Products and Easton's Theorem

1. Product Forcing

Let $\mathbb{P} = \langle P, \leqslant_P \rangle$, $\mathbb{Q} = \langle Q, \leqslant_Q \rangle$ be partial orders. We define their product by $\mathbb{P} \times \mathbb{Q} = \{\langle p, q \rangle : p \in P \wedge q \in Q\}$. This is ordered by $(p', q') \leqslant_{\mathbb{P}\times\mathbb{Q}} (p, q)$ iff $p' \leqslant_P p$ and $q' \leqslant_Q q$ (note: we will frequently use (p, q) instead of the more formal $\langle p, q \rangle$ when details of the pair coding are irrelevant).

For example, the forcing for adding two real, $\mathrm{FN}(\omega \times 2, 2)$ is isomorphic to the product $\mathrm{FN}(\omega, 2) \times \mathrm{FN}(\omega, 2)$ (which in this case is isomorphic to $\mathrm{FN}(\omega, 2)$ itself).

If $G \subseteq P$ and $H \subseteq Q$ are filters, then $G \times H \subseteq P \times Q$ is also easily a filter. Conversely, if $F \subseteq P \times Q$ is a filter, let $G = \{p \in P : \exists q \in Q\ (p, q) \in F\}$ and likewise $H = \{q \in Q : \exists p \in P\ (p, q) \in F\}$. Easily G and H are filters. If $(p, q) \in F$ then by definition $p \in G$ and $q \in H$, so $F \subseteq G \times H$. For the other direction, suppose $p \in G$ and $q \in H$. then $(p, q') \in F$ and $(p', q) \in F$ for some p', q'. Let $(r, s) \in F$ with $r \leqslant p, p'$, $s \leqslant q, q'$ (note: I've switched to the other definition of filter now). Since $(p, q) \leqslant (r, s) \in F$, $(p, q) \in F$. Thus, filters F in $P \times Q$ are precisely of the form $F = G \times H$ where F, G are filters in P, Q respectively. The relation between generics for $\mathbb{P} \times \mathbb{Q}$ and generics for $\mathbb{P}$, $\mathbb{Q}$ is clarified in the following lemma.

Lemma 1.1. *A filter $F = G \times H$ is M generic for $P \times Q$ iff G is M generic for P and H is $M[G]$ generic for Q.*

Remark 1.2. Of course, the situation is symmetrical with respect to P and Q, so we equally well say iff H is M generic for Q and G is $M[H]$ generic for P.

Proof. First suppose $G \times H$ is M generic for $P \times Q$. Let $D \subseteq P$, $D \in M$, be dense. Then $Q \times Q$ is dense in $P \times Q$, so let $(p, q) \in (G \times H) \cap (D \times Q)$. Thus, $p \in G \cap D$. This shows G is M generic for P. Let now $E \subseteq Q$, $E \in M[G]$, be dense in Q. Let $E = \tau_G$, where $\tau \in M^{\mathbb{P}}$. Let $p_0 \in P \cap G$ with $p \Vdash (\tau \text{ is dense in } \check{Q})$. Let

$$D = \{(p, q) \in P \times Q : (p \perp p_0) \vee (p \leqslant p_0 \wedge (p \Vdash \check{q} \in \tau))\}.$$

D is easily dense in $P \times Q$ [Let $(r, s) \in P \times Q$. If $r \perp p_0$, then $(r, s) \in D$. Otherwise, let $(r', s) \leqslant (r, s)$ with $r' \leqslant p_0$. Since $r' \Vdash (\tau \text{ is dense })$, $r' \Vdash \exists q \leqslant \check{s}\ (q \in \tau)$. Then there is a $p \leqslant r'$ and a $q \leqslant s$ with $p \Vdash (\check{q} \in \tau)$. Thus, $(p, q) \leqslant (r, s)$ and $(p, q) \in D$.] Let $(p, q) \in (G \times H) \cap D$. We must have $p \leqslant p_0$, and so $p \Vdash (\check{q} \in \tau)$. Since $p \in G$, $q \in E$, so $q \in H \cap E$. Thus, H is $M[G]$ generic for Q.

Conversely, suppose G is M generic for P and H is $M[G]$ generic for Q. Let $D \subseteq P \times Q$ be dense. Let $E = \{q \in Q : \exists p \in G\ (p, q) \in D\}$. Clearly $E \in M[G]$. It is enough to show that E is dense in Q for then we would have $q \in H \cap E$. By definition of E there would then be a $p \in G$ with $(p, q) \in D$. Hence, $(p, q) \in (G \times H) \cap D$. To see E is dense, let $s \in Q$. Let $A = \{p \in P : \exists q\ (q \leqslant s \wedge (p, q) \in D\}$. Clearly A is dense in P, and $A \in M$. So, let $p \in G \cap A$. Let $q \leqslant s$ with $(p, q) \in D$. Then $q \in E$ and $q \leqslant s$. □

Thus, if $\mathbb{P}$, $\mathbb{Q}$ are partial orders in M, forcing with the product $\mathbb{P} \times \mathbb{Q}$ is equivalent to doing a two-step forcing where we first force over M with $\mathbb{P}$ to get $M[G]$, and then force over $M[G]$ with $\mathbb{Q}$ to get $M[G][H]$. Note that $M[G][H] = M[G \times H]$, as G, H are definable from $G \times H$ and conversely.

