1$^{\text{st}}$ Midterm NAME:____________________

Math 2770

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1) A *Fermat prime* (pronounced fehr-mah) is a prime of the form $2^{2^n}+1$, for some natural number $n$. For example, taking $n=2$, $2^{2^n}+1=17$, which is prime, so 17 is a Fermat prime.

a) State a sentence of first-order logic that, interpreted over the natural numbers, expresses the claim "there are infinitely many Fermat primes".

b) Describe a game between Alice and Bob, playing natural numbers, such that Bob has a winning strategy if and only if there are infinitely many Fermat primes.

c) (Not for credit, just to think about.) Recall that a *Mersenne prime* is a prime of the form $2^p-1$, where $p$ is also prime. It has neither been proved nor refuted that there are infinitely many Mersenne primes, nor has it been proved or refuted that there are infinitely many Fermat primes. Which kind of prime do you think is more likely to have infinitely many examples? Why?

2) Let $R(x,y)$ denote "$xy=0$". What are the truth values of the sentences $\exists y\forall x R(x,y)$ and $\forall x\exists y Q(x,y)$

a) interpreted with the universe of discourse equal to $\mathbb{R}$?

b) interpreted with the universe of discourse equal to $\mathbb{N}$?

3) Let $A$, $B$, and $C$ be sets (and suppose that they are all subsets of some fixed universal set). Find another form for the expression $\overline{A \setminus (B \cap C)}$. In your new form, there should be only one overline, and it should appear only over the $A$.

(Hint: recall that $X \setminus Y = X \cap \overline{Y}$).

4) Suppose $f : A \to B$ and $g : B \to C$ are onto. Is $(g \circ f) : A \to C$ necessarily onto? Prove your answer.

5) Show that $\Sigma_{j=0}^{n} \sqrt{j}$ is $\Theta(n^{\frac{3}{2}})$.

6) Suppose $f(x)$ is $\Theta(x^2)$ (and assume $\forall x f(x) > 0$). Prove or refute: $\sqrt{f(x)}$ is $\Theta(x)$

7) Describe an algorithm that, given a finite sequence of integers *in increasing order* and a particular number $n$, returns the position $n$ occurs in the sequence.

8) Determine the worst-case time complexity of the algorithm above.

9) The *Mikebonacci sequence* $M_n$ is defined as follows:

$$M_n = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ M_{n-2} + M_{n-1} + n - 1, & \text{if } n > 1 \end{cases}$$

a) Compute $M_0$ through $M_8$. (Be careful here—it's easier to make a mistake on this one than on the Fibonaccis, and if you get the conjecture wrong it'll be hard to prove it.)

b) Can you find a pattern to which Mikebonacci numbers are odd and which are even?

c) Prove the conjecture you made above, by induction.