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(1)

Regularized theta lifts of harmonic Maass forms

- Generalize Borcherds lift
- $Z(m) \rightsquigarrow$ „Green function" $\Phi_m(z)$
- Converse theorem

Harmonic Maass forms

$\Gamma \leq \Gamma(1)$ finite index, $\tau = u + iv \in \mathbb{H}$.
For simplicity $k \in \mathbb{Z}$.

Def: A smooth fn $f: \mathbb{H} \to \mathbb{C}$ is called a harmonic Maass form of weight k for Γ if:

- $f|_k \gamma = f \quad \forall \gamma \in \Gamma$
- $\Delta_k f = 0$, where $\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$
- f has at most „linear exponential growth" at the cusps of Γ.

  [I.e. For $s \in \mathbb{P}^1(\mathbb{Q})$ and $\delta \in \Gamma(1)$ with $\delta\infty = s$, there is a $C > 0$ s.t.
  $$f_s(\tau) = f|_k \delta(\tau) = O(e^{Cv}), \quad v \to \infty$$
  ]

$\mathcal{H}_k(\Gamma) :=$ $\mathbb{C}$ vector space of these

Remark: $S_k(\Gamma) \subset M_k(\Gamma) \subset M_k^!(\Gamma) \subset \mathcal{H}_k(\Gamma)$

Ex: $E_2(\tau) = 1 - \frac{3}{\pi v} - 24 \sum_{n \geq 1} \sigma_1(n) q^n$

- Zagier: weight $\frac{3}{2}$ Eisenstein series
- Derivatives of incoherent wgt 1 Eisenstein series
- Completions of Mock theta fn's

__Facts__:

- $f \in H_k(\Gamma)$ has Fourier exp ($\tau \in \Gamma$)

$$f(\tau) = \underbrace{\sum_{n \geq n^+} a^+(n) q^n}_{f^+} + \underbrace{\sum_{n \leq n^-} a^-(n) W_{k,n}(v) q^n}_{f^-}$$

$$W_{k,n}(v) = \begin{cases} \Gamma(1-k, 4\pi n v), & n \neq 0 \\ v^{1-k}, & n = 0,\ k \neq 1 \\ \log(v), & n = 0,\ k = 1 \end{cases}$$

$\Gamma(a, x) = \int_x^\infty e^{-t} t^a \frac{dt}{t}$

- Have an antilinear map

$$\xi_k : H_k(\Gamma) \to M^!_{2-k}(\Gamma)$$

$$f \mapsto \xi_k(f) = v^{k-2} \overline{L_k f(\tau)} = 2i v^k \overline{\frac{\partial}{\partial \bar{\tau}} f}$$

$$\ker(\xi_k) = M^!_k(\Gamma)$$

Pf: Ex.

- There ex a $C > 0$ s.t.

$$a^+(n) = O(e^{C\sqrt{|n|}}), \quad n \to +\infty$$

$$a^-(n) = \quad '' \quad , \quad n \to -\infty$$

(analogous estimates at other cusps)

(Exercise)

Thm: The map $\xi_k : \mathcal{Y}_k(\Gamma) \to M^!_{2-k}(\Gamma)$ is surjective.

Pf:

$X_\Gamma = \Gamma \backslash \mathbb{H}^*$ compact mod curve ass. to Γ

$\pi : \mathbb{H} \to X_\Gamma$ can. map

$\mathcal{Y}_k$ = sheaf of mf's of wgt k on X_Γ

For $U \subset X_\Gamma$ open, we have

$\mathcal{Y}_k(U) = \{ f : \pi^{-1}(U) \to \mathbb{C} \,;\ f|_k \gamma = f \ \forall \gamma \in \Gamma,\ f \text{ hol at cusps} \}$

For a divisor D on X_Γ, let $\mathcal{O}_D$ be the corr sheaf, i.e.

$\mathcal{O}_D(U) = \{ s \in \mathcal{M}(U) \,;\ \mathrm{div}(s) \geq -D \}$

$\mathcal{E}^{p,q}$: sheaf of smooth diff forms of type (p,q) on X_Γ.