The following technical lemma combines lemmas **??** and **??**.

Lemma 1.3. *Let κ be a cardinal and $\mathbb{P}$ be κ^+-c.c. and $\mathbb{Q}$ be $\leqslant \kappa$ closed in a transitive model M of ZFC. Let $G \times H$ be M generic for $\mathbb{P} \times \mathbb{Q}$. Then any $f\colon \kappa \to M$ in $M[G][H]$ lies in $M[G]$.*

Proof. Let $f\colon \kappa \to M$ lie in $M[G][H]$. Let $\tau \in M^{\mathbb{P}\times\mathbb{Q}}$ with $f = \tau_{G\times H}$. For each $\alpha < \kappa$ let $D_\alpha = \{q \in Q\colon \forall p \in P\ \exists p' \leqslant p\ \exists x \in M\ ((p', q) \Vdash \tau(\check{\alpha}) = \check{x})\}$. We claim that D_α is dense in Q. To see this, let $q \in Q$. Let $p_0 \in P$, $q_0 \leqslant q$, and $x_0 \in M$ with $(p_0, q_0) \Vdash \tau(\check{0}) = \check{x}$. We construct $(p_0, q_0) \leqslant \cdots \leqslant (p_\beta, q_\beta) \leqslant$ as follows. Assume (p_γ, q_γ) is defined for $\gamma < \beta$, and $\beta < \kappa^+$. If $\{p_\gamma\}_{\gamma<\beta}$ is a maximal antichain on P, then we let stop the construction and let q extend all of the q_γ, which we can do as $\mathbb{Q}$ is $\leqslant \kappa$ closed. Otherwise, let p_β be incompatible with all of the p_γ, $\gamma < \beta$. Let q_β extend all of the q_γ for $\gamma < \beta$ and such that for some $x_\beta \in M$, $(p_\beta, q_\beta) \Vdash \tau(\check{\beta}) = \check{x_\beta}$. As $\mathbb{P}$ is κ^+-c.c. this construction cannot go on κ^+ times. Thus, for some $\beta < \kappa^+$, $\{p_\gamma\}_{\gamma<\beta}$ is a maximal antichain of P. The corresponding q (which extends all of the q_γ) lies in D_α.

Using again that $\mathbb{Q}$ is $\leqslant \kappa$ closed, we get that $D = \bigcap_{\alpha<\kappa} D_\alpha$ is dense in Q. Fix $q \in H \cap D$. Working in M we may define dense sets D_α, $\alpha < \kappa$, such that for all α and all $p \in D_\alpha$, there is an $x \in M$ such that $(p, q) \Vdash \tau(\check{\alpha}) = \check{x}$. In $M[G]$ we may then compute f, namely $f(\alpha) = x$ iff there is a $p \in D_\alpha \cap G$ such that $(p, q) \Vdash \tau(\check{\alpha}) = \check{x}$ (we are using the replacement and comprehension axioms in M to get that f is a set in M). □

As a warm-up for Easton's theorem, let us give another, perhaps more direct, proof of lemma **??**. So, let M satisfy GCH and $\kappa_1 < \cdots < \kappa_n$ be regular, and $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ with $\mathrm{cof}(\lambda_i) > \kappa_i$. Let $\mathbb{P}_i = \mathrm{FN}(\lambda_i, 2, \kappa_i)$ be the partial order for adding λ_i many subsets of κ_i. Let $\mathbb{P} = \mathbb{P}_1 \times \cdots \times \mathbb{P}_n$ be the product. Let $G = G_0 \times \cdots \times G_n$ be M generic for $\mathbb{P}$. First we show that $\mathbb{P}$ preserves all cardinalities and cofinalities. Let δ be a regular cardinal of M, but suppose $f\colon \rho \to \delta$ is cofinal where $\rho < \delta$ and $f \in M[G]$. Let $\mathbb{P}^- = \mathbb{P}_1 \times \cdots \times \mathbb{P}_i$ where i is maximal so that $\kappa_i \leqslant \rho$. Let $\mathbb{P}^+ = \mathbb{P}_{i+1} \times \cdots \times \mathbb{P}_n$. Clearly $\mathbb{P}^+$ is $\leqslant \rho$ closed in M. Also, $\mathbb{P}^-$ is $(2^{<\rho})^+ = \rho^+$-c.c. in M (note that P can be viewed as a subset of $\mathrm{FN}(\lambda_i, 2, \kappa_i)$ since $\sum_{j\leqslant i}(\kappa_j \times \lambda_j) \cong \lambda_i$; use then lemma **??**). From lemma 1.3 we have $f \in M[\mathbb{P}^-]$. From lemma **??** we then have that there is an $F\colon \rho \to \mathcal{P}(\delta)$, $F \in M$ with $|F(\alpha)| \leqslant \rho$ for all $\alpha < \delta$. This is a contradiction as δ is regular in M. Som $\mathbb{P}$ preserves all cofinalities and hence cardinalities (which the same argument also shows directly; thus the preservation of cardinals only requires M to satisfy ZF).

