**Other Probability Problems** [5]

## Joint Probability

Sometimes we want to know the probability of two events happening together. For example, what is the probability that it rains today *and* I am wearing blue jeans? Or what is the probability that today is Thursday *or* Friday? Finding such a *joint probability* depends on the relation of the two events to each other. Let's consider the possible relationships.

- The two events are alternate outcomes of the same experiment. For example, $A =$"Today is Thursday" and $B =$"Today is Friday". In this case, the probability of *both A and B* happening is zero (today can't be both Thursday and Friday), and the probability of *either A* or $B$ happening is the sum of the two original probabilities.
  Note: mutually exclusive events are called *disjoint.*

- The two events are independent of each other; that is, whether or not $A$ is true *has NO effect on* whether or not $B$ is true and vice versa. For example, $A =$"Today is Thursday" and $B =$"It's going to rain today" are independent. Take a second to reread the last line. Again, $A$ and $B$ are independent because it being Thursday does not affect whether it rains and vice versa.
  Note that if $B$ were instead "I'm wearing blue jeans today", $A$ and $B$ might or might not be independent, depending on whether or not Thursday's activities affect one's choice of clothes.
  If $A$ and $B$ are indeed independent, then the probability of both $A$ *and* $B$ happening is the *product* of their individual probabilities. The probability of $A$ *or* $B$ (or both) happening is the sum of their original probabilities *minus the overlap*: i.e., included inside the probability of $A$ is the probability for $A$ and $B$ happening. This probability is also included inside the probability for $B$ alone. Thus, in order to count the possibility only once, after you add the probabilities for $A$ and $B$, you should subtract back out the second copy of $P(A \text{ and } B)$.

- Of course there is the possibility that $A$ and $B$ are related (i.e., not independent) and not mutually exclusive. We will save this case for later.

Now consider the following scenarios:

1. Suppose I roll an unbiased six-sided die and flip an unbiased coin. What is the probability that:

   (a) I roll a 3 and the coin comes up heads?

   (b) I roll a prime number and the coin comes up heads?

   (c) I roll a prime number or the coin comes up heads (or both) [Henceforth, I will dispense with using the (or both)]?.

[5] Adapted from notes developed by Kribs-Zaleta, et al.

2. Suppose I roll two unbiased six-sided die. What is the probability that:

   (a) their sum is 8?

   (b) their product is 12?

   (c) their sum is 8 and their product is 12?

   (d) their sum is 8 or their product is 12?

**Conditional Probability**

*Conditional probabilities* investigate the effect of event $A$ on the probability of event $B$: given that situation $A$ has occurred, what is the probability that situation $B$ will occur also? For example, suppose that you roll two unbiased twelve-sided dice. Given that the sum of the two dice is 22, what is the probability that one of the two results is a 10? To answer this, we can observe that there are precisely three ways to get a sum of 22: (i) the first die shows a 10 and the second die shows a 12; (ii) the first and second dice both show 11; (iii) the first die shows a 12 and the second die shows a 10. Since each of these ways is equally likely, the probability of having a 10 showing is 2/3. Note: the common notation for conditional probability is written $P(A|B)$ and read "the probability of $A$ given $B$." In particular, we have just said that $P($ one of the results being 10 | sum of 22$)=2/3$.

1. If you roll two unbiased six-sided dice and get an 8, what is the probability that you got it by double 4s?

2. If you roll two unbiased six-sided dice and get an even total, what is the probability that you got it by double 1s?

Although conditional probability can be used for certain gambling situations, a more important application today is in testing situations. For example, suppose that there is a test for a certain disease. If a person with the disease takes the test, then the test will come back positive 95% of the time (like most medical tests it isn't foolproof). On the other hand, the test will show that you are positive 3.5% when you do not have the disease. An important question at this point is:

Given that I test positive, what is the chance that I have the disease?
We can answer this question in the situation described above by considering a chart which show the results for a typical group of 1000 people:

(a) Fill in the chart below with the appropriate numbers for a group of 1000 people if it is known that 20% of the population has the disease. Assume the 1000 people are exactly consistent with the probabilistic description of the text, e.g., 95% of those who have the disease test positive.

| | Have the disease | Don't have the disease |
|---|---|---|
| Test positive | | |
| Test negative | | |

(b) Given that someone in the sample tests negative, what is the probability that (s)he really does not have the disease?

(c) Given that someone in the sample tests positive, what is the probability that (s)he really does have the disease?

