Remark. These notes roughly follow the approach of D. Cohn’s "Measure Theory." Many thanks to J. Holshouser for producing the original draft of this document, which I have only edited.

Definition. If $G$ is a both a topological space and a group, then we say $G$ is a topological group if both multiplication and inversion are continuous.

Lemma. Let $G$ be a topological group with identity $e$, and let $U \subseteq G$ be open with $e \in U$. Then there is an open nhood $V \subseteq U$ such that $e \in V$ and $VV \subseteq U$.

Proof. Since multiplication is continuous, the set $O = \{(x, y) : xy \in U\}$ is open. So find $V_1, V_2 \subseteq G$ open such that $(e, e) \in V_1 \times V_2 \subseteq O$. Let $V = V_1 \cap V_2$. Then $e \in V$, and if $x, y \in V$, then $(x, y) \in O$ and thus $xy \in U$. □

Lemma. Let $G$ be a topological group, $K \subseteq G$ be compact and $U \subseteq G$ open with $K \subseteq U$. Then there is an open nhood $V$ of $e$ such that $KV \subseteq U$.

Proof. For each $x \in K$, let $W_x$ be an nhood of $e$ such that $xW_x \subseteq U$ and let $V_x$ be an nhood of $e$ such that $V_xV_x \subseteq W_x$. Then $\{xV_x : x \in K\}$ is an open cover of $K$, so choose $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \bigcup_{i \leq n} x_iV_{x_i}$$

Set $V = V_{x_1} \cap \cdots \cap V_{x_n}$. If $x \in K$, then there is an $i$ such that $x \in x_iV_i$. Thus

$$xV \subseteq (x_iV_{x_i})V_{x_i} \subseteq x_iW_{x_i} \subseteq U$$

Since $x$ was arbitrary, $KV \subseteq U$. □

Definition. A topological group $G$ is locally compact if there is a compact set $K \subseteq G$ which has nonempty interior.

Definition. A Borel measure $\mu$ on $G$ is regular if

1. If $K \subseteq G$ is compact, then $\mu(K) < \infty$.
2. If $U \subseteq G$ is open, then $\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ is compact}\}$. This condition is called inner regularity.
3. If $A \subseteq G$ is Borel, then $\mu(A) = \inf\{\mu(U) : U \supseteq A \text{ is open}\}$. This condition is called outer regularity.
**Definition.** A nonzero regular Borel measure on $G$ is a left Haar measure if it is left translation invariant. That is, if $A \subseteq G$ is Borel and $x \in G$, then $\mu(xA) = \mu(A)$.

**Remark.** The following theorem, which is the main focus of our talk, asserts the existence of a left Haar measure for any locally compact topological group. Though often called Haar’s theorem, Alfred Haar only proved the theorem for groups which are second countable, in 1933. The statement in its full generality was first proved by Andre Weil, using the Axiom of Choice. Later Henri Cartan furnished a proof which avoided the use of AC. The proof sketch that we present here will utilize choice in the form of Tychonoff’s theorem.

**Theorem.** Let $G$ be a locally compact Hausdorff topological group. Then there is a nonzero left Haar measure $\mu$ on $G$. Moreover, if $\mu$ and $\nu$ are both left Haar measures on $G$, then $\mu = c\nu$ for some $c \in \mathbb{R}$.

**Proof.** If $K \subseteq G$ is compact and $K^0 \neq \emptyset$, then we define $\#(K : V)$ to be the least $n$ such that there are $x_1, \cdots, x_n \in G$ with $K \subseteq x_1V \cup \cdots \cup x_nV$. Let $\mathcal{K}(G)$ be the set of all compact subsets of $G$, and let $\mathcal{U}$ be all neighborhoods of $e$.

Fix now and forever a compact $K_0 \subseteq G$ with $K_0^0 \neq \emptyset$. For each $U \in \mathcal{U}$, we define $h_U : \mathcal{K}(G) \to \mathbb{R}$ by

$$h_U(K) = \frac{\#(K : U)}{\#(K_0 : U)}$$

**Claim.** Let $U \in \mathcal{U}$. Then $h_U$ has the following properties:

1. $\#(K : U) \leq \#(K : K_0)\#(K_0 : U)$.
2. $0 \leq h_U(K) \leq \#(K : K_0)$.
3. $h_U(K_0) = 1$.
4. $h_U(xK) = h_U(K)$.
5. If $K \subseteq L$, then $h_U(K) \leq h_U(L)$.
6. $h_U(K \cup L) \leq h_U(K) + h_U(L)$
7. If $KU^{-1} \cap LU^{-1} = \emptyset$, then $h_U(K \cup L) = h_U(K) + h_U(L)$.

