∞-Borel codes in natural models of AD^+

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Abstract

We work under AD^+. The main result of this paper is that assuming $\mathsf{AD}_{\mathbb{R}}$, for every $\kappa < \Theta$, letting $X = \mathcal{P}(\kappa)$, every $A \subseteq X$ has an ∞-Borel code; furthermore, if $V = L(\mathcal{P}(\mathbb{R}))$ holds and $\kappa < \Theta$, every OD set $A \subseteq X$ has an OD ∞-Borel code. These results led us to the formulation of AD^{++}, which is the theory "$\mathsf{AD}^+ +$ "for every $\kappa < \Theta$, for every $A \subseteq \mathcal{P}(\kappa)$, A has an ∞-Borel code". It is not known whether AD^+ implies AD^{++}. AD^{++} has structural consequences that are not known to follow from AD^+. One such instance is the $ABCD$ Conjecture.

1 Introduction

This paper deals with the topic of ∞-Borel codes, which are generalizations of Borel codes for Borel sets. Borel codes are reals that canonically code a Borel set of reals. ∞-Borel codes are sets of ordinals that canonically code (often times) much more complicated sets of reals or elements of the space λ^κ for some ordinals κ, λ. ZFC implies that every set of reals is Suslin and therefore, has an ∞-Borel code; however, it is not known that the theory $\mathsf{ZF} + \mathsf{AD}$ implies this. The axiom AD^+, due to W. H. Woodin, is a strengthening of AD. Part of AD^+ stipulates that every set of reals has an ∞-Borel code. It is not known AD implies AD^+, but every known model of AD satisfies AD^+.

∞-Borel codes have a number of applications within the general AD^+ theory. For example, under ZF, suppose there are no uncountable sequences of distinct reals and every subset of $\mathcal{P}(\omega)$ has an ∞-Borel code, then every set of reals has the Ramsey property. In particular, AD^+ implies this regularity property for sets of reals. It is not known if AD implies this.

This paper gives partial answers to the following two questions about ∞-Borel codes under AD^+.

(i) Given a set A, can one construct an ∞-Borel code that is relatively simple (in definability) compared to the complexity of A?

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(ii) For a cardinal $\kappa > \omega$, are subsets of $\mathcal{P}(\kappa)$ ∞-Borel?

Regarding (i), Woodin has shown the following unpublished theorem, concerning the definability of ∞-Borel codes under AD^+.

Theorem 1.1 (Woodin). *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Suppose $X = \mathcal{P}(\omega)$ or ${}^\omega\omega$ and $A \subseteq X$ is OD. Then A has an OD ∞-Borel code. Suppose furthermore that $V = L(S, \mathbb{R})$ for some set $S \subset ON$, then every $OD(S)$ $A \subseteq X$ has an $OD(S)$ ∞-Borel code.*

Remark 1.2. *The proof of the "futhermore" clause of Theorem 1.1 can be easily adapted from a proof of a special case when $V = L(\mathbb{R})$ given in [Cha19] or the proof of the first part of the theorem. The main challenge is the proof of the first part of the theorem.*

In [IT18, Therem 5], Ikegami and the third author prove the following theorem.[1] We will outline a proof here. In the following, Θ is the supremum of ordinals α such that there is a surjection from $\mathbb{R}$ onto α and θ_0 is the supremum of ordinals α such that there is an OD surjection from $\mathbb{R}$ onto α

Theorem 1.3. *Assume AD^+. Suppose $\kappa < \Theta$, $X = {}^\omega\kappa$ and $A \subseteq X$, then A has an ∞-Borel code. Additionally, suppose $V = L(\mathcal{P}(\mathbb{R}))$ and either $\Theta = \theta_0$ or $\mathsf{AD}_\mathbb{R}$, then for any set of ordinals S, for every $OD(S)$ $A \subseteq X$, A has an $OD(S)$ ∞-Borel code.*

The above theorems have the following corollary.

Corollary 1.4.

1. *(Woodin) Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then for any $x \in \omega^\omega$, $\mathrm{HOD}_x = \mathrm{HOD}[x]$. Furthermore, suppose for some set of ordinals S, $V = L(S, \mathbb{R})$, then for any such x, $\mathrm{HOD}_{S,x} = \mathrm{HOD}_S[x]$.*
2. *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. Additionally, assume either $\Theta = \theta_0$ or $\mathsf{AD}_\mathbb{R}$, then for any set of ordinals S, for any $\kappa < \Theta$, for any $x \in \kappa^\omega$, $\mathrm{HOD}_{S,x} = \mathrm{HOD}_S[x]$.*

The proof of Corollary 1.4 gives a bit more than what's stated. See Remark 5.6. We will prove these theorems and use them to prove the following improvement. Theorem 1.5, partially answers (ii), is the main theorem of the paper.

Theorem 1.5. *Assume AD^+ and ω_1 is $\mathbb{R}$-supercompact. Suppose $\kappa < \Theta$, $X = \mathcal{P}(\kappa)$ and $A \subseteq X$, then A has an ∞-Borel code. Furthermore, assume additionally $V = L(\mathcal{P}(\mathbb{R})) \models \mathsf{AD}_\mathbb{R}$, suppose $A \subseteq X$ is OD_S for some set of ordinals S, then A has an $OD(S)$ ∞-Borel codes.*

Theorem 1.5 has the following corollary.

[1] The authors of [IT18] did not state the theorem this way. Furthermore, to prove the first clause of [IT18, Therem 5], one does not need the supercompactness of ω_1, strong compactness suffices.

Corollary 1.6. *Assume* $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$*. For any set of ordinals* S*, for any* $\kappa < \Theta$*, for any* $x \subseteq \kappa$*,* $\mathrm{HOD}_{S,x} = \mathrm{HOD}_S[x]$*.*

It is known that Woodin's theorem (Theorem 1.1) cannot be extended beyond ω in the situation where $\mathsf{AD}_{\mathbb{R}}$ fails. Woodin (unpublished) shows that if $\mathsf{AD}^+ + \neg\mathsf{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$ holds , then there is an uncountable κ (e.g. $\kappa = \omega_1$) and a $t \subseteq \kappa$ such that $\mathrm{HOD}_t \neq \mathrm{HOD}[t]$. Inspecting the proof of Corollary 1.6, one sees that this implies there is an OD set $A \subseteq \mathcal{P}(\kappa)$ that has no OD ∞-Borel codes.

[CJ] also used Theorem 1.1 to prove an analog of a result of Harrington-Slaman-Shore [HSS17] concerning the pointclass Σ^1_1: Assuming AD^+ and $V = L(\mathcal{P}(\mathbb{R}))$, if $H \subseteq \mathbb{R}$ has the property that there is a nonempty OD set $K \subseteq \mathbb{R}$ so that H is OD_z for all $z \in K$, then H is OD.

We propose the following principle that strengthens AD^+.

Definition 1.1 (AD^{++})**.** AD^{++} is the theory $\mathsf{AD}^+ +$ "for every $\kappa < \Theta$, for every $A \subseteq \mathcal{P}(\kappa)$, A has an ∞-Borel code".

Theorem 1.5 shows that $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ implies AD^{++}. In general, it is not known that AD^+ implies AD^{++}. AD^{++} seems to yield structural properties not known to follow from AD^+.

One such type of structural properties concerns distinguishing cardinalities of infinite sets under AD^+. This is a fundamental problem in set theory. Let X, Y be two sets. Cantor's original formulation of cardinalities states that X, Y have the same cardinality (denoted $|X| = |Y|$) if and only if there is a bijection $f : X \to Y$. $|X| \leq |Y|$ if and only if there is an injection of X into Y. And $|X| < |Y|$ if and only if $|X| \leq |Y|$ but $\neg(|Y| \leq |X|)$. The Axiom of Choice (AC) implies that every set is well orderable, and hence the class of cardinalities forms a wellordered class under the injection relation. Under AD, the class of cardinalities is not wellorderable; in fact, $\neg(|\mathbb{R} \leq |\omega_1|)$ and $\neg(|\omega_1| \leq |\mathbb{R}|)$. The following conjecture gives a sufficient and necessary condition for when the cardinalities of two sets of the form α^β, γ^δ for infinite cardinals $\alpha, \beta, \gamma, \delta$ are comparable.

Conjecture 1.7 (The $ABCD$ Conjecture)**.** *Assume* ZF*. Let* $\alpha, \beta, \gamma, \delta < \Theta$ *be infinite cardinals. Suppose* $\beta \leq \alpha, \delta \leq \gamma$*. Then*

$$|\alpha^\beta| \leq |\gamma^\delta| \text{ if and only if } \beta \leq \delta \text{ and } \alpha \leq \gamma.$$

Some remarks are in order about the conjecture. First, the conjecture implies in particular that if $\delta < \beta$ or if $\gamma < \alpha$, then α^β cannot inject into γ^δ. One easily sees that ZFC implies the failure of the $ABCD$ Conjecture; one can see that by, for instance, noticing that ZFC implies $|\omega^\omega| \geq |\omega_1^\omega|$[2]; in this case, $\gamma = \omega < \alpha = \omega_1$, yet ω_1^ω injects into ω^ω. The conjecture deals with the case $\beta \leq \alpha, \delta \leq \gamma$ being infinite cardinals, but the other cases either have been known to follow from AD^+ or can simply be reduced to the cases the conjecture deals

[2] If ZFC holds, Θ is the successor of the continuum and $\omega_1 < \Theta$.

with. For instance, if $\beta > \alpha$ and $\delta > \gamma$, then $|\alpha^\beta| = |\mathcal{P}(\beta)|$ and $|\gamma^\delta| = |\mathcal{P}(\delta)|$. AD^+ implies that $|\mathcal{P}(\beta)| \leq |\mathcal{P}(\delta)|$ if and only if $\beta \leq \delta < \Theta$. If $\beta > \alpha$ and $\delta \leq \gamma$, then we really compare $|\beta^\beta|$ and $|\gamma^\delta|$. It is important here that the cardinals in the conjecture are infinite and are $< \Theta$. For instance, when $\beta = 1$, α is an infinite cardinal $> \gamma \geq \delta$, then $|\alpha^\beta| = |\alpha|$ and AD^+ implies that α cannot inject into $\mathcal{P}(\gamma)$ and therefore cannot inject into γ^δ if $\alpha < \Theta$. On the other hand, $\alpha = 3$ can inject into $\mathcal{P}(\gamma)$ for $\gamma = 2$, or for example, $\alpha = \gamma^+$ and $\gamma \geq \Theta$, then α does inject into $\mathcal{P}(\gamma)$ if $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ holds. Also, if $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ holds, it is easy to see that $(\Theta^+)^\omega$ injects into Θ^Θ; this shows the failure of the conjecture for $\alpha = \Theta^+, \beta = \omega, \delta = \gamma = \Theta$.

The first author has recently shown that AD^{++} implies the $ABCD$ Conjecture. This result will appear in an upcoming paper. The result of this paper and the first author's aforementioned work show that the $ABCD$ Conjecture is a consequence of $\mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. It is not known that AD^+ implies the $ABCD$ Conjecture, though many specific instances of this conjecture have been established. See for example [CJT24, CJT22, CJT23, Woo06].

In Section 2, we review basic facts about AD^+ and ∞-Borel codes. In Section 3, we review homogeneous and weakly homogeneous sets in AD^+. In Section 4, we review Vopěnka algebras, which is a key tool in producing ∞-Borel codes in the AD^+ context. We prove 1.1–1.6 in Section 5. Some conjectures and open questions are presented in Section 6.

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2 AD^+ and ∞-Borel codes

We now review basic notions on determinacy axioms. For a nonempty set X, the **Axiom of Determinacy in X^ω** ($\mathsf{AD}_\mathbf{X}$) states that for any subset A of X^ω, in the Gale-Stewart game with the payoff set A, one of the players must have a winning strategy. We write AD for AD_ω. The ordinal $\boldsymbol{\Theta}$ is defined as the supremum of ordinals which are surjective images of $\mathbb{R}$. Under ZF+AD, Θ is very big, e.g., it is a limit of measurable cardinals while under ZFC, Θ is equal to the successor cardinal of the continuum $|\mathbb{R}|$. **Ordinal Determinacy** states that for any $\lambda < \Theta$, any continuous function $\pi\colon \lambda^\omega \to \omega^\omega$, and any $A \subseteq \omega^\omega$, in the Gale-Stewart game with the payoff set $\pi^{-1}(A)$, one of the players must have a winning strategy. In particular, Ordinal Determinacy implies AD while it is still open whether the converse holds under ZF+DC.

For $\lambda < \Theta$, we write $\mathcal{P}_\lambda(\mathbb{R})$ for the set of $A \subseteq \mathbb{R}$ such that the Wadge rank of A is $< \lambda$. For any set X, we write $\wp_{\omega_1}(X)$ for the set of countable subsets of X. We write $\mathcal{D}$ for the set of Turing degrees. For $x, y \in \omega^\omega$, we write $x \leq_T y$, $x \equiv_T y$ for x is Turing reducible to y and x is Turing equivalent to y respectively. A Turing degree has the form $[x]_T = \{y \in \omega^\omega : x \equiv_T y\}$.

We will introduce the notion of ∞-Borel codes. Before that, we review some terminology on trees. Given a set X, a **tree on** X is a collection of finite sequences of elements of X closed under initial segments. Given an element t of $X^{<\omega}$, $\mathrm{lh}(t)$ denotes its length, i.e., the domain or the cardinality of t. Given a tree T on X and elements s and t of T, s is an **immediate successor of** t **in** T if s is an extension of t and $\mathrm{lh}(s) = \mathrm{lh}(t) + 1$. Given a tree T on X and an element t of T, $\mathrm{Succ}_T(t)$ denotes the collection of all immediate successors of t in T. An element t of a tree T on X is **terminal** if $\mathrm{Succ}_T(t) = \emptyset$. For an element t of a tree T on X, $\mathrm{term}(T)$ denotes the collection of all terminal elements of T. Given a tree T on X, $[T]$ denotes the collection of all $x \in X^\omega$ such that for all natural numbers n, $x \upharpoonright n$ is in T. A tree T on X is **well-founded** if $[T] = \emptyset$. We often identify a tree T on $X \times Y$ with a subset of the set $\{(s,t) \in X^{<\omega} \times Y^{<\omega} \mid \mathrm{lh}(s) = \mathrm{lh}(t)\}$, and $\mathrm{p}[T]$ denotes the collection of all $x \in X^\omega$ such that there is a $y \in Y^\omega$ with $(x,y) \in [T]$.

Definition 2.1. Let λ, κ be non-zero ordinals.

1. An ∞**-Borel code in** λ^κ is a pair (T, ρ) where T is a well-founded tree on some ordinal γ, and ρ is a function from $\mathrm{term}(T)$ to $\kappa \times \lambda$.

2. Given an ∞-Borel code $c = (T, \rho)$ in λ^κ, to each element t of T, we assign a subset $B_{c,t}$ of λ^κ by induction on t using the well-foundedness of the tree T as follows:

   (a) If t is a terminal element of T, let $B_{c,t}$ be the basic open set $O_{\rho(t)}$ in λ^κ. Here $\rho(t)$ is a pair of ordinals $(\alpha, \beta) \in \kappa \times \lambda$ and $O_{\rho(t)}$ has the form $\{f \in \lambda^\kappa : \rho(t) \in f\}$.

   (b) If $\mathrm{Succ}_T(t)$ is a singleton of the form $\{s\}$, let $B_{c,t}$ be the complement of $B_{c,s}$ in the space λ^κ.

   (c) If $\mathrm{Succ}_T(t)$ has more than one element, then let $B_{c,t}$ be the union of all sets of the form $B_{c,s}$ where s is in $\mathrm{Succ}_T(t)$.

   We write B_c for $B_{c,\emptyset}$.

3. A subset A of λ^κ is ∞**-Borel** if there is an ∞-Borel code c in λ^κ such that $A = B_c$.

We will identify $\mathcal{P}(\lambda)$ with 2^λ. So an ∞-Borel code for $A \subseteq \mathcal{P}(\lambda)$ is an ∞-Borel code for a subset of 2^λ. We can generalize the above definitions of ∞-Borel codes in a number of ways. One way is we can replace λ in Definition 2.1 by a set of ordinals S. The definition of an ∞-Borel code for a set $A \in \mathcal{P}(S^\kappa)$ is modified in an obvious way from Definition 2.1. We can also generalize the definition of ∞-Borel codes in $\lambda_1^{\kappa_1} \times \cdots \times \lambda_n^{\kappa_n}$ for some $n \in \omega$ (with the product topology) in an obvious way. We leave the details to the reader.

