Applications of Martin's Axiom

1. Products of c.c.c. Spaces

We consider the question of when the product of two c.c.c. topological spaces is c.c.c.. The next lemma shows the question is the same for partial orders, topological spaces, and compact Hausdorff spaces. Recall that for any partial order $\mathbb{P} = \langle P, \leqslant \rangle$ there is an associated topological space X_P on P, where a neighborhood base at $p \in P$ consists of the single open set $N_p = \{q \in P : q \leqslant p\}$. Clearly this space is not (except for very trivial partial orders) even T_1.

Lemma 1.1. (*ZFC*) *Let κ be an infinite regular cardinal. Then the following are equivalent.*

(1) *There are κ-c.c. partial orders $\mathbb{P}$, $\mathbb{Q}$ such that $\mathbb{P} \times \mathbb{Q}$ is not κ-c.c.*
(2) *There κ-c.c. topological spaces X, Y such that the product $X \times Y$ is not κ-c.c.*
(3) *There are κ-c.c. compact Hasdorff spaces X, Y, such that the product $X \times Y$ is not κ-c.c.*
(4) *There are complete Boolean algebras $\mathcal{B}$, $\mathcal{C}$ such that their product $\mathcal{B} \times \mathcal{C}$ is not κ-c.c.*

Proof. Note first that for any partial order P, P is κ-c.c. iff X_P is κ-c.c. Also $X_{P \times Q} = X_P \times X_Q$. Suppose first that P, Q are κ-c.c. partial orders, but $P \times Q$ is not κ-c.c. Then X_P, X_Q are κ-c.c. topological spaces. Since $P \times Q$ is not κ-c.c., neither is $X_{P \times Q} = X_P \times X_Q$. So (1) implies (2).

To see (2) implies (1), let X, Y be two topological spaces which are κ-c.c., but $X \times Y$ is not κ-c.c. Let P be the collection of non-empty open sets in X ordered by $U \leqslant V$ iff $U \subseteq V$. Likewise define Q. Clearly P, Q are partial orders. P and Q are κ-c.c. partial orders since a κ antichain in P would be a κ collection of pairwise disjoint open sets in X. Let $\{W_\alpha\}_{\alpha<\kappa}$ be a κ family of pairwise disjoint open sets in $X \times Y$. Without loss of generality we may assume that $W_\alpha = U_\alpha \times V_\alpha$ is a product of basic open sets. Let $p_\alpha = (U_\alpha, V_\alpha) \in P \times Q$. The p_α form an antichain in $P \times Q$ (since if $(U, V) \leqslant (U_\alpha, V_\alpha), (U_\beta, V_\beta)$ then $U \subseteq U_\alpha \cap U_\beta$, $V \subseteq V_\alpha \cap V_\beta$, so $U \times V \subseteq W_\alpha \cap W_\beta$). So, $P \times Q$ is not κ-c.c.

(4) implies (1) trivially, since Boolean algebras are partial orders. For the other direction, let P, Q be as in (1). Let $\mathcal{B}$ be the regular open algebra of P, and likwise define $\mathcal{C}$. Thus, $\mathcal{B}$ and $\mathcal{C}$ are complete Boolean algebras. Since every element of $\mathcal{B}$ (i.e., every regular open subset of X_P) contains a basic open subset of X_P, it is clear that $\mathcal{B}$ is κ-c.c., and likewise for Q. For $p \in P$ let $B_p = \text{intcl}(N_p) \in \mathcal{B}$ and likewise define $C_q \in \mathcal{C}$ for $q \in Q$. Let $\{(p_\alpha, q_\alpha)\}_{\alpha<\kappa}$ be an antichain in $P \times Q$. Let $B_\alpha = B_{p_\alpha}$, and likewise for C_α. Then the (B_α, C_α) form an antichain in $\mathcal{B} \times \mathcal{C}$. For suppose $(B, C) \leqslant (B_\alpha, C_\alpha), (B_\beta, C_\beta)$. We may assume $B = B_p$, $C = C_q$. So, $B_p \subseteq B_{p_\alpha} \cap B_{p_\beta}$. This says N_{p_α} and N_{p_β} are dense below p. Thus, there is an $r \in P$ with $r \leqslant p_\alpha, p_\beta$. Similarly, q_α and q_β are compatible. Hence (p_α, q_α) and (p_β, q_β) are compatible, a contradiction.

