M_0^{0,n+1}\) is full. The resulting union has the desired property.

For \(i\) a limit ordinal we define everything by continuity. The successor stage is similar to the base case, with chains of length \(|i| + \aleph_0\) (see the proof of the existence of full models for details).
This is enough: Notice that by \((< \lambda, 2)\)-uniqueness, the model \(M^{i+1}_\ell\) is the unique full model over \(M^i_\ell \cup M^i_0\) (for \(\ell = 1, 2\)), as the relevant systems are full. We can therefore construct an increasing and continuous sequence of isomorphisms \(f_i : M^i_1 \to M^i_2\), which are the identity on \(M^0_0\), inductively. The union is the required isomorphism between \(M_1\) and \(M_2\) over \(M_0\).

For \(i = 0\), by \((\kappa_0, 1)\)-uniqueness, since both \(M^0_1\) and \(M^0_2\) are full over \(M^0_0\), there is an isomorphism \(f_0\) from \(M^1_1\) to \(M^1_2\), which is the identity on \(M^0_0\). At limit stages, we define \(f_i\) by continuity.

At successor stage, assume that \(f_i\) has been constructed. Let \(g\) be an elementary map extending \(f_i\), whose domain contains \(M^{i+1}_1\), which is the identity on \(M^{i+1}_0\). This is possible since \(M^i_1 \subseteq M^{i+1}_0\). Now \(g(M^{i+1}_1)\) and \(M^{i+1}_2\) are both full over \(M^{i+1}_0 \cup M^i_2\). Hence, by \((|i+1| + \kappa_0, 2)\)-uniqueness, \(g(M^{i+1}_1)\) and \(M^{i+1}_2\) are isomorphic over \(M^{i+1}_0 \cup M^i_2\). This isomorphism yields an isomorphism \(f_{i+1} : M^{i+1}_1 \to M^{i+1}_2\) extending \(f_i\) which is the identity on \(M^{i+1}_0\).

The next theorem now follows by induction on \(\lambda\) for all \(n < \omega\), using Lemma 3.15 and Proposition 3.16 just like in the proof of Theorem 3.9.

**Theorem 3.17.** Suppose that \(\mathcal{K}\) is excellent. Then \(\mathcal{K}\) has the \((\lambda, n)\)-uniqueness property for all cardinals \(\lambda\), and \(n < \omega\).

The next corollary is simply \((\lambda, 0)\)-uniqueness:

**Corollary 3.18.** For each cardinal \(\lambda\), there is a unique full model \(M \in \mathcal{K}\) of size \(\lambda\).

At this point, it may be helpful to examine our Hypothesis again. We have now shown that \(\mathcal{C}\) functions as a monster model; by uniqueness of full models the class of (small) models of \(\mathcal{K}\) corresponds exactly to the class of (small) elementary submodels of \(\mathcal{C}\). Notice, however, that, in Section 1 and 2, we have only used its \(\omega\)-homogeneity.

We can now present Shelah’s proof of categoricity (for a Baldwin-Lachlan style proof, see [Le3]). The strategy is as follows: There exists a full model in every cardinality, so the model in the categoricity cardinal is full. Any two full models of the same size are isomorphic, so if categoricity fails in some cardinal, there exists a non full model in that cardinality, which we use to construct a non full model in the categoricity cardinal, a contradiction.

**Theorem 3.19.** Let \(\mathcal{K}\) be excellent. Suppose that \(\mathcal{K}\) is \(\lambda\)-categorical for some uncountable \(\lambda\). Then \(\mathcal{K}\) is \(\mu\)-categorical for all uncountable \(\mu\).

**Proof.** We proved that there is a unique full model up to isomorphism in each cardinal (Theorem 3.12 and Corollary 3.17).
Assume, for a contradiction, that $\mu$ is the first uncountable cardinal such that $\mathcal{K}$ is not $\mu$-categorical. Thus, there exists $M \in \mathcal{K}$ of size $\mu$ which is not full. Then, there is a stationary $p \in S_{\text{at}}(c)$ for a finite $c \in M$, and $A \subseteq M$ of size $\kappa$ less than $\mu$, such that the unique free extension $q \in S_{\text{at}}(A)$ extending $p$ is not realised in $M$.

Construct an increasing and continuous sequence of models

$$(M_i : i < \kappa^+),$$

such that $A \subseteq M_0$, $M_i < M$, $M_{i+1} \neq M_i$. Choose $a_i \in M_{i+1} \setminus M_i$ and $b_i \in M_i$ such that $\text{tp}(a_i/M_i)$ does not split over $b_i$, for $i < \kappa^+$. By Fodor’s lemma, we may assume that each $b_i \in M_0$. Furthermore, by $\omega$-stability, we may assume that $\text{tp}(a_i/M_0)$ is constant for all $i < \kappa^+$. Finally, by the pigeonhole principle, we may assume that $b_i = b \in M_0$ for all $i < \kappa^+$.

We have therefore a type $r \in S_{\text{at}}(b)$ and $(a_i : i < \kappa^+)$ such that $\text{tp}(a_i/M_i)$ extends $r$ and does not split over $b$. In particular, $(a_i : i < \kappa^+)$ is an indiscernible sequence.

We can now construct an increasing chain of countable models

$$(N_n : n < \omega)$$

such that $bc \in N_0$, $N_n < M_n$, $a_n \in N_{n+1} \setminus N_n$ realises the nonsplitting extension of $r$ in $S_{\text{at}}(N_n)$. We can further choose $N_n$ so $N_n$ does not realise the type $q^* = q \upharpoonright A \cap N_0$.

We will construct a model $N_\lambda$ of size $\lambda$ (the categoricity cardinal) which omits $q^*$. Since $q^*$ is the unique free extension of the stationary type $p$, and so $N_\lambda$ is not full, contradicting categoricity in $\lambda$.

In order to do this, we continue $(N_n : n < \omega)$ to obtain an increasing and continuous sequence $(N_i : i < \lambda)$ of models such that $a_i \in N_{i+1}$ realises the unique nonsplitting extension of $r$ in $S_{\text{at}}(N_i)$ and $N_{i+1}$ is primary over $N_i a_i$, for $i \geq \omega$. This is possible by excellence, as there is a primary model over $N_i a_i$. Let $N_\lambda = \bigcup_{i<\lambda} N_i$. Then $N_\lambda$ has size $\lambda$ and is constructible over $N_\omega \cup \{a_i : i < \lambda\}$.

Suppose $d \in N_\lambda$ realises $q^*$. The type $\text{tp}(d/N \cup \{a_i : i < \lambda\}) \models q^*$, and is isolated over $a \cup a_{i_1} \ldots a_{i_m}$, with $a \in N_n$ and $n + 1 < i_1 < \ldots < i_m < \lambda$. By indiscernibility of $(a_i : n \leq i < \lambda)$ over $N_n$, we may assume that $i_1 < \cdots < i_m < \omega$. Hence, $q^*$ is realised in $N_{n+1}$, a contradiction. \hfill \Box

3.1. Shelah’s original approach. In the last part of this paper, we discuss Shelah’s original approach in [Sh 48], [Sh 87a], and [Sh 87b].

A good notion of independence, defined using $\omega$-stability, is the main prerequisite before formalising excellence. In [Sh 48], Shelah obtains $\omega$-stability from the set-theoretic assumption that $V = L$ and the model-theoretic assumption that
\( \mathcal{K} \) is categorical in \( \aleph_1 \). He does this by proving the amalgamation property over countable models; more specifically, he constructs \( 2^{\aleph_1} \) nonisomorphic models of size \( \aleph_1 \) using a diamond on \( \aleph_1 \) (which follows from \( V = L \)). In [Sh 87a], he weakens the set-theoretic assumption to \( 2^{\aleph_0} < 2^{\aleph_1} \) by using a weak diamond on \( \aleph_1 \) (weak diamonds were introduced by Devlin and Shelah [DeSh], where they prove the existence of weak diamonds in \( \aleph_1 \) from \( 2^{\aleph_0} < 2^{\aleph_1} \)). Our hypothesis on the existence of \( \mathcal{C} \) is the substitute for amalgamation over countable models; it allows us to obtain \( \omega \)-stability from categoricity in some uncountable cardinal, within ZFC, using Ehrenfeucht-Mostowski models.

In establishing that the rank induces a good independence relation, in particular for symmetry, Shelah [Sh 48] uses a many models argument to construct \( 2^{\aleph_1} \) nonisomorphic models of size \( \aleph_1 \) from the \( \omega_1 \)-order property (which follows from the failure of symmetry without additional set-theoretic assumption). Here, we use \( \mathcal{C} \) to extend the construction from the \( \omega_1 \)-order property to the \textit{bona fide} order property, and we use it to contradict \( \omega \)-stability. Once these two ingredients are established, the structure part stays within ZFC and uses no further model-theoretic assumption.

For the theory of excellence \textit{per se}, the reader noticed that we proved only the case \( n = 2 \) of Theorem 3.8 (this is the only hole, as we pointed out, Proposition 3.16 is for illustrative purposes only and is not needed in proving the uniqueness of full models). To prove this for general \( n \), Shelah uses the reflection principle to give a uniform method to obtain an increasing and continuous chain of \( (\lambda, \omega + 1) \)-independent systems whose union is any given \( (\lambda, \omega) \)-independent system. Checking that all the requirements are satisfied is then done by induction, using, among other things, the \textit{generalised symmetry lemma}. This lemma states that in order to check that a system of models is independent, it is enough to check that it is independent with respect to some enumeration (preserving the order \( \subset \)). It is a generalisation of the idea that to check the independence of a sequence over a set, it is enough to check that each new element in the sequence is independent from the previous elements over the set.

In this paper, we did not address how excellence is obtained from, say, categoricity in many cardinals. In [Sh 87b], Shelah shows that \((\aleph_0, n + 1)\)-existence for \( n \leq k \) follows from the assumptions that \( \mathcal{K} \) is categorical in \( \aleph_n \) and for each \( 1 \leq n \leq k \) (assuming that the \( 2^{\aleph_n} \)'s form a strictly increasing sequence). This is done by defining a strong negation of the \((\aleph_0, \omega)\)-uniqueness property implying the failure of categoricity in \( \aleph_{n+1} \). An important consequence is that the existence of arbitrarily large models (which follows from excellence) can now be derived from the behaviour of countable models. Hart and Shelah also showed that \((\aleph_0, n)\)-existence does not imply \((\aleph_0, n + 1)\)-existence, and that categoricity may fail at \( \aleph_{k+1} \), while holding for \( \aleph_n \), \( n \leq k \) [HaSh].
REFERENCES

[GrLe2] Rami Grossberg and Olivier Lessmann, Abstract decomposition theorem and applications. This volume.
[HaSh] Bradd Hart and Saharon Shelah, Categoricity over $P$ for first order $T$ or categoricity for $\phi \in L_{\omega_1,\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \ldots, \aleph_{k-1}$. Israel J. Math. 70 (1990), no. 2, 219–235.
[Ko] Alexei Kolesnikov, Dependence relations in non-elementary classes. This volume.


[Sh 666] Saharon Shelah, On what I do not understand (and have something to say) – model theory. *Fundamenta Math* 166 (2000) 1-82

[Sh 705] Saharon Shelah, Toward classification theory of good $\lambda$-frames and abstract elementary classes. Preprint


[Zi1] Boris Zilber, Covers of the multiplicative group of an algebraically closed field of characteristic 0. Preprint.


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Ultrafilters and Nuclear Spaces

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Abstract. We present an example of a separable non-nuclear space with the same probabilistic properties as nuclear spaces.

1. Introduction

This paper is connected with classical papers by Minlos [M] and Sazonov [S].

Gelfand [G] had introduced generalized random processes as continuous random linear functionals on the space $D(\mathbb{R})$ of all infinite differentiable functions with a compact support. He asked: does every generalized random process define a $\sigma$-additive probability on the space of distributions. The paper by Minlos answered affirmatively to this question in a more general context of nuclear spaces (introduced by Grothendieck [Gr]). In the same time Sazonov established an analog of the Bochner theorem for a Hilbert space. Kolmogorov [K] proposed a unified form of both these theorems (as a Bochner theorem for the multi-Hilbert spaces). He pointed out the importance of the field and described a simpler proof which uses an inequality of Prokhorov [P].

The author [Mu3] established some generalizations of the converse Minlos theorem.

The aim of the paper is to present here a counterexample complementing the converse theorems. The construction of the example uses essentially a non-trivial ultrafilter (1.1) with a special property in the set of natural numbers. The existence of such ultrafilter was proved by Choquet [Ch, 1967] (independently but much later in [Mu3, 1973]). Choquet called such ultrafilters quick.

Theorem 1.1. Assuming Choice Axiom and Continuum Hypothesis there is an ultrafilter, $\mathcal{F}$, in the set of natural numbers $\mathbb{N}$ satisfying the following conditions:

i) $\mathcal{F}$ contains only infinite sets;

ii) for any real sequence $(x_n) \to 0$ there exists $J \in \mathcal{F}$ such that

$$\sum_{n \in J} |x_n|^{1/2} < 1. \quad (1.1)$$

Proof. It is well known that the space $c_0$ of all real sequences convergent to 0 has the cardinality of the continuum. By the Cermelo theorem and Continuum Hypothesis the set $c_0$ can be well ordered in a way that every $\iota \in c_0$ is preceded by

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a countable set of elements. We use such ordering and will construct a basis of a filter \( \mathcal{F} \) consisting of infinite sets \( \{ \alpha(i) : i \in c_0 \} \).

Suppose that all \( \alpha(i) \subseteq \mathbb{N} \) with \( i < \kappa \) are constructed so that the intersection of a finite number of these sets is infinite. Denote

\[
\alpha(\iota_1, \iota_2, \ldots, \iota_n) = \bigcap_{i=1}^{n} \alpha(\iota_i).
\]

For every \( \kappa = (x_n) \in c_0 \) the set of all finite families \( \{(\iota_1, \iota_2, \ldots, \iota_n) : \iota_i < \kappa \) for all \( k,n \in \mathbb{N} \} \) is countable. We arrange this set into a sequence. Let the family \( (\iota_1, \iota_2, \ldots, \iota_n) \) occupies the \( k \)-th place in our sequence, then we extract from \( \alpha(\iota_1, \iota_2, \ldots, \iota_n) \) an infinite subset \( \beta = \beta(\iota_1, \iota_2, \ldots, \iota_n) \) so that

\[
\sum_{n \in \beta} |x_n|^{1/2} < 2^{-k}.
\]

Set \( \alpha(\kappa) = \bigcup \{ \beta(\iota_1, \iota_2, \ldots, \iota_n) : \iota_i < \kappa \) for all \( i \leq n \in \mathbb{N} \} \). We have extended our map to the element \( \kappa \) conserving the properties i) and ii). To conclude, we take as \( \mathcal{F} \) an ultrafilter containing all \( \alpha(i) \).

\[ \Box \]

**Remark 1.2.** The relation (1.1) is equivalent to the definition of quick ultrafilters by Choquet: for any sequence \( \{n_k\} \subseteq \mathbb{N} \) there exists an element \( \{m_k\} \in \mathcal{F} \) such that \( n_k \leq m_k \) for all \( k \in \mathbb{N} \).

We will notice that counterexamples of nonseparable spaces do not require ultrafilters. In the section 6 we will explain: why the separable spaces are of special interest in the field.

We cannot here prove all the results that we use. Moreover, a sketch of the proof will be given only for several theorems, but the example with the ultrafilter will be presented with a justification. A more detailed information may be found in [Mu4].

### 2. Some definitions

Let \( X \) be a real locally convex space (i.e., its topology is defined by a family \( \{p_\alpha\} \) of seminorms). We remind that a seminorm \( p \) on a real linear space \( X \) satisfies two conditions:

i) \( p(\lambda x) = |\lambda|p(x) \) for all \( x \in X, \lambda \in \mathbb{R} \),

ii) \( p(x + y) \leq p(x) + p(y) \) for all \( x,y \in X \).

A seminorm \( p \) is a norm if

i) \( p(x) = 0 \) iff \( x = 0 \).

We denote by \( X' \) the space of all continuous linear functionals on \( X \). The duality between \( X \) and \( X' \) defines the weak topologies on these spaces (\( \sigma(X,X') \) is the weakest topology on \( X \) such that all linear functionals from \( X' \) are continuous, \( \sigma(X',X) \) is the weakest topology on \( X' \) such that all linear functionals \( y \to y(x) \) from \( X \) are continuous). This duality also defines measurable structures on \( X \) and \( X' \). We will deal with the cylindrical algebra \( \mathcal{C}_X(X') \) on the space \( X' \) which is defined as the weakest algebra of sets in \( X' \) such that for all \( x \in X \) linear functionals \( y \to y(x) \) on \( X' \) are measurable.

We denote by \( \mathcal{E}_Y(X') \) (respectively, \( \mathcal{L}_Y(X') \)) the smallest algebra \( (\sigma\text{-algebra}) \) on \( X' \) such that all the functionals from \( Y \subset X \) are measurable.
DEFINITION 2.1. $\mathcal{C}_X(X')$ is the union of all $\sigma$-algebras $\mathcal{L}_{(x_1, \ldots, x_n)}(X')$ where $(x_1, \ldots, x_n)$ runs over the set of all finite subsets of $X$.

$\mathcal{L}_Y(X')$ is the smallest $\sigma$-algebra containing $\mathcal{C}_Y(X')$. Clearly,

$$\mathcal{L}_X(X') = \bigcup \{ \mathcal{L}_Y(X') : Y \subset X \text{ is countable} \}.$$  

All elements $C$ (cylindrical sets) of $\mathcal{C}_X(X')$ are represented in the form

$$C = C(x_1, \ldots, x_n; B) = \{ y \in X' : (y(x_1), \ldots, y(x_n)) \in B \}$$

where $(x_1, \ldots, x_n)$ runs over all finite collections in $X$ and $B$ runs over all Borel measurable sets in $\mathbb{R}^n$. We will call $B$ a base, $(x_1, \ldots, x_n)$ a generator of the cylindrical set.

DEFINITION 2.2. We call a finitely additive set function $\mu$ on $\mathcal{C}_X(X')$ a cylindrical probability provided the restriction of $\mu$ onto every sub-$\sigma$-algebra $\mathcal{L}_{(x_1, \ldots, x_n)}(X')$ is a probability measure.

A cylindrical probability may be defined by a random linear functional $\Phi : X \rightarrow L^0(\Omega, \mathfrak{A}, \mathbb{P})$, i.e., by a linear application with values in the space of all random variables defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Namely, a function $\mu_\Phi$ is defined by the relation

$$\mu_\Phi(C(x_1, \ldots, x_n; B)) = \mathbb{P}\{ \omega : (\Phi(x_1)(\omega), \ldots, \Phi(x_n)(\omega)) \in B \}.$$  

It is easy to verify that:

1) $\mu_\Phi$ is finitely additive,

2) a restriction of $\mu_\Phi$ to every sub-$\sigma$-algebra of $\mathcal{C}_X(X')$ consisting of cylindrical sets with a fixed generator $(x_1, \ldots, x_n)$ is $\sigma$-additive,

3) $\mu_\Phi$ is regular, i.e., for every $C \in \mathcal{C}_X(X')$ and $\varepsilon > 0$ there exist two cylindrical sets $F \subset C \subset G$ such that the base of $F$ is a closed set and the base of $G$ is an open set, $\mu_\Phi(G \setminus F) < \varepsilon$. The proof of this property consists in using the regularity property of probability measures on the Borel $\sigma$-algebra in $\mathbb{R}^n$.

A natural question is: when $\mu_\Phi$ is $\sigma$-additive?

There is a simple example where such $\mu_\Phi$ is not $\sigma$-additive. We recall that $\sigma$-additivity of a finitely additive set function defined on an algebra is equivalent to the continuity at $\emptyset$: $C_n \rightarrow \emptyset \Rightarrow \mu_\Phi(C_n) \rightarrow 0$.