We will also use the following characterization of ∞-Borelness:

Fact 2.1. *Let λ, κ be a non-zero ordinals and A be a subset of λ^κ. Then the following are equivalent:*

1. A is ∞-Borel, and

2. *for some formula ϕ and some set S of ordinals, for all elements x of λ^κ, x is in A if and only if $L[S,x] \vDash$ "$\phi(S,x)$".*

Proof. For the case $\lambda = 2$, one can refer to [Lar, Theorem 8.7]. The general case can be proved in the same way. □ □

Remark 2.2. *In fact, the second item of Fact 2.1 is equivalent to the following using Lévy's Reflection Principle:*

- *for some $\gamma > \lambda, \kappa$, some formula ϕ, and some set S of ordinals, for all elements x of λ^κ, x is in A if and only if $L_\gamma[S,x] \vDash$ "$\phi(S,x)$".*

Throughout this paper, we will freely use either of the equivalent conditions of ∞-Borelness.

We now introduce the axiom AD^+, and review some notions on Suslin sets. The axiom AD^+ states that (a) $\mathsf{DC}_{\mathbb{R}}$ holds, (b) Ordinal Determinacy holds, and (c) every subset of ω^ω is ∞-Borel. A subset A of ω^ω is **Suslin** if there are some ordinal λ and a tree T on $\omega \times \lambda$ such that $A = \mathrm{p}[T]$. A is **co-Suslin** if the complement of A is Suslin. An infinite cardinal λ is a **Suslin cardinal** if there is a subset A of ω^ω such that there is a tree on $\omega \times \lambda$ such that $A = \mathrm{p}[T]$ while there are no $\gamma < \lambda$ and a tree S on $\omega \times \lambda$ such that $A = \mathrm{p}[S]$. Under $\mathsf{ZF}+\mathsf{DC}_{\mathbb{R}}$, AD^+ is equivalent to the assertion that Suslin cardinals are closed below Θ in the order topology of $(\Theta, <)$. Another equivalence that is often useful in applications is the statement that $\mathsf{AD} + V = L(\mathcal{P}(\mathbb{R}))$ holds and every Σ_1 statement with Suslin co-Suslin sets as parameters true in V is true in a transitive model M of $\mathsf{ZF}^- + \mathsf{DC}_{\mathbb{R}}$ coded by a Suslin co-Suslin set of reals A. We call this **Σ_1-reflection into the Suslin co-Suslin sets** (or sometimes just Σ_1-reflection). Another form of Σ_1-reflection that is also useful is **Σ_1-reflection into the $\underset{\sim}{\Delta}^2_1$ sets**, which says that $\mathsf{AD}+ V = L(\mathcal{P}(\mathbb{R}))$ holds and every Σ_1 statement with $\underset{\sim}{\Delta}^2_1$ sets as parameters true in V is true in a transitive model M of $\mathsf{ZF}^- + \mathsf{DC}_{\mathbb{R}}$ coded by a $\underset{\sim}{\Delta}^2_1$ set of reals A.

The sequence $(\theta_\alpha : \alpha \leq \Omega)$ is called the **Solovay sequence** and is defined as follows. θ_0 is the supremum of ordinals α such that there is an OD surjection $\pi : \mathbb{R} \to \alpha$. For limit $\alpha \leq \Omega$, $\theta_\alpha = \sup_{\beta<\alpha}\theta_\beta$. Suppose θ_α has been defined for $\alpha < \Omega$, letting $A \subseteq \mathbb{R}$ be of Wadge rank θ_α, $\theta_{\alpha+1}$ is the supremum of α such that there is an $OD(A)$ surjection $\pi : \mathbb{R} \to \alpha$. $\Theta = \theta_\Omega$.

The following fundamental facts about AD^+ are due to Woodin.

Theorem 2.3 (Woodin)**.** *Assume $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. The following hold.*

1. *$V = L(J, \mathbb{R})$ for some set of ordinals J if and only if $\mathsf{AD}_{\mathbb{R}}$ fails.*

2. *For any real x, $HOD_x = L[Z]$ for some $Z \subseteq \Theta$.*

We will not prove Theorem 2.3. Instead, we will discuss some key ingredients that go into the proof. The proof of part (2) can be found in [Tra14b]. The set Z basically codes a Vopěnka algebra, to be discussed in the next section.

For part (1), let κ be the largest Suslin cardinal and $S(\kappa)$ be the class of all κ-Suslin sets. If $\mathsf{AD}_{\mathbb{R}}$ fails, $\kappa < \Theta$. In that case, let T be a tree projecting to a universal κ-Suslin set and define the equivalence relation $\equiv_T$ on $\mathbb{R}$ as: $x \equiv_T y$ iff $L[T,x] = L[T,y]$. We also define $x \leq_T y$ iff $x \in L[T,y]$. The measure μ_T on $\mathbb{R}/\equiv_T$ is defined as: $A \in \mu_T$ iff $\exists x\ \{y : x \leq_T y\} \subseteq A$. μ_T is non-principal and countably complete. Let $J = [x \mapsto T]_{\mu_T}$. One can show $V = L(J, \mathbb{R})$.

We end this section by proving a basic fact concerning supercompact measures on $\wp_{\omega_1}(X)$ for some set X. Assume $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$. Let X be a set such that there is a surjection $\pi : \mathbb{R} \to X$. Let μ be the Solovay measure. By a theorem of Solovay, cf. [Sol06], $\mathsf{AD}_{\mathbb{R}}$ implies μ exists and is the club filter on $\wp_{\omega_1}(\mathbb{R})$. Let μ_X be the measure on $\wp_{\omega_1}(X)$ induced by μ and π. This means μ_X is defined as: for any $A \subseteq \wp_{\omega_1}(X)$,

$$A \in \mu_X \Leftrightarrow \pi^{-1}[A] \in \mu.$$

By a theorem of Woodin (cf. [Woo83]), μ_X is the unique normal, fine, countably complete measure on $\wp_{\omega_1}(X)$. In fact, μ_X is just the club filter on $\wp_{\omega_1}(X)$.

Fact 2.4. *Assume $V = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$. The ultrapower $Ult(V, \mu_X)$ is well-founded.*

Proof. Suppose not. By Σ_1-reflection, there is a transitive model N of the form $L_\alpha(\mathcal{P}_\beta(\mathbb{R}))$ for $\alpha, \beta < \Theta$ that satisfies $\mathsf{ZF}^- + \mathsf{AD}_{\mathbb{R}}$, $\mathbb{R} \cup \wp_{\omega_1}(X) \subset N$, and $N \models$ "the ultrapower $M = \mathrm{Ult}(V, \mu_X)$ is ill-founded". Now since μ_X is the club measure on $\wp_{\omega_1}(X)$,

$$\mu_X^N = \mu_X \cap N.$$

Since $\mathsf{DC}_{\mathbb{R}}$ holds and that there is a surjection from $\mathbb{R}$ onto N, we can find a sequence $(f_n : n < \omega)$ such that $([f_n]_{\mu \cap N} : n < \omega)$ witnesses the ill-foundedness of the ultraproduct in N. Let $A_n = \{\sigma : f_{n+1}(\sigma) \in f_n(\sigma)\}$ for each n. Then $A_n \in \mu \cap N$ for each n. By countable completeness of μ, $\bigcap_n A_n \neq \emptyset$. Let $\sigma \in \bigcap_n A_n$. Then the sequence $(f_n(\sigma) : n < \omega)$ is a ϵ-descending sequence. Contradiction. $\square$

3 Homogeneously Suslin sets and applications

We summarize basic facts about (weakly) homogeneously Suslin sets. For a more detailed discussion, the reader should consult for example [Ste09]. Recall we identify the set of reals $\mathbb{R}$ with the Baire space ${}^\omega\omega$.

Given an uncountable cardinal κ, and a set Z, $meas_\kappa(Z)$ denotes the set of all κ-additive measures on $Z^{<\omega}$. If $\mu \in meas_\kappa(Z)$, then there is a unique $n < \omega$ such that $Z^n \in \mu$ by κ-additivity; we let this $n = dim(\mu)$. If $\mu, \nu \in meas_\kappa(Z)$, we say that μ *projects to* ν if $dim(\nu) = m \leq dim(\mu) = n$ and for all $A \subseteq Z^m$,

$$A \in \nu \Leftrightarrow \{u : u \restriction m \in A\} \in \mu.$$

For each $\mu \in meas_\kappa(Z)$, let $j_\mu : V \to Ult(V, \mu)$ be the canonical ultrapower map by μ. In this case, there is a natural embedding from the ultrapower of V by ν into the ultrapower of V by μ:

$$\pi_{\nu,\mu} : Ult(V,\nu) \to Ult(V,\mu)$$

defined by $\pi_{\nu,\mu}([f]_\nu) = [f^*]_\mu$ where $f^*(u) = f(u \restriction m)$ for all $u \in Z^n$. A tower of measures on Z is a sequence $\langle \mu_n : n < k \rangle$ for some $k \leq \omega$ such that for all $m \leq n < k$, $dim(\mu_n) = n$ and μ_n projects to μ_m. A tower $\langle \mu_n : n < \omega \rangle$ is *countably complete* if the direct limit of $\{Ult(V,\mu_n), \pi_{\mu_m,\mu_n} : m \leq n < \omega\}$ is well-founded. We will also say that the tower $\langle \mu_n : n < \omega \rangle$ is well-founded.

Definition 3.1. Given a tree T on $\omega \times \kappa$, a **homogeneity system for** T is a system $\langle \mu_s : s \in \omega^{<\omega} \rangle$ of countably complete measures on $\kappa^{<\omega}$ such that for all $s, t \in \omega^{<\omega}$ and $x \in \omega^\omega$, the following hold:

- $\mu_s(T_s) = 1$, here $T_s = \{t \in \kappa^{|s|} : (s,t) \in T\}$,
- $s \subseteq t \Rightarrow \mu_t$ projects to μ_s, and
- $x \in p[T] \Rightarrow \langle \mu_{x \restriction n} : n < \omega \rangle$ is wellfounded.

If such a system exists for T, we say that T is **homogeneous**.

$A = p[T]$ is κ**-homogeneous** if the measures $\langle \mu_s : s \in \omega^{<\omega} \rangle$ are κ-complete. A is $< \gamma$-homogeneous if it is κ-homogeneous for all $\kappa < \gamma$.

Definition 3.2. The tree T on $\omega \times \kappa$ is **weakly homogeneous** if there is a weak homogeneity system $\bar{\mu}$ associated with T, i.e. there is a system $\langle M_s : s \in \omega^{<\omega} \rangle$ such that the following hold:

- for each s, M_s is a countable set of countably complete measures on $\kappa^{<\omega}$ such that for each $\mu \in M_s$, $\mu(T_s) = 1$, and
- $x \in p[T] \Rightarrow$ there is a wellfounded tower $\langle \mu_n : n < \omega \rangle$ such that $\forall n\ \mu_n \in M_{x \restriction n}$.

$A = p[T] \subseteq \mathbb{R}$ is κ**-weakly homogeneous** iff the measures in the weak homogeneity system $\bar{\mu}$ associated with T are κ-complete. A is $< \gamma$-weakly homogeneous if it is κ-weakly homogeneous for all $\kappa < \gamma$.

Here are some facts about homogeneous sets and weakly homogeneous sets under AD and AD^+. Part (iii) of the theorem is an improvement of part (ii). We will only need part (i) of the theorem in this paper; but we state parts (ii) and (iii) for completeness.

Theorem 3.1. *(i) (Martin, [MS08]) Assume* AD *and suppose* $A \subseteq \mathbb{R}$ *is Suslin co-Suslin, then* A *is* $< \Theta$*-homogeneously Suslin.*

(ii) (Martin-Woodin, [MW08]) Assume $\mathsf{AD}_\mathbb{R}$*. Then every tree is* $< \Theta$*-weakly homogeneous.*

(iii) (Woodin, [Lar23]) Assume AD^+*. Then every tree* T *on* $\omega \times \kappa$ *for* κ *less than the largest Suslin cardinal is* $< \Theta$*-weakly homogeneous and hence every Suslin co-Suslin set of reals is* $< \Theta$*-weakly homogeneous.*

Theorem 3.1 allows us to prove the following facts.

Lemma 3.2. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$*. Then for any set C of reals that is Suslin co-Suslin, there is an $s \in \bigcup_{\gamma<\Theta} \gamma^\omega$ such that C is OD from s and that C is in* $\mathrm{HOD}_{\{s\}}(\mathbb{R})$*. If in addition* $\mathsf{AD}_{\mathbb{R}}$ *holds, then any set of reals C is OD from some $s \in \bigcup_{\gamma<\Theta} \gamma^\omega$ and that $C \in \mathrm{HOD}_{\{s\}}(\mathbb{R})$.*

Proof. See [IT23, Lemma 2.11].

□

We also have the following variation, in fact a refinement, of the above lemma that will be useful in Section 5.

Lemma 3.3. *Assume* $\mathsf{ZF} + \mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$*. Let* $\mathbb{C} = \mathrm{HOD}(\bigcup_{\gamma<\Theta} \gamma^\omega)$*. Let A be Suslin co-Suslin, then $A \in \mathbb{C}$.*

Proof. Let A be Suslin co-Suslin. By Theorem 3.1, $A = p[T]$ where T is homogeneously Suslin witnessed by the sequence $(\mu_u \mid u \in \omega^{<\omega})$ of measures on $\kappa^{<\omega}$ for some $\kappa < \Theta$. By a theorem of Kunen, all countably complete measures on $\kappa^{<\omega}$ are OD; in fact, there is an OD injection $f : meas_{\omega_1}(\kappa^{<\omega}) \to ON$. We let $s \in ON^\omega$ enumerate the parameters defining $(\mu_u \mid u \in \omega^{<\omega})$, that is $s = f[\{\mu_u : u \in \omega^{<\omega}\}]$. Now the set

$$R = \{(u, \alpha, \beta) : u \in \omega^{<\omega} \wedge j_{\mu_u}(\alpha) = \beta\}$$

is well-orderable, and $R \in \mathrm{HOD}[s]$. Now we can compute the Martin-Solovay tree T' of T inside of $\mathrm{HOD}[s]$ using R, where

$$(t, \vec{\alpha}) \in T' \Leftrightarrow \forall i < |t| \; j_{\mu_{t \restriction i+1}}(\vec{\alpha}(i)) > \vec{\alpha}(i+1).$$

Now for any $x \in \mathbb{R}$,

$$x \notin A \Leftrightarrow x \in p[T'] \Leftrightarrow \text{the tower } (\mu_{x\restriction n} : n < \omega) \text{ is ill-founded.}$$

The illfoundedness of the tower $(\mu_{x\restriction n} : n < \omega)$ can be computed in $\mathrm{HOD}[s][x]$ using T', R. So $\mathrm{HOD}[s][x]$ can decide whether $x \notin A$, equivalently whether $x \in A$.

The above sketch shows that $A, \neg A \in HOD[s](\mathbb{R})$ and hence $A \in \mathbb{C}$. □

4 Vopěnka algebras

We next introduce Vopěnka algebras and their variants we will use in this paper. In this section, all definitions assume the hypothesis $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$. The results of this section are all essentially due to W. H. Woodin. Recall the definition of forcing projection maps $\sigma : \mathbb{Q} \to \mathbb{P}$ between posets $\mathbb{Q}$ and $\mathbb{P}$ as defined in [Cha19, Section 7]. As a matter of notation, we write $1_{\mathbb{P}}$ for the weakest condition in $\mathbb{P}$.

Definition 4.1. Let γ be a non-zero ordinal $< \Theta$ and T be a set of ordinals.