(3) trivially implies (2). To finish we show that (1) implies (3). Given P and Q, let $\mathcal{B}$ and $\mathcal{C}$ again be their completions. Let X' be the Stone space of the Boolean algebra $\mathcal{B}$, that is, X' is the collection of ultrafilters on $\mathcal{B}$. X' is a topological space with basic open sets of the form $N_V = \{\mathcal{U} \in X' : V \in \mathcal{U}\}$ for V a regular open set in

X_P. X' is clearly Hausdorff, and it is a standard fact (see the following exercise) that X' is compact. X' is also κ-c.c., for if N_{V_α} were pairwisw disjoint open sets in X', then the V_α would be pairwise disjoint open sets in $\mathcal{B}$, contradicting $\mathcal{B}$ being κ-c.c. Likewise define Y'. To see $X' \times Y'$ is not κ-c.c., let (p_α, q_α) again be a κ size antichain in $P \times Q$. Let B_α, C_α be as before, and consider the basic open sets $N_{B_\alpha} \times N_{C_\alpha}$ in $X' \times Y'$. These are pairwise disjoint since if $N_{B_\alpha} \cap N_{B_\beta} \neq \varnothing$ then $B_\alpha \cap B_\beta \neq \varnothing$, and then (as argued above) p_α and p_β are compatible in P (and likewise for Q). □

We showed before that if there is a Suslin tree T, then T gives a partial order which is c.c.c. (i.e., T with the reverse of the tree ordering) but for which $T \times T$ is not c.c.c. We now show that CH also implies the non-productivity of c.c.c.

Theorem 1.2 (Galvin, Laver). *Assume ZFC + CH. Then there are two c.c.c. partial orders P, Q whose product $P \times Q$ is not c.c.c.*

Corollary 1.3. *Assuming CH, there are two c.c.c. compact Hausdorff spaces whose product is not c.c.c.*

Proof. Suppose $f\colon (\omega_1)^2 \to \{0,1\}$ is a partition of the pairs of countable ordinals. For $i \in \{0,1\}$ we define the partial order P_i to consist of all finite $p \subseteq \omega_1$ such that $f''p^2 = \{i\}$ (that is, P_i consists of finite sets which are homogeneous for color i). We order P_i by: $p \leqslant q$ iff $p \supseteq q$. No matter how we choose f, the product $P_0 \times P_1$ will not be c.c.c., since the elements (α, α) for $\alpha < \omega_1$ form an antichain. It remains to show that we can choose f so that P_0 and P_1 are c.c.c.

Let (using CH) $\{F_\alpha\}_{\alpha<\omega_1}$ enumerate all ω sequences $F_\alpha = (F_\alpha^0, F_\alpha^1, \dots, F_\alpha^n, \dots)$ of pairwise disjoint finite subsets of ω_1. Suppose we have defined $f(\alpha, \beta)$ when $\max\{\alpha, \beta\} < \gamma$. We define $f(\alpha, \gamma)$ for $\alpha < \gamma$ as follows. Let S_n, $n < \omega$, enumerate all objects of the form $S_n = \langle F_{\alpha_n}, X_n, i_n \rangle$ such that

(1) $\alpha_n < \gamma$, $i \in \{0,1\}$, and X_n is a finite subset of γ.
(2) $X_n \cap \bigcup_k F_{\alpha_n}^k = \varnothing$ and there are infinitely many $k \in \omega$ such that $f''(F_{\alpha_n}^k \cup X_n) = \{i\}$.

By diagonalizing, there is a sequence $\{Y_l\}_{l\in\omega}$ with each Y_l of the form $F_{\alpha_n}^k$ for some $n = n(l)$, $k = k(l)$, such that the Y_l are pairwise disjoint, for each n there are infinitely many l such that $n(l) = n$, and $f''(Y_l \cup X_{n(l)}) = \{i_{n(l)})\}$. For $\alpha \in Y_l$, define $f(\alpha, \gamma) = i_{n(l)}$. The definition of f gives us the following property.

(*): for any $\alpha < \gamma$, finite $X \subseteq \gamma$, and $i \in \{0,1\}$, if $\sup(\cup F_\alpha)) < \gamma$, $X \cap (\cup F_\alpha) = \varnothing$, and there are infiniely many n such that $f''(F_\alpha^n \cup X)^2 = \{i\}$, then there are infinitely many n such that $f''(F_\alpha^n \cup X)^2 = \{i\}$ and $f''(F_\alpha^n \cup \{\gamma\})^2 = \{i\}$.