EXAMPLE 2.3. Let $X = l^2$ be a Hilbert space of all square-summable real sequences with the norm $\| (x_n) \| = \left( \sum_n |x_n|^2 \right)^{1/2}$, $(e_n)$ be the canonical orthonormal basis in $l^2$ $(e_n = (0, \ldots, 0, 1, 0, \ldots)$, where $1$ has the $n$-th place), $\varepsilon_n$ be independent random variables such that $\mathbb{P}\{ \varepsilon_n = \pm 1 \} = 1/2$. We set

$$\Phi(x_n) = \sum_n \varepsilon_n x_n.$$  

Let us define $C_n = \{ y : y(e_i) = \pm 1 \text{ for } i \leq n \}$. Obviously, $\mu_\Phi(C_n) = 1$ for all $n$, but $\cap_n C_n = \emptyset$ (the sequences $(\pm 1, \pm 1, \ldots)$ do not belong to $l^2$).

REMARK 2.4. The linear hull $X_0$ of bases of a sequence $\{ C_n \}$ of cylindrical sets is a separable subspace of $X$. Thus, the following criterion is obvious.

PROPOSITION 2.5. Let $\Phi$ be a random linear functional on a locally convex space $X$. The $\mu_\Phi$ is $\sigma$-additive if and only if a restriction $\Phi_0$ of $\Phi$ to any separable subspace $X_0$ of $X$ defines a $\sigma$-additive cylindrical probability $\mu_\Phi_0$ on $X_0'$.  

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The famous Kolmogorov theorem affirms that $\mu_\Phi$ is $\sigma$-additive in case when $X = \mathbb{R}^{(T)}$ (the direct sum of $T$ copies of the real line) and $X' = \mathbb{R}^T$ (the direct product of $T$ copies of the real line). The following result is a corollary of the Kolmogorov theorem:

**Proposition 2.6.** Let $X$ be a real linear space, $X^\circ$ be the space of all real linear functionals (not necessarily continuous) on $X$. Then any random linear functional $\Phi$ on $X$ defines a $\sigma$-additive cylindrical probability on $\mathfrak{C}_X(X^\circ)$.

We use the notation $A^\circ = \{y \in X' : |y(x)| \leq 1 \text{ for all } x \in A\}.$

**Definition 2.7.** We call a cylindrical probability $\mu_\Phi$ on $\mathfrak{C}_X(X')$ tight if for every $\varepsilon > 0$ there exists a neighbourhood $U$ in $X$ such that

$$C \in \mathfrak{C}_X(X'), \ C \cap U^\circ = \emptyset \Rightarrow \mu_\Phi(C) < \varepsilon.$$  

**Proposition 2.8.** Every tight cylindrical probability $\mu$ is $\sigma$-additive.

For the proof we use the weak compactness of $U^\circ$. We suppose the contrary: there exists a sequence $C_n \in \mathfrak{C}_X(X')$ such that $C_n \not\subseteq \emptyset$ but $\mu(C_n) > \varepsilon > 0$. By using the regularity of $\mu$ we construct a sequence of closed cylindrical sets $F_n \subseteq C_n$ such that $F_n \not\subseteq \emptyset$ but $\mu(F_n) > \varepsilon/2$. By choosing $U^\circ$ in (1) for $\varepsilon = \varepsilon/4$, we obtain a contradiction with the compactness of $U^\circ$: $F_n \cap U^\circ \not\subseteq \emptyset$, but all this closed sets are non-empty.

Often, the $\sigma$-additivity arises as a consequence of the tightness.

**Remark 2.9.** If $X$ is a separable space then all the sets $U^\circ$ in the definition of tight cylindrical probabilities belong to the $\sigma$-algebra $\mathfrak{L}_X(X')$. If $X$ is metrizable then any $\sigma$-additive cylindrical probability on $\mathfrak{C}_X(X')$ is tight.

**Example 2.10.** Let us consider a continuous random linear functional $\Phi$ on a space of sequences $X = l^p$ with $X' = l^q$ where $1/p + 1/q = 1$ or $X = c_0$ with $X' = l^1$. In the both cases the space $X$ is separable and $\Phi$ is defined by its values $\xi_n = \Phi(e_n)$ on the elements $e_n$. In this cases the $\sigma$-additivity of $\mu_\Phi$ is equivalent to its tightness or to $\sum \xi_n^q < \infty$ a.s.

**Definition 2.11.** We call a locally convex space $X$ a space with the Sazonov property if there exists a topology $\mathfrak{T}$ on $X$ such that the continuity of a random linear functional $\Phi$ in the topology $\mathfrak{T}$ is equivalent to the tightness of the cylindrical probability $\mu_\Phi$. We will call a topology $\mathfrak{T}$ with such property an $S$-topology.

**Definition 2.12.** We call a locally convex space $X$ a space with the Minlos property if the initial topology on $X$ is an $S$-topology.

### 3. Nuclear spaces

All definitions and statements of this section may be found in [Pi].

**Definition 3.1.** A linear operator $T : X \rightarrow X_1$ between two Banach spaces is said to be nuclear if it admits a representation

$$T(x) = \sum_{n} y_n(x)x_n \text{ for all } x \in X, \text{ where } \sum \|x_n\|\|y_n\| < \infty, y_n \in X'.$$

**Remark 3.2.** A restriction of a nuclear operator to any infinite-dimensional subspace is not an isomorphic operator.
DEFINITION 3.3. A linear operator \( T : X \to X_1 \) between two Banach spaces is said to be \textit{absolutely summing} if for every unconditionally convergent series \( \sum x_n \) in \( X \) the series \( \sum \|Tx_n\| \) converges, too.

PROPOSITION 3.4. (Pietsch inequality). A linear operator \( T : X \to X_1 \) between two Banach spaces is absolutely summing iff there exists a finite \( \sigma \)-additive measure \( \mu \) on a ball \( U' = \{ y : \|y\| \leq 1 \} \) in \( X' \) such that
\[
\|T(x)\| \leq \int_{U'} |y(x)| \mu(dy) \text{ for all } x \in X.
\]

DEFINITION 3.5. A linear operator \( T : H \to H_1 \) between two Hilbert spaces is said to be \textit{Hilbert—Schmidt} if
\[
\sum_n \|Te_n\|^2 < \infty
\]
for any orthonormal basis \( (e_n) \) in \( H \).

All these definitions work also in case when the spaces are endowed with seminorms (instead of norms).

DEFINITION 3.6. A locally convex space \( E \) is said to be \textit{nuclear} if for every continuous seminorm \( p_1 \) on \( E \) there is another continuous seminorm \( p_0 \) such that the identical operator from \( (E,p_0) \) to \( (E,p_1) \) is nuclear (respectively, absolutely summing, Hilbert—Schmidt).

All three different definitions of nuclear spaces are equivalent [Pi].

REMARK 3.7. It follows from Remark 3.2 that a nuclear Banach space is a finite-dimensional space.

Now we can formulate the theorems by Minlos and Sazonov (in terms of Definitions 2.11 and 2.12).

4. Theorems by Minlos and Sazonov

THEOREM 4.1. \textit{Every nuclear space has the Minlos property.}

THEOREM 4.2. \textit{Any Hilbert space \( X \) has the Sazonov property. Namely, an S-topology on \( X \) may be defined as the topology defined by all seminorms \( x \to \|Ax\| \) where \( A \) runs over the set of all Hilbert — Schmidt operators on \( X \). (This topology is called \( \mathfrak{I}\)-topology in [S].)}

Kolmogorov had proposed the following non-Banach generalization of the Sazonov theorem. Often this statement is called the Minlos — Sazonov theorem.

THEOREM 4.3. \textit{Any multi-Hilbert space (i.e., the space \( X \) whose topology is defined by a family \( \{\langle \cdot, \cdot \rangle_\alpha \} \) of scalar products) has the Sazonov property. The S-topology (generalized \( \mathfrak{I}\)-topology) is defined by all seminorms \( x \to \|Ax\| \) where \( A \) runs over the set of all Hilbert — Schmidt operators with respect to scalar products \( \{\langle \cdot, \cdot \rangle_\alpha \} \).}

The Minlos theorem follows from Theorem 4.3 because a nuclear space is a multi-Hilbert space [Gr] and the initial topology in a nuclear space coincides with the generalized \( \mathfrak{I}\)-topology on it.
THEOREM 4.4. [M] Let a multi-Hilbert space $X$ have the Minlos property. Then $X$ is a nuclear space.

The proof deals only with Gaussian cylindrical probabilities. There is considered a Gaussian cylindrical probability $\mu$ defined by a characteristic functional

$$\hat{\mu}(x) = \int e^{iy(x)} \mu(dy) = \exp - (x, x)_\alpha.$$ 

By the Minlos property $\mu$ is tight, so for an $\varepsilon > 0$ there exists $\{(\cdot, \cdot)_\beta \}$ such that for some $C > 0$ there holds

$$\mu\{ y : (y, y)_\beta \leq C \} \geq 1 - \varepsilon.$$ 

This implies that the scalar product $\langle \cdot, \cdot \rangle_\alpha$ admits a representation

$$\langle y, y \rangle_\alpha = \langle Ay, Ay \rangle_\beta$$

where $A$ is a Hilbert — Schmidt operator in $(X, (\cdot, \cdot)_\beta)$. Thus $X$ is nuclear.

In order to formulate some more general results we need consider Banach spaces which embed into spaces of random variables.

5. Spaces of random variables

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ (where the probability measure $\mathbb{P}$ is not atomic) and the space $L^0(\Omega, \mathcal{A}, \mathbb{P})$ of all random variables on it. This space is endowed with the topology of the convergence in probability:

$$\xi_n \xrightarrow{\mathbb{P}} \xi \iff \mathbb{P}\{|\xi_n - \xi| \geq \varepsilon\} \to 0 \forall \varepsilon > 0.$$ 

If $\xi = 0$ this convergence is defined in terms of characteristic functions:

$$\xi_n \xrightarrow{\mathbb{P}} 0 \iff \varphi_{\xi_n}(t) = \int_\Omega \exp(it\xi_n)d\mathbb{P} \to 1 \forall t \in \mathbb{R}.$$ 

The space $L^0(\Omega, \mathcal{A}, \mathbb{P})$ is not locally convex but it contains subspaces isomorphic to Banach spaces. For example, $l^2$ is isomorphic to a subspace of $L^0(\Omega, \mathcal{A}, \mathbb{P})$ spanned on a sequence of independent standard Gaussian random variables $(\gamma_n)$. We define an isomorphic embedding $\Phi : l^2 \to L^0(\Omega, \mathcal{A}, \mathbb{P})$ in the following way:

$$\Phi \left( \sum_k a_k e_k \right) = \sum_k a_k \gamma_k.$$ 

Since $\varphi_{\gamma_n}(t) = \exp(-t^2/2)$, we have:

$$\sum_k a_{k}^{(n)} e_k \to 0 \iff \sum_k |a_{k}^{(n)}|^2 \to 0$$

$$\iff \varphi_{\Phi\left( \sum_k a_{k}^{(n)} \gamma_k \right)}(t) = \exp \left( - \sum_k |a_{k}^{(n)}|^2 t^2/2 \right) \to 1.$$ 

The spaces $l^p$ with $p \leq 2$ are also isomorphic to subspaces of some $L^0(\Omega, \mathcal{A}, \mathbb{P})$. In the construction of an embedding we use instead of $\gamma_n$ random variables $\gamma_p^n$ with a characteristic function $\exp(-|t|^p)$. If $p > 2$ then the function $\exp(-|t|^p)$ does not define a probability on the real line. The spaces $l^p$ with $p > 2$ and the space $c_0$ do not admit an isomorphic embedding into $L^0(\Omega, \mathcal{A}, \mathbb{P})$. The proof [Mu1] uses the Sazonov theorem and the following fundamental lemma [Kw]:
Lemma 5.1. If $\Phi : c_0 \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ is a random linear functional and for given $C > 0$, $\epsilon \in (0, 1)$ and for all $y \in c_0$ there holds

$$\mathbb{P}\{\omega : |\Phi(y)| > C \cdot \|y\|\} < \epsilon,$$

then there exists $\Omega'$ such that $\mathbb{P}(\Omega \setminus \Omega') < 8\epsilon$ and

$$\sum_n |\Phi(e_n)(\omega)|^2 < \infty \text{ a.s. for all } \omega \in \Omega'.$$

We will use in the sequel also another result [Sch] for sequences of random variables:

Lemma 5.2. Let $\xi_n$ be a sequence in the space $L^0(\Omega, \mathcal{A}, \mathbb{P})$ whose convex hull is a bounded set, $c_n \in \mathbb{R}$ satisfy

$$\sum_n |c_n||\log|c_n|| < \infty.$$

Then the series $\sum_n |c_n\xi_n|$ is convergent almost surely.

Mokobodzki [Mo] has connected the convergence in probability and almost sure limits with quick ultrafilters:

Theorem 5.3. $\xi_n \xrightarrow{\mathbb{P}} \xi \Rightarrow \lim_3 \xi_n = \xi$ a.s.

Let $X$ be a linear subspace of some $L^0(\Omega, \mathcal{A}, \mathbb{P})$. We will discuss the question: when two convergencies, in probability and almost surely, coincide on $X$. If $Y = L^0(\Omega, \mathcal{A}, \mathbb{P})$ then the answer is evident: this is a case when the probability $\mathbb{P}$ is atomic, so $L^0(\Omega, \mathcal{A}, \mathbb{P})$ is isomorphic to $\mathbb{R}^\mathbb{N}$. In the general situation we have the following statement [Mu2].

Theorem 5.4. In the linear subspace $X \subset L^0(\Omega, \mathcal{A}, \mathbb{P})$ the convergence in probability implies the a.s. convergence if and only if $X$ is nuclear in the topology of the convergence in probability.

The proof uses the Minlos theorem and Proposition 3.4, a more difficult part is to prove that the topology of the convergence in probability on $X$ is locally convex.

6. Generalizations and a counterexample

Now we will mention two results [Mu3] where the Minlos property implies the nuclearity (cf. Theorem 4.4):

Theorem 6.1. Every Fréchet space (i.e., a metrizable complete locally convex space) with the Minlos property is nuclear.

Theorem 6.2. Let a locally convex space have the Minlos property and be isomorphic to a subspace of a product of spaces of random variables. Then this space is nuclear.

The proof of both theorems uses the definition of nuclear spaces in terms of absolutely summing operators, the proof of the second one also uses Proposition 3.4.

Now we will construct an example of a locally convex space with the Minlos property which shows that the generalization of Theorem 6.1 to non-Fréchet spaces is false. We prefer to chose such counter-example in the class of separable spaces.
Proposition 2.5 shows that the problem of \(\sigma\)-additivity of cylindrical probabilities is reduced to the separable case, the sets \(U^0\) in the definition of tight cylindrical probabilities in case of a separable space \(X\) belong to the \(\sigma\)-algebra \(\mathcal{C}_X(X')\).

Note that every separable locally convex space contains a dense subset linearly isomorphic to \(\mathbf{R}^{(N)}\). Thus such space can be constructed as a completion of \(\mathbf{R}^{(N)}\) with respect to some net of seminorms.

This space could not be a Fréchet space (Theorem 6.1) and could not be isomorphic to a subspace of a product of spaces of random variables (Theorem 6.2).

**Theorem 6.3.** Let \(X_1\) be a completion of \(\mathbf{R}^{(N)}\) with respect to the following set of norms:

\[
\left\| \left( \sum_n z_n e_n \right) \right\|_{k,\alpha} = \sup_n |z_n a_n^k(\alpha)|, \text{ where } a_n^k(\alpha) = \begin{cases} 1 & \text{if } n \in \alpha; \\ n^{3k} & \text{otherwise} \end{cases}
\]

where \(k\) runs over \(\mathbf{N}\) and \(\alpha\) runs over elements of the ultrafilter \(\mathcal{F}\) from Theorem 1.1. Then

i) \(\{\| \cdot \|_{k,\alpha}\}\) is a net of seminorms;

ii) \(X_1\) is separable;

iii) \(X_1\) is not nuclear;

iv) \(X_1\) has the Minlos property.

**Proof.** Since \(\mathcal{F}\) is a filter, i) is true. Obviously, \(X_1\) contains a countable dense subset of the points \(\sum_{i \leq n} r_i e_i\), where \(n\) runs over \(\mathbf{N}\) and \(r_1\) run over the set of rationals.

The space \(X_1\) is not nuclear because intersections of finite number of sets in \(\mathcal{F}\) are infinite. Thus the space \(E = \text{Lin}\{e_n : n \in \alpha \cap \beta\}\) is infinite-dimensional but the restriction to \(E\) of the identical map from \((X_1, \| \cdot \|_{k,\alpha})\) to \((X_1, \| \cdot \|_{l,\alpha})\) is an isometry (cf. Remark 3.2).

Now we will prove iv), i.e., that every continuous random linear functional \(\Phi : X_1 \to L^0\) determines a tight cylindrical probability \(\mu_\Phi\). We point out that any linear functional continuous in the norm \(\| \cdot \|_{k,\alpha}\) is defined by a sequence \((y_n)\) such that \(\sum |y_n/a_n^k(\alpha)| < \infty\). For every \(\varepsilon \in (0, 1/16)\) there exist a norm \(\| \cdot \| = \| \cdot \|_{k,\alpha}\) (we will also denote \(a_n = a_n^k(\alpha))\) and \(C > 0\) such that \(\| y \| \leq 1\) implies \(\mathbb{P}\{|\Phi(y)| > C\} < \varepsilon\). By Lemma 5.1, there exists \(\Omega' \subset \Omega\) such that \(\mathbb{P}(\Omega \setminus \Omega') < 8\varepsilon\) and

\[
(6.1) \quad \sum_n \Psi^2 \left( \frac{e_n}{a_n} \right) < \infty \quad \text{a.s.}
\]

where \(\Psi = \Phi 1_{\Omega}\). Obviously

\[
(6.2) \quad |\mu_\Phi - \mu_\Psi| < 8\varepsilon.
\]

Thus we have only to prove the tightness of the cylindrical probability \(\mu_\Psi\).

Let us consider an extension \(\Psi_1\) of \(\Psi\) onto the space \(l^2(a_n)\) of sequences \((z_n)\) endowed with the norm

\[
\| (z_n) \|_{2} = \left( \sum_n a_n^2 y_n^2 \right)^{1/2}.
\]

By (6.1), the \(\Psi_1\) defines a tight cylindrical probability on \((l^2(a_n))'\). Thus, by the Sazonov theorem 4.2, \(\Psi_1\) is continuous in the \(\mathcal{F}\)-topology on \(l^2(a_n)\). More precisely,
the $\Psi_1$ is continuous in the topology defined by a sequence $q^{(s)}$ of seminorms defined by relations

$$q^{(s)}(z_n) = \|A^{(s)}(y_n)\|_2$$

where $A^{(s)}$ are Hilbert-Schmidt operators on $l^2(a_n)$. By the definition of $q^{(s)}$,

$$(6.3) \sum_n q^{(s)}(e_n/a_n)^2 < \infty.$$ 

We need only the following consequence from (6.3): for any $s \in \mathbb{N}$ a sequence $q^{(s)}(e_n/a_n)$ converges to 0. Thus, there exists a sequence of constants $p^{(s)}$ such that

$$b_n = \max_s p^{(s)} \cdot q^{(s)}(e_n/a_n) \to 0.$$ 

The sequence $\{e_n/a_n b_n\}$ is bounded in the topology defined by the norms $q^{(s)}$. Since $q^{(s)}$ are seminorms, the set $\text{Conv}\{e_n/a_n b_n\}$ is bounded, too. Hence, the set $\text{Conv}\{\Psi(e_n/a_n b_n) : n \in \mathbb{N}\}$ is bounded in $L^0(\Omega, \mathbb{P})$. We denote $b = (b_n)$ and take $\alpha(b) \in \mathfrak{B}$ from the proof of Theorem 1.1 and consider a sequence

$$c_n = \frac{a_n^k(\alpha) b_n}{a_n^{k+1}(\alpha \cap \alpha(b)).}$$

By the definition of $\alpha(b)$,

$$\sum_n c_n^{1/2} = \sum_{\alpha(b)} + \sum_{\mathbb{N}\setminus\alpha(b)} \leq \sum_n n^{-3/2} \max_n b_n^{1/2} + \sum_{\mathbb{N}\setminus\alpha(b)} b_n^{1/2} < \infty.$$ 

According to Lemma 5.2, the series

$$\sum_n \Psi\left(\frac{e_n}{a_n^{k+1}(\alpha \cap \alpha(b))}\right) = \sum_n c_n \Psi\left(\frac{e_n}{a_n b_n}\right)$$

converges almost surely. In other words, the cylindrical probability $\mu_\Psi$ is concentrated on linear functionals continuous by the norm $\| \cdot \|_{k+1, \alpha \cap \alpha(b)}$. Hence, by (6.3) $\mu_\Psi$ is tight.

