Practice for 2^{nd} Midterm — Test is 22 June 2004

Math 2770

19 June 2004

Dr. Michael Oliver

1) Let $a, b, c, d \in \mathbb{Z}$, and suppose $a|b$, $b|c$, and $a|d$. Can we conclude that $a|(5c + 2d)$? If so, explain how; if not, find a counterexample.

2) Suppose p and q are prime, a and b are integers, and pq divides ab. Which of the following scenarios are possible? (There may be more than one possible scenario.) Circle *all* that apply, and then explain why.

a) $p|a$ and $q|b$, but $p \nmid a$ and $q \nmid b$

b) $p|a$ and $q|a$, but $p \nmid b$ and $q \nmid b$

c) $p|a$, $p|b$, $q|a$, and $q|b$

d) $p|a$ and $p|b$, but $q \nmid a$ and $q \nmid b$

3)

a) Use Euclid's Algorithm to find gcd(15,27). At each stage, keep track of what linear combination of 15 and 27 is each of the new values that occurs.

b) Find integer values s and t such that

$$6 = 15s + 27t$$

4) For any integers a and b, show that

$$ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$$

(You may assume the Fundamental Theorem of Arithmetic. Also, you may write down without proof the formulas for gcd and lcm in terms of the prime factorizations of a and b.)

6) This time you may *not* assume the Fundamental Theorem of Arithmetic, but you may assume our "Key Lemma"; that is, if p is prime and $p|ab$, then either $p|a$ or $p|b$. Given that, prove the uniqueness part of the Fundamental Theorem of Arithmetic (that is, prove that no natural number has two distinct prime factorizations). Use either strong induction or the method of infinite descent. It is not necessary to be extremely formal, but you must show how induction or infinite descent is used.

7)

a) Does there exist an integer x such that

$$\begin{aligned} x &\equiv 2 \pmod{7} \\ x &\equiv 1 \pmod{14} \end{aligned}$$

Why or why not?

b) Does there exist an integer y such that

$$\begin{aligned} y &\equiv 2 \pmod{7} \\ y &\equiv 1 \pmod{13} \end{aligned}$$

Why or why not?

c) Exactly one of (a), (b) above has a "yes" answer. Find the integer requested (x or y, as the case may be)

8) What is the last digit of the decimal representation of 7^{1025}? No calculators, and you don't need one. Really.

9) Recall $\phi(n)$, the Euler totient function, is defined as the number of natural numbers less than n that are relatively prime to n.

a) If p is prime, what is $\phi(p^3)$? (Hint: Which numbers below p^3 are *not* relatively prime to p^3, and how many of them are there?

b) What is the value of $3^{100} \pmod{125}$? Hint: Apply part (a) and the extended Fermat's Little Theorem.