Clearly $(2^{\kappa_i}) \geqslant \lambda_i)^{M[\mathbb{P}]}$. To get the upper bound for $(2^{\kappa_i})^{M[\mathbb{P}]}$, write $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$ where $\mathbb{P}^- = \mathbb{P}_1 \times \cdots \times \mathbb{P}_i$ and $\mathbb{P}^+ = \mathbb{P}_{i+1} \times \cdots \times \mathbb{P}_n$. From lemma 1.3 we have $\mathcal{P}(\kappa_i)^{M[\mathbb{P}]} = \mathcal{P}(\kappa_i)^{M[\mathbb{P}^-]}$. Since $\mathbb{P}^-$ is κ_i^+-c.c. in M, there are at most $(\lambda_i^{\kappa_i})^{\kappa_i} = \lambda_i^{\kappa_i} = \lambda_i$ many nice names for a subset of κ_i in M (these computations are done in M; the last equality uses $\mathrm{cof}(\lambda_i) > \kappa_i$ and the GCH in M). Thus, $(2^{\kappa_i} \leqslant \lambda_i)^{M[\mathbb{P}]}$. We have thus shown that $(2^{\kappa_i} = \lambda_i)^{M[\mathbb{P}]}$ for all $i \leqslant n$.

2. Remarks on Class Forcing

In most applications we will be in the situation where $\mathbb{P} \in M$, that is, $\mathbb{P}$ is a set in M (this is what we have been considering up to this point). For some purposes, including Easton's theorem, we would like to generalize this to allow $\mathbb{P}$ being a class in M. Note that we are still assuming that M is a transitive set in V, thus there is no problem in quantifying over the classes of M (as statements in V). For this section, when we say a class of M, we mean a formula with set parameters from M.

Let M be a set which is a transitive model of ZF (or ZFC). Let $\mathbb{P} = \langle P, \leqslant \rangle$ where $\mathbb{P}, \leqslant \subseteq M$ are classes of M (i.e., definable in M from parameters in M), and such that $\mathbb{P}$ is a partial order. Note that M also satisfies that $\mathbb{P}$ is a partial order in the sense that, for example, $(\forall x, y, z\ [((x, y) \in \leqslant) \wedge ((y, z) \in \leqslant) \rightarrow ((x, z) \in \leqslant)])^M$. If $D \subseteq P$ is a class of M, we say D is dense just as before; if $\forall p \in P\ \exists q \in D (q \leqslant p)$. For a given D, this is expressible in M. We say $G \subseteq P$ is M generic for $\mathbb{P}$ exactly as before; if $G \cap D \neq \varnothing$ for all dense classes $D \subseteq P$ of M.

We define $M^{\mathbb{P}}$ essentially as before. Thus, $M^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{ON}^M} M^{\mathbb{P}}_\alpha$, where we take unions at limit ordinals and

$$M^{\mathbb{P}}_{\alpha+1} = \{\tau \in M \cap V_{\alpha+1} \colon (\tau \text{ is a relation }) \wedge \mathrm{dom}(\tau) \subseteq M^{\mathbb{P}}_\alpha \wedge \mathrm{ran}(\tau) \subseteq P\}.$$

Again, the transfinite recursion theorem shows that $M^{\mathbb{P}}$ is a well-defined class of M. Given a filter $G \subseteq P$, we define the evaluation map $\tau \rightarrow \tau_G$ exactly as before, and again define $M[G] = \{\tau_G \colon \tau \in M^{\mathbb{P}}\}$.

For $p \in P$, $\phi(x_1, \ldots, x_n)$ a formula, and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, we define the forcing relation $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ exactly as before. We again have that for all formuals $\phi(x_1, \ldots, x_n)$ that $\{(p, \tau_1, \ldots, \tau_n) \colon p \Vdash \phi(x_1, \ldots, x_n)\}$ is a class of M.