**Question 6.4.** Does such example exist without Continuum Hypothesis?

**Question 6.5.** The space $X_1$ is not barreled. Does there exist a barreled (see [B]) space with the properties i) - iv)?

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**References**


[Mu1] D.H. Mushtari, On a problem of Laurent Schwartz and of the realization of $l^p$ spaces by


1996.


[P] Yu.V. Prokhorov, Convergence of random processes and limit theorems in probability theory,
Probab. Th. and Appl., 1 (1956), 177-238.

[S] V.V. Sazonov, Remarks on characteristic functionals, Probab. Th. and Appl., 3 (1958),
201-205.

[Sch] L. Schwartz, Extension du théorème de Sazonov - Minlos à des cas non hilbertiens, C. r.

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Many quotient algebras of the integers modulo co-analytic ideals

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ABSTRACT. A family of $2^{\aleph_0}$ Borel ideals on a compact lattice are constructed such that no two quotient lattices by ideals from this family are isomorphic.

1. Introduction

In [Far02] Ilijas Farah asked the following question, "Are there infinitely (or even uncountably) many analytic ideals whose quotients are, provably in ZFC, pairwise non-isomorphic?" Even the question for $F_{\sigma\delta}$ ideals is not clear. A partial answer was by provided by Farah and Solecki in [FS03] Where a question of Just and Krawczyk is answered by showing that there are two $F_{\sigma\delta}$ ideals on $\mathbb{N}$ neither of which is $F_{\sigma}$ but whose quotient Boolean algebras are nonisomorphic (and homogeneous). In [Oli03] Mike Oliver has constructed $\aleph_1$ Borel ideals on $\mathbb{N}$ whose quotients are pairwise non-isomorphic$^1$. However, the complexity of these ideals is unbounded in $\omega_1$ in the sense that the $\alpha^{\text{th}}$ ideal has complexity $\Pi^0_\alpha$. This leaves open the following question: Is there a perfect family of $F_{\sigma\delta}$ ideals on $\mathbb{N}$ whose quotients are pairwise non-isomorphic?

This paper will broaden the scope of these investigations by taking the view that Borel ideals on $\mathcal{P}(\omega)$ are special cases of ideals on Polish lattices, whose precise definition is given in the next section. For the moment it suffices to remark that if $H([0,1])$ is the space of compact subsets of $[0,1]$ under the Hausdorff metric then $(H([0,1]), \subseteq)$ is a Polish lattice. It will be shown that there is a perfect family of $F_{\sigma\delta}$ ideals on this lattice whose quotients are pairwise non-isomorphic.

Inspiration for the partial orders used in the proof of Theorem 4.2 comes from the paper [RS] By Rosłanowski and Shelah.

2. Notation and Definitions

DEFINITION 2.1. By a Polish lattice will be meant a pair $(X, \leq)$ where $X$ is a Polish space, $\leq$ is a lattice ordering on $X$ and $\leq$ is a closed subset of $X \times X$.

Natural examples of Polish lattices are easily found:

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$^1$Recently he has improved this result to $2^{\aleph_0}$ pairwise non-isomorphic quotients. This will appear in his Ph.D. thesis (UCLA).
\begin{itemize}
  \item If \(\mathcal{P}(\omega)\) is given the usual Cantor topology then \((\mathcal{P}(\omega), \subseteq)\) is a Polish lattice.
  \item If \(\leq\) is defined coordinatewise then \((^\omega \omega, \leq)\) is a Polish lattice.
  \item If \(X\) is any Polish space and \(H(X)\) is the space of compact subsets of \(X\) under the Hausdorff metric then \((H(X), \subseteq)\) is also a Polish lattice.
\end{itemize}

**Definition 2.2.** If \((X, \leq)\) is a lattice and \(\wedge\) and \(\vee\) are the associated meet and join operators then \(\mathcal{I} \subseteq X\) is an ideal on \(X\) if and only if it is closed under \(\wedge\) and \(a \in \mathcal{I}\) and \(b \leq a\) implies that \(b \in \mathcal{I}\).

If \((X, \leq)\) is a lattice and \(\mathcal{I} \subseteq X\) is an ideal on \(X\) then define \(a \leq_\mathcal{I} b\) if and only if every \(c\) such that \(c \leq a\) and \(c \wedge b = 0\) belongs to \(\mathcal{I}\). Define \(a \sim_\mathcal{I} b\) if and only if \(a \leq_\mathcal{I} b\) and \(b \leq_\mathcal{I} a\). It is easy to check that \(\sim_\mathcal{I}\) is an equivalence relation. Let the equivalence class of \(a\) with respect to this relation be denoted by \([a]_\mathcal{I}\).

It is routine to check that \(\leq_\mathcal{I}\) induces a lattice partial order on \([a]_\mathcal{I} : a \in X\).

**Definition 2.3.** For any tree \(T\) and \(t \in T\) the notation \(T\langle t\rangle\) will be used to denote the tree \(\{s : t \sim s \in T\}\) where \(\sim\) denotes concatenation. If \(T\) is a tree, \(t \in T\) and \(D\) is a subtree of \(T\) define \(t \sim D = \{t \sim s : s \in D\}\). The sequence consisting of a single element \(a\) will be denoted by \(\hat{a}\) — in other words \(\{(0, a)\} = \hat{a}\).

Furthermore, if \(k\) is an integer then \(T[k]\) will denote \(\{t \in T : |t| = k\}\) and \(T[< k]\) will denote \(\{t \in T : |t| < k\}\) and \(T[\leq k]\) will denote \(\{t \in T : |t| \leq k\}\). For any closed subset \(C \subseteq ^\omega \omega\) the notation \(C[< \omega]\) will be used to denote the tree \(\{\sigma \upharpoonright k : \sigma \in C\) and \(k \in \omega\}\) and \(C[k]\) will be used to denote \(C[< \omega][k]\) and \(C[< k]\) will be used to denote \(C[< \omega][< k]\) and \(C[\leq k]\) will be used to denote \(C[< \omega][\leq k]\).

Finally, if \(T\) is a tree then let \(\overline{T}\) denote the closure of the tree \(T\) — in other words, \(f \in \overline{T}\) if and only if \(f \upharpoonright k \in T\) for each integer \(k\). For a closed set \(C \subseteq ^\omega \omega\) the notation \(C\langle t\rangle\) will denote \(C[< \omega]\langle t\rangle\) and, similarly, \(t \sim C\) will denote \(t \sim C[< \omega]\).

### 3. The Submeasures

**Definition 3.1.** Given a sequence of pairs \(S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N}\) define the tree \(T(S)\) to consist of all sequences \(s : k \rightarrow \mathbb{N}\) such that \(s(i) \in n_i\) for each \(i < k\). For an integer \(k\) the notation \(S^k\) will be used to denote the sequence defined by \(S^k = \{(p_{i+k}, n_{i+k})\}_{i \in \omega}\).

Now define a Banach space \(X(S, k)\) and a norm \(\|\cdot\|_{S,k}\) by induction on \(k\). Begin by setting \(X(S, 0) = \mathbb{R}\) and defining \(\|\cdot\|_{S,0}\) to be the absolute value function. Otherwise, set \(X(S, k) = X(S^1, k - 1)^{n_0}\) and let

\[
\|(x_1, x_2, \ldots, x_{n_0})\|_{S,k} = \left(\sum_{i=1}^{n_0} \|x_i\|_{S^1,k-1}^{p_0}\right)^{1/p_0}.
\]

and observe that the elements of \(X(S, k)\) are, essentially, real valued functions on \(T(S)[k]\). In particular, if \(f : T(S) \rightarrow \mathbb{R}\) then \(f \upharpoonright T(S)[k]\) can be associated, by recursion, with a unique element of \(X(S, k)\). To avoid excessive notation, from now on \(f \upharpoonright T(S)[k]\) will be identified with this element. Moreover, the notation \(\|f\|_{S,k}\) will be used instead of \(\|f \upharpoonright T(S)[k]\|_{S,k}\) when \(f : T(S) \rightarrow \mathbb{R}\) since there is no possibility of ambiguity.
Moreover, if \( f : T(S) \to \mathbb{R} \) then for any \( t \in T(S) \) there is a naturally defined function \( f(t) : T(S^{[t]}) \to \mathbb{R} \) obtained by setting \( f(t)(s) = f(t^{-s}) \). Notice that

\[
\|f\|_{S,k} = \left( \sum_{i=1}^{n_0} \left\| f^{(i)} \right\|_{S^1_k, k-1}^{p_0} \right)^{1/p_0}
\]

if \( k > 0 \). This can be easily established by induction on \( k \). The same notation will be used for functions \( f : T(S) \to \mathbb{R} \) — namely, for any \( t \in T(S) \) the function \( f(t) : T(S^{[t]}) \to \mathbb{R} \) is defined by setting \( f(t)(s) = f(t^{-s}) \).

**Lemma 3.2.** If \( S \) is as in Definition 3.1 and \( f : T(S)[k+1] \to \mathbb{R} \) and \( f' : T(S)[k] \to \mathbb{R} \) is defined by

\[
f'(s) = \left( \frac{\sum_{i=1}^{n_{k+1}} f(s^{-i})^{p_{k+1}}}{n_{k+1}} \right)^{1/p_{k+1}}
\]

then \( \|f\|_{S,k+1} = \|f'\|_{S,k} \).

**Proof.** This is a standard recursion argument. \( \square \)

**Lemma 3.3.** If \( x : T(S)[k] \to \mathbb{R}^+ \) and \( y : T(S)[k] \to \mathbb{R}^+ \) and \( x(s) \leq y(s) \) for each \( s \in T(S)[k] \) then \( \|x\|_{S,k} \leq \|y\|_{S,k} \).

**Proof.** Proceed by induction on \( k \). \( \square \)

**Definition 3.4.** If \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) define \( B(S) \) to be the space of all bounded, positive, real-valued functions on \( T(S) \). In order to define a crude approximation to an integral on \( B(S) \), for each \( f \in B(S) \) define \( f^* : T(S) \to \mathbb{R} \) by

\[
f^*(s) = \sup \{ f(\sigma) : \sigma \in T(S) \text{ and } s \subseteq \sigma \}
\]

and define \( \int f \, dS = \lim_{j \to \infty} \|f^*\|_{S,j} \).

Note that by Lemma 3.3 it follows that the sequence \( \{\|f^*\|_{S,j}\}_{j=0}^{\infty} \) is non-increasing and bounded above by the supremum of the image of \( f \) and below by 0. In other words, \( \int f \, dS \) exists for every \( f \in B(S) \). The integral notation here is simply meant to be suggestive. However, it is worth noting that if \( \beta \) is a constant and \( f \) and \( g \) are positive functions on \( T(S) \) then \( \int \beta f \, dS = \beta \int f \, dS \) and, by Lemma 3.3, \( \int (f + g) \, dS \leq \int f \, dS + \int g \, dS \).

**Definition 3.5.** Given \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) as in Definition 3.4 and \( f \in B(S) \) and \( s \in T(S) \) let \( \rho_{S,f}(s) = \int f(\sigma) \, dS^{[s]} \).

**Lemma 3.6.** If \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) is as in Definition 3.4 and \( f \in B(S) \) then

\[
\int f \, dS = \|\rho_{S,f}\|_{S,k}
\]

for each integer \( k \).

**Proof.** The case \( k = 0 \) follows from the equality \( \|\rho_{S,f}\|_{S,k} = |\rho_{S,f}(\emptyset)| = \int f \, dS \). Otherwise, using Identity 3.1

\[
\int f \, dS = \lim_{j \to \infty} \|f^*\|_{S,j} = \lim_{j \to \infty} \left( \sum_{m=1}^{n_0} \|f^{(m)}\|_{S^1_{j-1}}^{p_0} \right)^{1/p_0}
\]
\[
\left( \sum_{m=1}^{n_0} \lim_{j \to \infty} \| f^* (\hat{m}) \|_{S^1,j-1}^{p_0} \right)^{1/p_0} = \left( \sum_{m=1}^{n_0} \left( \int f (\hat{m}) dS^1 \right)^{p_0} \right)^{1/p_0}.
\]

By the induction hypothesis it follows that
\[
\int f dS = \left( \sum_{m=1}^{n_0} \| \rho_{S^1,f} (\hat{m}) \|_{S^1,k-1}^{p_0} \right)^{1/p_0}.
\]

However, observe that \( \rho_{S^1,f} (\hat{m}) = (\rho_{S,f}) (\hat{m}) \) and so, since \( k \geq 1 \), using Identity 3.1 it follows that \( \int f dS = \| \rho_{S,f} \|_{S,k} \) thus establishing the lemma. \( \square \)

An immediate corollary of Lemma 3.5 is worth noting for future reference: If \( t \in T(S) \) then
\[
(3.2) \quad \rho_{S,f}(t) = \| \rho_{S^1,f(t)} \|_{S^1,k}
\]
for any integer \( k \).

**NOTATION 3.7.** If \( C \subseteq \overline{T(S)} \) is closed the notation \( \rho_{S,\chi_C} \) will be replaced by \( \rho_{S,C} \). Also, \( \nu_S(C) \) will be used to denote \( \int \chi_C dS \) and \( \| \chi_C \|_{S,j} \) will be abbreviated to \( \|C\|_{S,j} \).

It is worth noting that Fremlin has shown that \( \nu_S \) can be extended to a submeasure on all Borel subsets of \( \overline{T(S)} \) that has inner and outer regularity properties. While this result will not be needed in the following, the reader is encouraged to see [Fre] for details.

**DEFINITION 3.8.** If \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) let \( H(S) \) be the Polish lattice of all closed \( C \subseteq T(S) \) under the Hausdorff metric and ordered by inclusion. Let \( \mathcal{I}(S) \) be the ideal on \( H(S) \) consisting of all \( C \in H(S) \) such that \( \nu_S(C) = 0 \). Let \( \mathbb{P}(S) = H(S) \setminus \mathcal{I}(S) \) ordered by inclusion.

Observe that if \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) then \( \mathcal{I}(S) \) is a \( \Pi_3^0 \) ideal or, in other words, an \( F_{\sigma \delta} \) ideal. To see this note that \( C \in \mathcal{I}(S) \) if and only if
\[
(\forall n)(\exists \delta)(\exists Y \subseteq T(S)[k])(\forall x \in C[< \omega])(\forall y \in T(S)[k] \setminus Y)(x \not\equiv y \text{ and } \|\chi_Y\|_{S,k} < 1/n)\]

**LEMMA 3.9.** Let \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) be as in Definition 3.4 and \( C \in \mathbb{P}(S) \) and \( \epsilon > 0 \) then there is \( s \in T(S) \) such that \( \rho_{S,C}(s) > 1 - \epsilon \).

**PROOF.** Suppose that \( S, C \) and \( \epsilon \) provide a counterexample. Using Definition 3.4 and the fact that \( \nu_S(C) > 0 \) choose \( j \) such that \( \|C\|_{S,j} < \nu_S(C)(1 + \epsilon) \). Observe that \( \rho_{S,C}(s) \leq (1 - \epsilon)\chi_C(s) \) for each \( s \in T(S)[j] \). It therefore follows from Lemma 3.3, Lemma 3.6 and the fact that \( \|\|_{S,j} \) is a norm that
\[
\nu_S(C) = \|\rho_{S,C}\|_{S,j} \leq \|(1 - \epsilon)\chi_C\|_{S,j} \leq (1 - \epsilon)\|C\|_{S,j} < (1 - \epsilon)(1 + \epsilon)\nu_S(C)
\]

implying that \( \nu_S(C) < (1 - \epsilon^2)\nu_S(C) \) which is impossible. \( \square \)

**LEMMA 3.10.** If \( S = \{(p_i, n_i)\}_{i \in \omega} \subseteq [1, \infty) \times \mathbb{N} \) is as in Definition 3.4 and \( C \in \mathbb{P}(S) \) and \( D \in \mathcal{I}(S) \) and \( \epsilon > 0 \) then there is \( E \subseteq C \) such that \( \nu_S(E) > \nu_S(C) - \epsilon \) and \( E \cap D = \emptyset \).

**PROOF.** Choose \( k \) such that \( \|D\|_{S,k} < \epsilon \). Let \( E \) be the set of all \( s \in C \) such that there is no \( t \in D[k] \) such that \( t \subseteq s \) and observe that \( E \) is closed and disjoint from \( D \). By noting that
\[
\rho_{S,C}(s) = \rho_{S,E}(s) + \rho_{S,C\setminus E}(s)
\]
for each \( s \in T(S)[m] \), it is seen by using Lemma 3.6 that
\[
\nu_S(E) = \|\rho_S,E||s,k \geq \|\rho_S,C||s,k - \|\rho_S,C\setminus E||s,k = \nu_S(C) - \|\rho_S,C\setminus E||s,k.
\]
However, \( \chi_D^s(s) \geq \rho_S,C\setminus E(s) \) for all \( s \in T(S)[m] \) and so \( \|\rho_S,C\setminus E||s,k \leq \|D||k < \epsilon \) and the result follows. \( \square \)

**Lemma 3.11.** Let \( C \in \mathbb{P}(S) \) and suppose that \( A \subseteq C[<\omega] \) is an antichain cover of \( C \) — in other words, a finite (by compactness) antichain such that for all \( \sigma \in C \) there is \( a \in A \) such that \( a \subseteq \sigma \). Suppose further that for each \( a \in A \) there is \( T_a \subseteq T(S)(a) \) such that \( \nu_S|a(T_a) > \theta \). Define \( \mu(A) \) to be the maximum of \( \{|a|\}_{a \in A} \). If
\[
D = \bigcup_{a \in A} a \sim T_a
\]
then \( \nu_S(D) \geq \theta\|C||s,\mu(A) \).

**Proof.** Proceed by induction on \( \mu(A) \). If \( \mu(A) = 0 \) then \( D = T_\theta \) and the assertion follows from the hypothesis and the fact that \( \|C||s,j \leq 1 \) for \( j \) and any tree \( C \subseteq T(S) \). Otherwise, it follows that \( \emptyset \notin A \) since \( A \) is an antichain. Using the induction hypothesis one can conclude that \( \nu_S(D\langle \hat{m} \rangle) \geq \theta\|C\langle \hat{m} \rangle||s,1,\mu(A) - 1 \) for each \( m \in n_0 \). To see that \( \nu_S(D) \geq \theta\|C||s,\mu(A) \) observe that
\[
\nu_S(D) = \|\rho_S,D||s,1 = \left( \frac{\sum_{m=1}^{n_0} \|\rho_S,D\langle \hat{m} \rangle||p_0}{n_0} \right)^{1/p_0} = \left( \frac{\sum_{m=1}^{n_0} \|\rho_S,D\langle \hat{m} \rangle||p_0}{n_0} \right)^{1/p_0}.
\]
by Lemma 3.6 and Identity 3.1. Furthermore \( \|\rho_S,D\langle \hat{m} \rangle||p_0 = \nu_S(D\langle m \rangle) \). Therefore
\[
\nu_S(D) \geq \theta \left( \frac{\sum_{m=1}^{n_0} \|C\langle \hat{m} \rangle||p_0}{n_0} \right)^{1/p_0} = \theta\|C||s,\mu(A).
\]
by Identity 3.1. \( \square \)

**Lemma 3.12.** If \( D \subseteq \mathbb{P}(S) \) is a dense subset, \( C \subseteq \mathbb{P}(S) \) and \( \epsilon > 0 \) then there is \( D \subseteq C \) and \( A \subseteq D[<\omega] \) such that
- \( \nu_S(D) \geq \nu_S(C)(1 - \epsilon) \)
- \( A \) is an antichain cover of \( D \)
- \( a \sim D(a) \in D \) for each \( a \in A \).