1. Let n be a natural number with $n \geq 1$ and $\mathcal{O}^T_{\gamma,n}$ be the collection of all nonempty subsets of $(\gamma^\omega)^n$ which are OD from T. Fix a bijection $\pi_n\colon \eta \to \mathcal{O}^T_{\gamma,n}$ which is OD from T, where η is some ordinal. Let $\mathbb{Q}^T_{\gamma,n}$ be the poset on η such that for each p, q in $\mathbb{Q}^T_{\gamma,n}$, we have $p \leq q$ if $\pi_n(p) \subseteq \pi_n(q)$. We call $\mathbb{Q}^T_{\gamma,n}$ the *Vopěnka algebra for adding an element of* $(\gamma^\omega)^n$ *in* $\mathrm{HOD}_{\{T\}}$.

2. For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $\sigma_{m,\ell}\colon \mathbb{Q}^T_{\gamma,m} \to \mathbb{Q}^T_{\gamma,\ell}$ be the natural map induced from π_ℓ and π_m, i.e., for all $p \in \mathbb{Q}^T_{\gamma,m}$, $\pi_l\big(\sigma_{m,\ell}(p)\big) = \{x \mid \exists y \in \pi_m(p)\ y \restriction \ell = x\}$. Then each $\sigma_{m,\ell}$ is a projection between posets. Let $\big(\mathbb{Q}^T_{\gamma,\omega}, (\sigma_n\colon \mathbb{Q}^T_{\gamma,\omega} \to \mathbb{Q}^T_{\gamma,n} \mid n < \omega)\big)$ be the inverse limit of the system $(\sigma_{m,\ell}\colon \mathbb{Q}^T_{\gamma,m} \to \mathbb{Q}^T_{\gamma,\ell} \mid 1 \leq \ell \leq m < \omega)$. We call $\mathbb{Q}^T_{\gamma,\omega}$ the *inverse limit of Vopěnka algebras for adding an element of* $(\gamma^\omega)^\omega$ *in* $\mathrm{HOD}_{\{T\}}$.

3. When $T = \emptyset$ or T is OD, then we omit it from our notation. Similarly, when $\gamma = 2$, we omit it from our notation. In particular, we denote $\mathbb{Q}_n$ the Vopěnka algebra for adding an element of $(2^\omega)^n$ in HOD.

Definition 4.2. Let γ be a non-zero ordinal $< \Theta$ and T be a set of ordinals.

1. Let n be a natural number with $n \geq 1$ and let $\mathbb{BC}^{T,*}_{\gamma,n}$ be the poset consisting of $OD(T)$ ∞-Borel codes for subsets of $(\gamma^\omega)^n$ with the ordering $p \leq q$ if $B_p \subseteq B_q$. We define the equivalence relation $\sim$ on $\mathbb{BC}^{T,*}_{\gamma,n}$ as follows: $p \sim q$ iff $B_p = B_q$. We let $\mathbb{BC}^T_{\gamma,n} = \mathbb{BC}^{T,*}_{\gamma,n}/\sim$.

2. For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $\sigma_{m,\ell}\colon \mathbb{BC}^T_{\gamma,m} \to \mathbb{BC}^T_{\gamma,\ell}$ be the natural map, i.e., for all $p \in \mathbb{BC}^T_{\gamma,m}$, $\sigma_{m,\ell}(p)$ is the equivalent class of Borel codes that code the set $\{x \in (\gamma^\omega)^l \mid \exists y \in B_p\ y \restriction \ell = x\}$. Assume each $\sigma_{m,\ell}$ is a well-defined projection between posets (see Remark 4.1). Let $\big(\mathbb{BC}^T_{\gamma,\omega}, (\sigma_n\colon \mathbb{BC}^T_{\gamma,n} \to \mathbb{BC}^T_{\gamma,\omega} \mid n < \omega)\big)$ be the inverse limit of the system $(\sigma_{m,\ell}\colon \mathbb{BC}^T_{\gamma,m} \to \mathbb{BC}^T_{\gamma,\ell} \mid 1 \leq \ell \leq m < \omega)$. We call $\mathbb{BC}^T_{\gamma,\omega}$ the *inverse limit of Vopěnka algebras of* ∞*-Borel codes for adding an element of* $(\gamma^\omega)^\omega$ *in* $\mathrm{HOD}_{\{T\}}$.

3. When $T = \emptyset$ or T is OD, then we omit it from our notation. Similarly, when $\gamma = 2$, we omit it from our notation as before.

Remark 4.1. *For each $m < \omega$, we can regard $\mathbb{BC}^T_{\gamma,m}$ as a sub-algebra of $\mathbb{Q}^T_{\gamma,m}$. In Definition 4.1, it is clear that the maps $\sigma_{m,\ell}$ are projections. However, in Definition 4.2, proving $\sigma_{m,\ell}$ is a well-defined forcing projection is non-trivial and uses*

$(*)$: *if $p \in \mathbb{BC}^T_{\gamma,m+1}$ then $\{t : \exists s \in B_p\ s \restriction m = t\}$ has an OD_T ∞-Borel code.*

If $(*)$ fails, then there is some m and some $p \in \mathbb{BC}^T_{\gamma,m+1}$ such that $\sigma_{m+1,m}(p)$ is not even defined. If $(*)$ holds, then all $\sigma_{m,l}$ are well-defined total functions. It is easy to check then they are all forcing projection maps (see [Cha19, Fact 7.14]).

The reader can see [Cha19, Section 7] for a more detailed discussion of these facts in the case $\gamma = \omega$. For $\gamma > \omega$, it is not clear to us if $(*)$ holds in general. We will prove some version of $(*)$ in the last section of the paper. The same remarks apply to the maps $\sigma_{m,\ell}$ in Definition 4.4.

The following lemmas will be useful in Section 5:

Lemma 4.2. *Assume $\mathsf{ZF} + \mathsf{AD}^+ +$ "$V = L(T, \mathbb{R})$" for some set T of ordinals.*

1. *$\mathbb{Q}_n^T$ is of size at most Θ and $\mathbb{Q}_n^T$ has the Θ-c.c. in $\mathrm{HOD}_{\{T\}}$ for all $n \leq \omega$. A similar statement holds for $\mathbb{BC}_n^T$ for all $n \leq \omega$.*
2. *For any condition $p \in \mathbb{Q}_\omega^T$, there is a $\mathbb{Q}_\omega^T$-generic filter H over $\mathrm{HOD}_{\{T\}}$ such that $p \in H$, and $V = \mathrm{L}(T, \mathbb{R}) \subseteq \mathrm{HOD}_{\{T\}}[H]$ and the set $\mathbb{R}^V$ is countable in $\mathrm{HOD}_{\{T\}}[H]$ and moreover, $\mathbb{R}^V$ is the symmetric reals of $\mathrm{HOD}_{\{}T\}[H]$.*
3. *Items (1) and (2) hold for $\mathbb{BC}_{\omega,\omega}^T$ if $\mathbb{BC}_{\omega,\omega}^T$ is well-defined (see Corollary 4.5).*

Lemma 4.3. *Assume $\mathsf{ZF} + \mathsf{AD}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$. Let $\gamma < \Theta$.*

1. *The posets $\mathbb{Q}_{\gamma,n}$ for $n \leq \omega$ are of size less than Θ in HOD.*
2. *Let $s \in (\gamma^\omega)^n$ for $n < \omega$, and $h_s = \{p \in \mathbb{Q}_{\gamma,n} \mid s \in \pi_n(p)\}$, where $\pi_n \colon \mathbb{Q}_{\gamma,n} \to \mathcal{O}_{\gamma,n}$ is as in Definition 4.1. Then the set h_s is a $\mathbb{Q}_{\gamma,n}$-generic filter over HOD such that $\mathrm{HOD}[h_s] = \mathrm{HOD}_{\{s\}}$.*
3. *(Woodin) For any condition $p \in \mathbb{Q}_{\gamma,\omega}$, there is a $\mathbb{Q}_{\gamma,\omega}$-generic filter H over HOD such that $p \in H$ and the set $(\gamma^\omega)^V$ is countable in $\mathrm{HOD}[H]$ and $\mathrm{HOD}(\gamma^\omega)$ is a symmetric extension.*
4. *Items (1) – (3) hold for $\mathbb{BC}_{\gamma,\omega}^T$ if $\mathbb{BC}_{\gamma,\omega}^T$ is well-defined (see Corollary 4.5).*

The proofs are standard; the reader can consult for instance [Ste08, Tra21]. The following generalization of the previous lemmas also holds. For more details, see [Tra21]. We first recall the definitions of the Vopěnka algebras for adding elements of $\bigcup_{\gamma<\Theta} \gamma^\omega$.

Definition 4.3. Let T be a set of ordinals.

1. Let n be a natural number with $n \geq 1$ and $\mathcal{O}_{\infty,n}^T$ be the collection of all nonempty subsets of $\gamma_1^\omega \times \cdots \times \gamma_n^\omega$ which are OD from T for some $\gamma_1, \ldots, \gamma_n < \Theta$. The order on $\mathcal{O}_{\infty,n}^T$ is defined as: for $p, q \in \mathcal{O}_{\infty,n}^T$, we say $p \leq q$ if for some $\gamma_1, \ldots, \gamma_n < \Theta$, $p, q \subseteq \gamma_1^\omega \times \cdots \times \gamma_n^\omega$ and $p \subseteq q$. Fix a bijection $\pi_n \colon \eta \to \mathcal{O}_{\infty,n}^T$ which is OD from T, where η is some ordinal. Let $\mathbb{Q}_{\infty,n}^T$ be the poset on η such that for each p, q in $\mathbb{Q}_{\infty,n}^T$, we have $p \leq q$ if $\pi_n(p) \subseteq \pi_n(q)$. We call $\mathbb{Q}_{\infty,n}^T$ the *Vopěnka algebra for adding an element of $(\bigcup_{\gamma<\Theta} \gamma^\omega)^n$ in $\mathrm{HOD}_{\{T\}}$*.

2. For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $\sigma_{m,\ell}\colon \mathbb{Q}^T_{\infty,m} \to \mathbb{Q}^T_{\infty,\ell}$ be the natural map induced from π_ℓ and π_m, i.e., for all $p \in \mathbb{Q}^T_{\infty,m}$, $\pi_l\big(\sigma_{m,\ell}(p)\big) = \{x \mid \exists y \in \pi_m(p)\ y \restriction \ell = x\}$. Then each $\sigma_{m,\ell}$ is a projection between posets. Let $\big(\mathbb{Q}^T_{\infty,\omega}, (\sigma_n\colon \mathbb{Q}^T_{\infty,\omega} \to \mathbb{Q}^T_{\infty,n} \mid n < \omega)\big)$ be the inverse limit of the system $(\sigma_{m,\ell}\colon \mathbb{Q}^T_{\infty,m} \to \mathbb{Q}^T_{\infty,\ell} \mid 1 \leq \ell \leq m < \omega)$. We call $\mathbb{Q}^T_{\infty,\omega}$ the *inverse limit of Vopěnka algebras for adding an element of* $(\bigcup_{\gamma<\Theta} \gamma^\omega)^\omega$ *in* $\mathrm{HOD}_{\{T\}}$.

3. When $T = \emptyset$ or T is OD, then we omit it from our notation. In particular, we denote $\mathbb{Q}_{\infty,n}$ the Vopěnka algebra for adding an element of $(\bigcup_{\gamma<\Theta} \gamma^\omega)^n$ in HOD.

Definition 4.4. Let T be a set of ordinals.

1. Let n be a natural number with $n \geq 1$ and let $\mathbb{BC}^{T,*}_{\infty,n}$ be the poset consisting of $OD(T)$ ∞-Borel codes for subsets of $\gamma_1^\omega \times \cdots \times \gamma_n^\omega$ for some $\gamma_1, \ldots, \gamma_n < \Theta$ with the ordering $p \leq q$ if $B_p \subseteq B_q$. We define the equivalence relation $\sim$ on $\mathbb{BC}^{T,*}_{\infty,n}$ as follows: $p \sim q$ iff $B_p = B_q$. We let $\mathbb{BC}^T_{\infty,n} = \mathbb{BC}^{T,*}_{\infty,n}/\sim$.

2. For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $\sigma_{m,\ell}\colon \mathbb{BC}^T_{\infty,m} \to \mathbb{BC}^T_{\infty,\ell}$ be the natural map, i.e., for all $p \in \mathbb{BC}^T_{\infty,m}$, $\sigma_{m,\ell}(p)$ is the equivalent class of Borel codes that code the set $\{x \in (\bigcup_{\gamma<\Theta} \gamma^\omega)^l \mid \exists y \in B_p\ y \restriction \ell = y\}$. Suppose each $\sigma_{m,\ell}$ is a well-defined projection between posets (see Remark 4.1). Let $\big(\mathbb{BC}^T_{\infty,\omega}, (\sigma_n\colon \mathbb{BC}^T_{\infty,\omega} \to \mathbb{BC}^T_{\infty,n} \mid n < \omega)\big)$ be the inverse limit of the system $(\sigma_{m,\ell}\colon \mathbb{BC}^T_{\infty,m} \to \mathbb{BC}^T_{\infty,\ell} \mid 1 \leq \ell \leq m < \omega)$. We call $\mathbb{BC}^T_{\infty,\omega}$ the *inverse limit of Vopěnka algebras of ∞-Borel codes for adding an element of* $(\bigcup_{\gamma<\Theta} \gamma^\omega)^\omega$ *in* $\mathrm{HOD}_{\{T\}}$.

3. When $T = \emptyset$ or T is OD, then we omit it from our notation. Similarly, when $\gamma = 2$, we omit it from our notation as before.

Definition 4.5. Let γ be a non-zero ordinal $< \Theta$ and T be a set of ordinals.

1. Let n be a natural number with $n \geq 1$ and $\mathcal{Q}^T_{\gamma,n}$ be the collection of all nonempty subsets of $\mathcal{P}(\gamma)^n$ which are OD from T. Fix a bijection $\pi_n\colon \eta \to \mathcal{Q}^T_{\gamma,n}$ which is OD from T, where η is some ordinal. Let $\mathbb{P}^T_{\gamma,n}$ be the poset on η such that for each p, q in $\mathbb{P}^T_{\gamma,n}$, we have $p \leq q$ if $\pi_n(p) \subseteq \pi_n(q)$. We call $\mathbb{P}^T_{\gamma,n}$ the *Vopěnka algebra for adding an element of* $\mathcal{P}(\gamma)^n$ *in* $\mathrm{HOD}_{\{T\}}$.

2. For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $\sigma_{m,\ell}\colon \mathbb{P}^T_{\gamma,m} \to \mathbb{P}^T_{\gamma,\ell}$ be the natural map induced from π_ℓ and π_m, i.e., for all $p \in \mathbb{P}^T_{\gamma,m}$, $\pi_l\big(\sigma_{m,\ell}(p)\big) = \{x \mid \exists y \in \pi_m(p)\ y \restriction \ell = x\}$. Then each $\sigma_{m,\ell}$ is a projection between posets. Let $\big(\mathbb{P}^T_{\gamma,\omega}, (\sigma_n\colon \mathbb{Q}^T_{\gamma,\omega} \to \mathbb{Q}^T_{\gamma,n} \mid n < \omega)\big)$ be the inverse limit of the system $(\sigma_{m,\ell}\colon \mathbb{P}^T_{\gamma,m} \to \mathbb{P}^T_{\gamma,\ell} \mid 1 \leq \ell \leq m < \omega)$. We call $\mathbb{P}^T_{\gamma,\omega}$ the *inverse limit of Vopěnka algebras for adding an element of* $\mathcal{P}(\gamma)^\omega$ *in* $\mathrm{HOD}_{\{T\}}$.

3. When $T = \emptyset$ or T is OD, then we omit it from our notation.

4. For $n \leq \omega$, we define $\overline{\mathbb{BC}}^T_{\gamma,n}$ be the "∞-Borel" version of $\mathbb{P}^T_{\gamma,n}$ the same way $\mathbb{BC}^T_{\gamma,n}$ is defined from $\mathbb{Q}^T_{\gamma,n}$ in Definition 4.2.

Definition 4.6. Let T be a set of ordinals.

