Exercise set 6
Math 6010
16 November 2004

The so-called *coin flipping measure* (call it ν) on 2^ω (i.e. ω-sequences of zeroes and ones) has the following properties:

a) ν is a measure on 2^ω (in the ordinary, real-analysis sense).

b) For every Borel set $A \subseteq 2^\omega$, A is in the domain of ν, and $0 \leq \nu(A) \leq 1$.

c) Suppose, for $n < \omega$ and $x \in 2^\omega$, we define $\mathrm{flip}_n(x)$ to be the sequence of zeroes and ones identical to x, except that the n^{th} bit is flipped (0 becomes 1, vice versa), and for $A \subseteq 2^\omega$, $\mathrm{flip}_n(A) = \{\mathrm{flip}_n(x) | x \in A\}$. Then, for every Borel A and every n, $\nu(\mathrm{flip}_n(A)) = \nu(A)$.

Intuitively, if we flip a coin ω times and count each occurrence of heads as a 1 and each tails as a 0, then $\nu(A)$ is the probability that the sequence will wind up in A.

1) Come up with an actual definition for coin-flipping measure and see what you can find out about its properties. Can you find a way to view (c) above as a kind of translation invariance, the way Lebesgue measure is invariant under adding a constant to every element of a set of reals?

2) Define an equivalence relation E_0 on 2^ω by

$$xE_0y \iff (\exists n)(\forall m > n)\ x(m) = y(m)$$

That is, x and y are E_0-equivalent just in case they differ in only finitely many places. Suppose $D \subseteq 2^\omega$ contains exactly one element of each E_0-equivalence class. What can you say about D, with regard to the coin-flipping measure?

3) How many equivalence classes $[x]_{E_0}$ are there ($x \in 2^\omega$)? Should there be an injection $f : 2^\omega/E_0 \to 2^\omega$? (By $2^\omega/E_0$ we mean $\{[x]_{E_0} | x \in 2^\omega\}$. By $[x]_{E_0}$ we mean $\{y | yE_0x\}$.)

4) Is it possible for an injection f as above to have a *lift* to a continuous function $g : 2^\omega \to 2^\omega$? I.e. can there be a continous such g such that, for every x, $g(x) = f([x]_{E_0})$? Hint: Given a basic open neighborhood in 2^ω, what do you know about the preimage of that neighborhood under g? Must it be measurable, by coin-flipping measure? If so, can it have a measure other than 0 or 1? Where can you go with that?