First: Assume Γ acts freely. ($\Rightarrow \mathcal{Y}_k$ is line bdl.)

Dolbeault resolution:

$$0 \to \mathcal{O} \to \mathcal{E}^{0,0} \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \to 0$$

Let $D = \sum_{s \in X_\Gamma \text{ cusp}} (s) \in \mathrm{Div}(X_\Gamma)$, $n \in \mathbb{Z}_{\geq 0}$

Tensor with the loc. free $\mathcal{O}$-module $\mathcal{Y}_k \otimes \mathcal{O}_{nD}$ ($\bar\partial$ is $\mathcal{O}$-linear)

$$0 \to \mathcal{Y}_k \otimes \mathcal{O}_{nD} \to \mathcal{E}^{0,0} \otimes \mathcal{Y}_k \otimes \mathcal{O}_{nD} \to \mathcal{E}^{0,1} \otimes \mathcal{Y}_k \otimes \mathcal{O}_{nD} \to 0$$

Dolbeault res for the hol vect bundle

Cohomology sequence

$$0 \to \mathcal{Y}_k \otimes \mathcal{O}_{nD}(X) \to \mathcal{E}^{0,0}(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD})$$
$$\xrightarrow{\bar\partial} \mathcal{E}^{0,1}(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD}) \to H^1(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD}) \to 0$$

(Used that $\mathcal{E}^{p,q} \otimes \mathcal{Y}_k \otimes \mathcal{O}_{nD}$ is fine $\Rightarrow$ acyclic)

Serre duality $\Rightarrow$

$$H^1(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD}) \cong H^0(X, \Omega \otimes \mathcal{Y}_k^\vee \otimes \mathcal{O}_{nD}^\vee)$$
$$\cong H^0(X, \mathcal{Y}_{2-k} \otimes \mathcal{O}_{-(n+1)D}) = 0$$

if $n \gg 0$.

$\Rightarrow$ For $n \gg 0$ get

(*) $$0 \to \mathcal{Y}_k \otimes \mathcal{O}_{nD}(X) \to \mathcal{E}^{0,0}(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD}) \xrightarrow{\bar\partial} \mathcal{E}^{0,1}(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD}) \to 0$$

Let $g \in M^!_{2-k}$.

Then $i v^{-k} \bar g \, d\bar z \in \mathcal{E}^{0,1}(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD})$

$(k,0), (0, 2-k), (0,-2)$

(*) $\Rightarrow \exists f \in \mathcal{E}^{0,0}(X, \mathcal{Y}_k \otimes \mathcal{O}_{nD})$ with growth at cusps s.t.

$$\bar\partial f = i v^{-k} \bar g \, d\bar z$$
$$\Rightarrow \xi_k f = g$$
$$\Rightarrow \Delta_k f = -\xi_{2-k} \xi_k f = -\xi_{2-k} g = 0$$

$\Rightarrow$ ~~harmonic~~ $\Rightarrow f \in H_k(\Gamma)$. $\square$

If Γ does not act freely, pass to normal finite index subgroup which does act freely, take invariants. $\square$

Def: $H_k(\Gamma) = \{ f \in \mathcal{H}_k(\Gamma) ; \ \xi_k(f) \in S_{2-k}(\Gamma) \}$.

Cor: Have exact sequences

$$0 \to M_k^!(\Gamma) \to \mathcal{H}_k(\Gamma) \to M_{2-k}^!(\Gamma) \to 0$$

$$0 \to M_k^!(\Gamma) \to H_k(\Gamma) \to S_{2-k}(\Gamma) \to 0$$

For simplicity from now on $\Gamma = \Gamma(1)$, $k \leq 2$.