Finally, the forcing theorem goes through as before. For example, consider the atomic case $\phi = (\tau_1 \in \tau_2)$. Suppose $p \in P$ and $p \Vdash \phi$. Then the class

$$D = \{q \in P \colon \exists \langle \sigma, r \rangle \in \tau_2\ (q \leqslant r \wedge r \Vdash (\tau_1 \approx \sigma))\}$$

is dense below p. D is now a class, not a set in M, but G being generic still implies $G \cap D \neq \varnothing$. If $q \in G \cap D$, let $\langle \sigma, r \rangle \in \tau_2$ be such that $q \leqslant r \wedge r \Vdash (\tau_1 \approx \sigma)$. So, $\sigma_G \in (\tau_2)_G$ and by induction $(\tau_1)_G = \sigma_G$. Hence, $(\tau_1)_G \in (\tau_2)_G$. The other direction is also as before.

Consider now which axioms of ZF hold in $M[G]$. Certainly founadation, extensionality, pairing and union hold in $M[G]$ (and again only require G to be a filter). We run into problems, though, when we try to show power set, replacement, and comprehension. The proofs of these axioms given previously for set forcing use the fact that $\mathbb{P}$ is a set in M. For example the proof of power set required us to consider $\{\sigma \colon \mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau) \wedge \mathrm{ran}(\sigma) \subseteq P\}$, where $\tau \in M^{\mathbb{P}}$ is fixed. If $\mathbb{P}$ is not a set in M, this will clearly be a proper class of M as well. Similarly, the previous proof of replacement required the apllication of replacement in M to the set $\mathrm{dom}(\tau) \times P$ for some $\tau \in M^{\mathbb{P}}$; here again this will be a proper class.

For general class forcing, the power set, replacement, and comprehension axioms may fail in $M[G]$. For example, power set will fail if we add ON many reals, or if we collapse ON to ω. Thus, some restiction on the forcing is necessary. The following gives a sufficient condition.

Theorem 2.1. *Let $\mathbb{P}$ be a class partial order of M, where M is a transitive model of ZF (or ZFC). Suppose that for arbitrarily large cardinals κ of M that we can write $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$ where $\mathbb{P}^-$ is κ^+-c.c. and $\mathbb{P}^+$ is $\leqslant \kappa$ closed. Let G be M generic for $\mathbb{P}$. Then $M[G]$ satisfies ZF (or ZFC).*

Proof. We show comprehension, power set, and replacement in $M[G]$.

To show comprehension, fix $a_1 = (\tau_1)_G, \ldots,, a_n = (\tau_n)_G$, $A = \tau_G$, and a formula $\phi(x_1, \ldots, x_n, y, z)$. We must show that $\{z \in A \colon \phi^{M[G]}(a_1, \ldots, a_n, A, z)\}$ exists as a set in $M[G]$. Let $\kappa > |\tau|$ be such that $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$ with $\mathbb{P}^-$ κ^+ c.c., and $\mathbb{P}^+ \leqslant \kappa$

closed. As in the proof of lemma 1.3, let $Q \subseteq P^+$ be those $q \in P^+$ such that for all $\langle \pi, p \rangle \in \tau$, there is a dense below p set of conditions $p' \in P^-$ such that (p', q) decides $\phi(\tau_1, \ldots, \tau_n, \tau, \pi)$. More precisely,

$$Q = \{q \in P^+ \colon \forall \pi \in \mathrm{dom}(\tau)\ \forall r \in P^-\ \ \exists s \in P^-\ [((s,q) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi)) \vee ((s,q) \Vdash \neg\phi(\tau_1, \ldots, \tau_n, \tau, \pi))].$$

As in the proof of lemma 1.3, it follows from the $\leqslant \kappa$ closure of $\mathbb{P}^+$ that Q is dense in P^+. Let $(p_0, q_0) \in G$ with $q_0 \in Q$. For $\pi \in \mathrm{dom}(\tau)$ let

$$D_\pi = \{p \in P^- \colon ((p, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi)) \vee ((p, q_0) \Vdash \neg\phi(\tau_1, \ldots, \tau_n, \tau, \pi))\}.$$

Thus, D_π is dense in P^-. Moreover, as in lemma 1.3 we may assume each D_π is an antichain which is a set of size $\leqslant \kappa$. Let

$$\sigma = \{\langle \pi, (p, q_0) \rangle \colon (\pi \in \mathrm{dom}(\tau)) \wedge (p \in D_\pi) \wedge ((p, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi))\}.$$

Note that σ is a valid name, that is, σ is a set in M.

If $x \in \sigma_G$, then $x = \pi_G$ where $(p, q_0) \in G$, and $(p, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi)$. Thus, $\phi^{M[G]}(a_1, \ldots, a_n, A, x)$. Suppose next that $x \in M[G]$ and $\phi^{M[G]}(a_1, \ldots, a_n, A, x)$. Then $x = \pi_G$ where $\pi \in \mathrm{dom}(\tau)$. Let $(p_1, q_1) \in G$ with $p_1 \in D_\pi$ and $(p_1, q_1) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi)$. Since $p_1 \in D_\pi$, either $(p_1, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi)$ or $(p_1, q_0) \Vdash \neg\phi(\tau_1, \ldots, \tau_n, \tau, \pi)$. The latter case is impossible as (p_1, q_0), (p_1, q_1) are compatible. This shows $\langle \pi, (p_1, q_0) \rangle \in \sigma$, and hence $x = \pi_G \in \sigma_G$ (note that $(p_1, q_0) \in G$, as G is a filter).