**Proof of Claim.** Part 2 follows from part 1, and everything else other than part 7 is clear from the definition of $h_U$. So suppose that $KU^{-1} \cap LU^{-1} = \emptyset$. Let $n = \#(K \cup L : U)$ and let $x_1, \cdots, x_n \in G$ such that $K \cup L \subseteq x_1U \cup \cdots \cup x_nU$. We wish to show that each $x_iU$ meets either $K$ or $L$, but not both. Suppose toward a contradiction that there is an $i$ such that $x_iU$ meets both $K$ and $L$. Let $y \in x_iU \cap K$. Then there is a $u \in U$ such that $y = x_iu$. So $x_i = yu^{-1}$ and thus $x_i \in KU^{-1}$. Similarly, if $z \in x_iU \cap L$, then $x_i \in LU^{-1}$. Thus $x_i \in KU^{-1} \cap LU^{-1}$, a contradiction. \hfill \Box

We now wish to take a “limit” of the $h_U$’s. For each $k \in \mathcal{K}(G)$, set $I_K = [0, \#(K : K_0)] \subseteq \mathbb{R}$. Set $X = \prod_{K \in \mathcal{K}(G)} I_K$. Then by Tychonoff, $X$ is compact and for each $U \in \mathcal{U}$, $h_U \in X$. For each $V \in \mathcal{U}$, set $S(V) = \{h_U : U \subseteq V\}$. Now suppose that $V_1, \cdots, V_n \in \mathcal{U}$. Let $V = V_1 \cap \cdots \cap V_n$. Then $V \in \mathcal{U}$, so $h_V \in \bigcap_{i \leq n} S(V_i)$. Thus the
collection \(\{S(V) : V \in \mathcal{U}\}\) of closed sets has the finite intersection property. Therefore, as \(X\) is compact, we can fix an \(h \in \bigcap_{V \in \mathcal{U}} S(V)\).

**Claim.** \(h\) has the following properties:

1. \(0 \leq h(K)\).
2. \(h(\emptyset) = 0\).
3. \(h(K_0) = 1\).
4. \(h(xK) = h(K)\).
5. If \(K \subseteq L\), then \(h(K) \leq h(L)\).
6. \(h(K \cup L) \leq h(K) + h(L)\).
7. If \(K \cap L = \emptyset\), then \(h(K \cup L) = h(K) + h(L)\).

**Proof of Claim.** The crucial fact of this proof is that the evaluation maps \(E_K : X \to \mathbb{R}\), given by \(E_K(f) = f(K)\), are continuous since \(X\) has the product topology. We present some of the parts, but leave the rest as exercise.

Let \(K \subseteq G\) be compact. Since \(E_K\) is continuous, \(E_K^{-1}([0, \infty))\) is closed. For each \(U \in \mathcal{U}\), \(h_U \in E_K^{-1}([0, \infty))\). Thus for each \(V \in \mathcal{U}\), \(S(V) \subseteq E_K^{-1}([0, \infty])\). So \(h \in E_K^{-1}([0, \infty])\), and thus \(h(K) \geq 0\). Since \(K\) was arbitrary, this proves part 1.

Again, let \(K \subseteq G\) be compact. Then \(\phi = E_{xK} - E_K\) is continuous, where the operation here is pointwise subtraction. Then \(\phi^{-1}(0)\) is closed, and we repeat the argument in the previous paragraph to see that \(h(xK) = h(K)\). Since \(K\) was arbitrary, this proves part 4.

We finish by proving part 7. Suppose that \(K \cap L = \emptyset\). Choose \(U_1, U_2\) which separate \(K\) and \(L\). By lemma 2, we can find \(V_1, V_2\) open nhoods of \(e\) such that \(KV_1 \subseteq U_1\) and \(LV_2 \subseteq U_2\). Let \(V = V_1 \cap V_2\). Then \(V\) is still an open nhood of \(e\). Now for any \(U \subseteq V^{-1}, U^{-1} \subseteq V\), and thus \(KU^{-1} \cap LU^{-1} = \emptyset\). By part 7 of claim 1, we then have that

\[
h_U(K) + h_U(L) - h_U(K \cup L) = 0
\]

Consider \(\phi = E_K + E_L - E_{K \cup L}\). Then \(\phi\) is continuous, so \(\phi^{-1}(0)\) is a closed set containing \(h_U\) for each \(U \subseteq V^{-1}\). Thus \(h \in S(V^{-1}) \subseteq \phi^{-1}(0)\). \(\square\)

Now we define an outer measure \(\mu^* : \mathcal{P}(G) \to [0, \infty]\), If \(U \subseteq G\) is open, define

\[
\mu^*(U) = \inf\{h(K) : K \subseteq U \text{ is compact}\}
\]

For \(A \in \mathcal{P}(G)\), we define

\[
\mu^*(A) = \inf\{\mu^*(U) : U \supseteq A \text{ is open}\}
\]
We aim to apply the Caratheodory condition to create a Borel measure. We must first verify that \( \mu^* \) is an outer measure, i.e. we must check the following:

1. \( \mu^* \) is non-negative,
2. \( \mu^* \) is monotone,
3. \( \mu^*(\emptyset) = 0 \),
4. \( \mu^* \) is countably subadditive.