To see this works, suppose that $\{p_\alpha\}_{\alpha<\omega_1}$ was an ω_1-antichain for P_0 (the other case being similar). Thinning to Δ-system, we may assume that $p_\alpha = r \cup G_\alpha$, where $G_\alpha \cap G_\beta = \varnothing$ for $\alpha \neq \beta$. To get a contradiction it suffices to find $\alpha \neq \beta$ such that $f''(G_\alpha \cup G_\beta) = \{0\}$. Consider the first ω many elements, and fix $\alpha < \omega_1$ such that $F_\alpha = (G_0, G_1, \dots, G_n, \dots)$. Let $\delta > \alpha$ and $\delta > \sup \bigcup_n G_n$. Fix one of the G_β with $G_\beta \cap \delta = \varnothing$ (which we can do as the G_β are pairwise disjoint). Say $G_\beta = \{\gamma_1, \dots, \gamma_m\}$. Using (*) m times (at step p we use $X = \{\gamma_1, \dots, \gamma_{p-1}\}$, $\gamma = \gamma_p$) we thin the G_n sequence to an infinite subsequence $(G_0', G_1', \dots)$ such that $f''(G_n' \cup G_\beta)^2 = \{0\}$. This contradicts the p_α being an antichain. □

We now show that MA + $\neg$CH implies that the product of two c.c.c. partial orders is c.c.c.

Definition 1.4. A partial order $\mathbb{P}$ is strongly c.c.c. if whenever $A \subseteq P$ with $|A| = \omega_1$, then there is a $B \subseteq A$ with $|B| = \omega_1$ such that for any finite $\{p_1, \dots, p_n\} \subseteq B$ there is a $p \in P$ with $p \leqslant p_1, \dots, p_n$.

Lemma 1.5. *Assume ZFC+ MA+ $\neg$CH. Then every c.c.c. partial order is strongly c.c.c.*

Proof. Let P be c.c.c., and $A = \{p_\alpha\}_{\alpha<\omega_1}$. First we claim that there is a $p \in A$ such that any $q \leqslant p$ is compatible with p_α for uncountably many α. For if not, then we could get $q_{\alpha_0} \leqslant p_{\alpha_0}$ (where $\alpha_0 = 0$) and α_1 such that q_{α_0} is incompatible with all p_β for $\beta \geqslant \alpha_1$. We could then get $q_{\alpha_1} \leqslant p_{\alpha_1}$ and α_2 so that q_{α_1} is incompatible with all p_β for $\beta \geqslant \alpha_2$. Continuing we define an ω_1 antichain $\{q_{\alpha_i}\}_{i<\omega_1}$.

We may assume p_0 is such that any $q \leqslant p_0$ is compatible with ω_1 many p_α.

Let Q be the partial order consisting of all finite subsets $u = \{q_1, \dots, q_n\} \subseteq P$ which contain p_0 and which are compatible, that is, there is a $q \in P$ with $q \leqslant q_1, \dots, q_n$. We order Q by $u \leqslant v$ iff $u \supseteq v$. We claim that Q is c.c.c. For suppose $\{u_\alpha\}_{\alpha<\omega_1}$ were an uncountable antichain in Q. For each $\alpha < \omega_1$, let $r_\alpha \in P$ with r_α extending all of the elements of u_α. Since P is c.c.c., there are $\alpha \neq \beta$ such that $r_\alpha \parallel r_\beta$. Let $r \leqslant r_\alpha, r_\beta$. Then r extends all elements of u_α and u_β, so $u_\alpha \cup u_\beta \in Q$, a contradiction. Thus, Q is c.c.c.

For each $\alpha < \omega_1$, let $D_\alpha \subseteq Q$ be those $u \in Q$ which contain a p_β for some $\beta > \alpha$. Then D_α is dense in Q. For given $u \in Q$, let $q \in P$ extend all of the elements of u. Since $p_0 \in u$, q is compatiblw with ω_1 many of the p_β. Let $\beta > \alpha$ with $p_\beta \parallel q$. Then $u \cup \{p_\beta\} \in D_\alpha$.