1. Let n be a natural number with $n \geq 1$ and $\mathcal{Q}^T_{\infty,n}$ be the collection of all nonempty $OD(T)$ subsets of $\mathcal{P}(\gamma_1) \times \cdots \times \mathcal{P}(\gamma_n)$ for some $\gamma_1, \ldots, \gamma_n < \Theta$. The order on $\mathcal{Q}^T_{\infty,n}$ is defined as: for $p, q \in \mathcal{Q}^T_{\infty,n}$, we say $p \leq q$ if for some $\gamma_1, \ldots, \gamma_n < \Theta$, $p, q \subseteq \mathcal{P}(\gamma_1) \times \cdots \times \mathcal{P}(\gamma_n)$ and $p \subseteq q$. Fix a bijection $\pi_n \colon \eta \to \mathcal{Q}^T_{\infty,n}$ which is OD from T, where η is some ordinal. Let $\mathbb{P}^T_{\infty,n}$ be the poset on η such that for each p, q in $\mathbb{P}^T_{\infty,n}$, we have $p \leq q$ if $\pi_n(p) \subseteq \pi_n(q)$. We call $\mathbb{P}^T_{\infty,n}$ the *Vopěnka algebra for adding an element of* $(\bigcup_{\gamma<\Theta} \mathcal{P}(\gamma))^n$ *in* $\mathrm{HOD}_{\{T\}}$.

2. For all natural numbers ℓ and m with $1 \leq \ell \leq m$, let $\sigma_{m,\ell} \colon \mathbb{P}^T_{\infty,m} \to \mathbb{P}^T_{\infty,\ell}$ be the natural map induced from π_ℓ and π_m, i.e., for all $p \in \mathbb{P}^T_{\infty,m}$, $\pi_l\big(\sigma_{m,\ell}(p)\big) = \{x \mid \exists y \in \pi_m(p)\ y \restriction \ell = x\}$. Then each $\sigma_{m,\ell}$ is a projection between posets. Let $\big(\mathbb{P}^T_{\infty,\omega}, (\sigma_n \colon \mathbb{P}^T_{\infty,\omega} \to \mathbb{P}^T_{\infty,n} \mid n < \omega)\big)$ be the inverse limit of the system $(\sigma_{m,\ell} \colon \mathbb{P}^T_{\infty,m} \to \mathbb{P}^T_{\infty,\ell} \mid 1 \leq \ell \leq m < \omega)$. We call $\mathbb{P}^T_{\infty,\omega}$ the *inverse limit of Vopěnka algebras for adding an element of* $(\bigcup_{\gamma<\Theta} \mathcal{P}(\gamma))^\omega$ *in* $\mathrm{HOD}_{\{T\}}$.

3. When $T = \emptyset$ or T is OD, then we omit it from our notation. In particular, we denote $\mathbb{P}_{\infty,n}$ the Vopěnka algebra for adding an element of $(\bigcup_{\gamma<\Theta} \gamma^\omega)^n$ in HOD.

4. For $n \leq \omega$, we define $\overline{\mathbb{BC}}^T_{\infty,n}$ be the "∞-Borel" version of $\mathbb{P}^T_{\infty,n}$ the same way $\mathbb{BC}^T_{\infty,n}$ is defined from $\mathbb{Q}^T_{\infty,n}$ in Definition 4.4.

For a given γ and T, if all $OD(T)$ subsets of γ^ω have $OD(T)$ ∞-Borel codes, then the posets $\mathbb{Q}^T_{\gamma,1}$ and $\mathbb{BC}^T_{\gamma,1}$ are isomorphic. In general, we do not know if this follows from AD^+. Let $t \in \gamma^\omega$, $h_t = \{p \in \mathbb{Q}^T_{\gamma,1} \mid t \in \pi_1(p)\} \subseteq \mathbb{Q}^T_{\gamma,1}$ be the HOD_T-generic for adding t, and $g_t = \{p \in \mathbb{BC}^T_{\gamma,1} \mid t \in B_p\} \subseteq \mathbb{BC}^T_{\gamma,1}$ be the HOD_T-generic for adding t. Clearly,

- $h_t \in \mathrm{HOD}_{T,t}$,
- $t \in \mathrm{HOD}_{T,h_t}$.

We do not know in general that $h_t \in \mathrm{HOD}_T[t]$. However, it is easy to see that (see [Cha19, Fact 7.6] for a proof)

- $t \in \mathrm{HOD}_T[g_t]$, and
- $g_t \in \mathrm{HOD}_T[t]$.

Therefore,

$$\mathrm{HOD}_T[t] = \mathrm{HOD}_T[g_t] \subseteq \mathrm{HOD}_{T,h_t} = \mathrm{HOD}_{T,t}. \tag{4.1}$$

A similar conclusion holds for $\mathbb{P}^T_{\gamma,1}$ and $\overline{\mathbb{BC}}^T_{\gamma,1}$.

A similar conclusion also holds for the forcings $\mathbb{Q}_{\infty,1}.\mathbb{P}_{\infty,1}$ and their ∞-Borel versions $\mathbb{BC}_{\infty,1}, \overline{\mathbb{BC}}_{\infty,1}$ (respectively). Let $t \in \gamma^\omega$ ($t \in \mathcal{P}(\gamma)$ respectively) for some $\gamma < \Theta$, $h_t \subseteq \mathbb{Q}_{\infty,1}$ ($h_t \subseteq \mathbb{P}_{\infty,1}$ respectively) be the generic over HOD for adding t, and $g_t \subseteq \mathbb{BC}_{\infty,1}$ ($g_t \subseteq \overline{\mathbb{BC}}_{\infty,1}$ respectively) be the generic over HOD for adding t. Then

$$\mathrm{HOD}[t] = \mathrm{HOD}[g_t] \subseteq \mathrm{HOD}_{h_t} = \mathrm{HOD}_t. \tag{4.2}$$

Equations 4.1 and 4.2 also hold for $t \in (\gamma^\omega)^n$ or $t \in \mathcal{P}(\gamma^n)$ for $n > 1$. Also note that the first equality of 4.1 and 4.2 also holds for any ZFC model containing the forcing. For instance, $L[\mathbb{BC}_{\infty,1}][t] = L[\mathbb{BC}_{\infty,1}][g_t]$. Some improvements of these will be presented in Section 5. The reader is advised to consult [Tra21, Cha19] for more detailed treatments of Vopěnka forcing and the variations discussed above.

We now address the extent to which the inverse limit $\mathbb{BC}^T_{\gamma,\omega}$ is well-defined and topics related to the forcings $\mathbb{BC}^T_{\gamma,n}$.

Lemma 4.4. $\mathsf{ZF} + \mathsf{AD}^+$. *Suppose $\gamma < \Theta$, $n < \omega$, and $A \subseteq (\gamma^\omega)^{n+1}$ has an ∞-Borel code (S, φ), then the set $B = \{g \in (\gamma^\omega)^n : \exists f (f, g) \in A\}$ has an $OD(S, \mu)$ ∞-Borel code for any fine, countably complete measure μ on $\wp_{\omega_1}(\gamma^\omega)$. If additionally either $\mathsf{AD}_\mathbb{R}$ holds or $\gamma = \omega$, then B has an $OD(S)$ ∞-Borel code.*

Proof. The first part of the lemma is proved in [IT18, Claim 2]. For the second part, if $\mathsf{AD}_\mathbb{R}$ holds then there exists a unique normal, fine, and countably complete measure μ on $\wp_{\omega_1}(\gamma^\omega)$ by results of Solovay [Sol06] and Woodin [Woo83]. Therefore, μ is OD and $OD(S, \mu) = OD(S)$. If $\gamma = \omega$, then there is an OD fine, countably complete measure μ on $\wp_{\omega_1}(\omega^\omega)$ is induced from the Martin measure as follows. First let ν be the Martin measure on $\mathcal{D}$ and $\pi : \mathcal{D} \to \wp_{\omega_1}(\omega^\omega)$ be defined as: $\pi([x]_T) = \{y \in \omega^\omega : y \leq_T x\}$. Clearly, π is an OD map. The measure μ is defined as: $A \in \mu$ iff $\pi^{-1}[A] \in \nu$. It is easy to verify that μ is an OD fine, countably complete measure on $\wp_{\omega_1}(\omega^\omega)$. Therefore, $OD(S, \mu) = OD(S)$. $\square$

Corollary 4.5. *Suppose $\mathsf{ZF} + \mathsf{AD}^+ + V = L(J, \mathbb{R})$ for some set of ordinals J. The following hold.*

(i) *The inverse limit $\mathbb{BC}^J_{\omega,\omega}$ is well-defined. In fact, $\mathbb{BC}^J_{\omega,\omega}$ is isomorphic to $\mathbb{Q}^J_{\omega,\omega}$.*

(ii) *For any $x \in \mathbb{R}$, any $OD(J, x)$ set $A \subseteq (\omega^\omega)^n$, A has an $OD(J, x)$ ∞-Borel code.*

(iii) $\mathrm{HOD}_J = L[J, \mathbb{BC}^J_{\omega,\omega}]$.

Proof. Part (i) is a consequence of Lemma 4.4 and [Cha19, Fact 7.14]. Lemma 4.4 implies $(*)$ for $\mathbb{BC}^J_{\omega,n+1}$ holds all $n<\omega$. The calculations in [Cha19, Fact 7.14] show that the inverse limit $\mathbb{BC}^J_{\omega,\omega}$ is well-defined because the maps $\sigma_{m,l}$ are all well-defined forcing projection maps. For part (ii), without loss of generality, let us fix an $OD(J,x)$ set $A\subseteq\omega^\omega$. The general case just involves more notations. Say $z\in A$ iff $L(J,\mathbb{R})\models\varphi[J,x,s,z]$ for some finite sequence of ordinals s. The following calculations produce an $OD(J,x)$ ∞-Borel code for A (see cf [Cha19, Corollary 7.20] for a similar calculation with more details). Work over $L[J,\mathbb{BC}^J_{\omega,\omega}]$, for any $z\in\mathbb{R}$, we let $g_{x,z}\subseteq\mathbb{BC}^J_{\omega,2}$ be the generic adding x,z. Let $\dot{\mathbb{R}}$ be the symmetric reals added by a generic $g\subset\mathbb{BC}^J_{\omega,\omega}$.

$$\begin{aligned} z\in A &\Leftrightarrow L[J,\mathbb{BC}^J_{\omega,\omega}][g_{x,z}]\vDash \text{“}1_{\mathbb{B}^J_{\omega,\omega}/g_{x,z}}\Vdash_{\mathbb{B}^J_{\omega,\omega}/g_{x,z}} L(J,\dot{\mathbb{R}})\models\varphi[J,x,s,z]\text{”}\\ &\Leftrightarrow L[J,\mathbb{BC}^J_{\omega,\omega},x,z]\vDash \text{“}1_{\mathbb{B}^J_{\omega,\omega}/g_{x,z}}\Vdash_{\mathbb{BC}^J_{\omega,\omega}/g_{x,z}} L(J,\dot{\mathbb{R}})\models\varphi[J,x,s,z]\text{”}. \end{aligned}$$

The fact that $L[J,\mathbb{BC}^J_{\omega,\omega}][g_{x,z}]=L[J,\mathbb{BC}^J_{\omega,\omega},x,z]$ follows from the remark after 4.2. The equivalences above follow from standard properties of the inverse limit of projections $\mathbb{BC}^J_{\omega,\omega}$ and calculations as in [Cha19, Theorems 7.18, 7.19]. The ∞-Borel code for A is the set of ordinals $S\in\mathrm{HOD}_{J,x}$ coding $(J,\mathbb{BC}^J_{\omega,\omega},x)$. Part (ii) immediately gives $\mathbb{BC}^J_{\omega,\omega}$ is isomorphic to $\mathbb{Q}^J_{\omega,\omega}$. Part (iii) now follows from calculations in [Cha19, Corollary 7.20, 7.21].

□

Lemma 4.6. *Assume* $\mathsf{ZF}+\mathsf{AD}^++\mathsf{AD}_{\mathbb{R}}+V=L(\mathcal{P}(\mathbb{R}))$.

1. *The posets* $\mathbb{Q}_{\infty,n},\mathbb{P}_{\infty,n}$ *for* $n\leq\omega$ *are* Θ*-cc in* HOD.
2. *Let* $s\in(\bigcup_{\gamma<\Theta}\gamma^\omega)^n$ *for* $n<\omega$ *and* $h_s=\{p\in\mathbb{Q}_{\infty,n}\mid s\in\pi_n(p)\}$*. Then the set* h_s *is a* $\mathbb{Q}_{\infty,n}$*-generic filter over* HOD *such that* $\mathrm{HOD}[h_s]=\mathrm{HOD}_{\{s\}}$*. A similar statement can be made for* $s\in(\bigcup_{\gamma<\Theta}\mathcal{P}(\gamma))^n$ *with regard to the forcing* $\mathbb{P}_{\infty,n}$.
3. *For any condition* $p\in\mathbb{Q}_{\infty,\omega}$*, there is a* $\mathbb{Q}_{\infty,\omega}$*-generic filter* H *over* HOD *such that* $p\in H$ *and the set* $(\bigcup_{\gamma<\Theta}\gamma^\omega)^V$ *is countable in* $\mathrm{HOD}[H]$*. Furthermore,* $V=\mathrm{HOD}((\bigcup_{\gamma<\Theta}\gamma^\omega)^V)$ *is the symmetric part of* $\mathrm{HOD}[H]$*. Similarly, for any condition* $p\in\mathbb{P}_{\infty,\omega}$*, there is a* $\mathbb{P}_{\infty,\omega}$*-generic filter* H *over* HOD *such that* $p\in H$ *and the set* $(\bigcup_{\gamma<\Theta}\mathcal{P}(\gamma))^V$ *is countable in* $\mathrm{HOD}[H]$*. Furthermore,* $V=\mathrm{HOD}((\bigcup_{\gamma<\Theta}\mathcal{P}(\gamma))^V)$ *is the symmetric part of* $\mathrm{HOD}[H]$.
4. *(1) – (3) above also hold for the posets* $\mathbb{BC}_{\infty,n}$ *(*$\overline{\mathbb{BC}}_{\infty,n}$ *respectively) and for* $\mathbb{BC}_{\infty,\omega}$ *(*$\overline{\mathbb{BC}}_{\infty,\omega}$ *) if these forcings are well-defined. In fact,* $\mathbb{BC}_{\infty,\omega}$ *is a well-defined inverse limit and is isomorphic to* $\mathbb{Q}_{\infty,\omega}$.

Proof sketch. We will not prove the lemma; instead, we sketch the main ideas here. (1) – (3) are standard calculations. The “Furthermore” clause of part (3) can be seen to follow from Lemma 3.3 and the fact that every set of reals is Suslin co-Suslin under $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$. For (4), first note that Lemma 4.4 implies $\mathbb{BC}_{\infty,\omega}$ is a well-defined inverse limit because the maps $\sigma_{m,l}$ are all well-defined forcing projection maps. To see $\mathbb{BC}_{\infty,\omega}$ is isomorphic to $\mathbb{Q}_{\infty,\omega}$, it suffices to show if $\gamma < \Theta$, $n < \omega$ and $A \subseteq (\gamma^\omega)^n$ is OD, then A has an OD ∞-Borel code. For ease of notation, we assume $n = 1$. For $f \in \gamma^\omega$, suppose $f \in A$ iff $V \models \varphi[s, f]$ for some finite set of ordinals s. As in the previous corollary, we can produce an OD ∞-Borel code for A as follows. Let Z be a set of ordinals such that $\mathrm{HOD} = L[Z]$[3] and $g_f \subseteq \mathbb{BC}_{\infty,1}$ be the generic adding f. Let $\mathbb{C} = \mathrm{HOD}((\bigcup_{\gamma<\Theta}\gamma^\omega)^V)$. We note that by Theorem 3.3, $\mathbb{C} = V$. Then

$$\begin{aligned} f \in A &\Leftrightarrow \mathrm{HOD}[g_f] \vDash \text{“}1_{\mathbb{B}_{\infty,\omega}/g_f} \Vdash_{\mathbb{B}_{\infty,\omega}/g_f} \mathbb{C} \models \varphi[s,f]\text{”} \\ &\Leftrightarrow L[Z,f] \vDash \text{“}1_{\mathbb{B}_{\infty,\omega}/g_f} \Vdash_{\mathbb{BC}_{\infty,\omega}/g_f} \mathbb{C} \models \varphi[s,f]\text{”}. \end{aligned}$$

The above calculations easily yield and OD ∞-Borel code for A. □

We will prove in Section 5 a version of Lemma 4.4 for $A \subseteq \mathcal{P}(\gamma)^{n+1}$ under $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ and use this to show that the inverse limit $\overline{\mathbb{BC}}_{\infty,\omega}$ is well-defined and is isomorphic to $\mathbb{P}_{\infty,\omega}$.