**Proof.** Let \( \delta = 1 - \sqrt{1 - \epsilon} \). Let \( D' \) be the set of all \( s \in C[<\omega] \) such that there is some \( D_s \subseteq C(s) \) such that \( s \sim D_s \in D \) and \( \nu_S|a(D_s) \geq 1 - \delta \). If \( E = \{ t \in C : (\forall d \in D')d \not\subset t \} \) then \( E \) is a closed subset of \( C \).

Moreover, \( \nu_S(E) = 0 \) because otherwise there is \( E' \subseteq E \) such that \( E' \in D \) and, by Lemma 3.9, there is \( s \in T(S) \) such that \( \nu_S|a(E'\langle s \rangle) = \rho_S,E'(s) \geq 1 - \delta \). It follows \( E'(s) \) witnesses that \( s \in D' \) and this contradicts the definition of \( E \) and that \( E \langle s \rangle \neq \emptyset \).

Now choose an integer \( j \) such that \( \|E\|s,j < \delta\nu_S(C) \). Observe that if \( s \in C[j] \setminus E[j] \) then for every \( t \in T(S) \) such that \( s \subseteq t \) there is some \( d \in D' \) such that \( d \subseteq t \). Therefore, there is a partition of \( C[j] \setminus E[j] \) into \( W \) and \( Y \) such that for each \( s \in W \) there is \( d_s \subseteq s \) such that \( d_s \in D' \) and for each \( s \in Y \) there is no such \( d_s \) and so, by compactness, there is a finite antichain cover \( A_s \) of \( C(s) \) such that \( s \sim a \in D' \) for each \( a \in A_s \). Let
\[
A = \{ d_s \}_{s \in W} \cup \bigcup_{s \in Y} s \sim A_s
\]
and let $D(a)$ witness that $d \in D'$ for each $a \in A$. Let $D = \bigcup_{a \in A} a \sim D(a)$. It remains to show that $\nu_S(D) \geq \nu_S(C)(1 - \epsilon)$.

To see this, let $C^* = \{ c \in C : c \upharpoonright j \notin E[j] \}$. From Lemma 3.11 it follows that

$$\nu_S(D) \geq (1 - \delta) \|C^*\|_{S, \mu(A)} \geq (1 - \delta) \nu_S(C^*) = (1 - \delta) \|\rho_{S,C^*}\|_{S,j}$$

the last equality following from Lemma 3.6. However, $\rho_{S,C^*} \upharpoonright T(S)[j] = \rho_{S,C} \upharpoonright T(S)[j] \setminus \rho_{S,C} \upharpoonright E[j]$ and so

$$\nu_S(D) \geq (1 - \delta) (\|\rho_{S,C} \|_{S,j} - \|\rho_{S,C} \upharpoonright E[j] \|_{S,j}) \geq (1 - \delta) (\nu_S(C) - \|E[j]\|_{S,j})$$

$$= (1 - \delta) (\nu_S(C) - \|E\|_{S,j}) \geq (1 - \delta) (\nu_S(C) - \nu_S(C)) \geq (1 - \epsilon) \nu_S(C)$$

as required. □

**Corollary 3.13.** If $D \subseteq P(S)$ is a dense subset, $C \in P(S)$, $k \in \mathbb{N}$ and $\epsilon > 0$ then there is $D \subseteq C$ and $A \subseteq D[< \omega]$ such that

- $\nu_S(D) \geq \nu_S(C)(1 - \epsilon)$
- $A$ is an antichain cover of $D$
- $D[k] = \{ s \in C[k] : \nu_S(C(s)) \neq 0 \}$
- $a \sim D(a) \in D$ for each $a \in A$.

**Proof.** Use Lemma 3.12 for each $s \in C[k]$ such that $\nu_S(C(s)) \neq 0$. □

While it is not essential for the argument, it may be helpful to some readers to think of the following lemma as establishing the $\omega^\omega$-bounding property for the partial order $P(S)$.

**Lemma 3.14.** If $C \models_{P(S)} \bar{x} : \omega \rightarrow \omega$ then there is $D \subseteq C$ such that for each $n \in \omega$ there is $J(n)$ such that for each $t \in D[J(n)]$ there is some $m \in \omega$ such that $t \sim D(t) \models_{P(S)} \bar{x}(n) = m$.

**Proof.** Use Lemma 3.12 for each $n$. In particular, inductively choose $D_n$ and $J(n)$ such that

$$D_{n+1} \subseteq D_n$$

$$J(n + 1) > J(n)$$

$$\forall t \in D_n[J(n)](\exists m)t \sim D_n(t) \models_{P(S)} \bar{x}(n) = m$$

$$D_{n+1}[J(n)] = D_n[J(n)]$$

$$\nu_S(D_n) > \nu_S(C) \prod_{i=2}^{n+2} (1 - 2^{-i})$$

Once this is done, simply let $D = \bigcap_{n \in \omega} D_n$. Note that Condition 3.6 guarantees that $D[J(n)] = D_n[J(n)]$ and, therefore, $\|D\|_{S,J(n)} = \|D_n\|_{S,J(n)}$ and so Condition 3.7 guarantees that $\|D\|_{S,J(n)} \geq \nu_S(C)/2$. Hence $D \in P(S)$. The conclusion of the Lemma follows from Condition 3.5.

To carry out the induction, suppose that $D_n$ and $J(n)$ have been chosen satisfying the induction hypotheses. Use Lemma 3.9 to find a maximal antichain $A$ above $J(n)$ and $D_{n+1}$ such that Conditions 3.3 and 3.6 are satisfied, for all $t \in A$ there is some $m$ such that $t \sim D_{n+1}(t) \models_{P(S)} \bar{x}(n) = m$ and such that...
\[ \nu_S(D_{n+1}) > \nu_S(D_n)(1 - 2^{n+3}). \] The last clause guarantees that Condition 3.7 is also satisfied. Finally, let \( J(n + 1) \) be greater than the maximal length of an element of \( A \). This guarantees that Condition 3.5 is satisfied. \( \square \)

4. Many Different Quotients

Now let \( n_0 = 2 \) and \( p_0 = 1 \) and then inductively choose \( p_i+1 > p_i \) such that

\[ \left( \frac{1}{\prod_{j=0}^{i} n_j} \right)^{1/p_i+1} \geq 1 - 2^{-(i+2)} \]

and then choose an integer \( n_{i+1} > n_i \) such that letting \( \xi \) be the constant function with value

\[ \left( \frac{\prod_{j=0}^{i} n_j}{n_{i+1}} \right)^{\frac{1}{p}+1} \]

the inequality

\[ \|\xi\|_{S,i} < 1/i \]

holds where \( S \) is the sequence \( \{(n_j, p_j)\}_{j=0}^{\infty} \).

Given a strictly increasing function \( F : \mathbb{N} \to \mathbb{N} \) let \( \tilde{F} \) be a name for the generic function satisfying \( \{r_G\} = \bigcap G \). Note that \( r_G \in \overline{T(\tilde{F})} \). It follows from Lemma 3.10 that

\[ \text{if } \nu_{\tilde{F}}(X) = 0 \text{ then } 1 \Vdash_{\mathbb{P}(\tilde{F})} "r_G \notin \check{X}" \]

where \( X \subseteq \overline{T(\tilde{F})} \) is any closed set from the ground model.

**Definition 4.1.** If \( F_1 \) and \( F_2 \) are two strictly increasing functions such that

\[ F_1(i) < F_2(i) < F_1(i+1) \]

for all but finitely many integers \( i \) this will be denoted by \( F_1 \prec F_2 \). Note that this is not transitive.

The following theorem, together with Fact 4.3, shows that if \( F_1 \prec F_2 \) then \( \mathbb{P}(\tilde{F}_1) \) and \( \mathbb{P}(\tilde{F}_2) \) are not isomorphic and, hence, neither are the quotient lattices \( H(\tilde{F}_1)/T(\tilde{F}_1) \) and \( H(\tilde{F}_2)/T(\tilde{F}_2) \). In order to see this, it suffices to observe the stronger assertion that \( \mathbb{P}(\tilde{F}_2) \) does no completely embed into \( \mathbb{P}(\tilde{F}_1) \). The reason that this is so is that, if it were the case that \( \mathbb{P}(\tilde{F}_2) \) did completely embed into \( \mathbb{P}(\tilde{F}_1) \) then, letting \( r_{G_2} \) be a name for the generic real added by \( \mathbb{P}(\tilde{F}_2) \), it would follow that \( r_{G_2} \) is also a \( \mathbb{P}(\tilde{F}_1) \)-name. Theorem 4.2 asserts that there is a set \( X \) such that \( \nu_{\tilde{F}_2}(X) = 0 \) and some condition in \( \mathbb{P}(\tilde{F}_1) \) which forces \( r_{G_2} \) to belong to \( X \) while Fact 4.3 asserts that this can not be so.

**Theorem 4.2.** Suppose that \( F_1 \prec F_2 \) and \( x \) is a \( \mathbb{P}(\tilde{F}_1) \)-name for an element of \( T(\tilde{F}_2) \). Then there is a closed set \( X \) such that \( \nu_{\tilde{F}_2}(X) = 0 \) and \( 1 \Vdash_{\mathbb{P}(\tilde{F}_1)} "x \notin \check{X}" \).

**Proof.** In order to see this, use Lemma 3.14 to find \( C \in \mathbb{P}(\tilde{F}_1) \) such that for each integer \( i \) there is some integer \( J(i) \) such that if \( t \in C[J(i)] \) then \( t^{-C(t)} \Vdash_{\mathbb{P}(\tilde{F}_1)} "x(i) = \check{x}_t" \) for some integer \( x_t \in n_{\tilde{F}_2(i)} \). Now choose \( K(0) \geq J(0) \) so large that \( F_1(i) < F_2(i) < F_1(i+1) \) for each \( i \geq K(0) \) and let \( C^0 = C \). Then, inductively construct integers \( K(m) \) and conditions \( C^m \in \mathbb{P}(\tilde{F}_1) \) satisfying the following conditions:

- \( \nu_{\tilde{F}_2}(X) = 0 \)
- \( 1 \Vdash_{\mathbb{P}(\tilde{F}_1)} "x \notin \check{X}" \)
- \( C^m \in \mathbb{P}(\tilde{F}_1) \) for each \( m \)
(1) \(\max(K(m) + 1, J(K(m))) = K(m + 1)\)
(2) if \(m \leq \bar{m}\) then \(C^m[K(m)] = C^m[K(m)]\)
(3) if \(t \in C^m[K(m)]\) then \(C^m(t) = C(t)\)
(4) if \(t \in C^m[K(m)]\) then there is some \(z_t \in T(\tilde{F}_2)[K(m)]\) such that
\[ t \sim C^{m+1}(t) \parallel_{P(\tilde{F}_1)} \text{"} z_t \subseteq x " \]
(5) for each \(t \in C^m[K(m - 1)]\)
\[ \rho_{\tilde{F}_1,C^m}(t) \geq \rho_{\tilde{F}_1,C}(t) \prod_{i=K(m-1)}^{K(m)+1} \left( 1 - 2^{-\left(F_2(i)+2\right)} \right) \]
with the convention that \(K(-1) = 0\).

Given that this can be accomplished, let \(C^* = \bigcap_{m \in \omega} C^m\). It will first be shown that:
\[ \nu_{\tilde{F}_1}(C^m) \geq \prod_{i=0}^{K(m)+1} \left( 1 - 2^{-\left(F_2(i)+2\right)} \right) \nu_{\tilde{F}_1}(C) \]  

(4.4)

To establish this proceed by induction on \(m\). The case \(m = 0\) follows from Condition 5, the fact that \(C^0 = C\) and that \(\nu_{\tilde{F}_1}(C) = \rho_{\tilde{F}_1,C}(\emptyset)\). If \(m > 0\) then it follows from Lemma 3.6 and Condition 5 that
\[ \nu_{\tilde{F}_1}(C^m) = \|\rho_{\tilde{F}_1,C^m}\|_{\tilde{F}_1,K(m-1)} \geq \|\rho\|_{\prod_{i=K(m-1)}^{K(m)+1} \left( 1 - 2^{-\left(F_2(i)+2\right)} \right)} \|\tilde{F}_1,K(m-1)\| \]

(4.5)

where the function \(\rho : T(\tilde{F}_1)[K(m - 1)] \to \mathbb{R}\) is defined by
\[ \rho(s) = \begin{cases} \rho_{\tilde{F}_1,C}(s) & \text{if } s \in C^{m-1}[K(m - 1)] = C^m[K(m - 1)] \\ 0 & \text{otherwise} \end{cases} \]

From the fact that \(C^{m-1} \subseteq C\) it follows that the expression 4.5 is at least as great as
\[ \prod_{i=K(m-1)}^{K(m)+1} \left( 1 - 2^{-\left(F_2(i)+2\right)} \right) \|\rho_{\tilde{F}_1,C^{m-1}}\|_{\tilde{F}_1,K(m-1)} = \]
\[ \prod_{i=K(m-1)}^{K(m)+1} \left( 1 - 2^{-\left(F_2(i)+2\right)} \right) \nu_{\tilde{F}_1}(C^{m-1}) \]

by Lemma 3.6. From the induction hypothesis it follows that the last expression is greater than or equal to
\[ \prod_{i=0}^{K(m)} \left( 1 - 2^{-\left(F_2(i)+2\right)} \right) \nu_{\tilde{F}_1}(C) \]

thus establishing Claim 4.4.

From Condition 2 it follows that \(C^*[K(m)] = C^m[K(m)]\) for each \(m\). Hence
\[ \nu_{\tilde{F}_1}(C^*) = \lim_{j \to \infty} \|C^*[\tilde{F}_1,j]\| = \lim_{m \to \infty} \|C^m\|_{\tilde{F}_1,K(m)} \]
\[ \lim_{m \to \infty} \nu_{\hat{F}_1}(C^m) \geq \lim_{m \to \infty} \prod_{i=0}^{K(m)-1} \left(1 - 2^{-\left(F_2(i)+2\right)}\right) \nu_{\hat{F}_1}(C) \]

by Claim 4.4. So \( \nu_{\hat{F}_1}(C^*) \geq \nu_{\hat{F}_1}(C)/2 > 0 \).

Furthermore, letting \( X \) be the closure of the tree generated by \( \{z_t : t \in C^*\} \) it follows that

\[ C^* \models_{\hat{F}_1} \text{“} x \in \hat{X} \text{”} \]

so it suffices to show that \( \nu_{\hat{F}_2}(X) = 0 \). In order to see this let \( X' : T(\hat{F}_2)[K(m) - 1] \to \mathbb{R} \) be defined by

\[ X'(s) = \left( \frac{|\{ i \in \hat{F}_2 : (\exists x \in X) s \models i \subseteq x \}|}{n_{K(m)}} \right)^{1/p_{K(m)}} \]

as in Lemma 3.2. Note that Lemma 3.2 implies that \( \|X\|_{\hat{F}_2,K(m)} = \|X'\|_{\hat{F}_2,K(m)-1} \).

Since

\[ X'(s) \leq \left( \frac{|C^m[K(m)]|}{n_{\hat{F}_2}[K(m)]} \right)^{1/p_{\hat{F}_2}(K(m))} \leq \left( \frac{|T(\hat{F}_1)[K(m)]|}{n_{\hat{F}_2}[K(m)]} \right)^{1/p_{\hat{F}_2}(K(m))} \]

\[ \leq \left( \frac{\prod_{i=0}^{K(m)} n_{\hat{F}_1}(i)}{n_{\hat{F}_2}(K(m))} \right)^{1/p_{\hat{F}_2}(K(m))} \leq \left( \frac{\prod_{i=0}^{K(m)-1} n_i}{n_{\hat{F}_2}(K(m))} \right)^{1/p_{\hat{F}_2}(K(m))} \]

the last inequality being a consequence of the fact that \( F_2(K(m)) > F_1(K(m)) \). It now follows from Hypothesis 4.2 that \( \|X\|_{\hat{F}_2,K(m)} \leq 1/F_2(K(m)) \leq 1/m \). Therefore \( \nu_{\hat{F}_2}(X) = 0 \) as required.

In order to carry out the induction, suppose that \( C^m \) and \( K(m) \) have been chosen and note that the choice of \( C^0 \) and \( K(0) \) satisfies all the induction hypotheses. Let \( K(m+1) = \max(K(m) + 1, J(K(m))) \) in order to satisfy Condition 1. Now proceed by induction on \( u \leq K(m+1) - K(m) \) to define \( C^{m,u} \subseteq C^m \) such that \( C^{m,u+1} \subseteq C^{m,u} \) and

(a) \( C^{m,u}[K(m+1) - u] = C^m[K(m+1) - u] \) (hence \( C^{m,0} = C^m \) and, by Condition 3, \( C^{m,0}(t) = C(t) \) for \( t \in C^{m,0}[K(m+1)] \))

(b) for each \( t \in C^{m,u}[K(m+1) - u] \)

(i) there is some \( z_t \in T(\hat{F}_2)[K(m)] \) such that \( t \models C^{m,u}(t) \models_{\hat{F}_1} \text{“} x \geq \hat{z}_t \text{”} \)

(ii) \( \rho_{\hat{F}_1,C^{m,u}}(t) \geq \rho_{\hat{F}_1,C}(t) \prod_{i=K(m) - u}^{K(m+1) - u - 1} (1 - 2^{-(F_2(i) + 2)}) \) (In the case \( u = 0 \) assume, as usual, that the empty product is equal to 1.)

(c) if \( t \in C^{m,u+1}[K(m+1) - u] \) then \( C^{m,u+1}(t) = C^{m,u}(t) \).

If this can be done then letting \( C^{m+1} = C^{m,K(m+1) - K(m)} \) will satisfy the induction hypotheses. Condition 1 has already been satisfied. Condition 2 follows immediately from (a). Condition 3 is a consequence of (c) and the remark at the end of (a) since it is easily seen that \( C^{m,i}(t) = C(t) \) for \( t \in C^{m,i}[K(m+1)] \) while Condition 4 follows from (b(ii)) for the case \( u = K(m+1) - K(m) \). In order to see that Condition 5 holds use the same case with (b(ii)).