Consider next power set. Let $x = \tau_G \in M[G]$, let $\kappa > |\tau|$ and again write $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$ as before. Let $\rho = \{\langle \sigma, \mathbb{1} \rangle \colon \mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau) \wedge \mathrm{ran}(\sigma) \subseteq \mathbb{P}^-\}$. We show that $\mathcal{P}(x) \subseteq \rho_G$. Fix $y \subseteq x$, say $y = \mu_G$. Arguing as in the previous case, we get a $(p_0, q_0) \in G$ such that for all $\pi \in \mathrm{dom}(\tau)$, the set $D_\pi \subseteq P^-$ is dense, where now

$$D_\pi = \{p \in P^- \colon ((p, q_0) \Vdash \pi \in \mu) \vee ((p, q_0) \Vdash \neg(\pi \in \mu))\}.$$

Let

$$\sigma = \{\langle \pi, (p, \mathbb{1}) \rangle \colon (\pi \in \mathrm{dom}(\tau)) \wedge (p \in D_\pi) \wedge ((p, q_0) \Vdash \pi \in \mu)\}.$$

Clearly $\sigma_G \subseteq \mu_G$ (note that if $(p, \mathbb{1}) \in G$, then $(p, q_0) \in G$, since $(p_0, q_0) \in G$). The other direction, $\mu_G \subseteq \sigma_G$ follows now exactly as in the previous case.

Consider replacement. The proof is again similar to the previous cases. Let $A = \tau_G$, $a_1 = (\tau_1)_G, \ldots,$ $a_n = (\tau_n)_G$, and $\phi(x_1, \ldots, x_n, y, z, w)$ be a formula. Assume that

$$\forall y \in A\ \exists z \in M[G]\ \phi^{M[G]}(a_1, \ldots, a_n, A, y, z).$$

Fix $\kappa > |\tau|$, and again write $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$. As in the previous cases (using the fact that $\mathbb{P}^+$ is $\leqslant \kappa$ closed and $\mathbb{P}^-$ is κ^+-c.c.) we get a $(p_0, q_0) \in G$ such that for each $\pi \in \mathrm{dom}(\tau)$ the set $D_\pi \subseteq \mathbb{P}^-$ is a dense set (which we may assume has size $\leqslant \kappa$), where

$$D_\pi = \{p \in P^- \colon \exists z \in M\ ((p, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi, \check{z})\}.$$

Using replacement in M, let S be a set in M such that for all $\pi \in \mathrm{dom}(\tau)$ and all $p \in D_\pi$, there is a $z \in S$ such that $(p, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi, \check{z})$. Let

$$\sigma = \{\langle \rho, \mathbb{1} \rangle \colon \rho \in S\}.$$

To see this works, let $y = \pi_G \in A = \tau_G$, where $\pi \in \mathrm{dom}(\tau)$. Let $(p_1, q_1) \in G$ with $p_1 \in D_\pi$. By definition of D_π, let $z \in M$ be such that $(p_1, q_0) \Vdash \phi(\tau_1, \ldots, \tau_n, \tau, \pi, \check{z})$.

From the definition of S it follows that $(p_1, q_0) \Vdash \phi(\tau_1, \dots, \tau_n, \tau, \pi, \check{z}')$ for some $z' \in S$. Hence, $\phi^{M[G]}(a_1, \dots, a_n, A, y, z')$ holds for some $z' \in \sigma_G$. This verifies replacement.

If M satisfies AC, then so does $M[G]$ by exactly the same argument as for set forcing, since if $A = \tau_G$, then τ is a set in M, and so there is map $f \in M$ from an ordinal α onto τ. Although G itself is no longer in $M[G]$ (as with set forcing), we nevertheless still get a map from α onto τ_G definable from f and $G \cap V_\beta$ for some β, which is a set in $M[G]$. Thus, in $M[G]$ we still define an F from α onto τ_G. □

If we assume the factoring hypothesis of theorem 2.1 holds for all regular cardinals, then the class forcing $\mathbb{P}$ also preserves all cardinals and cofinalities.