5 The existence of ∞-Borel codes

In this section, we prove Theorems 1.1, 1.3, 1.5 and their corollaries.

Proof of Theorem 1.1. Without loss of generality, we let $A \subseteq \mathcal{P}(\omega)$ and assume A is OD. First we assume $\mathsf{AD}_{\mathbb{R}}$ holds. Since $\mathsf{AD}_{\mathbb{R}}$ holds, $V = \mathbb{C}$ by Lemma 3.3. We show A has an OD ∞-Borel code. Suppose

$$x \in A \Leftrightarrow \mathbb{C} \vDash \varphi[x, s]$$

for some formula φ and some finite sequence of ordinals s.

We note that for any $f \in \bigcup_{\gamma<\Theta}\gamma^\omega$, letting $g_f \subseteq \mathbb{BC}_{\infty,1}$ be generic over HOD that adds f, then

$$\mathrm{HOD}[f] = \mathrm{HOD}[g_f]$$

by (4.2). Here is a brief sketch. First note that $f \in \mathrm{HOD}[g_f]$ so $\mathrm{HOD}[f] \subseteq \mathrm{HOD}[g_f]$; for the converse, we have that $g_f = \{c : f \in B_c\}$ and this calculation of g_f can be done over $\mathrm{HOD}[f]$. Here we use essentially here that conditions of the forcing are ∞-Borel codes.

Let $Z \subseteq \Theta$ be such that $\mathrm{HOD} = L[Z]$. Then we can produce an OD ∞-Borel code for A as follows, recall the definition of $\mathbb{BC}_{\infty,\omega}$ and related objects in Section 4.

$$\begin{aligned} x \in A &\Leftrightarrow \mathrm{HOD}[g_x] \vDash \text{“}1_{\mathbb{B}_{\infty,\omega}/g_x} \Vdash_{\mathbb{B}_{\infty,\omega}/g_x} \mathbb{C} \models \varphi[x,s]\text{”} \\ &\Leftrightarrow L[Z][x] \vDash \text{“}1_{\mathbb{B}_{\infty,\omega}/g_x} \Vdash_{\mathbb{B}_{\infty,\omega}/g_x} \mathbb{C} \models \varphi[x,s]\text{”}. \end{aligned}$$

[3]One can show Z can be taken to the the set of ordinals that canonically codes $\mathbb{BC}_{\infty,\omega}$.

Again, the main point is by Lemma 4.6, the inverse limit $\mathbb{BC}_{\infty,\omega}$ is well-defined. The above equivalence shows that $((Z,s),\psi)$ where $\psi(x,(Z,s))$ is the formula "$1_{\mathbb{B}_{\infty,\omega}/g_x} \Vdash_{\mathbb{B}_{\infty,\omega}/g_x} \mathbb{C} \models \varphi[x,s]$", is an OD ∞-Borel code for A.

Now assume $\mathsf{AD}_{\mathbb{R}}$ fails. By Theorem 2.3, $V = L(J,\mathbb{R})$ for some set of ordinals J. In fact, we can take $J = [d \mapsto T]_{\mu_T}$ where T is a tree projecting to a universal $S(\kappa)$ set, where κ is the largest Suslin cardinal, $S(\kappa)$ is the largest Suslin pointclass, and μ_T is the T-degree measure defined in Section 2. Now there are two cases.

Case 1: $\Theta = \theta_0$

We start with a claim.

Claim 5.1. *The largest Suslin pointclass is $\utilde{\Sigma}^2_1$.*

Proof. $\utilde{\Sigma}^2_1$ has the scale property by AD^+ (cf. [ST10]). By the fact that $\Theta = \theta_0$ is regular, we have that $\utilde{\Sigma}^2_1 = \Sigma_1(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$. To see this note that the $\subseteq$-direction is clear. To see the converse, let $A \subseteq \mathbb{R}$ be $\Sigma_1(x)$ for some real x; so let φ be a Σ_1-formula such that for any $y \in \mathbb{R}$, $y \in A \Leftrightarrow \varphi[y,x]$. Since Θ is regular, it is easy to see that there is a transitive $M = L_\alpha(\mathcal{P}_\beta(\mathbb{R}))$ for some $\alpha, \beta < \Theta$ such that for $y \in \mathbb{R}$,

$$y \in A \Leftrightarrow M \models \varphi[y,x].$$

So we can define $y \in A$ iff "there is a set of reals B coding a transitive structure M containing all reals such that $M \models \varphi[y,x]$". This is easily seen to be $\Sigma^2_1(x)$. So $A \in \utilde{\Sigma}^2_1$.

Now we finish proving the claim by noting that the set $C = \{(x,y) \in \mathbb{R}^2 : y \notin OD(x)\}$ is a Π_1-set that has no uniformization. This is a result by Martin, cf. [Ste83]. By the above, C is Π^2_1 and cannot be uniformized. This gives $\utilde{\Sigma}^2_1$ is the largest pointclass with the scale property as claimed. □

Therefore, we can take T and hence $J = [d \mapsto T]_{\mu_T}$ to be OD, where $T \in \mathrm{HOD}$ is a tree projecting to a universal Σ^2_1 set. Hence for some $Z \subseteq \Theta$ with $Z \in OD$,

$$\mathrm{HOD} = \mathrm{HOD}_J = L[Z].$$

We can produce an OD ∞-Borel code for A by the following calculations. Suppose

$$x \in A \Leftrightarrow V \models \varphi[x,s]$$

for some finite sequence of ordinals s. Letting $g_x \subseteq \mathbb{BC}_{\omega,1}$ be HOD-generic that adds x, then $\mathrm{HOD}[g_x] = \mathrm{HOD}[x]$. We note that by Corollary 4.5, the inverse limit $\mathbb{BC}_{\omega,\omega}$ is well-defined. We have

$$\begin{aligned} x \in A &\Leftrightarrow \mathrm{HOD}[g_x] \vDash \text{“}1_{\mathbb{B}_{\omega,\omega}/g_x} \Vdash_{\mathbb{B}_{\omega,\omega}/g_x} L(J,\mathbb{R}) \models \varphi[x,s]\text{”} \\ &\Leftrightarrow L[Z][x] \vDash \text{“}1_{\mathbb{B}_{\omega,\omega}/g_x} \Vdash_{\mathbb{BC}_{\omega,\omega}/g_x} L(J,\mathbb{R}) \models \varphi[x,s]\text{”}. \end{aligned}$$

The above equivalence easily gives an OD ∞-Borel code for A.

Case 2: $\Theta > \theta_0$

Let $M_0 = L(\mathcal{P}_{\theta_0}(\mathbb{R}))$.

Claim 5.2. *Let $\Gamma = \Sigma^2_1$. The following hold.*

(i) For any real x, $Env(\Gamma(x)) = Env^{M_0}(\Gamma(x))$.[4]

(ii) $M_0 \models \Theta = \theta_0$.

Proof. The proof of Claim 5.1 shows that $\underset{\sim}{\Sigma}^2_1$ is the largest Suslin pointclass below θ_0 in V. The fact that for each $x \in \mathbb{R}$, $Env(\Gamma(x)) \subset M_0$ follows from results in [Jac09]; see for instance Lemma 3.14. A set A is in $Env(\Gamma)(x)$ iff for each countable $\sigma \subseteq \mathbb{R}$, there is a $OD^{<\Gamma}(x)$ set B such that $A \cap \sigma = B \cap \sigma$ (cf. [Wil12]). This calculation is absolute between V and M_0. Part (i) follows. In M_0, $\underset{\sim}{\Sigma}^2_1$ is the largest Suslin pointclass and $\underset{\sim}{Env}(\Gamma) =_{def} \bigcup_{x\in\mathbb{R}} Env(\Gamma(x)) = \mathcal{P}(\mathbb{R})$; the last equality holds because the set $\{(x, y) : y \notin OD_x\}$ has no scale in M_0. This easily implies that in M_0, every set of reals A is OD from some real x. This means $M_0 \models \Theta = \theta_0$. This proves part (ii). $\square$

Claim 5.3. *Let $A \subseteq \mathbb{R}$ be OD. Then A is OD in M_0.*

Proof. Suppose A is OD, say $x \in A$ iff $\varphi[x, s]$ holds for some finite sequence of ordinals s. For each countable $\sigma \subset \mathbb{R}$, there is a transitive model M of $\mathsf{ZF}^- + \mathsf{DC}$ of the form $L_\alpha(\mathcal{P}_\beta(\mathbb{R}))$ that ordinal defines $A \cap \sigma$ via φ and $\{s, \sigma\}$, i.e.

$$\forall x \in \sigma \ x \in A \Leftrightarrow M \models \varphi[x, s, \sigma].$$

By Σ_1-reflection into $\underset{\sim}{\Delta}^2_1$, for each σ, there are $\alpha_\sigma, \beta_\sigma, \max(s_\sigma) < \underset{\sim}{\delta}^2_1$ such that

$$\forall x \in \sigma \ x \in A \Leftrightarrow L_{\alpha_\sigma}(\mathcal{P}_{\beta_\sigma}(\mathbb{R})) \models \varphi[x, s_\sigma, \sigma].$$

Note the Wadge rank of A is $< \theta_0$ and therefore, $A \in M_0$. Working in M_0, let μ be the fine, countably complete measure on $\wp_{\omega_1}(\mathbb{R})$ induced by the Turing measure via the canonical surjection $\pi : \mathcal{D} \to \wp_{\omega_1}(\mathbb{R})$, where $\pi(d) = \{x \in \mathbb{R} : x \leq_T d\}$. μ is OD. Let $\alpha = [\sigma \mapsto \alpha_\sigma]_\mu, \beta = [\sigma \mapsto \beta_\sigma]_\mu$, and $s^* = [\sigma \mapsto s_\sigma]_\mu$. We claim that A is definable in M_0 from (α, β, s^*). This is because for any $x \in \mathbb{R}$, $x \in A$ iff for any function $F_\alpha, F_\beta, F_{s^*}$ such that $[F_\alpha]_\mu = \alpha, [F_\beta]_\mu = \beta, [F_{s^*}]_\mu = s^*, \forall^*_\mu \sigma \ L_{F_\alpha(\sigma)}(\mathcal{P}_{F_\beta(\sigma)}(\mathbb{R})) \models \varphi[x, F_{s^*}(\sigma), \sigma]$. The above calculation finishes the proof of the claim.

$\square$

[4] Recall that that $\Gamma = \Sigma^2_1$ so $\underset{\sim}{\delta}^2_1 = o(\Gamma)$ is the Wadge ordinals of Γ. A set A is in $Env(\Gamma(x))$ iff for any countable $\sigma \subset \mathbb{R}$, $A \cap \sigma = B \cap \sigma$ for some $B \in OD^{<\Gamma}(x)$. Here B is $OD^{<\Gamma}(x)$ iff there are $\Gamma(x)$ sets $U, W \subseteq \mathbb{R} \times \mathbb{R}$ and a $\Gamma(x)$-norm φ, and an ordinal $\alpha < \underset{\sim}{\delta}^2_1$ such that $A = U_y = \neg W_y$ for every $y \in \mathrm{dom}(\varphi)$ with $\varphi(y) = \alpha$. [Wil12] shows that this notion of envelopes generalizes Martin's notion of envelopes $\underset{\sim}{\Lambda}(\underset{\sim}{\Gamma}, \underset{\sim}{\delta}^2_1)$ (cf [Jac09]), as it can be applied in situations where AD may not hold. Under AD these two notions are equivalent.

Using Claim 5.3 and Claim 5.2, we can quote the result of the $\Theta = \theta_0$ case to get that A has an OD ∞-Borel code.

This completes the proof of the first clause of Theorem 1.1. As mentioned in Remark 1.2, the "Furthermore" clause has a similar proof to the proof of Case 1, so we leave it to the kind reader.

$\square$

Remark 5.4. *We do not know if* $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ *implies that for an arbitrary set of ordinals* S*, every* $OD(S)$ *subset of* $\mathcal{P}(\omega)$ *has an* $OD(S)$ ∞*-Borel code.*

Proof of Theorem 1.3. The proof of [IT18, Claim 2] shows the following.

Lemma 5.5. *Assume* AD^+*. Suppose* $\kappa < \Theta$*,* $n < \omega$*, and* $A^* \subseteq (\kappa^\omega)^{n+1}$ *has* ∞*-Borel code* S^**. Let* μ *be a fine, countably complete measure on* $\wp_{\omega_1}(\kappa^\omega)$*. Then* $A = \{f : \exists x \in \kappa^\omega\ (x, f) \in A^*\}$ *has an* ∞*-Borel code* S *that is* $OD(S^*, \mu)$*.*

Let $\kappa < \Theta$ and $A \subseteq \kappa^\omega$. Then by basic AD^+ theory, there is a set of ordinals T such that $A \in L(T, \mathbb{R})$. To see this, first fix a pre-wellordering $\leq$ of $\mathbb{R}$ of order type κ and let S_0 be an ∞-Borel code for $\leq$. Using $\leq$ and the fact that one can canonically code an ω-sequence of reals by a real, one sees that κ^ω can be simply coded by $\leq$ and $\mathbb{R}$. Therefore, using $\leq$, one can code A by a subset $B \subseteq \mathbb{R}$. Let S_1 be an ∞-Borel code for B and let $T = (S_0, S_1)$. It is clear that $\kappa^\omega, A \in L(T, \mathbb{R})$.

Suppose $V = L(\mathcal{P}(\mathbb{R})) \models \mathsf{AD}^+ + \Theta = \theta_0$, then $V = L(T, \mathbb{R})$ for some OD set of ordinals T. Then for any $\kappa < \Theta = \theta_0$, there is an OD surjection $\pi : \mathbb{R} \to \kappa^\omega$. Let μ be the OD fine, countably complete measure on $\wp_{\omega_1}(\mathbb{R})$ in Claim 5.3. π, μ induce an OD fine, countably complete measure ν on $\wp_{\omega_1}(\kappa^\omega)$ by a standard procedure:

$$A \in \nu \Leftrightarrow \{\pi^{-1}[\sigma] : \sigma \in A\} \in \mu.$$

By the above discussion, every $OD = OD(T)$ subset of κ^ω has $OD(T, \mu) = OD$ ∞-Borel code. A similar argument also gives every $OD(S)$ subset of κ^ω has an $OD(S)$ ∞-Borel code for any set of ordinals S.

Suppose $V = L(\mathcal{P}(\mathbb{R})) + \mathsf{AD}_\mathbb{R} \models \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$. By [Woo83] and $\mathsf{AD}_\mathbb{R}$, there is a unique normal, fine measure μ_κ on $\wp_{\omega_1}(\kappa^\omega)$ for each $\kappa < \Theta$. So μ_κ is OD for each $\kappa < \Theta$. Let S be a set of ordinals and $A \subseteq \kappa^\omega$ be $OD(S)$. By Lemma 5.5 applied to the μ_κ's, we have that (*) holds and therefore $\mathbb{BC}^S_{\infty,\omega}$ is a well-defined limit. Let $\kappa < \Theta$ and $A \subseteq \kappa^\omega$ be $OD(S)$, so there is a formula φ and some finite set of ordinals $\vec{\beta}$ such that

$$f \in A \Leftrightarrow V \models \varphi[f, \vec{\beta}, S].$$

Let Z be an $OD(S)$ set of ordinals such that $\mathrm{HOD}_S = L[Z]$ and $g_f \subseteq \mathbb{BC}^S_{\infty,1}$ be the generic for adding f, we then have

$$
\begin{aligned}
f \in A \Leftrightarrow \mathrm{HOD}_S[g_f] &\vDash \text{“}1_{\mathbb{BC}_{\infty,\omega}/g_f} \Vdash_{\mathbb{BC}_{\infty,\omega}/g_f} \mathrm{HOD}(\bigcup_{\gamma<\Theta} \gamma^\omega) \models \varphi[f, \vec{\beta}, S]\text{”} \\
\Leftrightarrow L[Z][f] &\vDash \text{“}1_{\mathbb{BC}_{\infty,\omega}/g_f} \Vdash_{\mathbb{BC}_{\infty,\omega}/g_s} \mathrm{HOD}(\bigcup_{\gamma<\Theta} \gamma^\omega) \models \varphi[f, \vec{\beta}, S]\text{”}.
\end{aligned}
$$

The above calculations easily imply that A has an $OD(S)$ ∞-Borel code. $\square$

Proof of Corollary 1.4. For each $x \subseteq \omega$, let $g_x \subseteq \mathbb{BC}_{\omega,1}$ be the generic for adding x. Thus we have as before

$$\mathrm{HOD}[x] = \mathrm{HOD}[g_x].$$

Clearly $\mathrm{HOD}[x] \subseteq \mathrm{HOD}_x$. To see the converse, let $X \subseteq ON$ be $OD(x)$; say $X \subseteq \gamma$. Let φ be a formula defining X from x and some $s \in ON^{<\omega}$. So

$$\forall \beta < \gamma \ \beta \in X \Leftrightarrow \varphi(\beta, s, x)$$

and for each $\beta < \gamma$, let

$$T^*_\beta = \{a : \varphi(\beta, s, a)\}.$$

Note that T^*_β is OD for each β.