$f \in H_k := H_k(\Gamma(1))$ is uniquely determined by its principal part

$$P_f = \sum_{n \leq 0} a^+(n) q^n .$$

Define a bilinear pairing

$$M_{2-k} \times H_k \to \mathbb{C},$$

$$\{g, f\} := (g, \xi_k(f))_{Pet} = \int_{\Gamma \backslash \mathbb{H}} g(\tau) \overline{\xi_k(f)} v^{2-k} d\mu(\tau)$$

Prop: $g = \sum_{n \geq 0} b(n) q^n$,

$$f = \sum_{n \geq n^+} a^+(n) q^n + \sum_{n < 0} a^-(n) \Gamma(1-k, -4\pi n v) q^n,$$

then $\{g, f\} = \sum_{n \leq 0} b(n) a^+(-n)$.

Pf: $\eta := g(\tau) f(\tau) d\tau \in \mathcal{E}^{1,0}(X_\Gamma)$

$$d\eta = -g(\tau) \overline{\xi_k(f)} v^{2-k} d\mu(\tau)$$

$$\Rightarrow \{g, f\} = \int_F d\eta$$

Use Stoke's thm for the reg. int. □

Thm: The induced pairing

$$S_{2-k} \times H_k / M_k^! \longrightarrow \mathbb{C}$$

is non-degenerate.

Pf: Follows from the surjectivity of $\xi_k : H_k \longrightarrow S_{2-k}$. $\square$

Prop: i) For every Fourier pol $Q = \sum_{n<0} a^+(n) q^n$ there is a HMF $f \in H_k$ s.t. $P_f = Q + \text{const}$.

ii) f is weakly hol iff $\{g, f\} = 0 \quad \forall\, g \in S_{2-k}$.

Pf: Above thm + Serre duality. $\square$

A criterion for modularity:

Extend the pairing $M_{2-k} \times M_k^! \longrightarrow \mathbb{C}$, to $\mathbb{C}[[q]] \times M_k^! \longrightarrow \mathbb{C}$, $\{\sum b(m) q^m, \sum a(n) q^n\} = \sum_{m \geq 0} b(m) a(-m)$

Prop:

A formal power series $g = \sum_{m \geq 0} b(m) q^m \in \mathbb{C}[[q]]$ is the q-exp of a mf in M_{2-k} iff $\{g, f\} = 0 \quad \forall\, f \in M_k^!$.

Pf: Either use the prop. before or Serre duality again. $\square$

Automorphic Green functions

Consider the theta lift of $f \in H_k$.

For simplicity: begin with unimodular lattices.

(V,Q) quadr. space $/\mathbb{Q}$, sig $(n,2)$ $(2,n)$

$L \subset V$ even unimodular lattice ($\Rightarrow n \equiv 2\ (8)$)

$O(V)$

$\mathcal{D} = \{ z \subset V(\mathbb{R});\ \dim z = 2,\ Q|_z < 0,\ \text{oriented} \}$ $\cong \mathcal{H}^+ \cong \mathcal{K}^+$

$\Gamma \subset O(L)^+$ fin index

$X_\Gamma = \Gamma \backslash \mathcal{D}^+$

For $m \in \mathbb{Z}_{>0}$ have Heegner div

$$Z(m) = \sum_{\substack{\lambda \in L \\ Q(\lambda) = m}} \lambda^\perp$$

Have Siegel theta function

$$\Theta_L(\tau, z) = v \sum_{\lambda \in L} e\left(Q(\lambda_{z^\perp})\tau + Q(\lambda_z)\bar{\tau} \right)$$

wgt $1-\frac{n}{2}$

$$\Theta_L(\gamma\tau, z) = (c\tau + d)^{n/2 - 1}\, \Theta_L(\tau, z) \quad \forall \gamma \in SL_2(\mathbb{Z})$$

Def: If $f \in H_{1-n/2}$, we def its reg theta lift by

$$\Phi(z,f) = \int_{\mathcal{F}}^{reg} f(\tau)\, \Theta_L(\tau, z)\, d\mu(\tau) \ ,$$

that is, as the const term

in the Laurent exp at $s=0$ of the cont. of

$$\phi(z,f,s) = \lim_{T\to\infty} \int_{F_T} f(\tau)\, \Theta_L(\tau,z)\, v^{-s}\, d\mu(\tau).$$

- If $\mathrm{Re}(s) > 0$, the limit exists and defines a smooth fn in z on all of D.
- It has a merom. cont in s to $\mathbb{C}$.
- The fn $\phi(z,f)$ is def on all of D (!), but only smooth on the complement of the div

$$Z(f) = \sum_{m>0} c^+(-m)\, Z(m),$$

where $c^+(n)$ denote the coeff's of f^+.