In order to construct \( C^{m,u+1} \) given \( C^{m,u} \), Hypothesis 4.1 will be used. In particular, let \( t \in C^{m}[K(m+1) - (u+1)] = C^{m,u}[K(m+1) - (u+1)] \). Then
for each \( w \in n_{\tilde{F}_1(K(m+1) - u)} \) such that \( t \sim \tilde{w} \in C^{m,u}[K(m+1) - u] \) there is \( z_t \sim \tilde{w} \in T(\tilde{F}_2)[K(m)] \) such that
\[
t \sim \tilde{w} \sim C^{m,u}(t \sim \tilde{w}) \|_{\tilde{F}_1} \text{ "} \tilde{z}_t \sim \tilde{w} \subseteq x".
\]
For each \( z \in T(\tilde{F}_2)[K(m)] \) let \( U_z(t) = \{ w : z_t \sim \tilde{w} = z \} \). Then
\[
\rho_{\tilde{F}_1, C^{m,u}}(t) = \nu_{\tilde{F}_1}(C^{m,u}(t)) = \| \rho_{\tilde{F}_1}[t, C^{m,u}(t)] \|_{\tilde{F}_1[t, 1]} = \frac{\left( \sum_{j \in n_{\tilde{F}_1(K(m+1) - u)}} \rho_{\tilde{F}_1, C^{m,u}}(t_j) \rho_{\tilde{F}_1}(K(m+1) - u) \right)^{\frac{1}{PF_1(K(m+1) - u)}}}{n_{\tilde{F}_1}(K(m+1) - u)}
\]
by Lemma 3.6. Since \( \{ U_z(t) \} z \in T(\tilde{F}_2)[K(m)] \) forms a partition of \( n_{\tilde{F}_1(K(m+1) - u)} \) it follows that
\[
\rho_{\tilde{F}_1, C^{m,u}}(t) = \left( \frac{\sum_{z \in T(\tilde{F}_2)[K(m)]} \sum_{j \in U_z(t)} \rho_{\tilde{F}_1, C^{m,u}}(t \sim \tilde{j}) \rho_{\tilde{F}_1}(K(m+1) - u) \right)^{\frac{1}{PF_1(K(m+1) - u)}}
\]
and, therefore,
\[
\left( \frac{\sum_{j \in U_z(t)} \rho_{\tilde{F}_1, C^{m,u}}(t \sim \tilde{j}) \rho_{\tilde{F}_1}(K(m+1) - u) \right)^{\frac{1}{PF_1(K(m+1) - u)}} \geq \frac{\rho_{\tilde{F}_1, C^{m,u}}(t)}{\left| T(\tilde{F}_2)[K(m)] \right|^{1/\|PF_1(K(m+1) - u)}}
\]
for some \( z(t) \in T(\tilde{F}_2)[K(m)] \). Since \( \left| T(\tilde{F}_2)[K(m)] \right| \leq \prod_{j=0}^{F_2(K(m+1) - (u+1))} n_j \) it follows that
\[
\left( \sum_{j \in U_z(t)} \rho_{\tilde{F}_1, C^{m,u}}(t \sim \tilde{j}) \rho_{\tilde{F}_1}(K(m+1) - u) \right)^{\frac{1}{PF_1(K(m+1) - u)}} \geq \frac{\rho_{\tilde{F}_1, C^{m,u}}(t)}{\left( \prod_{j=0}^{F_2(K(m+1) - (u+1))} n_j \right)^{1/\|PF_1(K(m+1) - u)}}
\]
and the right hand side of the inequality is greater than or equal to
\[
\rho_{\tilde{F}_1, C^{m,u}}(t) \left( 1 - 2^{-(F_2(K(m+1) - (u+1)) + 2)} \right)
\]
by Hypothesis 4.1. Notice that it is indeed possible to apply Hypothesis 4.1 in this situation because \( F_2(K(m+1) - (u+1)) < F_1(K(m+1) - u) \) by the choice of \( K(0) \).

Let \( U(t) = U_z(t) \) and let \( C^{m,u+1} \) be
\[\{ s \in C^{m,u} : s(K(m+1) - u) \in U(s \uparrow (K(m+1) - (u+1))) \}.\]

Observe that if \( t \in C^{m,u+1}[K(m+1) - (u+1)] \) then \( \rho_{\tilde{F}_1, C^{m,u+1}}(t) \) is equal to the left hand side of inequality 4.6. Combining this with expression 4.7 and the induction hypothesis yields Condition (b(ii)). Condition (a) follows from the fact that \( U(t) \) is defined for each \( t \in C^{m}[K(m+1) - (u+1)] \) since, by the induction hypothesis, \( C^{m,u}[K(m+1) - (u+1)] = C^{m}[K(m+1) - (u+1)] \). Conditions (b(i)) and (c) are immediate consequences of the construction.
In order to obtain \(2^{\aleph_0}\) pairwise non-isomorphic quotients of the integers by \(\Pi_3^0\) ideals it suffices to produce a perfect family of strictly increasing functions \(\{F_\theta\}_{\theta \in 2^\omega}\) such that if \(\theta \neq \zeta\) then \(F_\theta \preceq F_\zeta\) or \(F_\zeta \preceq F_\theta\). For \(U \subseteq \mathbb{N}\) define
\[
F_U(m) = \frac{m^2 + m}{2} + |U \cap (m + 1)|
\]
and note that if \(U \not
subseteq V\) then \(F_U \preceq F_V\). Hence if \(\mathcal{W}\) is any family of subsets of \(\mathbb{N}\) of size \(2^{\aleph_0}\) which is linearly ordered by strict inclusion then the family \(\{F_W : W \in \mathcal{W}\}\) has the desired properties.

References


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What does the automorphism group of a free abelian group \( A \) know about \( A \)?

Vladimir Tolstykh

ABSTRACT. Let \( A \) be an infinitely generated free abelian group. We prove that the automorphism group \( \text{Aut}(A) \) first-order interprets the full second-order theory of the set \(|A|\) with no structure. In particular, this implies that the automorphism groups of two infinitely generated free abelian groups \( A_1, A_2 \) are elementarily equivalent if and only if the sets \(|A_1|, |A_2|\) are second-order equivalent.

Introduction

In his paper [6] of 1976 Shelah proved that the elementary theories of the endomorphism semi-groups of free algebras of ‘large’ infinite ranks had very strong expressive power. More precisely, let \( V \) be an arbitrary variety of algebras and \( F_\kappa(V) \) be a free algebra from \( V \) with \( \kappa \geq \aleph_0 \) free generators. Then the endomorphism semi-group \( \text{End}(F_\kappa(V)) \) first-order interprets the full-second theory \( \text{Th}_2(\kappa) \) of the cardinal \( \kappa \) (viewed as a set with no structure), provided that \( \kappa \) is greater than the cardinality of the language of \( V \).

That remarkable result naturally leads to the following problem: what are the varieties of algebras for which the automorphism groups of free algebras are logically strong in a similar sense? Shelah himself formulated this problem in the cited paper [6] and then after more than 20 years mentioned it again in his survey [7]: Problem 3.14 from [7] suggested to classify the varieties of algebras \( V \) such that the automorphism groups \( \text{Aut}(F_\kappa(V)) \) first-order interpret the theory \( \text{Th}_2(\kappa) \) for all (or all sufficiently large) infinite cardinals \( \kappa \).

The results on symmetric groups obtained by Shelah before the publication of the paper [6] implied that, for instance, the variety of all sets with no structure and the variety of all semi-groups were the examples of, say, ‘negative’ kind. Indeed, according to [5], the symmetric group of an infinite cardinal \( \kappa \), in other words, the automorphism group of the set \( \kappa \) with no structure, first-order interprets the theory \( \text{Th}_2(\kappa) \) only if the cardinal \( \kappa \) is ‘small’ (namely, at most \( 2^{\aleph_0} \)).

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The author found in [8]—as a byproduct of his study of the elementary types of infinite-dimensional classical groups—that for any variety of vector spaces the automorphism groups of free algebras are as logically strong as the endomorphism semi-groups. A bit informally, one of the results from [8] can be quoted in the following form: if $\kappa$ is an infinite cardinal, then the general linear group $GL(\kappa, D)$ over a division ring $D$ first-order interprets $Th_2(\kappa)$, provided that $\kappa > |D|$. Thus varieties of vector spaces give examples of ‘positive’ kind as to Shelah’s problem.

In the papers [9] and [10] the author studied Shelah’s problem for classical group varieties. It turned out that the variety of all groups and any variety $\mathfrak{N}_c$ of nilpotent groups of class $c \geq 2$ meet the requirements of Shelah’s problem: if $F$ is an infinitely generated free or free nilpotent group, then the group $Aut(F)$ first-order interprets the theory $Th_2(|F|) (= Th_2(\text{rank } F))$. In the present paper we examine the case of the variety of all abelian groups. The main result of the paper states that the variety in question also meets requirements of Shelah’s problem.

Let $A$ denote an infinitely generated free abelian group; clearly, $A$ can be considered as a free $\mathbb{Z}$-module. One of the standard approaches to understanding of the nature of the automorphism groups of modules is an investigation of possibility of generalization for these groups of the methods developed for general linear groups, the automorphism groups of vector spaces. In the first section of the paper we, like in [8], work to reconstruct by means of first-order logic in $Aut(A)$ some geometry of the $\mathbb{Z}$-module $A$. Namely, we interpret in $Aut(A)$ the family $\mathcal{D}^1(A)$ consisting of all direct summands of $A$ having rank or corank one. To make comparison, the first-order interpretation in the general linear group $GL(V)$ of an infinite-dimensional vector space $V$ of the family of all lines and hyperplanes of $V$ done in [8] is much longer. However, both interpretations have much in common and both originated from the well-known works on classical groups.

In principle, the reconstruction of $\mathcal{D}^1(A)$ can be extended to the reconstruction in $Aut(A)$ of the family $\mathcal{D}(A)$ of all direct summands of $A$ followed by the first-order interpretation in the structure $\langle Aut(A), \mathcal{D}(A) \rangle$ of the endomorphism semi-group $End(A)$ of $A$ (similarly to [8]). We, however, prefer a shorter way, making in Section 2 an effort to reconstruct in $Aut(A)$ the general linear group of some vector space of dimension $|A|$. Namely, using the action of $Aut(A)$ on $\mathcal{D}^1(A)$ we prove $\mathcal{D}$-definability in $Aut(A)$ of the principal congruence subgroup $\Gamma_2(A)$ of level two. The quotient subgroup $Aut(A)/\Gamma_2(A)$ is isomorphic to the general linear group of the vector space $A/2A$ over the field $\mathbb{Z}_2$. Thus the group $Aut(A)$ first-order interprets the group $GL(|A|, \mathbb{Z}_2)$. The latter group, as it has been said above, first-order interprets the theory $Th_2(|A|)$. As a consequence, we have that the automorphism groups $Aut(A_1)$ and $Aut(A_2)$, where $A_1, A_2$ are infinitely generated free abelian groups, are elementarily equivalent if and only the cardinals $|A_1|$ and $|A_2|$ are second-order equivalent as sets.

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1. Definable geometric properties of automorphisms

Let $A$ denote a free abelian group of infinite rank. As it has been said in the Introduction, our aim in this section is a first-order reconstruction in $Aut(A)$ of
the family of direct summands of $A$ of rank or corank one (we say that a direct summand $B$ of $A$ has corank $m$, if any direct complement of $B$ to $A$ is of rank $m$.)

We shall essentially exploit the structure of involutions (the elements of the order two) in the group $\text{Aut}(A)$ given by the following theorem.

**Theorem 1.1.** Let $G$ be a free abelian group. Every involution $\varphi \in \text{Aut}(G)$ has a basis $\mathcal{B}$ of $G$ such that for any $b \in \mathcal{B}$ either $\varphi b = \pm b$, or $\varphi b \in \mathcal{B}$.

The theorem was first established for the groups of finite rank by Hua and Reiner [3, Lemma 1]; in general, the result is proven in [11]. Let us call a basis of $A$ on which $\varphi$ acts in a way described in the Theorem a canonical basis for $\varphi$.

Let $2A$ denote the group of even elements of $A$:

$$2A = \{2a : a \in A\}.$$  

The natural homomorphism $A \to A/2A$ induces the homomorphism of the automorphism groups $\text{Aut}(A) \to \text{Aut}(A/2A)$ which we will denote by $\sim$. The fact that the group $A/2A$ can be viewed as a vector space over $\mathbb{Z}_2$ will be extensively used in this paper.

**Remark 1.2.** Take an involution $\varphi \in \text{Aut}(A)$ and some its canonical basis $\mathcal{B}$. The (cardinal) number $p(\mathcal{B})$ of unordered pairs $\{b, \varphi b\}$, where $b \in \mathcal{B}$ and $\varphi b \neq \pm b$ is an invariant of $\varphi$. Indeed, $p(\mathcal{B})$ equals the residue of the induced linear transformation $\varphi$ of the vector space $A/2A$ over $\mathbb{Z}_2$:

$$p(\mathcal{B}) = \text{res}(\varphi) = \dim \text{Res}(\varphi).$$

(here $\text{Res}(\varphi)$ is the image of the linear transformation $1 - \varphi$, see [4]). This implies that if $(\varphi_1, B_1), (\varphi_2, B_2)$ are pairs similar to the pair $(\varphi, B)$ and $\varphi_1, \varphi_2$ are conjugate in $\text{Aut}(A)$, then $p(B_1) = p(B_2)$.

Let $\varphi$ be an involution in $\text{Aut}(A)$; we let $A^+_{\varphi}$ and $A^-_{\varphi}$ denote the subgroups

$$\{a : \varphi a = a\} \text{ and } \{a : \varphi a = -a\}$$

respectively; clearly, $\varphi$ is diagonalizable if and only

$$A = A^+_{\varphi} \oplus A^-_{\varphi}.$$ 

It is helpful to remember that two diagonalizable involutions from $\text{Aut}(A)$ are commuting if and only if there is a basis of $A$ in which they both diagonalizable.

We shall call a diagonalizable involution $\varphi$ a $\gamma$-involution, where $\gamma$ is a cardinal, if

$$\gamma = \text{rank } A^-_{\varphi} < \text{rank } A^+_{\varphi}. $$

1-involutions, like in linear group theory, will be called extremal involutions.

A number of facts on definability of certain families of involutions in the automorphism groups of infinitely generated free abelian groups has been proved implicitly in the author's paper [11]. Because of that we shall give only sketches of proofs for the next two statements, Lemma 1.3 and Lemma 1.4; the reader is referred to the proof of Proposition 2.4 in [11] to find there the omitted details.

For an involution $\varphi$ in the group $\text{Aut}(A)$ we shall denote by $K(\varphi)$ the conjugacy class of $\varphi$ in $\text{Aut}(A)$. The set $K^2(\varphi) = K(\varphi)K(\varphi)$ is the family of all products $\varphi_1\varphi_2$, where $\varphi_1, \varphi_2 \in K(\varphi)$.

**Lemma 1.3.** The family of all diagonalizable involutions is $\varnothing$-definable in $\text{Aut}(A)$.
PROOF. We claim that $\varphi$ is diagonalizable if and only if the set $K^2(\varphi)$ contains no elements of order three.

Using Theorem 1.1 one checks that the diagonalizable involutions are exactly involutions in the kernel of the homomorphism $\sim : \text{Aut}(A) \to \text{Aut}(A/2A)$. On the other hand, the images under $\sim$ of all elements of order three from $\text{Aut}(A)$ are non-trivial. This implies that if $\varphi$ is diagonalizable, then there are no elements of order three in $K^2(\varphi)$.

Conversely, for any non-diagonalizable involution $\psi \in \text{Aut}(A)$ we can easily find a conjugate $\psi'$ of $\psi$ such that the automorphism $\psi\psi'$ is of order three. □

LEMMA 1.4. The families of extremal involutions (1-involutions), 2-involutions and 4-involutions are all $\varnothing$-definable in $\text{Aut}(A)$.

PROOF. A diagonalizable involution $\varphi$ is an extremal involution if and only if all involutions in $K^2(\varphi)$ are conjugate and $\varphi$ is not a square in $\text{Aut}(A)$.

Indeed, if $\varphi$ is an extremal involution, then the only involutions in the set $K^2(\varphi)$ are 2-involutions. In particular, all involutions in $K^2(\varphi)$ are conjugate. Applying Theorem 1.1, we can demonstrate that the latter property holds also only for diagonalizable involutions $\rho$ such that

$$\text{rank } A^+_{\rho} = 1.$$ 

But any such an involution is a square in $\text{Aut}(A)$, whereas any 1-involution is not.

The 2-involutions are the only involutions from $K^2(\varphi)$, where $\varphi$ is an arbitrary 1-involution. Let $\theta$ be a 2-involution. Then 4-involutions are those involutions in $K^2(\theta)$ that are not conjugate to $\theta$. □

We need also a family of non-diagonalizable involutions $\{\pi\}$ whose elements satisfy the condition

$$\text{rank } A^+_{\pi} = 1 \text{ or } \text{rank } A^-_{\pi} = 1.$$ 

For any canonical basis $B$ for a non-diagonalizable involution $\pi$ with (1.1) we have that

(a) $B$ contains exactly one pair of distinct elements, say, $b, c$ taken by $\pi$ to one another (Remark 1.2);

(b) $\pi$ either inverts all elements in $B \setminus \{b, c\}$, or fixes all these elements (otherwise, both subgroups $A^+_{\pi}$ and $A^-_{\pi}$ were of rank $> 1$).

Thus either $\pi \sim \pi'$, or $\pi \sim -\pi'$ for every pair of non-diagonalizable involutions $\pi, \pi'$ with (1.1), where $\sim$ denotes the conjugacy relation. Keeping in mind (a), we shall call non-diagonalizable involutions with (1.1) by 1-permutations.

LEMMA 1.5. The following statements are equivalent:

(i) $\pi$ is a 1-permutation;

(ii) $\pi$ is not diagonalizable and the set $K^2(\pi)$ contains no 4-involutions.

In particular, the family of 1-permutations is $\varnothing$-definable in $\text{Aut}(A)$.

PROOF. Let $\pi$ be a non-diagonalizable involution, which is not a 1-permutation, and let $B$ be a canonical basis for $\pi$. One then can readily find $\pi'$, a conjugate of $\pi$, whose product with $\pi$ is a 4-involution. Indeed, suppose first that $p(B) > 1$ (the notation was introduced in Remark 1.2). Then $B$ contains distinct elements $b_1, b_2, b_3, b_4$ such that $\pi b_1 = b_2$ and $\pi b_3 = b_4$. 
The second case is the case when \( p(\mathcal{B}) = 1 \). Here \( \pi b_1 = b_2 \) for some distinct \( b_1, b_2 \in \mathcal{B} \) and, since \( \pi \) is not a 1-permutation, two such elements \( b_3 \) and \( b_4 \) can be found in \( \mathcal{B} \) that

\[
\pi b_3 = b_3 \text{ and } \pi b_4 = -b_4.
\]

Then, for both of the cases under consideration, we construct \( \pi' \) as follows:

\[
\pi' b_i = -\pi b_i \text{ for } i = 1, \ldots, 4 \text{ and } \pi' b = \pi b \text{ for all } b \in \mathcal{B} \setminus \{b_1, b_2, b_3, b_4\}.
\]

Conversely, suppose \( \pi \) is a 1-permutation. We may assume that \( \text{rank } A^-_\pi = 1 \). Let \( \pi_1, \pi_2 \) be conjugates of \( \pi \). Then \( \text{Im}(1 - \pi_1) \) and \( \text{Im}(1 - \pi_2) \) are subgroups of rank 1. Since

\[
1 - \pi_1 \pi_2 = (1 - \pi_2) + (1 - \pi_1)\pi_2,
\]

we have

\[
\text{Im}(1 - \pi_1 \pi_2) \subseteq \text{Im}(1 - \pi_1) + \text{Im}(1 - \pi_2),
\]

and so \( \text{rank Im}(1 - \pi_1 \pi_2) \leq 2 \). Then \( \pi_1 \pi_2 \) is not a 4-involution because for any 4-involution \( \psi \) we have \( \text{rank Im}(1 - \psi) = 4 \). \( \square \)

Until the end of this section we fix some 2-involution \( \theta^* \). In order to mark somehow one special type of commutativity with \( \theta^* \), we say that an extremal involution \( \psi \) (resp. a 1-permutation \( \psi \)) commutes with \( \theta^* \) properly, if \( \psi \sim \theta^* \psi \).