Fix an OD injection $\pi^* : OD \cap \mathcal{P}(\omega) \to \mathrm{HOD}$ as in the definition of the usual Vopěnka forcing $\mathbb{Q}_{\omega,1}$, where π^* maps the algebra $\mathcal{O}_{\omega,1}$ of OD subsets of $\mathcal{P}(\omega)$ into its isomorphic copy $\mathbb{Q}_{\omega,1}$ in HOD. We can assume that $\pi^*[\mathcal{O}_{\omega,1}] = \mathbb{BC}_{\omega,1}$ because we have shown every OD subset of $\mathcal{P}(\omega)$ has an OD ∞-Borel code. We have that

$$Z = \{(\beta, \pi^*(T^*_\beta)) : \beta < \gamma\} \in \mathrm{HOD}$$

and that for $\beta < \gamma$,

$$\beta \in X \Leftrightarrow (\beta, \pi^*(T^*_\beta)) \in Z \wedge \pi^*(T^*_\beta) \in g_x.$$

The above equivalence implies

$$X \in \mathrm{HOD}[g_x] = \mathrm{HOD}[x].$$

So we have shown

$$\mathrm{HOD}_x = \mathrm{HOD}[x].$$

For the “furthermore” clause of part (1), we use the “furthermore” clause of Theorem 1.1, which states that if $A \subseteq \mathcal{P}(\omega)$ is OD_S for some set of ordinals S then A has an OD_S ∞-Borel code when $V = L(S, \mathbb{R})$. By an argument similar to the above, we get for each real x,

$$\mathrm{HOD}_{S,x} = \mathrm{HOD}_S[x].$$

This completes the proof of part (1).

The proof of part (1) can be adapted to prove part (2). But we will give here a different proof of part (2) that cannot be used to prove part (1). We assume $\mathsf{AD}_{\mathbb{R}}$ and use the forcing $\mathbb{BC}_{\infty,\omega}$ and related objects as in Section 4 to prove part (2) holds for any $s \in \gamma^\omega$ for $\gamma < \Theta$. Again, we have that for any $s \in \gamma^\omega$ for some $\gamma < \Theta$, letting $g_s \subseteq \mathbb{BC}_{\infty,1}$ be the generic adding s,

$$\mathrm{HOD}[s] = \mathrm{HOD}[g_s].$$

Furthermore, by Lemma 4.6, $V = \mathrm{HOD}(\bigcup_{\gamma<\Theta} \gamma^\omega)$ is the symmetric extension of HOD induced by a generic $H \subseteq \mathbb{BC}_{\infty,\omega}$. Note here that by Theorem 1.3, $\mathbb{BC}_{\infty,\omega}$ is well-defined. Let $X \in \mathrm{HOD}_s$ be a set of ordinals. So there is a formula φ and a finite sequence of ordinals t such that

$$\alpha \in X \Leftrightarrow V \models \varphi[\alpha, s, t].$$

Now we have:

$$\begin{aligned}\alpha \in X &\Leftrightarrow \mathrm{HOD}[g_s] \models \text{“}1_{\mathbb{BC}_{\infty,\omega}/g_s} \Vdash_{\mathbb{BC}_{\infty,\omega}/g_s} \mathrm{HOD}(\bigcup_{\gamma<\Theta} \gamma^\omega) \models \varphi[\alpha, s, t]\text{”} \\ &\Leftrightarrow \mathrm{HOD}[s] \models \text{“}1_{\mathbb{BC}_{\infty,\omega}/g_s} \Vdash_{\mathbb{BC}_{\infty,\omega}/g_s} \mathrm{HOD}(\bigcup_{\gamma<\Theta} \gamma^\omega) \models \varphi[\alpha, s, t]\text{”}.\end{aligned}$$

The above calculations show that $X \in \mathrm{HOD}[s]$. So $\mathrm{HOD}[s] = \mathrm{HOD}_s$. The argument above can be easily adapted to work for an arbitrary set of ordinals S by running the argument above over HOD_S using $\mathbb{BC}^S_{\infty,\omega}$.

Suppose now $\Theta = \theta_0$. Then since $\mathsf{AD}_{\mathbb{R}}$ fails, by Theorem 2.3, there is a set of ordinals T such that $V = L(T, \mathbb{R})$. Since $\Theta = \theta_0$, we can in fact take T to be OD. Therefore, $\mathrm{HOD}_T = \mathrm{HOD}$. Furthermore, for any $\gamma < \Theta$, $V = \mathrm{HOD}(\gamma^\omega)$ is a symmetric extension of HOD induced by a generic $H \subseteq \mathbb{BC}_{\gamma,\omega}$. The fact that $\mathbb{BC}_{\gamma,\omega}$ is well-defined follows from Theorem 1.3. The rest of the proof is the same as in the $\mathsf{AD}_{\mathbb{R}}$ case with $\mathbb{BC}_{\gamma,\omega}$ used in place of $\mathbb{BC}_{\infty,\omega}$. □

Remark 5.6. *(i) The main reason we need a different proof for part (1) of Corollary 1.4 is because we do not have an analogue of Lemma 4.6 in the situation of part (1), where $\mathsf{AD}_{\mathbb{R}}$ may fail.*

(ii) One can easily modify the proof above and [IT18, Claims 2 and 3] to show that if $V = L(T, \mathbb{R}) \models \mathsf{AD}^+$ for some set of ordinals T, and $f \in \wp_{\omega_1}(\kappa)$ for some uncountable cardinal $\kappa < \Theta$, then

$$\mathrm{HOD}_{T,f} = \mathrm{HOD}_T[f] = \mathrm{HOD}_T[g_f]$$

where g_f is HOD_T generic for the variation of the Vopenka algebra in HOD_T consisting of $OD(T)$ subsets of $\wp_{\omega_1}(\kappa)$ with $OD(T)$ ∞-Borel codes. The main point is that there is an $OD(T)$ fine, countably complete measure on $\wp_{\omega_1}(\wp_{\omega_1}(\kappa))$.

Proof of Theorem 1.5. We assume AD^+ and let ν witness ω_1 is $\mathbb{R}$-supercompact. Let $\kappa < \Theta$ and $A \subseteq \mathcal{P}(\kappa)$. Let $\leq$ be a prewellordering of the reals of length κ and let

$$\hat{A} = \{x \in \mathbb{R} : x \text{ codes } C_x \in A\}.$$[5]

Let

$$A^* = \{(x, C_x) : x \in \hat{A}\}.$$

In other words, $(x,f) \in A^*$ iff $x \in \hat{A}$ and $f = C_x$. We claim that A^* has an ∞-Borel code. Note that $\hat{A} \subseteq \mathbb{R}$ and hence by AD^+, $\hat{A}$ has an ∞-Borel code; similarly, $\leq$ has an ∞-Borel code. We fix ∞-Borel codes S_1, S_2 for $\leq, \hat{A}$ respectively. If $\kappa < \theta_0$ and $A \subseteq \mathcal{P}(\kappa)$ is OD, then we can in fact assume $\leq$ is OD and hence can take $S_1, S_2 \in \mathrm{HOD}$ by Theorem 1.1.

Let $T = (S_1, S_2)$. We work in $L(T, \mathbb{R})$. Let $\dot{R} \in \mathrm{HOD}$ be the canonical $\mathbb{BC}^T_{\omega,\omega}$-name for the symmetric reals added by a $\mathbb{BC}^T_{\omega,\omega}$-generic over HOD_T and $Z \subseteq ON$ be such that $\mathrm{HOD}_T = L[Z]$. We note that the system $(\mathbb{BC}^T_{\omega,\omega}, (\mathbb{BC}^T_{\omega,n}, \sigma_{n,m} : n \geq m))$ is a well-defined inverse limit system and satisfies Lemma 4.2 in $L(T, \mathbb{R})$ (see Remark 4.1 and Theorem 1.1). We then have the following equivalence, where $g_x \subseteq \mathbb{BC}^T_{\omega,1}$ is the generic adding x:

$$\begin{aligned}
(x,f) \in A^* \Leftrightarrow \ & \mathrm{HOD}_T[x,f] \vDash \text{“}x \in B_{S_2} \wedge \forall \alpha < \kappa \\
& \qquad \alpha \in f \Leftrightarrow \text{“}\mathrm{HOD}_T[x] \models 1_{\mathbb{BC}^T_{\omega,\omega/g_x}} \Vdash_{\mathbb{BC}^T_{\omega,\omega/g_x}} L(S_1, S_2, \dot{\mathbb{R}}) \vDash \\
& \qquad\qquad \exists y(|y|_{\leq} = \alpha \wedge y \in C_x)\text{””} \\
\Leftrightarrow \ & L[Z][x,f] \vDash \text{“}x \in B_{S_2} \wedge \forall \alpha < \kappa \\
& \qquad \alpha \in f \Leftrightarrow \text{“}L[Z][x] \models 1_{\mathbb{BC}^T_{\omega,\omega/g_x}} \Vdash_{\mathbb{BC}^T_{\omega,\omega/g_x}} L(S_1, S_2, \dot{\mathbb{R}}) \vDash \\
& \qquad\qquad \text{“}\exists y(|y|_{\leq} = \alpha \wedge y \in C_x)\text{””}.
\end{aligned}$$

The above calculations easily produce an $OD(S_1, S_2, \mu)$ ∞-Borel code for A^*, noting that the clause “$|y|_{\leq} = \alpha \wedge y \in C_x$” can easily be written as a formula $\varphi(S_1, S_2, \mu, y, \alpha)$.

Next, we want to produce an ∞-Borel code for A from an ∞-Borel code for A^*. This is accomplished by proving the following lemma.

Lemma 5.7. *Assume AD^+ and suppose there is a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$. Suppose $A^* \subseteq \mathcal{P}(\omega) \times \mathcal{P}(\kappa)$ is ∞-Borel for some $\kappa < \Theta$. Then $\exists^{\mathbb{R}} A^* = \{f : \exists x \ (x,f) \in A^*\}$ is ∞-Borel.*

Proof. Let $c = (T, \rho)$ be an ∞-Borel code for A^* and $A = \exists^{\mathbb{R}} A^*$. We may assume c is coded as a subset of $\lambda < \Theta^V$. We sketch an argument here. Suppose $c = (T, \rho)$ be an ∞-Borel code for A^* and suppose $\sup(c) \geq \Theta^V$. Work in $L(c, \mathbb{R})$; we want to find an ∞-Borel code c^* for A^* such that $\sup(c^*) < \Theta$. We may assume c codes a prewellordering $\leq$ of $\mathbb{R}$ of order type κ. So in $L(c, \mathbb{R})$, $\kappa < \Theta$; furthermore, $L(c, \mathbb{R}) \models$ “Θ is regular”. Let $\xi >> \sup(c)$ be a regular

[5] The coding $x \mapsto C_x$ is via the Coding Lemma relative to $\leq$.

cardinal and X be the Skolem hull of $L_\xi(c,\mathbb{R})$ from parameters $\mathbb{R}\cup\{A,A^*,c\}$ and let $\pi : M \to X$ be the uncollapse map. Then M has the form $L_\gamma(c^*,\mathbb{R})$, there is a surjection from $\mathbb{R}$ onto M, and π is an elementary embedding. Furthermore, $A, A^* \in M$, γ. Since there is a surjection from $\mathbb{R}$ onto M, $\sup(\mathrm{c}^*) < \Theta$. Since π is elementary,

$$M \models \text{“}c^* \text{ is an } \infty\text{-Borel code for } A^*\text{”}.$$

It is clear that c^* is indeed an ∞-Borel code for A^* since $\mathcal{P}(\omega)\times\mathcal{P}(\kappa) \subset M$. c^* is the desired ∞-Borel code for A^*.

Let ν be a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$. Let μ be the supercompact measure on $\wp_{\omega_1}(\mathcal{P}(\kappa)\cup\lambda)$ induced by ν and some surjection $\pi : \mathbb{R} \to \mathcal{P}(\kappa)\cup\lambda$.[6] By coding (S_1, S_2) into c, we may assume $\kappa < \Theta^{L(c,\mathbb{R})}$.

In the following, for a $\sigma \in \wp_{\omega_1}(\mathcal{P}(\kappa)\cup\lambda)$ the tree T^σ is defined as $T\cap\sigma$ and the assignment ρ^σ is defined as: for a terminal element t of T^σ, $\rho^\sigma(t) = \rho(t)\cap\sigma$. The code $c^\sigma = (T^\sigma, \rho^\sigma)$ will yield the set B_{c^σ} by induction as follows:

- if t is a terminal element of T^σ, let $B_{c^\sigma,t}$ be the basic open set $O_{\rho^\sigma(t)}$ in the space $2^\omega \times 2^{\sigma\cap\kappa}$. In the case $\rho^\sigma(t) = \emptyset$, then we let $B_{c^\sigma,t} = 2^\omega \times 2^{\sigma\cap\kappa}$.
- If $\mathrm{Succ}_{T^\sigma}(t)$ is a singleton of the form $\{s\}$, let $B_{c^\sigma,t}$ be the complement of $B_{c^\sigma,s}$ in the space $2^\omega \times 2^{\sigma\cap\kappa}$.
- If $\mathrm{Succ}_{T^\sigma}(t)$ has more than one element, then let $B_{c^\sigma,t}$ be the union of all sets of the form $B_{c^\sigma,s}$ where s is in $\mathrm{Succ}_{T^\sigma}(t)$.
- $B_{c^\sigma} = B_{c^\sigma,\emptyset}$.

Claim 5.8. *For any $(x,f) \in \mathcal{P}(\omega)\times\mathcal{P}(\kappa)$,*

$$(x,f) \in A^* \Leftrightarrow \forall^*_\mu\sigma\ (x, f\cap\sigma) \in B_{c^\sigma}.$$

Proof. The proof is by induction on the ranks of the nodes in T. If t is a terminal node of T, it is easy to see that $(x,f) \in O_{\rho(t)}$ iff $\forall^*_\mu\sigma\ (x, f\cap\sigma) \in O_{\rho^\sigma(t)}$. Here note that by fineness, $\forall^*_\mu\sigma\ x, f \in \sigma$. The case $\mathrm{Succ}_T(t)$ is a singleton $\{s\}$ is easy and we leave it to the reader. We verify the case $\mathrm{Succ}_T(t)$ has more than one element and hence $B_{c,t} = \bigcup_{s\in\mathrm{Succ}_T(t)} B_{c,s}$. If $(x,f) \in B_{c,t}$, then there is an $s \in \mathrm{Succ}_T(t)$ such that $(x,f) \in B_{c,s}$; since $\forall^*_\mu\sigma\ s \in \mathrm{Succ}_{T^\sigma}(t)$ by fineness, by the inductive hypothesis, $\forall^*_\mu\sigma\ (x, f\cap\sigma) \in B_{c^\sigma,s}$. So $\forall^*_\mu\sigma\ (x, f\cap\sigma) \in B_{c^\sigma,t}$. Conversely, suppose $\forall^*_\mu\sigma\ (x, f\cap\sigma) \in B_{c^\sigma,t}$. Then for each such σ, there is $s^\sigma \in \mathrm{Succ}_{T^\sigma}(t)$ such that $(x, f\cap\sigma) \in B_{c^\sigma,s^\sigma}$. By normality of μ, there is a fixed s such that

$$\forall^*_\mu\sigma\ s \in \mathrm{Succ}_{T^\sigma}(t) \wedge (x, f\cap\sigma) \in B_{c^\sigma,s}.$$

This implies $s \in \mathrm{Succ}_T(t)$ and $(x,f) \in B_{c,t}$ as desired. $\square$

[6] $A \in \mu$ iff $\{\pi^{-1}[\sigma] : \sigma \in A\} \in \nu$.