<u>Recall:</u> Incomplete Γ-fn

$$\Gamma(0,t) = \int_t^\infty e^{-v} \frac{dv}{v} \qquad (t>0).$$

Extend it to $\mathbb{R}_{\geq 0}$ by putting

$$\tilde{\Gamma}(0,t) = CT_{s=0}\left[\Gamma(-s,t)\right] = CT_{s=0}\left[\int_t^\infty e^{-v} v^{-s} \frac{dv}{v}\right]$$

$$= \begin{cases} \Gamma(0,t), & t>0 \\ 0, & t=0 \end{cases}$$

Note $\Gamma(0,t) = -\log(t) + \Gamma'(1) + O(t^{1-\varepsilon}),\ t\to 0$

<u>Theorem</u> (Borcherds, B.)

For any $z_0 \in \mathbb{D}$ there is a neighborhood $U \subset \mathbb{D}$ s.t.

$$\phi(z,f) - \sum_{\lambda \in L \cap z_0^\perp} c^+(-Q(\lambda))\, \Gamma(0, 4\pi Q(\lambda_z))$$

$L \cap z_0^\perp$ → pos def lattice rank $\leq n$

is smooth on U.

In particular, $\phi(z,f)$ has a logarithmic singularity along $-4Z(f)$.

<u>Thm</u> (B.)

The diff form

$\eta(z,f) := \partial\bar\partial \phi(z,f)$

extends to a smooth $(1,1)$-form on X_Γ.

For any $(n-1,n-1)$-form β on X_Γ with compact support we have

$$\int_{X_\Gamma} \phi(z,f) \wedge (\partial\bar\partial \beta) + \int_{Z(f)} \beta = \int_{X_\Gamma} \eta(z,f) \wedge \beta .$$

- So $\phi(z,f)$ is a logarithmic Green fn for $Z(m)$ → Arakelov Geom.
- $\eta(z,f)$ represents Chern class of $Z(f)$ in $H^2(X_\Gamma, \mathbb{C})$
- Can also show that $\eta(z,f)$ is harmonic and square integrable.
- Interesting question: $\phi(z,f)$ at the boundary? → [BBK], [BHY].

Picture:

Get commutative diagram:

$$\begin{tikzcd}
M^!_{1-\frac{k}{2},0} \arrow[r] \arrow[d] & \{\log|F|,\ F \in \tilde{\mathcal{O}}(X_\Gamma)\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
H_{1-\frac{k}{2},0} \arrow[r, "f \mapsto \Phi(z,f)"] \arrow[d, "\xi"'] & \{\text{(harmonic) log Green fn's}\} \arrow[r, "\partial\bar\partial"] & Z^{1,1}(X_\Gamma) \\
S_{1+\frac{k}{2}} \arrow[urr] & &
\end{tikzcd}$$

harm. (1,1) forms

Cor: Obtain map

$$\Lambda : S_{1+\frac{k}{2}} \longrightarrow Z^{1,1}(X_\Gamma)$$

$$g \longmapsto f \in H_{1-\frac{k}{2},0} \text{ s.t. } \xi(f) = g \longmapsto \partial\bar\partial\,\Phi(z,f)$$

Ex: $K/\mathbb{Q}$ real quadratic

$$L = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_K) ;\ X^t = X^\sigma \right\}$$

$$= \mathcal{O}_K \oplus \mathbb{Z}^2 \qquad (\text{not unimodular})$$

$$Q = \det$$

Sig $(2,2)$.

X_Γ Hilbert modular surface, $Z(m) =$ HZ div's.