We fix also an extremal involution \( \varphi^* \) and a 1-permutation \( \pi^* \) both properly commuting with \( \theta^* \) such that

\[
(\pi^* \varphi^*)^2 = \theta^*.
\]

Let \( B \) denote the subgroup \( A^-_{\theta^*} \). Since both \( \varphi^* \) and \( \pi^* \) commute with \( \theta^* \), they both preserve \( B \):

\[
\varphi^* B = \pi^* B = B.
\]

Since, further, \( \varphi^* \) and \( \pi^* \) commute with \( \theta^* \) properly, their restrictions to \( B \) are an extremal involution and a 1-permutation of \( \text{Aut}(B) \), respectively. Let

\[
f^* = \varphi^*|_B \text{ and } p^* = \pi^*|_B.
\]

We have that

\[
f^* p^* f^* p^* = -\text{id}_B
\]

and then \( p^* f^* p^* = -f^* \). This implies that \( p^* \) takes to each other the subgroups \( A^+_{f^*} \) and \( A^-_{f^*} \):

\[
p^* A^+_{f^*} = A^-_{f^*}.
\]

If then \( e_1 \) is a basis element of \( A^+_{f^*} \), then \( e_2 = p^* e_1 \) is a basis element of \( A^-_{f^*} \). Summing up, we see that in the basis \( \{e_1, e_2\} \) of \( B \) the automorphisms \( f^* \) and \( p^* \) have the matrices

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

respectively.

Next is the proof of \( \mathcal{O} \)-definability of certain transvections. Recall that a unimodular element of \( A \) (a primitive element in a more general context) is an element of \( A \) that can be included in some basis of \( A \). Let \( \delta : A \to \mathbb{Z} \) be a non-zero homomorphism of abelian groups; in this case the ker \( \delta \) is a direct summand of \( A \) of corank 1. Fix a unimodular element \( x \) in ker \( \delta \). Then the mapping

\[
\tau a = a + \delta(a)x,
\]
is an automorphism of $A$, which is called a \textit{transvection}. If $\tau$ is a transvection determined by a homomorphism $\delta$, then one may correctly associate with $\tau$ a natural number defining it via

$$m(\tau) = |\delta(y)|$$

where $y \in A$ satisfies $A = \langle y \rangle \oplus \ker \delta$. It can be easily seen that for every pair $\tau_1, \tau_2$ of transvections $m(\tau_1) = m(\tau_2)$ if and only if $\tau_1$ and $\tau_2$ are conjugate. We shall call a transvection $\tau$ an $m$-\textit{transvection}, if $m(\tau) = m$.

**Lemma 1.6.** (i) Among the conjugates $\rho$ of $\pi^*$ properly commuting with $\theta^*$ there are exactly four ones different from $\pi^*$ that satisfy the equation

$$(\pi^* \rho)^3 = \text{id}_A;$$

(ii) The automorphisms $(\varphi^* \rho)^2$, where $\rho$ is any of 1-permutations described in (i), are all 2-transvections.

**Proof.** Let $\rho$ be a 1-permutation satisfying the conditions from (i). First note that due to the proper commutativity with $\theta^*$, the restriction of $\rho$ to $A^+_\theta$ must be equal to that one of $\pi^*$.

We denote by $R$ the matrix of the restriction of $\rho$ on $B = A^-_{\theta}$ in the above described basis $\{e_1, e_2\}$.

Since the condition $(\pi^* \rho)^3 = \text{id}$ can be rewritten as

$$\pi^* \rho \pi^* = \rho \pi^* \rho,$$

we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = R \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R.$$ (1.2)

Let now

$$R = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $a, b, c \in \mathbb{Z}$ (the trace of $R$ should be equal to zero like the trace of any non-central involution in $\text{GL}(2, \mathbb{Z})$, Theorem 1.1). It follows from (1.2) that

$$\begin{align*}
- a &= a(b + c), \\
 c &= b^2 - a^2, \\
 b &= c^2 - a^2.
\end{align*}$$ (1.3)

According to (1.3), there are two cases for study: $a = 0$ and $b + c + 1 = 0$.

In the first case we have that $b = c = 1$ and then $\rho = \pi^*$, which is impossible.

The second case: we use the condition $\det R = -1$ ($\rho$ is a conjugate of $\pi^*$).

Then

$$\det R = -1 = -a^2 - bc = -a^2 - b(-b - 1)$$

or

$$a^2 = b^2 + b + 1.$$  

The only $b \in \mathbb{Z}$ for which the number $b^2 + b + 1$ is a square are $b = 0, -1$.

Thus, there are indeed at most four possibilities for $R$:

$$\begin{pmatrix} e & 0 \\ -1 & -e \end{pmatrix}, \begin{pmatrix} e & -1 \\ 0 & -e \end{pmatrix},$$  (1.4)
where \( e = \pm 1 \). One easily verifies that for all four 1-permutations \( \rho \) that correspond to the matrices in (1.4) and such that \( \pi^* e = \rho c \) for all \( c \in A^+_\theta \), the conditions from (i) of the Lemma are true.

The statement in (ii) is now a consequence of the following observations:

\[
\begin{align*}
\left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \left( \begin{array}{cc} e & 0 \\ -1 & -e \end{array} \right) \left( \begin{array}{cc} e^2 & 0 \\ 2e & e^2 \end{array} \right) \\
\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} e & -1 \\ 0 & -e \end{array} \right) \left( \begin{array}{cc} e^2 & -2e \\ 0 & e^2 \end{array} \right) ,
\end{align*}
\]

where \( e = \pm 1 \).

\( \square \)

**Lemma 1.7.** The family of all 2m-tranvectors (where \( m \) runs over \( \mathbb{N} \)) is \( \varnothing \)-definable in \( \text{Aut}(A) \).

**Proof.** We shall continue to use the parameters picked up above. One more parameter will be serviceable, however: a 2-transvection \( \tau^* \), one of the four 2-transvectors described in Lemma 1.6 (ii).

Let us consider the set \( S \) of automorphisms \( \{ \varphi^* \rho \} \), where \( \rho \) is an extremal involution or a 1-permutation properly commuting with \( \theta^* \). If the matrix of the restriction of \( \tau^* \) on \( B \) is, for instance,

\[
\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
\]

then only those elements from \( S \) commute with \( \tau^* \) whose restrictions on \( B \) have matrices

\[
\begin{pmatrix} e & b \\ 0 & e \end{pmatrix}
\]

where \( e = \pm 1 \) and \( b \in \mathbb{Z} \). The squares of the matrices of the form (1.5) are matrices

\[
\begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix}
\]

Thus the set consisting of squares of elements of \( S \) is a set that for each natural number \( m \) contains a 2m-tranvection. This implies that a suitably chosen existential formula defines the 2m-tranvectors. \( \square \)

**Lemma 1.8.** Two distinct extremal involutions \( \varphi_1, \varphi_2 \) have the mutual (eigen) subgroup, that is, either

\[ A^+_{\varphi_1} = A^+_{\varphi_2}, \text{ or } A^-_{\varphi_1} = A^-_{\varphi_2} \]

if and only if the product \( \varphi_2 \varphi_1 \) is a 2m-tranvection for some non-zero natural \( m \).

**Proof.** Assume that subgroups \( A^-_{\varphi_1} \) and \( A^-_{\varphi_2} \) are generated by unimodular elements \( x_1 \) and \( x_2 \) respectively, and write \( B_1 \) and \( B_2 \) for \( A^+_{\varphi_1} \) and \( A^+_{\varphi_2} \). Let also \( \tau \) denote the product \( \varphi_2 \varphi_1 \).

\((\Leftarrow)\). Suppose \( B_1 \neq B_2 \). Then the intersection \( B_1 \cap B_2 \) is of corank 2. The fixed-point subgroup \( C \) of \( \tau \) has corank 1 and contains (a direct summand of \( A \)) \( B_1 \cap B_2 \); then there is a unimodular element \( y \in A \) such that

\[ C = \langle y \rangle \oplus (B_1 \cap B_2) \]

We have \( \varphi_2 \varphi_1 y = y \) and then

\[ \varphi_1 y - y = \varphi_2 y - y \]
The above element is non-zero, since otherwise \( y \in B_1 \cap B_2 \). Thus \( \langle x_1 \rangle \cap \langle x_2 \rangle \neq 0 \), or \( \langle x_1 \rangle = \langle x_2 \rangle \), since both \( x_1, x_2 \) are unimodular.

(\( \Rightarrow \)). (i) Suppose that \( B_1 \neq B_2 \), but \( \langle x_1 \rangle = \langle x_2 \rangle \). Since \( B_1 \cap B_2 \) is a direct summand of \( A \) of corank 2, then for some unimodular \( z \)

\[ B_1 = \langle z \rangle \oplus (B_1 \cap B_2); \]

the element \( z \) can be expressed as \( mx_2 + b_2 \), where \( m \in \mathbb{Z} \) and \( b_2 \in B_2 \). We then have

\[ \tau z = \varphi_2 \varphi_1 z = \varphi_2 (mx_2 + b_2) = -mx_2 + b_2 = mx_2 + b_2 - 2mx_2 = z - 2mx_2. \]

Taking into account that \( \tau x_2 = x_2 \), we see that \( \tau \) is a 2m-transvection.

(ii) Suppose that \( B = B_1 = B_2 \) and \( \langle x_1 \rangle \neq \langle x_2 \rangle \). The element \( x_1 \) can be then written as

\[ x_1 = ex_2 + b = ex_2 + mc, \]

where \( b = mc \) is an element of \( B \) and \( c \) is a unimodular. Hence

\[ \tau x_1 = \varphi_2 \varphi_1 x_1 = \varphi_2 (-ex_2 - mc) = ex_2 - mc = x_1 - 2mc \]

and \( \tau \) is a 2m-transvection.

PROPOSITION 1.9. Let \( D^1(A) \) be the family of all direct summands of \( A \) having rank or corank one. Then the action of the group Aut(A) on the family \( D^1(A) \) is first-interpretable in Aut(A) without parameters.

PROOF. In view of Lemma 1.4, Lemma 1.7 and Lemma 1.8 all we have to do is to explain when two pairs of extremal involutions \( (\varphi_1, \varphi_2) \) and \( (\psi_1, \psi_2) \) both having mutual subgroups determine the same direct summand of \( A \). It is easy: we just say that for all \( i, j \) either \( \varphi_i = \psi_j \), or \( \varphi_i \psi_j \) is a 2m-transvection.

In the conclusion of the section we present a purely algebraic observation due to Oleg Belegradek who had found it while reading the first draft of the paper.

PROPOSITION 1.10. Let \( A_1, A_2 \) be infinitely generated free abelian groups. The groups \( \text{Aut}(A_1) \) and \( \text{Aut}(A_2) \) are isomorphic if and only if the cardinals rank \( A_1 \) and rank \( A_2 \) are equal.

PROOF. Let \( A \) be an infinitely generated free abelian group. It is easy to show that the cardinality of any maximal family of pairwise commuting 1-involutions in \( \text{Aut}(A) \) is equal to rank of \( A \). Since, by Lemma 1.4, the 1-involutions are \( \varnothing \)-definable in \( \text{Aut}(A) \) uniformly in \( A \), and isomorphisms preserve first-order formulae, the result follows.

2. Definability of the congruence subgroup of level two

Let \( m > 1 \) be a natural number. Write \( \Gamma_m(A) \) for the subgroup of \( \text{Aut}(A) \) consisting of the automorphisms of \( A \) that act trivially (in the natural way) on the group \( A/mA \). The subgroups \( \Gamma_m(A) \) are natural analogues of the principal congruence subgroups of the groups \( \text{SL}(n, \mathbb{Z}) \).

We are going to prove \( \varnothing \)-definability of the subgroup \( \Gamma_2(A) \), the principal congruence subgroup of \( \text{Aut}(A) \) of level two. As it has been said in the Introduction this will imply a possibility of first-order interpretation in \( \text{Aut}(A) \) of the general linear group of the vector space \( A/2A \) over the field \( \mathbb{Z}_2 \).

THEOREM 2.1. The subgroup \( \Gamma_2(A) \) is \( \varnothing \)-definable in \( \text{Aut}(A) \).
PROOF. We shall use properties of the group SL(3, \(\mathbb{Z}\)) and with this idea in mind we are going to fix somehow some three direct summands of rank one in \(A\). To achieve that we use certain definable parameters. First, we take three pairwise commuting extremal involutions \(\varphi_1^*, \varphi_2^*, \varphi_3^*\) in \(\text{Aut}(A)\) such that any product \(\varphi_i^* \varphi_j^*\), where \(i \neq j\) is a 2-involution. There exists a basis \(B\) of \(A\) in which \(\varphi_1^*, \varphi_2^*, \varphi_3^*\) are all diagonalizable. Let \(e_i\) denote the element of \(B\) that \(\varphi_i^*\) \((i = 1, 2, 3)\) sends to the opposite.

Second, we need two 1-permutations \(\pi_1^*\) and \(\pi_2^*\) to provide a suitable action on \(\{e_1, e_2, e_3\}\); our requirements on \(\pi_1^*\) and \(\pi_2^*\) are therefore as follows:

(i) \(\pi_1^* \varphi_1^* \pi_1^* = \varphi_2^*\) and \(\pi_1^*\) commutes with \(\varphi_2^*\);
(ii) \(\pi_2^* \varphi_1^* \pi_2^* = \varphi_3^*\) and \(\pi_1^*\) commutes with \(\varphi_3^*\);
(iii) \(\pi_1^*\) and \(\pi_2^*\) are conjugate and their product is of order three.

In the following statement we simultaneously introduce and characterize some transvections we are going to deal with.

CLAIM 1. The elementary transvections which act trivially on \(B \setminus \{e_1, e_2, e_3\}\) and whose matrices in \(\{e_1, e_2, e_3\}\) (more precisely, matrices of the corresponding restrictions) are of the form \(E + nE_{ij}\), where \(1 \leq i, j \leq 3, i \neq j\) and \(E_{ij}\) are the matrix units, are definable with parameters \(\varphi_1^*, \varphi_2^*, \varphi_3^*\) and \(\pi_1^*, \pi_2^*\).

We choose a 2-transvection \(\tau_1^*\), one of the four 2-transvections that satisfy the condition (ii) of Lemma 1.6 for the 2-involution \(\theta_1^* = \varphi_1^* \varphi_2^*\) and the 1-permutation \(\pi_1^*\). Without loss of generality we may suppose that the matrix of \(\tau_1^*\) in \(\{e_1, e_2, e_3\}\) is

\[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

It is easy to see that among the automorphisms \(\varphi_1^* \rho\), where \(\rho\) is either an extremal involution, or a 1-permutation properly commuting with \(\theta_1^*\) there are exactly four automorphisms whose square is \(\tau_1^*\). The reason is that there are two solutions to the matrix equation

\[
X^2 = \begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix}
\]

in \(\text{SL}(2, \mathbb{Z})\), namely,

\[
X = \pm \begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

and that any automorphism properly commuting with \(\theta_1^*\) must act on its fixed-point subgroup, say, \(C\), either as the \(\text{id}_C\), or \(-\text{id}_C\). Let us denote the said four automorphisms by \(\sigma_1, \sigma_2, \sigma_3, \sigma_4\) and let us further agree that \(\sigma_1\) is the only transvection among the automorphisms \(\sigma_i\).

The matrices of the automorphisms \(\sigma_i\) in the basis \(\{e_1, e_2, e_3\}\) are

\[
\pm \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \pm \begin{pmatrix}
-1 & -1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

(the reader may as well imagine the diagonals of the matrices stretched up to infinity filled with units, but there is actually no need in that, since already three coordinates do the job.)
Let $\sigma$ be one of our automorphisms $\sigma_i$. We consider the conjugate $\sigma' = \pi \sigma \pi^{-1}$ of $\sigma$ by the automorphism $\pi = \pi_2^* \pi_1^*$. Then the matrix of the commutator $[\sigma, \sigma'] = \sigma \sigma' \sigma^{-1} \sigma'^{-1}$ is either the matrix

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \text{ or } 
\begin{pmatrix}
1 & 2 & -3 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{pmatrix}.
$$

Thus only in the case when $\sigma = \sigma_1$ we have the commutator $[\sigma, \sigma']$ conjugate to $\sigma$. Really, as to the automorphisms $\sigma_2, \sigma_3, \sigma_4$ they all have eigen value $-1$, while none of the commutators $[\sigma_i, \sigma_i']$ with $i = 2, 3, 4$ has this eigen value. Summing up, we see that $\sigma_1$, a 1-transvection, is definable over the chosen parameters.

Like in the proof of Lemma 1.7 we see that the elementary transvections whose matrices in $\{e_1, e_2, e_3\}$ are

$$
(2.1)
$$

are definable with parameters $\varphi_1^*, \varphi_2^*, \pi_1^*$ and $\tau_1^*$. Then elementary transvection with the matrices

$$
\begin{pmatrix}
1 & n & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

are also definable with the parameters $\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*$, since they are none the other than either the transvections with matrices (2.1), or the products of the transvections with (2.1) and the elementary transvection with the matrix

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

which is now known to be definable over $\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*$. The other required elementary transvections are conjugates of the tranvections with (2.1) by suitable automorphisms acting on $\{e_1, e_2, e_3\}$ as permutations, definable products of $\pi_1^*$ and $\pi_2^*$. Claim 1 is proved.

Let us note in passing that definability of 1-transvections with definable parameters we have just proved immediately implies the following proposition.

**Proposition 2.2.** Let $A$ be an infinitely generated free abelian group. Then
(i) the family of all transvections is $\mathcal{O}$-definable in $\text{Aut}(A)$;
(ii) Let $m \geq 1$ be a natural number. The family of all $m$-transvections is $\mathcal{O}$-definable in $\text{Aut}(A)$.

Next is the construction of some set which is contained in $\Gamma_2(A)$ and which is definable with our parameters.

**Claim 2.** There is a set $D$ definable with parameters $\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*, \tau_2^*$ such that
(i) the automorphisms from $D$ act trivially on $B \setminus \{e_1, e_2, e_3\}$ and their matrices in $\{e_1, e_2, e_3\}$ are congruent modulo 2 to the identity matrix;
(ii) \( D \) contains all automorphisms with (i) whose matrices in \( \{e_1, e_2, e_3\} \) are of the form
\[
\begin{pmatrix}
  a & b & 0 \\
  c & d & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]
where
\[ a \equiv d \equiv 1 \pmod{2} \text{ and } b \equiv c \equiv 0 \pmod{2}. \]

The argument is based upon the remarkable observation made in the paper [2] by Carter and Keller:

each matrix of the form
\[
\begin{pmatrix}
  a & b & 0 \\
  c & d & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]
from the (matrix) group \( \text{SL}(3, \mathbb{Z}) \) is a product of at most 41 elementary transvections.

Suppose that \( t_1, \ldots, t_{41} \) are elementary transvections, matrices from \( \text{SL}(3, \mathbb{Z}) \). One corresponds to the product
\[ t_1 t_2 \ldots t_{41} \]
a sequence
\[
(2.2) \quad (\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{41})
\]
where \( \tilde{t} \) is the image of \( t \) in \( \text{SL}(3, \mathbb{Z}_2) \) under the natural homomorphism \( \text{SL}(3, \mathbb{Z}) \to \text{SL}(3, \mathbb{Z}_2) \). There are of course finitely many sequences of the form (2.2). Some of them determine the identity matrix in \( \text{SL}(3, \mathbb{Z}_2) \), some do not; we appreciate the former sequences, say `good' ones. Clearly, the image \( \tilde{t} \) of an elementary transvection \( t \) is trivial in \( \text{SL}(3, \mathbb{Z}_2) \) if and only if \( t \) is a square of an elementary transvection in \( \text{SL}(3, \mathbb{Z}) \). So the fact that a sequence \( (\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{41}) \) is `good' can be translated into a disjunction of statements each of which says for every \( i = 1, \ldots, 41 \) that the \( i \)th transvection \( t_i \) is or is not a square.

Having the elementary transvections with respect to the basis \( \{e_1, e_2, e_3\} \) (this time automorphisms of \( A \)) definable in \( \text{Aut}(A) \) with the parameters introduced above, we may realize the above considerations for the group \( \text{Aut}(A) \). This completes the proof of Claim 2.

Let now \( \chi(\overline{\varphi}) \) be a first-order formula that describes the parameters \( \varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*, \tau_2^* \). Suppose that \( \overline{\varphi} \) is any tuple of elements of \( \text{Aut}(A) \) that satisfies \( \chi \); we then denote by \( D(\overline{\varphi}) \) the family of automorphisms constructed over \( \overline{\varphi} \) in the same way as \( D \) is constructed over our parameters.