Let $f \subseteq \kappa$, then

$$\begin{aligned} f \in A &\Leftrightarrow \exists x \ (x, f) \in A^* \\ &\Leftrightarrow \exists x \ \forall^*_\mu \sigma \ (x, f \cap \sigma) \in B_{c^\sigma} \\ &\Leftrightarrow \forall^*_\mu \sigma \ \exists x \in \sigma \ (x, f \cap \sigma) \in B_{c^\sigma}. \end{aligned}$$

The last equality uses the normality of μ when restricted to the non-wellordered part, i.e. the normality of the measure induced by μ on $\wp_{\omega_1}(\mathcal{P}(\kappa))$. The proof of Claim 5.8 also uses the normality of μ, but only on the ordinal part.

Claim 5.9. *Let j_μ be the ultrapower embedding induced by μ. Letting $M_\sigma = L[c](\sigma)$[7] for each $\sigma \in \wp_{\omega_1}(\mathcal{P}(\kappa) \cup \lambda)$ and $M = \prod_\sigma M_\sigma/\mu$ be the ultraproduct of the M_σ's by μ, then:*

(a) Los's theorem holds for the ultraproduct M.

(b) $\mathbb{R} \subseteq M$ and $M \models \mathsf{AD}^+$. Therefore, by (a), $\forall^*_\mu \sigma \ M_\sigma \models \mathsf{AD}^+$.

(c) The ultraproduct M is well-founded.

(d) If $f \subset \kappa$ then $\forall^*_\mu \sigma \ f \cap \sigma \in M_\sigma$. In particular, $\forall^*_\mu \sigma \ c^\sigma \in M_\sigma$.

(e) For any $f \subseteq \kappa$, $[\sigma \mapsto f \cap \sigma]_\mu = j_\mu[f]$. In particular, if $x \subseteq \omega$, then $x = [\sigma \mapsto x]_\mu$.

(f) $\forall^*_\mu \sigma \ \mathbb{R} \cap M_\sigma = \mathbb{R} \cap \sigma$.

Proof. Part (a) is a standard argument using the normality of μ. The reader can see for example [Tra14a, Lemma 2.4] for a proof. For part (b), let $x \in \mathbb{R}$, then by fineness of μ, $\forall^*_\mu \sigma \ x \in \sigma$. So $\forall^*_\mu \sigma \ x \in M_\sigma$. Furthermore, $x = [\sigma \mapsto x]_\mu$ by countable completeness of σ, so $x \in M$. That $M \models \mathsf{AD}^+$ follows immediately from the fact that $\mathbb{R} \subset M$. The proof of part (c) is the same as that of Fact 2.4. Part (d) is clear since for each σ, $f, \sigma \in M_\sigma$ and M_σ is a model of ZF. For part (e), let $f \subseteq \kappa$. For $\alpha \in \kappa$, $j_\mu(\alpha) = [\sigma \mapsto \alpha]_\mu$. Therefore, if $\alpha \in f$, then by fineness, $\forall^*_\mu \sigma \ \alpha \in \sigma \cap f$. We have shown $j_\mu[f] \subseteq [\sigma \mapsto f \cap \sigma]_\mu$. Suppose g is such that $\forall^*_\mu \sigma \ g(\sigma) \in f \cap \sigma$. By normality of μ, there is an α such that $\forall^*_\mu \sigma \ g(\sigma) = \alpha$. So $[g]_\mu = j_\mu(\alpha) \in j_\mu[f]$. If $x \subseteq \omega$, then $\forall^*_\mu \sigma \ x \cap \sigma = x$ and since $j[\omega] = \omega$, $x = [\sigma \mapsto x]_\mu$. This completes the proof of (e). To see (f), first note that $\mathbb{R} \cap \sigma \subseteq \mathbb{R} \cap M_\sigma$ by part (b) and Los' theorem. For the converse, it suffices to see that $[\sigma \mapsto \sigma \cap \mathbb{R}]_\mu = \mathbb{R}^V$; this is because by part (b), $\mathbb{R}^V = \mathbb{R}^M$ and by Los' theorem, $\mathbb{R}^M = [\sigma \mapsto M_\sigma \cap \mathbb{R}]_\mu$. But this follows from (e). Indeed, for any $x \in \mathbb{R}$, $x = [\sigma \mapsto x]_\mu$ and $\forall^*_\mu \sigma \ x \in \sigma$, therefore, $x \in [\sigma \mapsto \sigma \cap \mathbb{R}]_\mu$. So $[\sigma \mapsto \sigma \cap \mathbb{R}]_\mu = [\sigma \mapsto M_\sigma \cap \mathbb{R}]_\mu = \mathbb{R}^V$; by Los' theorem, $\forall^*_\mu \ \mathbb{R} \cap M_\sigma = \mathbb{R} \cap \sigma$ as desired.

□

[7] $L[c](\sigma)$ is the minimal model of ZF containing $ON \cup \{c\} \cup \{A \cap \sigma : A \in \sigma\}$.

Remark 5.10. *We note that part (f) generally cannot be improved for subsets f of κ for $\kappa > \omega$. In general, if $f \subseteq \kappa$, $\forall^*_\mu \sigma\ f \cap \sigma \in M_\sigma$ follows from (d), but it is not true that $\forall^*_\mu \sigma\ f \cap \sigma \in \sigma$.*

For each $\sigma \in \wp_{\omega_1}(\mathcal{P}(\kappa) \cup \lambda)$ and $f \subseteq \kappa$, let

$$H_{\sigma,f} = \mathrm{HOD}^{L(c,\mathbb{R})}_{c,\{\sigma\},f\cap\sigma} \text{ and } K_{\sigma,f} = \mathrm{HOD}^{M_\sigma}_{c,\{\sigma\},f\cap\sigma}.$$

Let $\mathbb{Q}_{\sigma,f}$ be the Vopenka algebra for adding a real whose conditions are $OD^{M_\sigma}(c, \{\sigma\}, f \cap \sigma)$-$\infty$ Borel codes for subsets of $\mathbb{R} \cap M_\sigma$. The following is the key claim.

Claim 5.11. *Let $f \subseteq \kappa$.*

(i) $\forall^*_\mu \sigma\ H_{\sigma,f} = \mathrm{HOD}^{L(c,\mathbb{R})}_{c,\{\sigma\}}[f \cap \sigma]$.

(ii) $\forall^*_\mu \sigma$ *$K_{\sigma,f}$ is uniformly definable in $H_{\sigma,f}$ from parameters $\{\sigma\}, f \cap \sigma, c$.*

(iii) *$f \in A$ iff* $\forall^*_\mu \sigma\ H_{\sigma,f} \models$ "$K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f}\ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x\ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}$."

Proof. Part (i) follows from the proof of Corollary 1.4 and Remark 5.6. For part (ii), first note that $\forall^*_\mu \sigma\ f \cap \sigma \in M_\sigma$ by Claim 5.9. Letting $W_{\sigma,f}$ be the Vopenka algebra $\mathbb{BC}^{c,\{\sigma\},f\cap\sigma}_{\omega,\omega}$ defined in M_σ, then by Corollary 4.5,

$$K_{\sigma,f} = L[W_{\sigma,f}, c, f \cap \sigma]$$

is definable over $L(c,\mathbb{R})$ uniformly from parameters $\{\sigma\}, c, f \cap \sigma$. Let $X_{\sigma,f}$ be a set of ordinals that canonically codes $W_{\sigma,f}, c, f \cap \sigma$. Then there is a fixed formula ψ such that

$$x = X_{\sigma,f} \Leftrightarrow L(c,\mathbb{R}) \models \psi[x, f \cap \sigma, \{\sigma\}, c].$$

ψ induces a formula ψ^* with the property that: $H_{\sigma,f} \models \psi^*[x, c, \{\sigma\}, f \cap \sigma]$ if and only if $x = X_{\sigma,f}$. Here letting $\mathbb{P}_{\sigma,f} = \mathbb{BC}^{c,\{\sigma\},f\cap\sigma}_{\omega,\omega}$ defined in $L(c,\mathbb{R})$ and $\dot{\mathbb{R}}$ the symmetric name for the symmetric reals added by $\mathbb{P}_{\sigma,f}$, then $\psi^*[x, c, \{\sigma\}, f \cap \sigma]$ is the statement "$1 \Vdash_{\mathbb{P}_{\sigma,f}} L(\check{c}, \dot{\mathbb{R}}) \vDash \psi[x, f \cap \sigma, \{\sigma\}, c]$".

Now we prove (iii). First we note that by (ii), the statement "$K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f}\ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x\ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}$" is absolute between V and $H_{\sigma,f}$. Furthermore, there is a fixed formula φ such that $H_{\sigma,f} \models \varphi[\sigma, f \cap \sigma, c]$ if and only if $K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f}\ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x\ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}$. Now suppose $f \in A$. Then $\forall^*_\mu \sigma \exists x \in \sigma (x, f \cap \sigma) \in B_{c^\sigma}$. So for each such σ, $M_\sigma \models$ "$\exists x \in \sigma (x, f \cap \sigma) \in B_{c^\sigma}$". Fix such a σ and let $x \in \sigma$ be a witness and $g_x \subseteq \mathbb{Q}_{\sigma,f}$ be the corresponding generic adding x over $K_{\sigma,f}$, then $K_{\sigma,f}[x] = K_{\sigma,f}[g_x] \models (x, f \cap \sigma) \in B_{c^\sigma}$. By the forcing theorem, we get that $K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f}\ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x\ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}$, so we have obtained the right hand side of the equivalence. For the converse, assume

$$\forall^*_\mu \sigma\ H_{\sigma,f} \models \text{“}K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f}\ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x\ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}.\text{”}$$

So

$$\forall^*_\mu \sigma \ K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f} \ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x \ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}.$$

For each such σ, let p_σ be the $K_{\sigma,f}$-least condition in $\mathbb{Q}_{\sigma,f}$ such that $p_\sigma \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x(x, f \cap \sigma) \in \dot{B}_{c^\sigma}$. Let $g \subseteq \mathbb{Q}_{\sigma,f}$ $g \in M_\sigma$ be any generic over $K_{\sigma,f}$ and $p_\sigma \in g$, then $K_{\sigma,f}[g] \models \exists x(x, f \cap \sigma) \in B_{c^\sigma}$. Since $\forall^*_\mu \sigma$, $K_{\sigma,f}[g] \subseteq M_\sigma$ and $\mathbb{R} \cap M_\sigma = \mathbb{R} \cap \sigma$, we have that $\forall^*_\mu \sigma \ \exists x \in \sigma \ (x, f \cap \sigma) \in B_{c^\sigma}$. By normality, fix x such that $\forall^*_\mu \sigma \ (x, f \cap \sigma) \in B_{c^\sigma}$. By Claim 5.8, $(x, f) \in B_c$ and therefore, $f \in A$.
$\square$

Using Claim 5.11, we produce an ∞-Borel code for A by the following calculations. First let Z_σ be a set of ordinals such that $\mathrm{HOD}^{L(c,\mathbb{R})}_{c,\{\sigma\}} = L[Z_\sigma]$. Let $K_\infty = [\sigma \mapsto K_{\sigma,f}]_\mu$, $\mathbb{Q}_\infty = [\sigma \mapsto \mathbb{Q}_{\sigma,f}]_\mu$, and $Z_\infty = [\sigma \mapsto Z_\sigma]_\mu$. Let $W = j_\mu \restriction \kappa$.

$$\begin{aligned}
f \in A &\Leftrightarrow \forall^*_\mu \sigma \ H_{\sigma,f} \models \text{“}K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f} \ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x \ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}\text{”} \\
&\Leftrightarrow \forall^*_\mu \sigma \ L[Z_\sigma][f \cap \sigma] \models \text{“}K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f} \ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x \ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}\text{”} \\
&\Leftrightarrow L[Z_\infty][j_\mu[f]] \models \text{“}K_\infty \models \exists p \in \mathbb{Q}_\infty \ p \Vdash_{\mathbb{Q}_\infty} \exists x \ (x, j_\mu[f]) \in \dot{B}_{j_\mu[c]}\text{”} \\
&\Leftrightarrow L[Z_\infty, W][f] \models \text{“}L[Z_\infty][j_\mu[f]] \models \text{“}K_\infty \models \exists p \in \mathbb{Q}_\infty \ p \Vdash_{\mathbb{Q}_\infty} \exists x \ (x, j_\mu[f]) \in \dot{B}_{j_\mu[c]}\text{””}.
\end{aligned}$$

The first two equivalences are from Claim 5.11. The third equivalence follows from Claim 5.9. The last equivalence follows from the fact that one can easily compute $j_\mu[f]$ from W and f for any $f \subseteq \kappa$. We note here that by Claim 5.11(ii), there is a fixed formula φ such that $H_{\sigma,f} \models \varphi[\sigma, f \cap \sigma, c]$ if and only if $K_{\sigma,f} \models \exists p \in \mathbb{Q}_{\sigma,f} \ p \Vdash_{\mathbb{Q}_{\sigma,f}} \exists x \ (x, f \cap \sigma) \in \dot{B}_{c^\sigma}$.
$\square$

Lemma 5.7 and the discussion above show that an ∞-Borel code S for A can be found and furthermore, S is $OD(\mu, c)$, where c is an ∞-Borel code for A^*.

We now prove the "Furthermore" clause. We first start with the following key lemma.

Lemma 5.12. *Assume AD^+. Suppose $n \geq 1$ and $A^* \subseteq \mathcal{P}(\kappa)^{n+1}$ is ∞-Borel for some $\kappa < \Theta$. Suppose $c \subseteq \lambda$ is an ∞-Borel code for A^* and μ is a supercompact measure on $\wp_{\omega_1}(\mathcal{P}(\kappa) \cup \lambda)$, then an $OD(\mu, c)$ ∞-Borel code for $\exists^{\mathcal{P}(\kappa)} A^*$ can be found.*

Proof. Without loss of generality, we assume $n = 1$. Let $c = (T, \rho)$ be an ∞-Borel code for A^* and $A = \exists^{\mathcal{P}(\kappa)} A^*$. As in Lemma 5.7, we may assume T is coded as a subset of $\lambda < \Theta$.

Let μ be a supercompact measure on $\wp_{\omega_1}(\mathcal{P}(\kappa) \cup \lambda)$.[8] Let $j = j_\mu$ be the ultrapower embedding associated with μ. Let us define the objects $M_\sigma, H_{\sigma,f}, K_{\sigma,f}$ as above.

[8]If $\mathsf{AD}_\mathbb{R}$ holds, then such a μ exists and is unique. The existence (and uniqueness) of μ follows from $\mathsf{AD}_\mathbb{R}$ by the discussion in Section 2.

In the following, for a $\sigma \in \wp_{\omega_1}(\mathcal{P}(\kappa) \cup \lambda)$ the tree T^σ is defined as $T \cap \sigma$ and the assignment ρ^σ is defined as: for a terminal element t of T^σ, $\rho^\sigma(t) = \rho(t) \cap \sigma$. The code $c^\sigma = (T^\sigma, \rho^\sigma)$ will yield the set B_{c^σ} by induction as follows:

- if t is a terminal element of T^σ, let $B_{c^\sigma,t}$ be the basic open set $O_{\rho^\sigma(t)}$ in the space $2^{\sigma\cap\kappa} \times 2^{\sigma\cap\kappa}$. In the case $\rho^\sigma(t) = \emptyset$, then we let $B_{c^\sigma,t} = 2^{\sigma\cap\kappa} \times 2^{\sigma\cap\kappa}$.
- If $\mathrm{Succ}_{T^\sigma}(t)$ is a singleton of the form $\{s\}$, let $B_{c^\sigma,t}$ be the complement of $B_{c^\sigma,s}$ in the space $2^{\sigma\cap\kappa} \times 2^{\sigma\cap\kappa}$.
- If $\mathrm{Succ}_{T^\sigma}(t)$ has more than one element, then let $B_{c^\sigma,t}$ be the union of all sets of the form $B_{c^\sigma,s}$ where s is in $\mathrm{Succ}_{T^\sigma}(t)$.
- $B_{c^\sigma} = B_{c^\sigma,\emptyset}$.