Λ can be identified with

the Doi-Naganuma lifting

$$S_2(\Gamma_0(D), \chi_D) \longrightarrow S_2(SL_2(\mathcal{O}_K))$$

Thm (B.): Let F a merom mf for Γ of wt r. Assume

$$\operatorname{div}(F) = \sum_{m>0} a(m)\, Z(m).$$

Then $\log\|F\| = \log |F(z)|\, Q(y)^{r/2} = -\frac{1}{4}\Phi(z,f) + \text{const}$

where $f \in H_{1-\frac{n}{2}}$ is the unique HMF with $P_f = \sum_{m>0} a(m) q^{-m}$. ("weak converse thm")

Pf: · wlog $r=0$.

· $\log \|F\| + \frac{1}{4}\phi(z,f)$ is smooth fn on X_Γ

· show it is harmonic and in $L^{1+\varepsilon}(X_\Gamma)$

$\overset{\text{Yau}}{\Longrightarrow}$ it is constant. □

Q: Is f actually in $M^!_{1-\frac{n}{2}}$?

(x) The Gross-Kohnen-Zagier Thm

Hirzebruch-Zagier: intersection #'s of HZ-divisors on Hilbert mod surfaces are coefficients of wt 2 elliptic mf's

GKZ: Positions of Heegner div's on $X_0(N)$ in $J_0(N)$ are coefficients of wt $\frac{3}{2}$ mf's

Kudla-Millson: $O(n,2)$ in cohomology

Borcherds: $O(n,2)$ in CH^1 (or Pic).

Thm (HZ, GKZ, KM, Borcherds)

The generating series

$$A(\tau) = c(\mathcal{M}_{-1/2}) + \sum_{m>0} Z(m) q^m \in \mathbb{C}[[q]] \otimes_{\mathbb{Z}} CH^1(X_\Gamma)$$

is a modular form in $M_{1+n/2}$.
I.e. for every $\lambda \in CH^1(X_\Gamma)^\vee$, we have

$$\lambda(A)(\tau) = \lambda(c(\mathcal{M}_{-1/2})) + \sum_{m>0} \lambda(Z(m))\, q^m \in M_{1+\frac{n}{2}}.$$

Pf: To show:

$$\{A, f\} = 0 \in CH^1(X_\Gamma)_{\mathbb{C}} \quad \forall f \in M^!_{1-\frac{n}{2}}$$

Suffices: $\forall f \in M^!_{1-n/2}(\mathbb{Z})$
$= \sum c(n) q^n$

For such an f, we have to show

$$(*) \qquad c(0)\, c(\mathcal{M}_{-\frac{1}{2}}) + \underbrace{\sum_{m>0} c(-m) Z(m)}_{Z(f)} = 0 \in CH^1(X_\Gamma)_{\mathbb{C}}$$

$\Psi(f, z)$ is rational section of $\mathcal{M}_{c(0)/2}$ with $\operatorname{div} \Psi(f, z) = \sum_{m>0} c(-m) Z(m)$

So $\quad Z(f) = c(\mathcal{M}_{c(0)/2}) = -c(0)\, c(\mathcal{M}_{-\frac{1}{2}})$ $= Z(f)$

$\Rightarrow (*)$ $\qquad\in CH^1(X_\Gamma)_{\mathbb{C}}$ $\quad\square$

Ex for λ: $\quad \lambda(Z) = \operatorname{vol}(Z) = \int_Z \Omega^{n-1}$

volume of a divisor.

$$\operatorname{vol}(A)(\tau) = \underbrace{\operatorname{vol}(c(\mathcal{M}_{-\frac{1}{2}}))}_{= -\frac{1}{2}\operatorname{vol} X_\Gamma} + \sum_{m>0} \operatorname{vol}(Z(m))\, q^m$$

$$\overset{\text{KM, BK}}{=} -\frac{1}{2} \operatorname{vol} X_\Gamma \cdot E_{1+\frac{n}{2}} \in M_{1+\frac{n}{2}}$$

The converse theorem

Q: Is every merom. mf for $O(n,2)$ with divisor on Heegner divisors the Borcherds lift of a weakly hol. mf of wt $1-\frac{n}{2}$?

