MATH 3510 EXAM # 1 Answers
03/08/2016

Name:__________________________ ID: __________________________

Directions: This is a closed-book, closed-notes exam. You have 80 minutes to finish the exam. Read instructions for each problem carefully and answer all questions.

1. Consider the following binary operation on $\mathbb{R}$:

$$a * b = a + \frac{1}{2}b.$$

(a) Is $*$ associative? Why or why not?

Associativity means $(a * b) * c = a * (b * c)$ for all $a, b, c \in \mathbb{R}$. We compute:

$$(a * b) * c = (a + \frac{1}{2}b) * c = (a + \frac{1}{2}b) + \frac{1}{2}c = a + \frac{1}{2}b + \frac{1}{2}c$$

$$a * (b * c) = a * (b + \frac{1}{2}c) = a + \frac{1}{2}(b + \frac{1}{2}c) = a + \frac{1}{2}b + \frac{1}{4}c$$

If they were equal we would have $\frac{1}{2}c = \frac{1}{4}c$, which is not the case when $c = 1$. So $*$ is not associative.

(b) Let $\mathbb{R}^+$ be the set of all positive real numbers. Is $\mathbb{R}^+$ closed under $*$? Why or why not?

Translating the statement,

$$\begin{aligned} & \mathbb{R}^* \text{ is closed under } * \\ \iff & \text{for all } a, b \in \mathbb{R}^+,\ a * b \in \mathbb{R}^* \\ \iff & \text{for all positive reals } a, b,\ a + \tfrac{1}{2}b \text{ is positive} \end{aligned}$$

From the last equivalent statement, we see that it is correct.

2. (a) Describe the Klein 4-group V by giving a group table.

`This was one of two parts for which everybody got right.`

(b) Is the group abelian? Explain your answer briefly based on the group table (i.e. what feature of the group table tells you if the group is abelian or not).

`This was the other part.`

(c) Is the group cyclic? Justify your answer.

The group is not cyclic because there is not a single element which generates the whole group. For every element $a \in V$ we have $a^2 = e$ and $\langle a \rangle = \{e, a\} \neq V$.

3. Consider the 2×2 matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

(a) Compute A^2, A^3 and A^{-1}, A^{-2}.

For any $n \in \mathbb{Z}$,

$$A^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}.$$

(b) What is the cyclic subgroup $\langle A \rangle$ generated by A in $SL(2, \mathbb{R})$?

$$\begin{aligned} \langle A \rangle &= \{A^n \,|\, n \in \mathbb{Z}\} \\ &= \left\{ \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\} \end{aligned}$$

(Problem 3 continued)

(c) Find an isomorphism between the groups $\mathbb{Z}$ and $\langle A \rangle$. Justify your answer (i.e. define an isomorphism and then prove that it is an isomorphism).

Define $\phi : \mathbb{Z} \to \langle A \rangle$ by

$$\phi(n) = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}.$$

We need to show that ϕ is one-to-one, onto, and a homomorphism.

To show that ϕ is one-to-one, we let $\phi(n) = \phi(m)$ and show that $n = m$. Suppose $\phi(n) = \phi(m)$, then

$$\begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2m & 1 \end{bmatrix}.$$

It must be that $2n = 2m$ (in order for the two matrices to be equal), and therefore $n = m$.

To show that ϕ is onto, we need to show that any matrix in $\langle A \rangle$ is $\phi(n)$ for some $n \in \mathbb{Z}$. From part (b), we know that any matrix in $\langle A \rangle$ is of the form

$$\begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}$$

which is $\phi(n)$ by our definition of ϕ.

To show that ϕ has the homomorphism property, we need to verify that

$$\phi(n+m) = \phi(n)\phi(m)$$

that is,

$$\begin{bmatrix} 1 & 0 \\ 2(n+m) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2m & 1 \end{bmatrix}.$$

This can be seen from matrix multiplication.

4. Give a proof of the following statement:

> If G is a group, H is a subgroup of G, and K is a subgroup of H, then K is a subgroup of G.

Since G is a group, we use $*$ to denote the group operation on G, e to denote the identity element of G, and for any $a \in G$, a^{-1} to denote the inverse of a in G. To show that K is a subgroup of G, we need to show the following three statments:

(1) for any $a, b \in K$, $a * b \in K$ (this is what we called ''closure of K under $*$'')

(2) $e \in K$ (this is what we called the ''identity'' condition)

(3) for any $a \in K$, $a^{-1} \in K$ (this is what we called the ''inverse'' condition).

To show (1), suppose $a, b \in K$. Because K is a subgroup of H, we know that $a, b \in H$, and because H is a subgroup of G, we know that $a, b \in G$. Now, because H is a subgroup of G and $a, b \in H$, we have $ab \in H$ by the ''closure of H under $*$''. Furthermore, because K is a subgroup of H and $a, b \in K$, we have $ab \in K$.

To show (2), we see that $e \in H$ since H is a subgroup of G, and since K is a subgroup of H, we have $e \in K$.

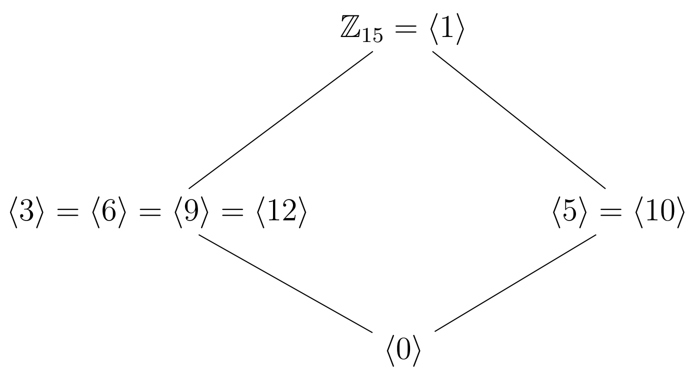
To show (3), suppose $a \in K$. Of course $a \in H$ since K is a subgroup of H. Now because H is a subgroup of G and $a \in H$, we have $a^{-1} \in H$. Because K is a subgroup of H and $a \in K$, $a^{-1} \in K$.

5. Consider the group $\mathbb{Z}_{15}$.

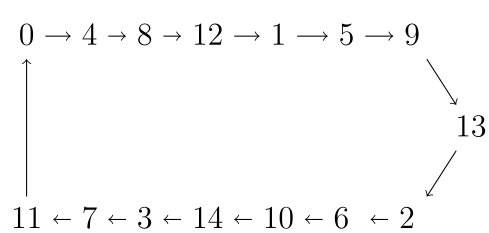
(a) List all the generators of $\mathbb{Z}_{15}$. How many are there?

The generators are all numbers from 1 through 14 that are relatively prime with 15. They are 1, 2, 4, 5, 7, 8, 11, 13, 14. There are eight of them.

(b) Draw the subgroup diagram for the group $\mathbb{Z}_{15}$.



(c) Let $S = \{4\}$. Draw the Cayley digraph of $\mathbb{Z}_{15}$ with respect to S.



6. (a) Give a complete list of cyclic groups up to isomorphism.

$\langle \mathbb{Z}, + \rangle$, $\langle \mathbb{Z}_n, +_n \rangle$ for all $n \in \mathbb{Z}^+$

(b) Give an example of a non-abelian group. Justify your answer.

$SL(2, \mathbb{R}) =$ all 2×2 matrices of nonzero determinant under matrix multiplication. It is a group. It is not abelian because matrix multiplication is not commutative. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and they are not equal.

Another example is the permutation group S_3.