Parts one through three are clear from the definition on \( \mu^* \) and the properties of \( h \). We thus need to confirm subadditivity. Because we have defined \( \mu^* \) to be outer regular, it suffices to check open sets. Let \( U = \bigcup_{i \in \omega} U_i \) be a countable union of open sets. Let \( K \subseteq U \) be compact. Then by a standard fact about compact sets, there are compact sets \( K_1, \ldots, K_n \) such that \( K = K_1 \cup \cdots \cup K_n \) and \( K_i \subseteq U_i \) for each \( i \leq n \).

Then

\[
h(K) \leq \sum_{i \leq n} h(K_i) \leq \sum_{i \leq n} \mu^*(U_i) \leq \sum_{i \in \omega} \mu^*(U_i)
\]

Since this is true for all compact \( K \subseteq U \), \( \mu^*(U) \leq \sum_{i \in \omega} \mu^*(U_i) \).

Let \( \mu \) be \( \mu^* \) restricted to the \( \mu^* \)-measurable sets. By Caratheodory’s theorem, \( \mu \) is a measure on these sets. It remains to show that the Borel sets of \( G \) are \( \mu^* \)-measurable. It suffices to show that the open sets of \( G \) are \( \mu^* \)-measurable as the \( \mu^* \)-measurable sets are a \( \sigma \)-algebra. Moreover, since \( \mu^* \) is outer regular, it again suffices to check only the \( \mu^* \)-measurability of open sets. In other words, it suffices to check that

\[
\mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \cap U^c)
\]

whenever \( U \) and \( V \) are open in \( G \). To see this, first fix \( \epsilon > 0 \), and use the definition of \( \mu^* \) to find a compact set \( K \subseteq V \cap U \) for which \( h(K) > \mu^*(V \cap U) - \epsilon \). Next choose a compact set \( L \subseteq V \cap K^c \) for which \( h(L) > \mu^*(V \cap K^c) - \epsilon \). Since \( K \) and \( L \) are disjoint, we have

\[
\mu^*(V \cap U) + \mu^*(V \cap U^c) - 2\epsilon < h(K) + h(L) = h(K \cup L) \leq \mu^*(V).
\]

Since \( \epsilon \) was chosen arbitrarily, we have the inequality we want. Thus \( \mu \) is a measure on a \( \sigma \)-algebra of sets which includes the Borel sets of \( G \).

We will now show that \( \mu \) is a nonzero left Haar measure. We begin with regularity. It is clear from definition that \( \mu \) is outer regular. Moreover if \( K \) is compact and \( U \) is open containing \( K \), then \( h(K) \leq \mu(U) \); now taking the infimum over all such \( U \), we see that \( h(K) \leq \mu(K) \). This inequality, together with the definition of \( \mu \), show that \( \mu \) is also inner regular. So we need to check that compact sets get finite measure. Let \( K \subseteq G \) be compact. Choose \( U \subseteq G \) open such that \( K \subseteq U \) and \( \overline{U} \) is compact. We can do this as \( G \) is locally compact, \( K \) is compact, and a finite union of compact sets is compact. Then

\[
\mu(K) \leq \mu(U) \leq h(\overline{U}) < \infty
\]
The second inequality follows from the monotonicity of $h$. Thus $\mu$ is regular. Now $\mu$ is a Haar measure as $h$ is invariant under left translations. Finally $\mu$ is nonzero as $1 = h(K_0) \leq \mu(K_0)$.

It remains now to show that $\mu$ is unique up to constant multiplication, i.e. if $\nu$ is another left Haar measure, then $\nu = c\mu$. To see this, fix any nonzero nonnegative continuous function $g : G \to \mathbb{R}$ with compact support, and let $f : G \to \mathbb{R}$ be an arbitrary continuous function with compact support. The integrals of both functions with respect to $\mu$ and $\nu$ will be finite and the integrals of $g$ will also be nonzero. We wish to show that the following ratio holds:

$$\frac{\int f d\nu}{\int g d\nu} = \frac{\int f d\mu}{\int g d\mu}$$

which will imply that $\int f d\nu = c\int f d\mu$ for the constant $c = \int g d\nu / \int g d\mu$. Since this will hold for each $f$, then the Riesz representation theorem will imply the result we want.

So let’s prove that the ratio holds. First consider an arbitrary function $h : G \times G \to \mathbb{R}$ with compact support. Using Fubini’s theorem to reverse the order of integration as necessary, and the translation-invariance of $\mu$ and $\nu$ to make appropriate variable substitutions, we find that

$$\int \int h(x, y) d\nu(y) d\mu(x) = \int \int h(y^{-1}x, y) d\mu(x) d\nu(y) = \int \int h(y^{-1}, xy) d\nu(y) d\mu(x).$$

(We replace $x$ with $y^{-1}x$ for the first equality and $y$ with $xy$ for the second.) Now apply this identity to the function $h(x, y) = f(x)g(yx)/\int g(tx) d\nu(t)$ and we get:

$$\int f(x) d\mu(x) = \int g(x) d\mu(x) \int \frac{f(y^{-1})}{\int g(ty^{-1}) d\nu(t)} d\nu(y).$$

Thus the ratio of $\int f d\mu$ to $\int g d\mu$ depends on $f$ and $g$, but not on $\mu$, which proves the claim. \qed