Setting (vector valued)

(V,Q) quadr. space, sig $(n,2)$ $\mathbb{Z}, L$

$L \subset V$ even lattice

$A = L'/L$

$\Gamma \subset O(L)$ s.t. A is fixed

$X_\Gamma = \Gamma \backslash \mathbb{D}^+$

Heegner divisors

$h \in A$, $m \in \mathbb{Q}_{>0}$.

$$Z(m,h) = \sum_{\substack{\lambda \in L+h \\ Q(\lambda)=m}} \lambda^\perp$$

Weil rep. $\quad \rho = \rho_A : Mp_2(\mathbb{Z}) \longrightarrow GL(\mathbb{C}[A])$

subrep of Weil rep on $S(V(\mathbb{A}_f))$

Thm (Borcherds)

Let $f = \sum_{h \in A} \sum_m c(m,h) q^m e_h$ $\in M^!_{1-n/2, \bar\rho}$ and assume $c(m,h) \in \mathbb{Z}$ $\forall m<0$. There ex a merom. mf $\Psi(z,f)$ for Γ s.t.

i) $wt(\Psi) = c(0,0)/2$

ii) $div(\Psi) = Z(f) = \frac{1}{2} \sum_{h \in A} \sum_{m>0} c(-m,h)\, Z(m,h)$

iii) Π - exp.

Q: Let F be a merom. mf for Γ with $\operatorname{div}(F) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{m<0} c(m,\beta) Z(m,\beta)$.

Is there an $f \in M^!_{1-n/2,\rho}$ s.t. $\Psi(f) = F$?

Known:

- No in general for $n=1$
  - GZ-formula: $L'(G,1)=0 \Rightarrow$ vanishing of Heegner div
  - Mock theta fn's
- B. (2000): Yes, if $L \cong \Delta \oplus U \oplus U$, $U = II_{1,1} = (\mathbb{Z}^2, x_1 x_2)$

Thm 1: „Yes", if $L \cong K \oplus U$, where K is isotropic of sig $(n-1,1)$, $n \geq 3$.

- Explain criterion, idea of proof
- newform theory.

(*) Recall the picture:

$$
\begin{tikzcd}
 & M^!_{1-\frac{n}{2},\rho,0} \arrow[r] \arrow[d] & \{0\} \arrow[d] & \\
N_{1-\frac{n}{2},L} \arrow[r, hook] \arrow[d] & H_{1-\frac{n}{2},\rho,0} \arrow[r, "f \mapsto \partial\bar\partial \Phi(z,f)"] \arrow[d, "\xi"] & Z^{1,1}(X_\Gamma) & (\longrightarrow H^2(X_\Gamma, \mathbb{C})) \\
\xi(N_{1-\frac{n}{2},L}) \arrow[r, hook] & S_{1+\frac{n}{2},\rho} \arrow[ur, "\Lambda"'] & &
\end{tikzcd}
$$

$N_{1-\frac{n}{2},L} = \{ f \in H_{1-\frac{n}{2},\rho} \,;\, Z(f) = 0 \in \operatorname{Div}(X_\Gamma) \}$

- If $f \in N_{1-\frac{k}{2},L}$, then

$\phi(z,f) = \text{const} \Rightarrow \partial\bar\partial\phi(z,f) = 0$

$\Rightarrow \Lambda(\xi(f),z) = 0$

$\Rightarrow \xi(N_{1-\frac{k}{2},L}) \subset \ker(\Lambda)$

Def $S^+_{1+\frac{k}{2},L} := \xi(N_{1-\frac{k}{2},L})^\perp \subset S_{1+\frac{k}{2},\rho}$

$\Lambda^+ =$ restriction of Λ to $S^+_{1+\frac{k}{2},L}$.

Theorem: Suppose $n \geq 2$ and $n >$ Witt rank (L). TFAE:

i) $\Lambda^+ : S^+_{1+\frac{k}{2},L} \longrightarrow \mathcal{H}^{1,1}(X_\Gamma)$ injective

ii) Every merom mf ^{F} for Γ with div (F) as in ② is the Borcherds lift of an $f \in M^!_{1-\frac{k}{2},\bar\rho}$.