Theorem 2.2. *Let $\mathbb{P}$ be a class partial order of M, where M is a transitive model of ZFC. Suppose that for all regular cardinals κ of M that we can write $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$ where $\mathbb{P}^-$ is κ^+-c.c. and $\mathbb{P}^+$ is $\leqslant \kappa$ closed. Then all cardinals and cofinalites are preserved in forcing with $\mathbb{P}$.*

Proof. Let δ be a regular cardinal of M, and suppose $\rho = (\mathrm{cof}(\delta))^{M[G]} < \delta$. We again use the argument of lemma 1.3. Write $\mathbb{P} = \mathbb{P}^- \times \mathbb{P}^+$ where $\mathbb{P}^-$ is ρ^+-c.c. and $\mathbb{P}^+$ is $\leqslant \rho$ closed. Let $f\colon \rho \to \delta$ be onto, $f \in M[G]$. Fronm lemma 1.3, $f \in M[G^-]$, where $G = G^- \times G^+$. Since $\mathbb{P}^-$ is ρ^+ c.c., there is an $F \in M$, $F\colon \rho \to \delta$ with $\mathrm{ran}(f) \subseteq \mathrm{ran}(F)$. This contradicts δ being regular in M. □

We are now ready to give Easton's theorem.

Definition 2.3. An *Easton* function is a class function F of M with domain a class of regular cardinals of M and range in cardinals of M satisfying:

(1) If $\lambda_1 < \lambda_2$ are in $\mathrm{dom}(F)$, then $F(\lambda_1) \leqslant F(\lambda_2)$.
(2) $\forall \lambda \in \mathrm{dom}(F)$ $(\mathrm{cof}(F(\lambda) > \lambda)$.

If F is an Easton function for M, we define the Easton forcing $\mathbb{P}_F$ as follows. Condition $p \in \mathbb{P}_F$ are functions with domain $\mathrm{dom}(F)$, and for $\lambda \in \mathrm{dom}(F)$, $p(\lambda) \in \mathrm{FN}(F(\lambda), 2, \lambda)$. Further, we require p to satisfy the *Easton condition*: for all regular κ of M, $\{\lambda < \kappa \colon p(\lambda) \neq \mathbb{1}\}$ has size $< \kappa$.

Thus, the forcing is just the product of the forcings to make 2^λ at least $F(\lambda)$, except we add the Easton condition which restricts the size of the domains of the conditions. Note that the Easton condition is only non-trivial when κ is a weakly inaccessible cardinal.

Theorem 2.4 (Easton)**.** *Let M be a transitive model of ZFC+ GCH, and F a class of M which is an Easton function. Assume G is M generic for the Easton forcing $\mathbb{P}_F$. Then $M[G]$ satisfies ZFC, all cardinalities and cofinalities are preserved from M to $M[G]$, and for all regular cardinals λ of $M[G]$ we have $(2^\lambda = F(\lambda))^{M[G]}$.*

Proof. Fix a regular cardinal λ of M (equivalently, of $M[G]$). Write $\mathbb{P}_F = \mathbb{P}^{\leqslant\lambda} \times \mathbb{P}^{>\lambda}$ where $\mathbb{P}^{\leqslant\lambda}$ consists of those $p \in \mathbb{P}_F$ with $\mathrm{dom}(p) \subseteq \lambda + 1$, and $\mathbb{P}^{>\lambda}$ those p with $\mathrm{dom}(p) \subseteq \mathrm{CARD} - (\lambda + 1)$. Clearly $\mathbb{P}^{>\lambda}$ is $\leqslant \lambda$ closed. We show that $\mathbb{P}^{\leqslant\lambda}$ is λ^+ c.c. For every regular $\kappa \leqslant \lambda$, $\mathrm{FN}(F(\kappa), 2, \kappa)$ is $(2^{<\kappa})^+ = \kappa^+$ c.c. in M, since M satisfies the GCH. Suppose $\{p_\alpha\}_{\alpha<\lambda^+}$ were an antichain of size λ^+ in $\mathbb{P}^{\leqslant\lambda}$. Let $d_\alpha = \mathrm{dom}(p_\alpha)$. Since $|d_\alpha| < \lambda$ (by the Easton condition if λ is limit, otherwise trivially), there are only $\lambda^{<\lambda} = \lambda$ many choices for d_α. So, we may assume that all of the p_α have the same domain d. By regularity of λ, each p_α may be viewed as a function from $d \times F(\lambda) \to \{0,1\}$ with domain of size $< \lambda$. Since $\mathrm{FN}(F(\lambda), 2, \lambda)$ is $(2^{<\lambda})^+ = \lambda^+$ c.c. in M, this is a contradiction. Thus, $\mathbb{P}^{\leqslant\lambda}$ is λ^+ c.c.

From lemmas 2.1, 2.2 we know that $M[G]$ satisfies ZFC and all cardinals and cofinalities are preserved from M to $M[G]$. We clearly have for all regular cardinals of $M[G]$ that $(2^\lambda \geqslant F(\lambda))^{M[G]}$. To see the other direction, fix a regular cardinal λ of M (equivalently, of $M[G]$) and consider $\mathbb{P}_F = \mathbb{P}^{\leqslant\lambda} \times \mathbb{P}^{>\lambda}$ as above.

