Practice for 1st Midterm — Test is 10 June 2004

Math 2770

8 June 2004

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1) Suppose Alice and Bob play the following game:

Alice	$m =$	
Bob		$n =$

The rules are: Alice and Bob must play natural numbers. Bob wins if $n > m$ and $2^n - 1$ is prime; otherwise, Alice wins.

a) Write a sentence of quantified first-order logic that is true (interpreted with the natural numbers as the universe of discourse) if and only if Bob has a winning strategy for the game.

b) Explain what the above sentence "says", in English.

2) Let $Q(x, y)$ denote "$x+y = 0$". What are the truth values of the sentences $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$

a) interpreted with the universe of discourse equal to $\mathbb{R}$?

b) interpreted with the universe of discourse equal to $\mathbb{N}$?

3) Let A, B, and C be sets (and that they are all subsets of some fixed universal set). Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

4) Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2\left\lfloor \frac{n}{2} \right\rfloor$.

a) Is f one-to-one (injective)?

b) Is f onto (surjective)?

Be sure to explain your answers.

5) Show that $\Sigma_{j=0}^{n} j^3$ is $\Theta(n^4)$. (Hint: You need to show two things – that it's $O(n^4)$, and that it's $\Omega(n^4)$.) (Second hint: It should *not* be necessary to find an explicit formula for$\Sigma_{j=0}^{n} j^3$).

6) Suppose $f(x)$ is $O(h(x))$, and $g(x)$ is $\Omega(k(x))$, and suppose that $\forall x(k(x) > 0)$. Prove or refute:

$$\frac{f(x)}{g(x)} \text{ is } O\left(\frac{h(x)}{k(x)}\right)$$

7) Describe an algorithm that, given a finite sequence of integers, returns the smallest integer in the sequence.

8) Determine the worst-case time complexity of the algorithm above.

9) The *Fibonacci sequence* F_n is defined as follows:

$$F_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F_{n-2} + F_{n-1}, & \text{if } n > 1 \end{cases}$$

a) Compute F_0 through F_{11}. Hint: if this takes more than thirty seconds, you've missed something.

b) Can you find a pattern to which Fibonacci numbers are odd and which are even? For example, could it be that, for every k, F_{4k}, F_{4k+1}, and F_{4k+3} are all even, but F_{4k+2} is odd? Hint: no, that's not the answer, but it looks something like that.

c) Prove the conjecture you made above, by induction.