By MA (and the fact that $\omega_1 < 2^\omega$ from $\neg$CH) there is a filter G on Q meeting all of the D_α. Let $B = \cup G$, so $B \subseteq A$. Since G meets all of the D_α, $|G| = \omega_1$. Since G is a filter, for any finite $\{u_1, \dots, u_n\} \subseteq G$, there is a $p \in G$ extending all of the elements in $\bigcup_{i=1}^n u_i$. It follws that finite number of elements from B have a common extension in P. □

Lemma 1.6. *If P is strongly c.c.c.and $\mathbb{Q}$ is c.c.c., then $P \times Q$ is c.c.c.*

Proof. Suppose (p_α, q_α) were an uncountable antichain in $P \times Q$. Since P is strongly c.c.c., we may thin the sequence so that for any finite subset u of $\{p_\alpha\}_{\alpha<\omega_1}$ the elements of u have a common extension (actually, all we need is that any two are compatible). Since Q is c.c.c., get now $\alpha < \beta$ such that $q_\alpha \parallel q_\beta$. Then $p_\alpha \parallel p_\beta$ and $q_\alpha \parallel q_\beta$, so $(p_\alpha, q_\alpha) \parallel (p_\beta, q_\beta)$, a contradiction. □

Theorem 1.7. *Assume ZFC + MA + $\neg$CH. Then the product of two c.c.c. partial orders is c.c.c. Likewise the product of two c.c.c. topological spaces is c.c.c.*

Proof. Immediate from lemmas 1.5 and 1.6. □

As a consequence, we have the following topological result.

Theorem 1.8. *Assume ZFC+MA+$\neg$CH. Then if $(X_\alpha, \tau_\alpha)_{\alpha\in I}$ are c.c.c. topological spaces, then their product $X = \prod_{\alpha\in I} X_\alpha$ is also c.c.c.*

Proof. Suppose $\{V_\alpha\}_{\alpha<\omega_1}$ is an uncountable antichain (i.e., pairwise disjoint open sets) in X. Without loss of generality we many assume the V_α are basic open, say $V_\alpha = U_{\beta_1^\alpha} \times \cdots \times U_{\beta_n^\alpha}$, where $U_{\beta_i^\alpha}$ is open in X_{β_i} and n depends on α. We call $\{\beta_1^\alpha, \dots, \beta_n^\alpha\}$ the support of V_α. We may thin the antichain so that the supports form a Δ-system with say root $r = \{\beta_1, \dots, \beta_m\}$. then clearly $\{W_\alpha\}$ also must form an antichain, where W_α is the product of the $U_{\beta_i^\alpha}$ for $\beta_i^\alpha \in r$. This is a contradiction, as theorem 1.7 gives that $X_{\beta_1} \times \cdots \times X_{\beta_m}$ is c.c.c. □

2. Almost Disjoint Sets

If $x, y \subseteq \omega$, we say x and y are *almost disjoint* if $x \cap y$ is finite.

Lemma 2.1. *There is a 2^ω sequence $\{x_\alpha\}_{\alpha<2^\omega}$ of infinite subsets of ω such that if $\alpha \neq \beta$ then x_α, x_β are pairwise disjoint.*

Proof. Let $\{y_\alpha\}_{\alpha<2^\omega}$ be distinct infinite subsets of ω. Let $f\colon \omega^{<\omega} \to \omega$ be one-to-one. Define $x_\alpha = \{f(y_\alpha \cap n)\colon n \in \omega\} \subseteq \omega$. Let $n \in y_\alpha - y_\beta$, say Then for any $m > \max\{f(y_\alpha \cap 1), \dots, f(y_\alpha \cap n)\}$, if $m \in x_\alpha$ then $m \notin x_\beta$. □

Theorem 2.2. *Assume ZFC+ MA. Then $\forall \kappa < 2^\omega$ $(2^\kappa = 2^\omega)$. In particular, 2^ω is regular.*

Proof. We use the almost disjoint sets forcing. Fix a sequence $\{x_\alpha\}_{\alpha<2^\omega}$ of infinite, almost disjoint subsets of ω. Fix $\kappa < 2^\omega$, and we use this sequence to code subsets of κ. Let $A \subseteq \kappa$. We claim that there is an $x \subseteq \omega$ such that for all $\alpha < \kappa$ if $\alpha \in A$ then $x \cap x_\alpha$ is finite, and if $\alpha \notin A$ then $x \cap x_\alpha$ is infinite. This clearly suffices to show that $2^\kappa \leqslant 2^\omega$.

To see the claim, consider the partial order $\mathbb{P}$ which consists of all pairs $p = (s, F)$ where $s \in 2^{<\omega}$ and $F \subseteq A$ is finite. We regard s as the characteristic function of the subset we are trying to build. If $p' = (s', F')$, then we define $p' \leqslant p$ iff s' extends s, $F \subseteq F'$, and for all $n \in \mathrm{dom}(s') - \mathrm{dom}(s)$, if $s'(n) = 1$ then for all $\alpha \in F$ we have $x_\alpha(n) = 0$.