**Claim 3.** The following are equivalent:
(a) \( \sigma \in \text{Aut}(A) \) is an element of \( \Gamma_2(A) \);
(b) there is a direct summand \( B \) of \( A \) of rank or corank 1 such that for every direct summand \( C \) isomorphic via some automorphism from \( \text{Aut}(A) \) to \( B \) there exist a tuple \( \overline{\varphi} \) satisfying \( \chi \) and \( \rho \in D(\overline{\varphi}) \) with
\[
\sigma C = \rho C.
\]
Let consider the implication (b) $\Rightarrow$ (a). Suppose that the direct summand $B$ mentioned in (b) is of rank one and $e$ a unimodular element of $A$. Then for suitable parameters $\varphi$ there is $\rho \in D(\varphi)$

$$\sigma(e) = \rho(e).$$

By Claim 2 the set $D(\varphi)$ is contained in $\Gamma_2(A)$ and hence

$$\sigma e = \pm \rho e \equiv \pm e \equiv e (\mod 2A).$$

It then follows that $\sigma \in \Gamma_2(A)$.

Suppose now that $B$ is of corank 1. Let $e$ be a unimodular element of $A$ and let $\{e, e_0, e_1, \ldots, e_n, \ldots\}$ be a basis of $A$. According to the condition $\sigma$ moves the direct summand

$$C_0 = \langle e, e_1, e_2, \ldots, e_n, \ldots \rangle$$

exactly as some $\rho \in \Gamma_2(A)$ does:

$$\sigma C_0 = \rho C_0.$$ 

This implies that $\sigma e$ is congruent modulo $2A$ to some element of $C_0$:

$$(2.3) \quad \sigma e \equiv ke + k_1 e_1 + k_2 e_2 + \ldots + k_n e_n + \ldots (\mod 2A).$$

The same argument can be applied to the subgroup

$$C_1 = \langle e, e_0, e_2, \ldots, e_n, \ldots \rangle$$

of which $e$ is also a member; this leads to

$$\sigma e \equiv le + l_0 e_0 + l_2 e_2 + \ldots + l_n e_n + \ldots (\mod 2A).$$

One deduces then that

$$(k - l)e - l_0 e_0 + k_1 e_1 + (k_2 - l_2)e_2 + \ldots + (k_n - l_n)e_n + \ldots \equiv 0 (\mod 2A).$$

The images of $e, e_0, e_1, e_2, \ldots$ under the natural homomorphism $A \to A/2A$ must be linearly independent over $\mathbb{Z}_2$ and therefore

$$l_0 \equiv k_1 \equiv 0 (\mod 2).$$

Continuing in a similar fashion, we see that all (non-zero) coefficients $k_i$ in (2.3) are even; the coefficient $k$ must therefore be odd. Thus $\sigma$ is in $\Gamma_2(A)$, as required.

The implication (a) $\Rightarrow$ (b). Suppose that $\sigma \in \Gamma_2(A)$ and $e$ is a unimodular element of $A$. Then for a basis $\{e, e_0, e_1, \ldots, e_n, \ldots\}$ of which $e$ forms a part we have

$$\sigma e = e + 2(k e + \sum_i k_i e_i).$$

Suppose that $s$ is the greatest common divisor of non-zero elements $k_i$. Then

$$\sigma e = (1 + 2k)e + 2s(\sum_i k'_i e_i).$$

Clearly, $\gcd(1 + 2k, 2s) = 1$ (since $\sigma e$ is unimodular) and the element $g = \sum_i k'_i e_i$ is unimodular. If so, there are $b, d \in \mathbb{Z}$ such that the matrix

$$\begin{pmatrix}
1 + 2k & b & 0 \\
2s & d & 0 \\
0 & 0 & 1
\end{pmatrix}$$
from $\operatorname{SL}(3, \mathbb{Z})$ is congruent to the identity matrix modulo 2. This implies that there exist a tuple $\varphi$ satisfying $\chi$ and some $\rho \in D(\varphi)$ such that

$$
\sigma(e) = \rho(e).
$$

Claim 3 is proved.

Since we know how to interpret in $\operatorname{Aut}(A)$ by means of first-order logic the direct summands of $A$ of rank/corank 1, the conditions in (ii) of Claim 3 are easily translated into first-order formulae. The proof of Theorem 2.1 is now completed. \qed

**Remark 2.3.** Very recently Bardakov proved that the principal congruence subgroups of the groups $\operatorname{SL}(n, \mathbb{Z})$, where $n \geq 3$ all have finite width with respect to elementary transvections (unpublished; personal communication). Recall that the width of a group $G$ relative to a generating set $S$ with $S^{-1} = S$ is either the minimal natural number $k$ such that every element of $G$ is a product of at most $k$ elements of $S$, or $\infty$ otherwise.

The result by Bardakov could be used then to simplify the proof of Theorem 2.1.

**Theorem 2.4.** Let $A$ be an infinitely generated free abelian group. Then the group $\operatorname{Aut}(A)$ first-order interprets the second-order theory $\operatorname{Th}_2(\dim A)$, uniformly in $A$.

**Proof.** The proof is based on Theorem 2.1 and the following important theorem from the paper [1] by Bryant and Macedonska.

**Theorem.** Let $F$ be a free group of infinite rank and let $V$ be a characteristic subgroup of $F$ such that $F/V$ is nilpotent. Then every automorphism of $F/V$ is induced by an automorphism of $F$.

Let $A$ stand for the free abelian group $F/[F,F]$. As a corollary of the result by Bryant–Macedonska we have that the natural homomorphism

$$
\mu : \operatorname{Aut}(A) \to \operatorname{Aut}(A/2A)
$$

(induced by the natural homomorphism $A \to A/2A$) is surjective. Indeed, according to the Theorem, the natural homomorphisms

$$
\mu_1 : \operatorname{Aut}(F) \to \operatorname{Aut}(A) \text{ and } \mu_2 : \operatorname{Aut}(F) \to \operatorname{Aut}(A/2A)
$$

are both surjective. On the other hand,

$$
\mu_2 = \mu \circ \mu_1,
$$

and then $\mu$ must be surjective, too.

Adding this to the fact that $\Gamma_2(A)$, the kernel of $\mu$, is $\emptyset$-definable in $\operatorname{Aut}(A)$, we get that the group $\operatorname{Aut}(A)$ first-order interprets the group $\operatorname{Aut}(A/2A)$:

$$
\operatorname{Aut}(A)/\ker \mu = \operatorname{Aut}(A)/\Gamma_2(A) \cong \operatorname{Aut}(A/2A).
$$

The group $\operatorname{Aut}(A/2A)$ is the general linear group of the vector space $A/2A$ over the field $\mathbb{Z}_2$. On the other hand, the general linear group $\operatorname{GL}(V)$ of a infinite-dimensional vector space $V$ over a field $D$ first-order interprets $\operatorname{Th}_2(\dim_D V)$, see [8, Theorem 11.4]. Therefore the elementary theory of the group $\operatorname{Aut}(A/2A)$ first-order interprets the second-order theory

$$
\operatorname{Th}_2(\dim \mathbb{Z}_2 A/2A) = \operatorname{Th}_2(\dim A),
$$
and the result follows.

**Corollary.** Let $A_1, A_2$ be infinitely generated free abelian groups. The groups Aut($A_1$) and Aut($A_2$) are elementarily equivalent if and only if the cardinals $|A_1|$ and $|A_2|$ (viewed as sets with no structure) are second-order equivalent.

**Proof.** The necessity part is a consequence of Theorem 2.4. To prove the converse, one syntactically interprets in the second-order theory Th$_2$(ϰ), where ϰ is an infinite cardinal, the elementary theory of the automorphism group of a free abelian group with ϰ as the domain (rather easy; cf. [9, Theorem 4.1] where a similar interpretation is done in quite full detail for the case of the elementary theory of the automorphism group a free group over ϰ.)

**References**


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A categoricity theorem for quasi-minimal excellent classes

Boris Zilber

Introduction This work stemmed from attempts to prove a canonicity of a certain type of structures, which we called pseudo-analytic (see [Z1]). These attempts, even for a particular class of structures, fields with pseudo-exponentiation, involved a substantial work in field arithmetic, and even after most of the algebraic problems were sorted out in [Z2], a problem remained, which we realized was of a pure model-theoretic nature. This is a categoricity statement for certain classes of structures which we now call quasi-minimal excellent. Initially we formulated the assumptions for the theorem as sufficient conditions applicable for some, and hopefully for all, pseudo-analytic structures, without being aware of Shelah’s theory of excellent classes. Later, after discussing this with B.Hart, R.Grossberg and O.Lessmann, we realized that the assumptions are essentially a special case of Shelah’s definition of excellent classes but formulated in the spirit of the Baldwin-Lachlan categoricity proof. There is no doubt that the proofs of two our main theorems follow Shelah’s [S], yet (as was pointed out by the referee) their statements can not be derived directly from Shelah’s particularly because they treat classes more general than $L_{\omega_1,\omega}$. This special form of a categoricity theorem has a much easier formulation customized for applications, and there is a rich and mathematically meaningful class of examples satisfying this special definition.

The proof does not use any previous theory, though the technique is essentially Shelah’s. The reader is encouraged to compare the present setting with Shelah’s original work and carry out a further analysis, especially with regards to pseudo-analytic and analytic examples. We discuss shortly the applications in the second section of the paper.

Since the first version of the paper (July 2001) we have realized that a richer class of examples requires a slightly stronger version of the main theorem, this is a categoricity theorem for almost quasi-minimal excellent classes presented in the last section.

I am grateful to R.Grossberg, O.Lessmann and J.Baldwin for many helpful discussions on the topic of the paper and to O.Roche for suggesting a correction to the statement of Assumption 2.

1. Quasi-minimal excellent structures

Let $C$ be a class of structures closed under isomorphisms and satisfying the following assumptions 1-3:

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ASSUMPTION 1 (pregeometry). For any \( H \in C \) for any \( X \subseteq H \) there is a superset \( X \subseteq \text{cl}(X) \subseteq H \) satisfying the following:

(i) \( \text{cl}(X) \in C \) as a substructure of \( H \);

(ii) \( \text{cl}(Y) = \bigcup \{ \text{cl}(X) : X \subseteq Y, \ X \text{ finite} \} \);

(iii) \( X \rightarrow \text{cl}(X) \) is a monotone idempotent operator.

We are also often interested in two extra conditions:

**the countable closure property**: \( \text{cl}(X) \) is countable for \( X \) finite;

**the exchange principle**: \( z \in \text{cl}(Xy) \setminus \text{cl}(X) \Rightarrow y \in \text{cl}(Xz) \).

**DEFINITION 1.1.** Let \( H, H' \in C \) and \( G \subseteq H, G \subseteq H' \). Then a (partial) mapping, identical on \( G \), \( \varphi : H \rightarrow H' \) is called a \( G \)-**monomorphism** if it preserves quantifier-free formulas over \( G \).

**DEFINITION 1.2.** A non-empty subset \( X \subseteq H \) is called **cl-independent over** \( G \) if

\[
\text{cl}(G) \cap X = \emptyset
\]

and for any \( Y_1, Y_2 \subseteq X \)

\[
\text{cl}(Y_1 \cup G) \cap \text{cl}(Y_2 \cup G) = \text{cl}(Y_1 \cap Y_2 \cup G).
\]

\( X \) is said to be just cl-independent, if it is cl-independent over \( \emptyset \).

**ASSUMPTION 2 (\( \omega \)-homogeneity over a submodel).** Let \( G \subseteq H, G \subseteq H', G \in C \) or \( G = \emptyset \). Then

(i) if \( X \) and \( X' \) are cl-independent subsets over \( G \) in some \( H, H' \in C \), correspondingly, then any bijection \( \varphi : X \rightarrow X' \) is a \( G \)-monomorphism;

(ii) if a partial \( \varphi : H \rightarrow H' \) is a \( G \)-monomorphism, \( \text{Dom} \varphi = X \), with \( X \) finite, then for any \( y \in \text{cl}(X \cup G) \) there is an extension \( \varphi' \) of \( \varphi \) with \( \text{Dom} \varphi' = X \cup \{ y \} \).

(iii) if \( \varphi : X \cup \{ y \} \rightarrow X' \cup \{ y' \} \) is a monomorphism, then

\[
y \in \text{cl}(X) \text{ iff } y' \in \text{cl}(X').
\]

**DEFINITION 1.3.** Given \( X, C \subseteq H \) we say that the **type of \( X \) over \( C \)** is **defined over** \( C_0 \) if any \( \varphi : X \rightarrow H' \) which is a \( C_0 \)-monomorphism is also a \( C \)-monomorphism.
Definition 1.4. A subset \( C \subseteq H \) will be called **special** if there is a closed-independent \( A \subseteq H \) and \( A_1, \ldots, A_k \subseteq A \) such that
\[
C = \bigcup_i \text{cl}(A_i).
\]

Assumption 3. Suppose \( C \subseteq H \) is special and \( X \) is a finite subset of \( \text{cl}(C) \). Then the type of \( X \) over \( C \) is defined over a finite subset \( C_0 \subseteq C \).

Remark 1.5. It follows from assumptions 1 and 2, by the Ehrenfeucht-Fraïssé criterion, that, given finite \( X \subseteq H \), any two points in \( H \setminus \text{cl}(X) \) are \( L_{\omega_1, \omega} \)-equivalent. If also an uncountable structure \( H \in C \) satisfies the countable closure property then any \( L_{\omega_1, \omega} \)-definable subset of \( H \) is either countable or has a countable complement.

A class \( C \) of structures satisfying assumptions 1-3 will be called in this paper **quasi-minimal excellent**. Also, each member of the class is said to be quasi-minimal excellent.

Below we usually use letters \( A, A_0, \ldots \) to denote infinite independent sets, while \( A, A_0, \ldots \) will stand for their finite subsets.

Lemma 1.1. Let \( A_0 \subseteq H, A_0' \subseteq H' \) be independent subsets and suppose \( G = \text{cl}(A_0) \) and \( G' = \text{cl}(A_0') \) are countable. Then, given a bijection \( \varphi_0 : A_0 \rightarrow A_0' \), there is an isomorphism \( \varphi : G \rightarrow G' \), extending \( \varphi_0 \).

Proof. Assuming \( A_0 = \{ a_i : i \in \mathbb{N} \} \), \( A_0' = \{ a_i' : i \in \mathbb{N} \} \) and \( \varphi_0(a_i) = a_i' \), by induction on \( i \) assume that there is an isomorphism
\[
\psi_i : \text{cl}(a_1, \ldots, a_i) \rightarrow \text{cl}(a_1', \ldots, a_i'),
\]
sending \( a_j \) to \( a_j' \) for all \( j \leq i \). Then by definition \( \{ a_{i+1} \} \) is cl-independent over \( \text{cl}(a_1, \ldots, a_i) \) and so is \( \{ a_{i+1}' \} \) over \( \text{cl}(a_1', \ldots, a_i') \). It follows from the \( \omega \)-homogeneity over a submodel and the countability assumption that \( \psi_i \) can be extended to
\[
\psi_{i+1} : \text{cl}(a_1, \ldots, a_{i+1}) \rightarrow \text{cl}(a_1', \ldots, a_{i+1}').
\]
Then \( \varphi = \bigcup_i \psi_i \) satisfies the requirements. \( \square \)

Lemma 1.2. Let \( A \) be a cl-independent subset in \( H, A_0 \subseteq A \) infinite and \( A_0, \ldots, A_m \subseteq A \) finite subsets. For \( X \subseteq H \) denote \( \text{cl}^*(X) = \text{cl}(A_0 \cup X) \), and let \( B_0 \subseteq \text{cl}^*(A_0) \) finite, \( b_i \in \text{cl}^*(A_i), i = 1, \ldots, m \), and \( p(x_1, \ldots, x_m) \) a quantifier-free type over \( B_0 \), such that \( \models p(b_1, \ldots, b_m) \). Then there is a mapping
\[
\pi : \bigcup A_i \cup B_0 \cup \{ b_1, \ldots, b_m \} \rightarrow \text{cl}^*(A_0),
\]
which is identity on \( B_0 \), \( b_1^\pi \in \text{cl}^*(A_1^\pi) \subseteq \text{cl}^*(A_i \cap A_0) \) and \( \models p(b_1^\pi, \ldots, b_m^\pi) \).

Proof. Let \( A^* \) be a finite subset of \( A_0 \), such that \( B_0 \subseteq \text{cl}(A_0 \cup A^*) \) and \( b_i \in \text{cl}(A_i \cup A^*) \) for all \( i = 1, \ldots, m \).

Let \( G_0 = \text{cl}(A^* \cup A_0) \) and \( \pi_0 \) be the identity map on \( G_0 \). Then the extension of \( \pi_0 \) to an injection
\[
\pi_1 : G_0 \cup \bigcup_{1 \leq i \leq m} A_i \setminus (A_0 \cup A^*) \rightarrow G_0 \cup A_0 \setminus (A_0 \cup A^*)
\]
is a monomorphism by the assumption 2(i). By \( \omega \)-homogeneity extend \( \pi_1 \) to a monomorphism \( \pi \) with a domain covering \( \{ b_1, \ldots, b_m \} \). Then \( \models p(b_1^\pi, \ldots, b_m^\pi) \), since \( \pi \) is a monomorphism over \( B_0 \). Also, \( b_1^\pi \in \text{cl}(A_1^\pi \cup A^*) \subseteq \text{cl}^*(A_1^\pi) \) by the construction, and \( \text{cl}^*(A_1^\pi) \subseteq \text{cl}_1^*(A_i \cap A_0) \) by the definition of \( \pi_1 \). \( \square \)
THEOREM 1. Let $C$ be a quasi-minimal excellent class, $H, H' \in C$, both with the  
countable closure property, $A \subseteq H$, $A' \subseteq H'$, independent and $\text{cl}(A) = H$,  
$\text{cl}(A') = H'$. Suppose that there is a bijection $\psi_0 : A \to A'$.

Then $\psi_0$ extends to an isomorphism $\psi : H \to H'$.

PROOF. If $A$ is finite or countable, the statement is proved in Lemma 1.1. So we assume $A$ is infinite and uncountable, and so contains a proper countable subset $A_0$. Let $\psi_0(A_0) = A'_0 \subseteq A'$. W.l.o.g. we may assume that $\psi_0$ is identity on $A_0$. By Lemma 1.1 we may also identify $\text{cl}_H(A_0) = \text{cl}_{H'}(A'_0) = G$.

We continue the proof of the theorem through a series of intermediate statements.

We now change the language by fixing $G$ pointwise and so "monomorphism" from now on means "monomorphism over $G$" and $\text{cl}(X)$ means $\text{cl}(X \cup A_0)$, the 
same as $\text{cl}^*$ in Lemma 1.2 above. Notice that in the new language assumptions 1-3 still hold for structures containing $A_0$.

LEMMA 1.3. $\psi_0$ is a monomorphism.

PROOF. First notice that $A \setminus A_0$ along with $A' \setminus A_0$ are independent. Then apply assumption 2(i).

LEMMA 1.4. If $X \subseteq A$ is finite, then there is an isomorphism $\text{cl}(X) \to \text{cl}(\psi_0(X))$ extending $\psi_0|_X$.

PROOF. Immediate from the countable closure property and the assumptions 2 by the back-and-forth method.

NOTATION 1.6. For $X \subseteq H$ denote

$$\text{cl}^{-}(X) = \bigcup_Y \text{cl}(Y).$$

LEMMA 1.5. Let $A_0, \ldots, A_m \subseteq A \subseteq A$, $A$ finite and $A_0A_i$ for $i > 0$. Then any 
two tuples $D, D'$ in $\text{cl}(A_0)$ which are isomorphic over $\text{cl}^{-}(A_0)$, are also isomorphic 
over $\text{cl}^{-}(A_0) \cup \bigcup_{1 \leq i \leq m} \text{cl}(A_i)$.