The following claim mirrors Claim 5.8. The difference is in Claim 5.13, $x \subseteq \kappa$ and as mentioned above, $\forall^*_\mu\sigma\ x \cap \sigma \in M_\sigma$ but $x \cap \sigma$ is not necessarily in σ. Fortunately, we do not care whether $x \cap \sigma \in \sigma$ in the following arguments, but we do use the fact that $x \cap \sigma \in M_\sigma$.

Claim 5.13. *For any $(x, f) \in \mathcal{P}(\kappa)^{n+1}$,*

$$(x, f) \in A^* \Leftrightarrow \forall^*_\mu\sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma} \Leftrightarrow (j[x], j[f]) \in B_{j[c]}.$$

Proof. Again, we assume $n = 1$ here. We prove the first equivalence. The proof is by induction on the ranks of the nodes in T just like in Claim 5.8. If t is a terminal node of T, it is easy to see that $(x, f) \in O_{\rho(t)}$ iff $\forall^*_\mu\sigma\ (x \cap \sigma, f \cap \sigma) \in O_{\rho^\sigma(t)}$. Here note that by fineness, $\forall^*_\mu\sigma\ x, f \in \sigma$. The case $\mathrm{Succ}_T(t)$ is a singleton $\{s\}$ is easy and we leave it to the reader. We verify the case $\mathrm{Succ}_T(t)$ has more than one element and hence $B_{c,t} = \bigcup_{s\in\mathrm{Succ}_T(t)} B_{c,s}$. If $(x, f) \in B_{c,t}$, then there is an $s \in \mathrm{Succ}_T(t)$ such that $(x, f) \in B_{c,s}$; since $\forall^*_\mu\sigma\ s \in \mathrm{Succ}_{T^\sigma}(t)$ by fineness, by the inductive hypothesis, $\forall^*_\mu\sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma,s}$. So $\forall^*_\mu\sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma,t}$. Conversely, suppose $\forall^*_\mu\sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma,t}$. Then for each such σ, there is $s^\sigma \in \mathrm{Succ}_{T^\sigma}(t)$ such that $(x \cap \sigma, f \cap \sigma) \in B_{c^\sigma,s^\sigma}$. By normality of μ, there is a fixed s such that

$$\forall^*_\mu\sigma\ s \in \mathrm{Succ}_{T^\sigma}(t) \wedge (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma,s}.$$

This implies $s \in \mathrm{Succ}_T(t)$ and $(x, f) \in B_{c,t}$ as desired.

The second equivalence follows from Los' Theorem and the fact that: $[\sigma \mapsto \sigma \cap f]_\mu = j[f], [\sigma \mapsto \sigma \cap x]_\mu = j[x], [\sigma \mapsto \sigma \cap c^\sigma]_\mu = j[c]$. See Claim 5.9. □

Claim 5.14. *Let $f \subseteq \kappa$, then*

$$f \in A \Leftrightarrow \forall^*_\mu\sigma\ \exists x \in \sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma}.$$

Proof. Fix $f \subseteq \kappa$. We have the following equivalences.

$$\begin{aligned} f \in A &\Leftrightarrow \exists x\ (x, f) \in A^* \\ &\Leftrightarrow \exists x\ \forall^*_\mu\sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma} \\ &\Leftrightarrow \forall^*_\mu\sigma\ \exists x \in \sigma\ (x \cap \sigma, f \cap \sigma) \in B_{c^\sigma}. \end{aligned}$$

The first equivalence is by definition. The second equivalence follows from Claim 5.13. The last equality uses the normality of μ when restricted to the non-wellordered part, i.e. the normality of the measure induced by μ on $\wp_{\omega_1}(\mathcal{P}(\kappa))$. The proof of Claim 5.13 also uses the normality of μ, but only on the ordinal part.

$\square$

For each σ and f, let $\mathbb{P}^n_{\sigma,f} \in K_{\sigma,f}$ be the poset isomorphic to the algebra of $OD(c,\{\sigma\}, f\cap\sigma)^{M_\sigma}$ subsets of σ^n in M_σ for each $n<\omega$ and $\mathbb{P}^\infty_{\sigma,f}$ be the inverse limit of the $\mathbb{P}^n_{\sigma,f}$ via the canonical projection maps (see Definition 4.5). Let $\dot\sigma$ be the $\mathbb{P}^\infty_{\sigma,f}$-symmetric name that whenever $G\subseteq Col(\omega,\sigma)$ is generic over M_σ, letting $g\subseteq \mathbb{P}^\infty_{\sigma,f}$ be the $K_{\sigma,f}$-generic induced by G, then $\dot\sigma_g=\sigma$ and M_σ is the corresponding symmetric extension of $K_{\sigma,f}$. Note that $\mathbb{P}^\infty_{\sigma,f}$ is countable in $L(c,\mathbb{R})$. By the previous claim, we now have the following equivalence:[9]

(†) $\quad f\in A \Leftrightarrow \forall^*_\mu\sigma\ H_{\sigma,f}\models$ "$K_{\sigma,f}\models 1_{\mathbb{P}^\infty_{\sigma,f}} \Vdash_{\mathbb{P}^\infty_{\sigma,f}}$ "$\exists x\in\dot\sigma\ (x\cap\dot\sigma, f\check{\cap}\sigma)\in \dot B_{c^\sigma}$"".

To see the equivalence, first suppose $f\in A$. By Claim 5.14,

$$\forall^*_\mu\sigma\ \exists x\in\sigma\ M_\sigma\models \text{“}(x\cap\sigma, f\cap\sigma)\in \dot B_{c^\sigma}\text{”}.$$

By the forcing theorem and homogeneity of $\mathbb{P}^\infty_{\sigma,f}$, we have $\forall^*_\mu\sigma$,

$$K_{\sigma,f}\models 1_{\mathbb{P}^\infty_{\sigma,f}} \Vdash_{\mathbb{P}^\infty_{\sigma,f}} \text{“}\exists x\in\dot\sigma\ (x\cap\dot\sigma, f\check{\cap}\sigma)\in \dot B_{c^\sigma}\text{”}.$$

But then the right hand side of the equivalence follows from Claim 5.11.

For the converse, assume the right hand side. For each such σ, there is a $g\in V$ such that $g\subseteq\mathbb{P}^\infty_{\sigma,f}$ is generic over $K_{\sigma,f}$ and $\dot\sigma_g=\sigma$, then

$$\exists x\in\sigma\ (x\cap\sigma, f\cap\sigma)\in B_{c^\sigma}.$$

By normality of μ, there is an x such that $\forall^*_\mu\sigma\ x\in\sigma\wedge(x\cap\sigma, f\cap\sigma)\in B_{c^\sigma}$. By Claim 5.14. we get $f\in A$. (†) has been verified.

Now we produce an ∞-Borel code for A by similar calculations as before, using (†). First let Z_σ be a set of ordinals such that $\mathrm{HOD}^{L(c,\mathbb{R})}_{c,\{\sigma\}}=L[Z_\sigma]$. Let $K_\infty=[\sigma\mapsto K_{\sigma,f}]_\mu$, $\mathbb{P}_\infty=[\sigma\mapsto\mathbb{P}^\infty_{\sigma,f}]_\mu$, $c_\infty=[\sigma\mapsto c]_\mu$, $\dot\sigma_\infty=[\sigma\mapsto\dot\sigma]_\mu$, and $Z_\infty=[\sigma\mapsto Z_\sigma]_\mu$. Let $W=j_\mu\restriction\kappa$.

$$\begin{aligned}
f\in A &\Leftrightarrow \forall^*_\mu\sigma\ H_{\sigma,f}\models \text{“}K_{\sigma,f}\models 1_{\mathbb{P}^\infty_{\sigma,f}}\Vdash_{\mathbb{P}^\infty_{\sigma,f}}\\
&\qquad \text{“}\exists x\in\dot\sigma\ (x\cap\dot\sigma, f\check{\cap}\sigma)\in\dot B_{c^\sigma}\text{””}\\
&\Leftrightarrow \forall^*_\mu\sigma\ L[Z_\sigma][f\cap\sigma]\models\text{“}K_{\sigma,f}\models 1_{\mathbb{P}^\infty_{\sigma,f}}\Vdash_{\mathbb{P}^\infty_{\sigma,f}}\\
&\qquad \text{“}\exists x\in\dot\sigma\ (x\cap\dot\sigma, f\check{\cap}\sigma)\in\dot B_{c^\sigma}\text{””}\\
&\Leftrightarrow L[Z_\infty][j[f]]\models\text{“}K_\infty\models 1_{\mathbb{P}_\infty}\Vdash_{\mathbb{P}_\infty}\\
&\qquad \text{“}\exists x\in\dot\sigma_\infty\ (x\cap\dot\sigma_\infty, \check{j[f]})\in\dot B_{j[c]}\text{””}\\
&\Leftrightarrow L[Z_\infty,W][f]\models\text{“}L[Z_\infty][j[f]]\models\text{“}K_\infty\models 1_{\mathbb{P}_\infty}\Vdash_{\mathbb{P}_\infty}\\
&\qquad \text{“}\exists x\in\dot\sigma_\infty\ (x\cap\dot\sigma_\infty, \check{j[f]})\in\dot B_{j[c]}\text{”””}.
\end{aligned}$$

[9]We use the canonical name $f\check{\cap}\sigma$ for $f\cap\sigma$.

As before, the above calculations show that A is ∞-Borel and an $OD(c,\mu)$ ∞-Borel code for A can be found. $\square$

We now assume $V = L(\mathcal{P}(\mathbb{R}))$ and $\mathsf{AD}_{\mathbb{R}}$. As mentioned above, we have a unique supercompact measure μ on $\wp_{\omega_1}(X)$ for any set X that is a surjective image of $\mathbb{R}$. Lemma 5.12 applied to the unique, hence OD measures μ, shows that the inverse limit $\overline{\mathbb{BC}}_{\infty,\omega}$ is well-defined. Let $X = \mathcal{P}(\kappa)$ and $A \subseteq X$ be arbitrary. We give an alternative proof that A has an ∞-Borel code. Let φ, f define A where $f \in \gamma^\omega$ for some $\gamma < \Theta$, i.e. for any $x \subset \kappa$,

$$x \in A \Leftrightarrow V \models \varphi[x, f].$$

We can construe f as a countable subset of γ. As in the argument of [Cha19, Corollary 7.20] but now using Lemmata 5.12 and 4.6, we have, letting Z be such that $\mathrm{HOD} = L[Z]$ and $g_{f,x} \subseteq \overline{\mathbb{BC}}_{\infty,2}$ be the generic adding the pair (f,x),

$$\begin{aligned} x \in A &\Leftrightarrow \mathrm{HOD}[g_{f,x}] \models 1_{\overline{\mathbb{BC}}_{\infty,\omega}/g_{f,x}} \Vdash_{\overline{\mathbb{BC}}_{\infty,\omega}/g_{f,x}} \mathrm{HOD}(\bigcup_{\beta<\Theta} \mathcal{P}(\beta)) \models \varphi[x,f] \\ &\Leftrightarrow L[Z,f,x] \models 1_{\overline{\mathbb{BC}}_{\infty,\omega}/g_{f,x}} \Vdash_{\overline{\mathbb{BC}}_{\infty,\omega}/g_{f,x}} \mathrm{HOD}(\bigcup_{\beta<\Theta} \mathcal{P}(\beta)) \models \varphi[x,f] \end{aligned}$$

The second equivalence, as mentioned before, follows from 4.2. The above easily yields an $OD(f)$ ∞-Borel code for A. In particular, if A is OD, then A has an OD ∞-Borel code. The argument given easily generalizes to show that for any set of ordinals S, for any $\kappa < \Theta$, if a set $A \subseteq \mathcal{P}(\kappa)$ is $OD(S)$, then A has an $OD(S)$ ∞-Borel code. This completes the proof of the theorem. $\square$

Remark 5.15. *Theorem 1.5 shows that $\overline{\mathbb{BC}}_{\infty,\omega}$ is isomorphic to $\mathbb{P}_{\infty,\omega}$ if $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$ holds.*

Proof of Corollary 1.6. Using $\overline{\mathbb{BC}}^S_{\infty,\omega}$, one can show as in the proof of Corollary 1.4 that for any set of ordinals S, any $x \subseteq \kappa$ for any $\kappa < \Theta$, we have

$$\mathrm{HOD}_S[x] = \mathrm{HOD}_{S,x}.$$

The key points are that the limit $\overline{\mathbb{BC}}^S_{\infty,\omega}$ is well-defined by Theorem 1.5 and that $V = \mathrm{HOD}_S(\bigcup_{\gamma<\Theta} \mathcal{P}(\gamma))$ is a symmetric extension of HOD_S induced by a generic $H \subseteq \overline{\mathbb{BC}}^S_{\infty,\omega}$. $\square$

6 Questions

We collect a few questions left open from the above analysis.

Question 6.1. *(i) Assume AD^+. Suppose μ is an arbitrary countably complete measure on some set X. Must $Ult(V,\mu)$ be well-founded?*

(ii) *Assume* AD^+. *Suppose* μ *is an arbitrary supercompact measure on* $\wp_{\omega_1}(X)$ *for some set* X. *Suppose* $(M_\sigma : \sigma \in \wp_{\omega_1}(X))$ *is such that for each* σ, M_σ *is a transitive model of* ZF^-. *Must Los's theorem holds for the ultraproduct* $\prod_\sigma M_\sigma/\mu$*?*

(iii) *Does* AD^+ *and* ω_1 *is* $\mathbb{R}$*-supercompact imply there must be a unique normal, fine measure on* $\wp_{\omega_1}(\mathbb{R})$*?*

Regarding 6.1(i), Solovay [Sol06] shows that $\mathrm{cof}(\Theta) > \omega + \mathsf{DC}_\mathbb{R} + \neg\mathsf{DC}_{\mathcal{P}(\mathbb{R})}$ implies there is a countably complete measure μ on $\mathrm{cof}(\Theta)$ such that $\mathrm{Ult}(V, \mu)$ is ill-founded. We do not know if a model of AD^+ satisfying the hypothesis Solovay's proof requires can exist. Regarding (iii), by results of Solovay and Woodin, $\mathsf{AD}_\mathbb{R} + \mathsf{DC}_\mathbb{R}$ implies that there is a unique normal, fine measure on $\wp_{\omega_1}(\mathbb{R})$. The minimal model of the theory "AD^+ and ω_1 is $\mathbb{R}$-supercompact" also satisfies the uniqueness of such a measure (cf [Tra15] and [RT18]). It is known that the conclusion of (iii) is false in the absence of AD^+.

Question 6.2. *Does* AD^+ *imply* AD^{++}*?*

By Theorem 1.5, Question 6.2 has a positive answer if we additionally assume $\mathsf{AD}_\mathbb{R}$. We do not know even in $L(\mathbb{R})$, every subset of $\mathcal{P}(\omega_1)$ has an ∞-Borel code. However, it is known that Question 6.2 has a positive answer in various $\mathsf{AD}^+ + \neg\mathsf{AD}_\mathbb{R}$ models not of the form $V = L(\mathcal{P}(\mathbb{R}))$. For instance, in the model of the form $L(\mathbb{R}, \mu)$ that satisfies AD^+ + "μ is a normal fine measure on $\wp_{\omega_1}(\mathbb{R})$", for every $\kappa < \Theta$, every $A \subseteq \mathcal{P}(\kappa)$ has an ∞-Borel code. Even if AD^{++} is not a consequence of AD^+, one can still conjecture.

Conjecture 6.3 (AD^+)**.** *The ABCD Conjecture holds.*

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