Exercise set 2
Math 6010
17 September 2004

1) Let X be a Polish space, and let $A \subseteq X$ be given. Fix a particular enumeration $\{\mathcal{V}_i | i \in \omega\}$ of the basic open neighborhoods of X. I and II play the following game: before the first round, set the variable U, which will always represent an open subset of X, to be all of X. Now at each round, I names two basic open neighborhoods $\mathcal{V}$ and $\mathcal{V}'$, subject to the following conditions:

- $\mathcal{V}$ and $\mathcal{V}'$ are disjoint.
- $\mathcal{V}$ and $\mathcal{V}'$ are both subsets of U.
- The diameters of $\mathcal{V}$ and $\mathcal{V}'$ are both at most half the diameter of U.
- The topological closures of $\mathcal{V}$ and $\mathcal{V}'$ are both contained in U.

The above constitutes I's play for the round. Then II plays by selecting either $\mathcal{V}$ or $\mathcal{V}'$, which then becomes U for the next round.

In the end, take the intersection of all the basic open neighborhoods selected by II from the choices offered by I; assuming I has followed all the rules, this intersection will be a singleton; call it $\{x\}$. I wins if and only if

- He has obeyed the rules at each step, and
- $x \in A$.

So finally we get to the problems:

a) Suppose A is countable. Who wins? Find a winning strategy for that player.

b) (Stop supposing A is countable.) Suppose I has a winning strategy. What can you say about the set A?

c) (Stop supposing I has a winning strategy.) Suppose II has a winning strategy, call it τ – that is, τ is a function from the set of *positions* in the game to the set of II's possible choices. Here a position is the entire history of the game up to some finite stage—all the offers made by I and all the choices made by II.

Now, given $x \in X$ and p a position with I about to move, say that x is *rejected by τ at position p* if and only if two things happen:

- x is an element of the open set U, after playing up to position p (that is, x is an element of the last open neighborhood accepted by II)
- but, if I's next move is two basic open neighborhoods, one of which contains x, then τ will tell II to pick the neighborhood *not* containing x.

Prove the following:

- For any $x \in A$, x is rejected by τ at some position
- For any position p, there is at most one x rejected by τ at p