Exercise set 4
Math 6010
12 October 2004

1) Show that the decoding of Borel codes is $\underset{\sim}{\boldsymbol{\Delta}}^1_1$ on the $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ set on which it makes sense. That is, letting BC be the set of all Borel codes, and for $c \in BC$ letting A_c be the Borel set coded by c, and finally letting

$$R = \{(x, c) | c \in BC \wedge x \in A_c\}$$

show that there are $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$ and $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ sets R_Σ and R_Π (respectively), both subsets of $\omega^\omega \times \omega^\omega$, such that

$$R_\Sigma \cap (\omega^\omega \times BC) = R_\Pi \cap (\omega^\omega \times BC) = R$$

(Remark: actually R_Σ is Σ^1_1 lightface; mutatis mutandis for R_Π)

First hint: First work on R_Σ. It will be enough to show that there is a continuous map $c \mapsto T_c$ from ω^ω to $(\omega \times \omega)^{<\omega}$ such that, whenever $c \in BC$, T_c is a tree and $A_c = p[T_c]$. Figure out why this is enough, and how to get the map.

Second hint: To get R_Π, figure out how to get from a Borel code c to a Borel code for the *complement* of A_c. Now you have a $\underset{\sim}{\boldsymbol{\Sigma}}^1_1$ way of decoding the complement, which should turn into a $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ way of decoding A_c. Fill in the details.

2) Show that the decoding from the previous problem *cannot* be Borel, even on the $\underset{\sim}{\boldsymbol{\Pi}}^1_1$ set on which it makes sense. I.e. show that there is no Borel $R_B \subseteq \omega^\omega \times \omega^\omega$ such that $R = R_B \cap (\omega^\omega \times BC)$.

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3) Suppose *I* and *II* play natural numbers as usual, and let $x = \langle x_0, x_1, \cdots \rangle$ be *I*'s play and $y = \langle y_0, y_1, \cdots \rangle$ be *II*'s. Each player is trying to play a larger countable ordinal than the other. If both players play countable ordinals (i.e. if both X and y are in *WO*), then the player who plays the larger ordinal wins. Further conditions:

- *II* wins ties (i.e. if the players play *equal* countable ordinals, then *II* wins).
- If one player succeeds in playing a countable ordinal (i.e. his play is an element of *WO*) and the other one doesn't, then the player who plays a countable ordinal, wins.
- If neither player plays a countable ordinal, then *I* wins.

Problems:

a) Show that, *if* the game is determined, then *I* wins. Hint: Fix a strategy τ for *II*, and show that *I* can beat it. Two cases: either for every play by *I*, τ gives a countable ordinal as *II*'s play, or else not. In the second case, how can *I* win? In the first case, how can *I* win? At some point you need to use $\underset{\sim}{\mathbf{\Sigma}}^1_1$-boundedness.

b) Forget the determinacy hypothesis, and just show directly that *I* has a winning strategy. Hint: if *II* fails to produce a wellordering of ω then he loses in any case, so *I*'s strategy may assume that *II*'s play will be a wellordering. Thus he can try to produce a longer one, say an isomorphic copy of *II*'s ordering plus some more stuff put on the end. Fill in the details.

c) Suppose we make the game harder for *I*, by choosing an arbitrary increasing function $f : \omega_1 \to \omega_1$ and demanding of *I* not just that he play a larger ordinal than *II*'s (assuming *II* plays an ordinal) but that *I* play an ordinal larger than f of *II*'s ordinal. The other rules remain the same.

   (a) Is it ever possible for *II* to have a winning strategy?

   (b) Can you find conditions on f that will allow you to show directly (for games involving f satisfying those conditions) that *I* has a winning strategy, as before?