Exercise set 1
Math 6010
10 September 2004

1) Make precise "continuous=to get finitely much info about output, need only finitely much info about input". E.g., for a particular finite sequence $s = \langle y_0, y_1, \ldots, y_k \rangle$ and a continuous f (say, from Baire space to Baire space), will it always be the case that there's a *fixed* M so that, if you know the first M coordinates of x, then you know whether $f(x)$ begins with s? If so, prove it; if not, can you reformulate it so it's right? When I say "whether or not $f(x)$ begins with s", do I mean that whatever finite information you know about x that determines the answer, must determine it in *either* direction, or is it only true if $f(x)$ really *does* begin with s? Why?

2) Find an explicit bijection between the open interval (0,1) (in the "real reals") and the closed interval [0,1]. Find an explicit bijection between $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$.

3) For x an element of Baire space, let X_x be the set of all elements of Baire space that are pointwise less than x (that is, for $y \in X_x$ and any natural number n, $y(n) < x(n)$.

a) Show X_x is compact without appealing to Tychonoff by finding an explicit continuous surjection from 2^ω onto X_x.

b) Why does that imply X_x is compact?

c) The original version of this exercise set said "show that X_x is pre-compact directly (i.e. without appealing to Tychonoff) by finding an explicit homeomorphic embedding of X_x into the Cantor space 2^ω". Why is this nonsense?

4)

a) Prove *König's Lemma*: Any finite-splitting tree with infinitely many nodes has an infinite branch.

b) Use this to give an alternative proof that X_x from problem (3) is compact.

5)

a) Find a homeomorphism between Baire space and the irrational reals between 0 and 1 (Hint: I'll give you the homeomorphism the exercise is really to prove it really *is* a homeomorphism. To an element of Baire space $\langle x_0, x_1, x_2, x_3, \ldots \rangle$ assign the value of the continued fraction $\frac{1}{1+x_0+\frac{1}{1+x_1+\frac{1}{1+x_2+\frac{1}{1+x_3+\cdots}}}}$. You need to check that this is a bijection between Baire space and the irrationals between 0 and 1, and that both it and its inverse are continuous.)

b) Describe the order on Baire space that corresponds to the natural order on the irrationals, pulled back by the above homeomorphism. (Hint: actually, you probably needed to do this anyway to prove continuity in part (a).)

Here follow the exercises I've assigned informally as they came up in lecture:

1) Show that any two wellorderings are comparable—that is, either there's an order-isomorphism from the first to an initial segment of the second, or vice versa.

2) In class we sketched a proof (using AC) that every set may be wellordered: Given a set X, let $F : \mathcal{P}(X)\setminus\{\emptyset\} \to X$ be a choice function (for any $A \subseteq X$, A nonempty, we have $F(A) \in A$). Now define by transfinite recursion

$$x_\alpha = F(X \setminus \{x_\beta : \beta < \alpha\})$$

Finish the proof by showing that:

a) The induction must stop somewhere (can't go on for *every* ordinal).

b) When the induction stops, the set of all x_α enumerated equals X itself

c) This defines a wellorder of X

3) Characterize the compact subsets of Baire space.

4) Prove the Cantor-Bendixson theorem.

5) Prove that there's a Borel isomorphism between any two perfect Polish spaces (we pretty well finished this off in class, but it wouldn't hurt to see if you can write out a proof yourself, to check if you followed.