To see $\mathbb{P}$ is c.c.c. simply note that (p, F) is compatible with (p', F') if $s = s'$, and there are only countably many choices for s.

For $\alpha \in A$, let $D_\alpha = \{(s, F)\colon x_\alpha \in F\}$. Clearly D_α is dense as we may extend any (s, F) to $(s, F \cup \{x_\alpha\})$. For $\alpha < \kappa$, $\alpha \notin A$, and $k \in \omega$, let $D_{\alpha,k} = \{(s, F)\colon \exists m \geqslant k\ (m \in x_\alpha \wedge s(m) = 1)\}$. To see this is dense, note that x_α is almost disjoint from all the x_β for $\beta \in F$. Hence there is am $m \geqslant k$ such that $m \in x_\alpha - \bigcup_{\beta\in F} x_\beta$. Extend s to s' where $s'(m) = 1$ and $s'(l) = 0$ for all other $l \in \mathrm{dom}(s') - \mathrm{dom}(s)$. Then $(s', F) \leqslant (s, F)$ and $(s', F) \in D_{\alpha,k}$.

By MA, let G be a filter on $\mathbb{P}$ meeting all of the D_α for $\alpha \in A$ and all of the $D_{\alpha,k}$ for $\alpha \notin A$. Let $x = \bigcup\{s\colon \exists F\ (s, F) \in G\}$ be the real determined from G. Suppose first that $\alpha \in A$. Since D_α is dense, let $(s, F) \in G \cap D_\alpha$. Suppose $(s', F') \in G$ as well. Since G is a filter, there is some $(s'', F'') \in G$ with $(s'', F'') \leqslant (s, F), (s', F')$. Thus, for all $n \in \mathrm{dom}(s'') - \mathrm{dom}(s)$ we have $s''(n) = 1 \to (n \notin x_\alpha)$. Hence $x \cap x_\alpha \subseteq \{n\colon s(n) = 1\}$, so x is almost disjoint from x_α. Suppose now that $\alpha \notin A$. For any $k \in \omega$ there is an $(s, F) \in G \cap D_{\alpha,k}$. Thus there is an $m \in x \cap x_\alpha$ with $m \geqslant k$. This shows $x \cap x_\alpha$ is infinite. □

3. Measure and Category

We first show that MA $+ \neg$CH implies the $< 2^\omega$ additivity of measure and category. First we consider category. Recall a subset of a Polish space is meager if it is contained in the countable union of closed nowhere dense sets. A set A is comeager if its complement is meager, equivalently if A contains a countable intersection of dense open sets. The Baire category theorem (for complete metric spaces) says that every comeager set is dense, in particular non-empty. It is immediate that a countable union of meager sets is meager (equivalently, a countable intersection of comeager sets is comeager). The following result shows that with MA we may extend this to $< 2^\omega$ size unions.

Theorem 3.1. *Assume MA. Then the union of* $< 2^\omega$ *meager sets in a Polish space is meager.*

Proof. Let $\kappa < 2^\omega$ and fix dense open sets U_α, $\alpha < \kappa$. We show that there are dense open sets V_n such that $\bigcap_n V_n \subseteq \bigcap_{\alpha<\kappa} U_\alpha$. Fix a base $\{B_n\}_{n\in\omega}$ for the Polish space. Let $\mathbb{P}$ consist of all finite sequences $p = \langle (W_0, F_0), \dots, (W_n, F_n)\rangle$ where each $F_i \subseteq \kappa$ is finite, W_i is a finite union of basic open sets, and $W_i \subseteq \bigcap_{\alpha\in F_i} U_\alpha$. If $p' = \langle (W_0'', F_0), \dots, (W_m', F_m')\rangle$ then we define $p' \leqslant p$ iff $m \geqslant n$ and for all $i \leqslant n$ we have $F_i' \supseteq F_i$ and $W_i' \supseteq W_i$.