Every subset of λ in $M[G]$ is in $M[G^{\leqslant\lambda}]$, where $G = G^{\leqslant\lambda} \times G^{>\lambda}$, from lemma 1.3. Since $\mathbb{P}^{\leqslant\lambda}$ is λ^+ c.c., $(2^\lambda)^{M[G^{\leqslant\lambda}]} \leqslant (|\mathbb{P}^{\leqslant\lambda}|^{\lambda\cdot\lambda})^M$. Also, $|\mathbb{P}^{\leqslant\lambda}| \leqslant F(\lambda)^{<\lambda} 2^{<\lambda} = F(\lambda)$ since $\mathrm{cof}(F(\lambda)) > \lambda$ and the GCH in M (these computations are done in M). Thus, $(2^\lambda)^{M[G^{\leqslant\lambda}]} \leqslant F(\lambda)^\lambda = F(\lambda)$. □

Finally, we use class forcing to get a model of GCH.

Theorem 2.5. *Let M be a transitive model of ZFC. Then there is a class partial order $\mathbb{P}$ of M such that if G is M-generic for $\mathbb{P}$ then $M[G]$ satisfies ZFC + GCH.*

Proof. Let M be a transitive model of ZFC. Let $\alpha \to \beth_\alpha$ be the beth function of M (for this proof, $\beth_\alpha$ always denotes $(\beth_\alpha)^M$). For each ordinal α of M, let $\mathbb{P}_\alpha = \mathrm{coll}(\beth_\alpha^+, \beth_{\alpha+1})^M = \mathrm{FN}(\beth_\alpha^+, \beth_{\alpha+1}, \beth_\alpha^+)^M$. Note that $\mathbb{P}_\alpha$ is $\beth_\alpha$ closed and $(\beth_{\alpha+1}^{\beth_\alpha})^+ = \beth_{\alpha+1}^+$ c.c. ($\beth_{\alpha+1}^{\beth_\alpha} = (2^{\beth_\alpha})^{\beth_\alpha} = 2^{\beth_\alpha} = \beth_{\alpha+1}$).

Let $\mathbb{P}$ be the Easton product of the $\mathbb{P}_\alpha$. That is, $\mathbb{P}$ consists of functions p with domain a subset of ordinals, $p(\alpha) \in \mathbb{P}_\alpha$ for all $\alpha \in \mathrm{dom}(p)$, and p satisfies the Easton condition: for all inaccessible λ, $\{\alpha < \lambda \colon p(\alpha) \neq \mathbb{1}\}$ has size $< \lambda$. For $\alpha \in \mathrm{ON}^M$, let $\mathbb{P}^{<\alpha}$ denote those $p \in \mathbb{P}$ with $\mathrm{dom}(p) \subseteq \alpha$. Likewise, $\mathbb{P}^{\geqslant\alpha}$ denotes those p with $\mathrm{dom}(p) \cap \alpha = \varnothing$. Clearly $\mathbb{P} = \mathbb{P}^{<\alpha} \times \mathbb{P}^{\geqslant\alpha}$ (at least, up to isomorphism).

First we show that $M[G]$ satisfies ZFC. For α a successor ordinal of M, consider $\mathbb{P} = \mathbb{P}^{<\alpha} \times \mathbb{P}^{\geqslant\alpha}$. Easily $\mathbb{P}^{\geqslant\alpha}$ is $\leqslant \beth_\alpha$ closed. Any $p \in \mathbb{P}^{<\alpha}$ can be viewed as a function from $\beth_{\alpha-1}^+$ to $\beth_\alpha$ of size $\leqslant \beth_{\alpha-1}$. Since $\mathrm{FN}(\beth_{\alpha-1}^+, \beth_\alpha, \beth_{\alpha-1}^+)$ is $\beth_\alpha^+$ c.c., it follows that $\mathbb{P}^{<\alpha}$ is $\beth_\alpha^+$ c.c. From lemma 2.1 it now follows that $M[G]$ satisfies ZFC.