Pf i $\Rightarrow$ ii: (ii $\Rightarrow$ i Exercise)

Assume Λ^+ is inj.

Let F be a mf for Γ with div as in ②.

Wlog $wt(F) = 0$.

„Weak converse thm" $\Rightarrow \exists f \in H_{1-\frac{k}{2},\bar\rho,0}$ s.t. $\phi(z,f) = \log|F| + \text{const}$.

$\Rightarrow \Lambda(\xi(f)) = \partial\bar\partial\phi(z,f) = \partial\bar\partial\log|F| = 0$

$\overset{i)}{\Rightarrow} \xi(f) \in \xi(N_{1-\frac{k}{2},L})$

$\Rightarrow \exists f' \in N_{1-\frac{k}{2},L}$ s.t. $\xi(f) = \xi(f')$.

$\Rightarrow f - f' \in M^!_{1-\frac{k}{2},\bar\rho}$ with $Z(f-f') = Z(f)$

$\Rightarrow$ (ii) $\square$

The Fourier expansion of Λ

Thm: Assume $L \cong K \oplus U$. Let
$g = \sum_{\gamma} \sum_{m} b(m,\gamma) q^m e_\gamma \in S_{1+\frac{k}{2},S}$. For $\lambda \in K'$ the λ-th coeff of $\Lambda(g)$ is

$$\sum_{d \mid \lambda} d^{k-1} \, b\left(Q(\lambda)/d^2, \lambda/d\right).$$

Cor: If $g \in \mathrm{Ker}(\Lambda)$, then $b(Q(\lambda),\lambda) = 0$
$\forall \lambda \in K'$.

$\Rightarrow$ many coeff's of g vanish

- To show: All coeff's vanish!
  (Note: Assumption on L
  $\Rightarrow S_{1+\frac{k}{2},S} = S^{+}_{1+\frac{k}{2},L}$)
- OK if $K \cong D \oplus U$ $\Rightarrow$ old thm.
- Not clear otherwise. E.g. $K = K_0(N)$
  $\Rightarrow Q(K) \in N\mathbb{Z}$.

Newform theory for $M_{k,S}$

Show that $g = 0$ if sufficiently many coeff's vanish.

Let $H \subset A = L'/L$ be isotropic subgroup.
$B = H^\perp/H$ is disc. form with $\bar{Q}$.
$|B| = |A|/|H|^2$.

Prop: There are maps

$M_{k,S_B} \longrightarrow M_{k,S_A}$, $g \mapsto g\uparrow^A_H = \sum_{\mu \in H^\perp} g_{\mu+H} e_\mu$.

[$g\uparrow^A_H$ is supported on $H^\perp$]

1) $M_{k,\rho_A} \to M_{k,\rho_B}$, $\quad f \mapsto f\downarrow^A_H = \sum_{\mu \in H^\perp} f_\mu e_{\mu+H}$

Pf: Exercise.

Basic newform lemma:

Prop: Let $f \in M_{k,A}$ and assume f is supported on $H^\perp$ (i.e. $f_\mu = 0$ if $\mu \notin H^\perp$). Then $f_{\mu+\mu'} = f_\mu$ $\forall \mu \in A, \mu' \in H$ and

$$f = \frac{1}{|H|} f \downarrow^A_H \uparrow^A_H .$$

[$f \in M_k(\Gamma_0(N))$, $a_f(n) = 0$ unless $p \mid n$ $\Rightarrow$ $p \mid N$ and $f = f_0(pz)$]

Generalization:

Thm: Let $H_1, \dots, H_m \subset A$ be isotropic subgroups of prime order $p_i = |H_i|$ with $p_i \neq p_j$ for $i \neq j$.
If $f \in M_{k,\rho_A}$ is supported on $H_1^\perp \cup \dots \cup H_m^\perp$, then

$$f = \sum_{\emptyset \neq S \subset \{1,\dots,m\}} (-1)^{|S|+1} \frac{1}{|H_S|} f \downarrow^A_{H_S} \uparrow^A_{H_S} ,$$

where $H_S = \sum_{i \in S} H_i$ (isotr. subgroup).

