Exercise set 5
Math 6010
9 November 2004

1) Show that, for any cardinal κ, $\operatorname{cof}(2^\kappa) > \kappa$.

2) Show that if λ is a limit ordinal such that $\operatorname{cof}(\lambda) > \aleph_0$, then the club filter on λ is $\operatorname{cof}(\lambda)$-complete.

3) Let λ be a cardinal on which there is an $\aleph_1$-complete nonprincipal ultrafilter $\mathcal{U}$. Let κ be the "completeness" of $\mathcal{U}$—that is, κ is least such that there is a size-κ collection $\{A_\alpha | \alpha < \kappa\}$, where each $A_\alpha \in \mathcal{U}$, but $\bigcap_{\alpha<\kappa} A_\alpha \notin \mathcal{U}$.

a) Show that there is a size-κ collection $\{B_\alpha | \alpha < \kappa\}$ such that each $B_\alpha \notin \mathcal{U}$, but $\bigcup_{\alpha<\kappa} B_\alpha \in \mathcal{U}$.

b) Show that the B_α above may be taken to be pairwise disjoint.

c) Define $\mathcal{V} \subseteq \mathcal{P}(\kappa)$ as follows: Given $A \subseteq \kappa$,

$$A \in \mathcal{V} \iff \bigcup_{\alpha \in A} B_\alpha \in \mathcal{U}$$

- Is $\mathcal{V}$ a filter on κ?
- Is it nonprincipal?
- Is it an ultrafilter?
- What is its completeness?