Clearly $(|\beth_{\alpha+1}| \leqslant (\beth_\alpha)^+)^{M[G]}$. Thus, $|\beth_\alpha|^{M[G]} \leqslant \aleph_\alpha^{M[G]}$. First we show that $|\beth_\alpha|^{M[G]} = \aleph_\alpha^{M[G]}$, and for this it suffices to show that $\beth_\alpha^+$ is still a cardinal of $M[G]$. First assume that α is a successor and write $\mathbb{P} = \mathbb{P}^{<\alpha} \times \mathbb{P}^{\geqslant\alpha}$ as above. Thus, $\mathbb{P}^{\geqslant\alpha}$ is $\leqslant \beth_\alpha$ closed and $\mathbb{P}^{<\alpha}$ is $\beth_\alpha^+$ c.c. If $\rho < \beth_\alpha^+$ and $f \colon \rho \to \beth_\alpha^+$, then from lemma 1.3 it follows that $f \in M[G^{<\alpha}]$, where $G = G^{<\alpha} \times G^{\geqslant\alpha}$. Since $\mathbb{P}^{<\alpha}$ is $\beth_\alpha^+$ c.c., this gives an onto $F \colon \beth_\alpha \to \beth_\alpha^+$ in M, a contradiction. Suppose next that α is limit. We consider cases as to whether $\beth_\alpha$ is regular (i.e., inaccessible). Suppose first that $\beth_\alpha$ is regular. Again write $\mathbb{P} = \mathbb{P}^{<\alpha} \times \mathbb{P}^{\geqslant\alpha}$. $\mathbb{P}^{\geqslant\alpha}$ is $\leqslant \beth_\alpha$ closed. From the Easton condition, any $p \in \mathbb{P}^{<\alpha}$ has domain bounded in α. This gives that $\mathbb{P}^{<\alpha}$ is $\beth_\alpha^+$ c.c., since if there were an antichain in $\mathbb{P}^{<\alpha}$ of size $\beth_\alpha^+$ we could assume the domains of the conditions were constant, and a simple computation would then show there are $< \beth_\alpha$ many conditions in the antichain. If $f \colon \beth_\alpha \to \beth_\alpha^+$ were onto and in $M[G]$, lemma 1.3 would give a function $F \colon \beth_\alpha \to \beth_\alpha^+$ in M which was also onto, a contradiction. Suppose then that $\beth_\alpha$ is singular, say $\rho = \mathrm{cof}(\beth_\alpha) < \beth_\alpha$. Let $\{\alpha_i\}_{i<\rho}$ be increasing cofinal in $\beth_\alpha$ with $\beth_{\alpha_0} > \rho$. Let D be those $p \in \mathbb{P}$ such that there is a sequence $\{A_i^\gamma\}$ for $i < \rho$, $\gamma < \beth_{\alpha_i}$, each A_i a maximal antichain of $\mathbb{P}^{<\alpha_i}$, such that for all $i < \rho$, $\gamma < \beth_{\alpha_i}$, and all $q \in A_i^\gamma$ there is an ordinal β such that $(q, p^{\geqslant\alpha_i}) \Vdash \tau(\check{\gamma}) = \check{\beta}$. Iterating the argument of lemma 1.3 ρ times shows that D is dense. Fix $p \in \mathbb{P} \cap D$, and let A_i^γ be the corresponding antichains. From the

A_i^γ we may construct in M a set of size $|\bigcup_{i,\gamma} A_i^\gamma|$ which contains the range of f. Thus in M we have a set of size $\beth_\alpha$ which contains $\beth_\alpha^+$, a contradiction.

We now know that $\beth_\alpha$ has cardinality $\aleph_\alpha^{M[G]}$ in $M[G]$. To show the GCH in $M[G]$ it is thus enough to show that there are at most $\beth_{\alpha+1}$ many subsets of $\beth(\alpha)$ in $M[G]$. Suppose first that α is a successor. Write $\mathbb{P} = \mathbb{P}^{<\alpha} \times \mathbb{P}^{\geqslant\alpha}$. Every subset of $\beth_\alpha$ in $M[G]$ lies in $M[G^{<\alpha}]$, so it is enough to count these. In this case $\mathbb{P}^{<\alpha}$ has size $\beth_\alpha$, so there are at most $\beth_\alpha^{\beth_\alpha} = \beth_{\alpha+1}$ many nice names for subsets of $\beth_\alpha$. Suppose next that $\beth_\alpha$ is innaccessible. Again write $\mathbb{P} = \mathbb{P}^{<\alpha} \times \mathbb{P}^{\geqslant\alpha}$. Every subset of $\beth_\alpha$ in $M[G]$ again lies in $M[G^{<\alpha}]$. From the Easton condition, $\mathbb{P}_{<\alpha}$ has size $\beth_\alpha$ in M. So again there are at most $\beth_\alpha^{\beth_\alpha} = \beth_{\alpha+1}$ many nice names for subsets of $\beth_\alpha$. Finally, suppose α is limit and $\beth_\alpha$ is singular, say $\rho = \mathrm{cof}(\beth_\alpha) < \beth_\alpha$. Let $\beta < \alpha$ be a successor with $\beth_\beta > \rho$. Proceeding inductively, we may assume the GCH holds in $M[G]$ below $\beth_\alpha$. Thus it suffices to show that $((\beth_\alpha)^\rho \leqslant \beth_{\alpha+1})^{M[G]}$. Write $\mathbb{P} = \mathbb{P}^{<\beta} \times \mathbb{P}^{\geqslant\beta}$. Every $f \in (\beth_\alpha)^\rho \cap M[G]$ lies in $M[G^{<\beta}]$. Since $|\mathbb{P}^{<\rho}| \leqslant \beth_\beta$, there are at most $(\beth_\beta^{\beth_\alpha})^\rho \leqslant 2^{\beth_\alpha} = \beth_{\alpha+1}$ many nice names for functions $f \in (\beth_\alpha)^\rho$. $\square$