PROOF. Assume not, and $D, D'$ provide a counterexample. Then there is a 
quantifier-free formula $q(x, y)$ over $\text{cl}^{-}(A_0)$, such that for some $B \subseteq \bigcup_{1 \leq i \leq m} \text{cl}(A_i)$ both $q(D, B)$ and $\neg q(D', B)$ hold in $H$. By assumption 1.2 there is a map

$$\pi : \bigcup_{1 \leq i \leq m} A_i \cup D \cup D' \cup B \to \text{cl}(A_0),$$

identical on $D \cup D'$, such that

$$\models \neg q(D', B^\pi) \& q(D, B^\pi)$$

and

$$\pi(B \cap \text{cl}(A_i)) \subseteq \text{cl}(A_0) \cap \text{cl}(A_i^\pi) \subseteq \text{cl}(A_0 \cap A_i) \subseteq \text{cl}^{-}(A_0).$$

Thus $B^\pi \subseteq \text{cl}^{-}(A_0)$. This contradicts the assumptions on $D, D'$.

DEFINITION 1.7. Given a finite $A \subseteq A \setminus A_0$ we call a subset $D \subseteq \text{cl}(A)$ **perfect**

over $A$ in $H$ if (an inductive definition)

(i) $A = \emptyset$, or

(ii) for any $XA$ the sets $D_X = D \cap \text{cl}(X)$ are perfect over $X$, and the type of $D$ over $\text{cl}^{-}(A)$ is defined over $\bigcup_{X \subseteq A} D_X$. 


Lemma 1.6. Let $A \subseteq A \setminus A_0$ and $D \subseteq \text{cl}(A)$ be finite, $n \geq 0$ and assume that for each $X \subseteq A$ with $|X| \leq n + 1$ $D_X = D \cap \text{cl}(X)$ is perfect over $X$ and $\varphi_X : D_X \to H'$ is a monomorphism. Denote $\varphi = \bigcup_{X \subseteq A, |X| = n+1} \varphi_X$, and suppose also that the restriction of $\varphi$ on $\bigcup_{Y \subseteq A, |Y| = n} D_Y$ is a monomorphism. Then $\varphi$ is a monomorphism.

Proof. Let

$$\{X \subseteq A, |X| = n + 1\} = \{A_i : i = 0, \ldots, m\}, \text{ distinct.}$$

Notice that since, by $\text{cl}$-independence, for any $i \neq j$

$$D \cap \text{cl}(A_i) \cap \text{cl}(A_j) \subseteq \bigcup_{Y \subseteq A, |Y| = n} D_Y,$$

$\varphi$ is a well-defined mapping.

Remember that

$$\bigcup_{Y \subseteq A, |Y| = n} D_Y = \bigcup_{i \leq m} D_{A_i} \cap \text{cl}^{-}(A_i),$$

and apply Lemma 1.5 to see that $\varphi$ preserves the quantifier-free formulas on

$$D_{A_0} \cup \bigcup_{0 < i \leq m} D_{A_i} \cap \text{cl}^{-}(A_i).$$

The same lemma by induction on $l \leq m$ yields that $\varphi$ preserves the quantifier-free formulas on

$$\bigcup_{i \leq l} D_{A_i} \cup \bigcup_{l < i \leq m} D_{A_i} \cap \text{cl}^{-}(A_i).$$

For $l = m$ we have the statement of the present lemma. \qed

Lemma 1.7. Given a finite $A \subseteq A \setminus A_0$ for any finite $D \subseteq \text{cl}(A)$ there is a finite $\tilde{D}$, $D \subseteq \tilde{D} \subseteq \text{cl}(A)$, perfect over $A$.

More precisely,

$$\tilde{D} = D \cup \bigcup_{X \subseteq A} \tilde{D}_X, \text{ where } D \cap \text{cl}(X) \subseteq \tilde{D}_X \subseteq \text{cl}(X).$$

Moreover, if we relpace $\tilde{D}_X$ by $\tilde{D}'_X \supseteq \tilde{D}_X$, perfect over $X$, and $D$ by $D' \subseteq D$, then

$$\tilde{D}' = D' \cup \bigcup_{X \subseteq A} \tilde{D}'_X,$$

is still perfect over $A$.

Proof. (i) Induction on $|A|$. The statement is obvious for $A = \emptyset$. In the general case by assumption 3 the type of $D$ over $G \cup \text{cl}^{-}(A)$ is defined over a finite subset, say $\bigcup_{X \subseteq A} D_X$, and $G$ with each $D_X \subseteq \text{cl}(X)$. By induction there are finite $\tilde{D}_X$ perfect over $X$. Put

$$\tilde{D} = D \cup \bigcup_{X \subseteq A} \tilde{D}_X,$$

which is perfect over $A$ by the definition.

The 'moreover' statement is immediate by the definition. \qed

Proof of the theorem. We proceed by induction on $n$ assuming that there is a family

$$D_n = \{D_{A,i} \subseteq \text{cl}(A) : A \subseteq A \setminus A_0, |A| \leq n, i \in \mathbb{N}\}$$
of finite sets perfect over $A$, such that

$$\text{cl}(A) = \bigcup_{i \in \mathbb{N}} D_{A,i},$$

for each $B \subseteq A$

$$D_{A,i} \cap \text{cl}(B) = D_{B,i},$$

and $D_m \subseteq D_n$ for $m \leq n$.

We also assume that there are monomorphisms for all $k \in \mathbb{N}$:

$$\psi_{n,k} : \bigcup_{AA,|A| \leq n} D_{A,k} \to H', \quad \psi_0 \subseteq \psi_{n,k} \subseteq \psi_{n,k+1}$$

and for each $A \subseteq A$ of size $\leq n$

$$\psi_A = \bigcup_{k \in \mathbb{N}} \psi_{n,k}|_{\text{cl}(A)} : \text{cl}(A) \to \text{cl}(\psi_0(A))$$

is a bijection.

Notice that for $D_{\emptyset,k}$ we can take any ascending chain of finite subsets covering $\text{cl}()$, which is a substructure of common $G$, and $\psi_{0,k}$ be the identity mappings on the appropriate subsets.

Suppose that a chain $D_{X,i}$ has been constructed for each $X \subseteq A \setminus A_0$ of size $\leq n$ as well as monomorphisms $\psi_{n,k}$ for all $k \in \mathbb{N}$. For each $|A| = n + 1$ put $D_{A,0} = \bigcup_X D_{X,0}$, $\text{cl}^0(A) = \text{cl}(A) \setminus \text{cl}^-(A)$, and for each $i \in \mathbb{N}$ by induction on $i$:

$$D_{A,i+1} = \begin{cases} D_{A,i} \cup \bigcup_X D_{X,i+1} \cup \min\{\text{cl}^0(A) \setminus D_{A,i}\} & \text{if this is perfect over } A \\ D_{A,i} \cup \bigcup_X D_{X,i+1} & \text{otherwise} \end{cases}$$

Since, given $a \in \text{cl}^0(A)$, for some $j > i$ the type of $\{a\} \cup D_{A,i}$ over $G \cup \text{cl}^-(A)$ is defined over $G \cup \bigcup_X D_{X,j}$, we get by Lemma 1.7 $a \in D_{A,j}$, and thus $\bigcup_i D_{A,i} = \text{cl}(A)$. Also for all $j$ $D_{A,j} \setminus \bigcup_X D_{X,j} \subseteq \text{cl}^0(A)$, so for any $B \subseteq A$, by the induction hypothesis, $D_{A,j} \cap \text{cl}(B) = D_{B,j}$.

By the $\omega$-homogeneity for each $k \in \mathbb{N}$ we can extend $\psi_{n,k}|_{\text{cl}(A)}$, whose domain is obviously a subset of $\bigcup_{X \subseteq A,|X| = n} D_{X,k}$, to

$$\psi_{A,k} : D_{A,k} \to H'.$$

More precisely, when $\psi_{A,k}$ is constructed, $A' = \psi_0(A)$, and $a'$ is the first element in $\text{cl}^0(A')$, of quantifier free type, say $p$, over $\psi_{A,k}(D_{A,k})$, let $j \geq k$ be such that the corresponding type $\psi_{A,k}^{-1}(p)$ over $D_{A,k}$ is realised in $D_{A,j}$, say by $a$. Extend by the $\omega$-homogeneity $\psi_{A,k}$ to $\psi_{A,j} : D_{A,j} \to H'$ so that $\psi_{A,j}(a) = a'$. By doing this we have also defined $\psi_{A,m}$ for $k \leq m \leq j$ as the corresponding restrictions of $\psi_{A,j}$. Also, this process guarantees that the image of $\bigcup_{i \in \mathbb{N}} \psi_{A,i}$ is the whole of $\text{cl}(A')$.

Define

$$\psi_{n+1,k} = \bigcup_{AA,|A| = n+1} \psi_{A,k}.$$

Since for any distinct $A_1, A_2 \subseteq A$ of size $n + 1$

$$D_{A_1,i} \cap D_{A_2,i} = D_{A_1 \cap A_2,i},$$

and $|A_1 \cap A_2| \leq n$, the definition is consistent.
Given a finite $B \subseteq A \setminus A_0$, the mapping $\varphi = \psi_{n+1,k}|_{\text{cl}(B)}$, by Lemma 1.6, is a monomorphism. So $\psi_{n+1,k}$ is a a monomorphism.

Finally, put $\psi = \bigcup_{n \in \mathbb{N}} \psi_{n,n}$.
Obviously, the domain of $\psi$ is $\text{cl}(A) = H$ and the range is
$$\bigcup_{A' \subseteq A' \text{ finite}} \text{cl}(A') = H'.$$

It finishes the construction of an isomorphism. □

**Corollary 1.** If under the assumptions of the theorem $\text{cl}$ satisfies also the exchange principle, then $C$ is categorical in every uncountable cardinality.

**Proof.** Under the exchange principle $\text{cl}$-independence is equivalent to the independence in the usual pregeometry sense. In other words $A$ is just a basis of $H$, which exists for any structure $H$, and by assumptions 1(ii) and 1(iv) has the same cardinality as $H$ modulo $\aleph_0$. It follows from the theorem that any $H$ and $H'$ in $C$ of the same uncountable cardinality are isomorphic. □

**Theorem 2.** Suppose a quasi-minimal excellent class $C^#$ is axiomatizable by an $L_{\omega_1,\omega}$-sentence $\Sigma$ and the relations $y \in \text{cl}(x_1, \ldots, x_n)$ are $L_{\omega_1,\omega}$-definable for all $n$. Suppose also that there is an $H \in C^#$ containing an infinite $\text{cl}$-independent subset $A$.

Then for any infinite cardinal $\kappa$ there is a unique structure $H_\kappa \in C^#$ of cardinality $\kappa$, satisfying the countable closure property.

**Proof.** Let $C$ be the subclass of $C^#$ satisfying the countable closure property. Choose $A_0 \subseteq A$ countable. By assumption 1 $\text{cl}(A_0) \in C^#$. By the downward Lowenheim-Skolem theorem choose a countable submodel $H_0 \subseteq \text{cl}(A_0)$ of $\Sigma$. $H_0$ obviously satisfies the countable closure property. So, we may assume $\text{cl}(A_0) = H_0 \in C$.

Suppose now $\text{cl}(A) = H \in C$ for some $A$ of cardinality $\lambda \geq \aleph_0$. Notice that by assumption 2 any $\text{cl}$-independent set is indiscernible. Let $a$ be a point in $A$, $A' = A \setminus \{a\}$ and $H' = \text{cl}(A')$.

Obviously, $H'H$ and by Theorem 1 a bijection between $A'$ and $A$ extends to an isomorphism $H' \cong H$. This gives rise to a sequence
$$H' = H_0 \subseteq H_1 \subseteq \ldots H_\alpha \subseteq \ldots$$
of isomorphic structures and embeddings of the form $H_\alpha = \text{cl}(A_\alpha)$, for $\alpha$ ordinals of cardinality $\lambda$, and

$$A' = A_0 \subseteq A_1 \subseteq \ldots A_\alpha \subseteq \ldots$$

with unions on limit steps.
Taking
$$H_{\lambda^+} = \bigcup_{\alpha < \lambda^+} H_\alpha,$$
we get a structure of cardinality $\lambda^+$, which is still in $C^#$, since by the $\omega$-homogeneity and the Ehrenfeucht-Fraisse criterion
$$H \equiv_{L_{\omega_1,\omega}} H_{\lambda^+}.$$
Also the countable closure property is satisfied for $H_{\lambda^+}$ since

$$H_{\lambda^+} = \bigcup_{\alpha < \lambda^+} \text{cl}(A_{\alpha}).$$

Thus $H_{\lambda^+} \in C$.

We can then carry on the same process up to $\kappa$. □

**Remark 1.8.** The countable closure property is not $L_{\omega_1, \omega}$-axiomatizable (yet, it is $L_{\omega_1, \omega}(Q)$ in many cases) so, the theorem proves categoricity for not necessarily $L_{\omega_1, \omega}$-axiomatizable subclass of $C^\omega$.

The following question has been suggested by the referee.

Does Theorem 2 hold when $\Sigma$ is an $L_{\omega_1, \omega}(Q)$-sentence?

**2. Examples and applications**

Let $F$ be an algebraically closed field of characteristic 0 and $F^\times$ its multiplicative group. Then, by obvious group theoretic reasons there exists an abelian divisible torsion free group $H$ and a homomorphism $\text{ex}$ such that the sequence

$$0 \to \mathbb{Z} \to H \to \text{ex} F^\times \to 0$$

is exact. We call this sequence a **cover of the multiplicative group of the field** $F$. In logical terms the sequence is a two-sorted structure $(H, F)$ with a group structure on sort $H$ and a field structure on $F$ and the mapping $\text{ex}$ between the sorts.

It is more convenient in the context of the present paper to represent the sequence of the form (2.1) as a one sorted structure

$$H = (H, +, E(x, y), S(x, y, z), P(x, y, z))$$

with domain $H$, a group operation $+$, a binary equivalence relation $E$ interpreted as the pullback of the equality on $F$ by $\text{ex}$, and ternary relations $S$ and $P$ which are pullbacks of the basic ternary additive and multiplicative equations on $F$. In fact, $P$ is redundant, since $\text{ex}(x) \cdot \text{ex}(y) = \text{ex}(z)$ is a consequence of $x + y = z$ in the group $H$, so we omit it.

In particular, for the classical structure, the complex numbers with exponentiation: $H = F = \mathbb{C}$ and $\text{ex} = \exp$, we have

$$E(x, y) \equiv x - y \in 2\pi i \mathbb{Z} \text{ and } S(x, y, z) \equiv e^x + e^y = e^z.$$  

It can easily be seen, as in [Z2], that

*There is an $L_{\omega_1, \omega}$-sentence $\Sigma$ (which is a finite diagram in Shelah’s sense) such that any model $H$ of $\Sigma$ represents a sequence (2.1) with some algebraically closed field $F$, and conversely, any $H$ corresponding to a sequence of the form (2.1) is a model of $\Sigma$.*

The main result of [Z2] is the following

**Theorem 3.** The class of models of $\Sigma$ is quasi-minimal excellent. Moreover, the class satisfies the exchange principle and the countable closure property.

Uncountable models of $\Sigma$ are not homogeneous.
COROLLARY 2. \( L_{\omega_1,\omega} \) theory of the structure
\[ (\mathbb{C}, +, e^x + e^y = e^{x+y}) \]
is categorical in all uncountable cardinalities.

**Fields with pseudo-exponentiation.** These are structures of the form 
\((F, +, \cdot, \text{ex})\), where \((F, +, \cdot)\) is an algebraically closed field of characteristic zero. Further assumptions are:

1. ex is a homomorphism from the additive group onto the multiplicative group of the field, with cyclic kernel.

2. "The Schanuel conjecture" holds: for any additively independent \(x_1, \ldots, x_n \in F\) the transcendence degree of \(x_1, \ldots, x_n, \text{ex}(x_1), \ldots, \text{ex}(x_n)\) is at least \(n\).

3. Any normal, free system of equations over \(F\) has a generic solution in \(F\). (We skip here the definitions of 'normal' and 'free'.)

The class of fields with pseudo-exponentiation was studied in [Z3]. A closure operator \(\text{cl}\) was introduced, and the main result of the paper states that

*The class satisfying the \(L_{\omega_1,\omega}\)-conditions (1)-(3) above is quasi-minimal excellent.*

The countable closure property (which can be formulated by a sentence of \(L_{\omega_1,\omega}(Q)\)) does not follow from (1)-(3). But by Theorem 2 we get that in any uncountable cardinality there is a unique field with pseudo-exponentiation satisfying the countable closure property. The one of cardinality continuum is a very close analogue of the complex numbers with exponentiation \((\mathbb{C}, +, \cdot, \exp)\) and, conjecturally, is isomorphic to it.

3. **Almost quasi-minimal classes**

We slightly generalize here the main result of the paper by extending the main definition. A good reason for this is the generalizations of the example of the previous section studied in [Z4], group covers of semi-Abelian varieties. Under certain arithmetic assumptions group covers of semi-Abelian varieties are almost quasi-minimal in the sense below.

Let \(B\) be a class of structures with a closure operator \(\text{cl}\) satisfying Assumption 1 and with an extra unary predicate \(U\) such that

*for all \(H \in B\) the closure operator \(\text{cl}\) on \(H\) has the property*

\[ \text{cl}(U(H)) = H. \]

We call such a class **almost quasi-minimal excellent** if it satisfies Assumptions 1-3 with the following correction to Assumption 2(i) (with the same meaning of \(H, H'\) and \(G\)):

(i) if \(X \subseteq U(H)\) and \(X' \subseteq U(H')\) are cl-independent over \(G\) in some \(H, H' \in B\), correspondingly, then any bijection \(\varphi : X \rightarrow X'\) is a \(G\)-monomorphism;
Part (ii) and (iii) of the assumption remain unchanged.

We also redefine

**Definition 3.1.** A subset $C \subseteq H$ is called **special** if there is a cl-independent $A \subseteq U(H)$ and $A_1, \ldots A_k \subseteq A$ such that

$$C = \bigcup_i \text{cl}(A_i).$$

With this definition in mind Assumption 3 stays unchanged.

**Theorem 4.** Let $B$ be an almost quasi-minimal excellent class, $H, H' \in B$, both with the countable closure property, $A \subseteq U(H)$, $A' \subseteq U(H')$, independent and $\text{cl}(A) = H$, $\text{cl}(A') = H'$. Suppose that there is a bijection $\psi_0 : A \rightarrow A'$.

Then $\psi_0$ extends to an isomorphism $\psi : H \rightarrow H'$.

**Proof.** Same as for Theorem 1. \qed

**Theorem 5.** Suppose a quasi-minimal excellent class $B^\#$ is axiomatizable by an $L_{\omega_1, \omega}$-sentence $\Sigma$ and the relations $y \in \text{cl}(x_1, \ldots, x_n)$ are $L_{\omega_1, \omega}$-definable for all $n$. Suppose also that there is an $H \in B^\#$ containing an infinite cl-independent subset $A \subseteq U(H)$.

Then for any infinite cardinal $\kappa$ there is a unique structure $H_\kappa \in B^\#$ of cardinality $\kappa$, satisfying the countable closure property.

**Proof.** Same as for Theorem 2. \qed

**References**


[S] S. Shelah, Classification theory for non-elementary classes I: the number of uncountable models of $\psi \in L_{\omega_1, \omega}$, Parts A,B, Isr. J. Math., v.46, (1983), 212-240, 241-273


[Z2] B. Zilber, Covers of the multiplicative group of an algebraically closed field of characteristic zero. To appear in JLMS

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Two conferences, Logic and Its Applications in Algebra and Geometry and Combinatorial Set Theory, Excellent Classes, and Schanuel Conjecture, were held at the University of Michigan (Ann Arbor). These events brought together model theorists and set theorists working in these areas. This volume is the result of those meetings. It is suitable for graduate students and researchers working in mathematical logic.