3. A routine blood test during your physical comes up positive for LSI (LazyStudentItis). You are shocked, since the incidence rate in your relatively low-risk group is only 1 in 7,000. The doctor tells you that the test is 99.9% accurate in both directions (that is, if you have the disease, the test will correctly diagnose its presence in 99.9% of cases and if you don't have the disease, the test will correctly diagnose its absence in 99.9% of cases).

   (a) How worried should you be? To determine this, answer the question, "Given that you have tested positive once, what is the probability that you actually have the disease?"

   (b) You go back for a second test and it again comes up positive. Given this, what is the probability that you have the disease? Assume the two tests are independent.

**Independence**

In some situations, knowledge that an event happened can affect the probability of another event happening. Sometimes, the knowledge that an event happened will not affect the probability of another event happening. To see this, let's look at an example.

Suppose Curly Que is getting married on Saturday. You don't know her or even that she is getting married. Suppose further that for any given Saturday, the probability that you will go to the movies is .25. Will Curly Que's wedding on a particular Saturday affect your movie making decision? No, and the probability of you going to the movies is still .25. We would say that the events Curly Que getting married on a Saturday and you going to the movies on the same Saturday are *independent* because knowledge of one occurring does not change the probability of the other occurring.

However, suppose instead that *you* were getting married on a Saturday. This would likely change the probability of you going to the movies. For the sake of this argument, I'll assume the probability of you going to the movies is now 0. Thus, we would say that the events you getting married on a Saturday and you going to the movies on the same Saturday are *not* independent.

In general, two events $A$ and $B$ are independent if $P(A|B) = P(A)$.

1. For the following problem, use the information below:

   A single card is selected from a standard 52-card deck.
   $B$ = the drawn card is black; $R$ = the drawn card is red
   $Q$ = the drawn card is a queen; $F$ = the drawn card is a face card (i.e. a jack, queen, or king)

   (a) Determine if $B$ and $R$ are independent.

   (b) Determine if $R$ and $Q$ are independent.

   (c) Determine if $Q$ and $F$ are independent.

2. Is having a girl first independent of having 1 boy and 1 girl in a two-child family?

3. Suppose that someone makes the claim that smoking cigarettes is independent of getting good grades.

   (a) What do they mean?

   (b) What questions should be answered to determine if their claim is correct?

**Pay to Play**
Suppose that you were going to play each of the following games repeatedly. How much should you be willing to pay in order to break even - that is, in order for the expected value of your winnings to match exactly the cost of playing (so that neither you nor "the house" comes out ahead).

1. Flip a coin. If heads, shows you win \$3. If tails shows, you win \$7.

2. A spinner is labeled in such a way that you have 30% chance of getting a 1, a 25% chance of getting a 2 and an equal chance of getting a 3 or 4. You win the amount of the number that the spinner lands on.

3. Roll two unbiased dice repeatedly, until at least one of them is a six. If the other die is also a six, then you win \$36; otherwise you win the amount shown on the other die.

**Fair Games**
A game is considered *fair* if it has an expected value of zero. Determine if the following games are fair.

1. Roll two dice at once. If their sum is 2, 3, 5, 10, 11, or 12, player A wins \$1; otherwise, player A loses \$1.

2. Roll two dice at once. If their difference is odd, player A wins \$1; otherwise, player A loses \$1.

3. Roll two dice at once. If their product is odd, player A wins; otherwise, player A loses \$1.

4. American roulette wheels typically have 38 equally spaced slots numbered 0, 00, and 1 through 36. A player who bets \$1 that an odd number comes up (0 and 00 are considered neither even nor odd) wins \$1 and gets the original \$1 bet back. If the player loses, the \$1 goes to the house (or casino).

NAME ______________________

NAME ______________________

NAME ______________________

NAME ______________________

NAME ______________________

**Group Number** ________________

Turn in one copy of your answers to this assignment per group. Record your answers to the questions below.

1. Suppose I roll two unbiased six-sided die. What is the probability that:
   (a) their sum is 9?
   (b) their product is 20?
   (c) their sum is 9 and their product is 20?
   (d) their sum is 9 or their product is 20?

2. A routine blood test during your physical comes up positive for LSI (LazyStudentItis). You are shocked, since the incidence rate in your relatively low-risk group is only 1 in 100,000. The doctor tells you that the test is 99.9% accurate in both directions (that is, if you have the disease, the test will correctly diagnose its presence in 99.9% of cases and if you don't have the disease, the test will correctly diagnose its absence in 99.9% of cases).

   So, to determine how upset you should be, you decide to answer the following question: Given that you tested positive for the disease, what is the probability that you have the disease?

3. Is having at most one boy independent of having all of the children of the same gender in a two-child family? in a three child family?

4. What does it mean for a game to be fair?