Combine this with the result on the FE of Λ and the criterion to get

Thm: Assume that $L \cong D \oplus U(N) \oplus U$ for some $N \in \mathbb{Z}_{>0}$ and some pos def lattice D of dim $n-2 > 0$.

Thm: $\Lambda : S_{1+\frac{n}{2}, \rho_L} \longrightarrow \mathcal{H}^{1,1}(X_\Gamma)$ is inj and the converse thm holds.

Cor: Assume $L \cong K \oplus U$ for some even lattice K of sig $(n-1,1)$ where $n \geq 3$. Let $M \subset K$ be a sublattice of finite index that splits a $U(N)$ over $\mathbb{Z}$ and put $B = M'/M$. Then every merom mf F for $\Gamma = \Gamma(L)$ with divisor as in Q is the Borcherds lift of some $f \in M^!_{1-\frac{n}{2}, \rho_B}$.

Rem: There always exists such an M (with $\mathrm{level}(M) = \mathrm{level}(K)$).

Pf of Thm: Let $g = \sum_{\mu \in A} g_\mu e_\mu = \sum_{\mu, m} b(m,\mu) q^m e_\mu$

We only prove the case $N = p =$ prime, i.e. $L \cong D \oplus U(p) \oplus U$.

Identify L with the r.h.s., put $K = D \oplus U(p)$, $A = L'/L = K'/K$.

Let $e = \frac{1}{p}(0,1) \in U(p)' \subset K'$, $e' = \frac{1}{p}(1,0) \in U(p)' \subset K$

Let $\mu \in A$ with $(\mu, e) = \frac{1}{p} + \mathbb{Z}$.

$\Rightarrow \exists\, \lambda \in \mu + K$ s.t. $(\lambda, e) = \frac{1}{p}$.

$\Rightarrow$ For $a \in \mathbb{Z}$ we have

$\lambda + ape \in \mu$, $Q(\lambda + ape) = Q(\lambda) + a$.

Thm $\Rightarrow$ $b(Q(\lambda)+a, \mu) = 0 \ \forall a$

$\Rightarrow$ $g_\mu = 0$.

Use action of $(\mathbb{Z}/p\mathbb{Z})^* \subset O(L)^+$
and equivariance of the lift Λ
to get

$g_\mu = 0 \quad \forall\, \mu \in A$ with $(\mu, e) = \frac{r}{p} + \mathbb{Z}$,
$r \in (\mathbb{Z}/p\mathbb{Z})^*$.

$\Rightarrow g$ is supported on $I^\perp \subset A$,
where $I = \langle e \rangle \subseteq A$ isotr.

$\Rightarrow \quad g = \tilde{g} \uparrow_I^A \quad$ for $\tilde{g} \in S_{1+\frac{k}{2}, \rho_B}$, $B = I^\perp / I$,

Let $L_0 = \underbrace{D \oplus \mathbb{Z}pe' \oplus \mathbb{Z}e}_{\cong K} \oplus U$
Then $\Lambda_L(g) = \Lambda_{L_0}(\tilde{g})$
$\Rightarrow \tilde{g} = 0 \Rightarrow g = 0$

in particular

(*) $\quad g_{\mu+\mu'} = g_\mu \quad \forall\, \mu' \in I$.

For $\lambda = \lambda_D + pa e' + e \in K' \quad (\lambda_D \in D',\ a \in \mathbb{Z})$
we have
$$\bar{\lambda} = \bar{\lambda}_D + \bar{e} \in I^\perp \subset A$$
and $Q(\lambda) = Q(\lambda_D) + a$

$\Rightarrow \quad c(Q(\lambda_D) + a,\ \bar{\lambda}_D + \bar{e}) = 0 \quad \forall\, a$

$\Rightarrow \quad g_{\bar{\lambda}_D + \bar{e}} = 0$

$\Rightarrow \quad g_\mu = 0 \quad \forall\, \mu \in I^\perp$.

$\Rightarrow \quad g = 0$ $\square$