$\mathbb{P}$ is c.c.c. since if $m = n$ and $W_i = W_i'$, then p is compatible with p'. For each $\alpha < \kappa$, let D_α be those p such that $\alpha \in F_i$ for some i. Clearly each D_α is dense. For each i and basic open set B_j, let $E_{i,j}$ be those p such that $W_n \cap B_j \neq \emptyset$. Easily each $E_{i,j}$ is also dense. From MA, let G be a filter on $\mathbb{P}$ meeting all of these dense sets. Define V_i to be the union of all the W_i such that (W_i, F_i) is the i^{th} coordinate of a $p \in G$. Since G meets each $E_{i,j}$, each V_i is dense open. Let $x \in \bigcap_i V_i$. Fix $\alpha < \kappa$ and we show that $x \in W_\alpha$. Let $p \in G \cap D_\alpha$, say $p = \langle (W_0, F_0), \dots, (W_n, F_n)\rangle$ with $\alpha \in F_i$. If $q = \langle (W_0', F_0'), \dots, (W_m', F_m')\rangle$ is also in G, then (if $m \geqslant i$) $W_i' \subseteq U_\alpha$ as well, since p is compatible with q (any common extension will have i^{th} coordinate (W, F) where $\alpha \in F$ and $W \supseteq W_i \cup W_i'$). Thus, $x \in V_i \subseteq U_\alpha$. □

We now prove the analogous result for measure.

Theorem 3.2. *Assume MA. Let* μ *be a* σ*-finite Borel measure on a separable metric space* X*. Then the union of* $< 2^\omega$ *subsets of* X *of* μ *measure* 0 *has* μ *measure* 0.

Proof. We may assume that μ is finite, that is, a probability measure. Recall that any Borel probability measure on a metric space is regular, that is, for every Borel set $B \subseteq X$ and every $\epsilon > 0$, there is a closed set F and an open set U with $F \subseteq B \subseteq U$ and $\mu(U - F) < \epsilon$. Recall also that a set $A \subseteq X$ is defined to have measure 0 if it is contained in a Borel set of measure 0, and a set is measurable if is equal to a Borel set modulo a set of measure 0. The measurable sets form a σ-algebra containing the Borel sets, and μ extends to a countably additive measure on this algebra.

Fix $\kappa < 2^\omega$, and suppose A_α, for $\alpha < \kappa$, are measure zero sets. Fix $\epsilon > 0$, and we show that there is an open set U containing $A = \bigcup_{\alpha<\kappa} A_\alpha$ of measure $< \epsilon$. Let $\mathbb{P}$ consist of all open sets in X of measure $< \epsilon$. Define $p \leqslant q$ iff $p \supseteq q$. We show that $\mathbb{P}$ is c.c.c. Suppose $\{p_\alpha\}_{\alpha<\omega_1}$ were an uncountable antichain in $\mathbb{P}$. By thinning, we may assume that for some $\epsilon_1 < \epsilon$ that $\mu(p_\alpha) < \epsilon_1$ for all α. For each α there is a finite union of basic open sets (with respect to a fixed countable base for X) $U_\alpha \subseteq p_\alpha$ such that $\mu(p_\alpha - U_\alpha) < (\epsilon - \epsilon_1)/2$. Fix $\alpha \neq \beta$ such that $U_\alpha = U_\beta$. Then $\mu(p_\alpha \cup p_\beta) \leqslant \mu(U_\alpha) + \mu(p_\alpha - U_\alpha) + \mu(p_\beta - U_\alpha) < \epsilon$.

For each $\alpha < \kappa$, let D_α be those p_α containing A_α. Easily each D_α is dense. By MA, let G be a filter on $\mathbb{P}$ meeting all of the D_α. Let V be the union of all the open sets $p \in G$. Clearly V contains each A_α. We claim that $\mu(V) \leqslant \epsilon$. Since G is a filter, any finite union of open sets $p \in G$ has measure $< \epsilon$. By countable additivity, any countable union of sets $p \in G$ has measure $\leqslant \epsilon$. However, in any second countable space the union of a family of open sets is given by a union of a countable subfamily. Hence, $\mu(V) \leqslant \epsilon$. □

As a corollary we get a little more:

Corollary 3.3. *Assume MA. Let μ be a σ-finite Borel measure on a separable metric space X. Then the algebra $\mathcal{M}$ of measurable sets is closed under $\kappa < 2^\omega$ unions and intersections for all $\kappa < 2^\omega$. Also μ is κ-additive for all $\kappa < 2^\omega$.*

Proof. We prove this by induction on $\kappa < 2^\omega$. Suppose $\{A_\alpha\}_{\alpha<\kappa}$ are given with $A_\alpha \in \mathcal{M}$. We show that $A = \bigcup_{\alpha<\kappa} A_\alpha$ is in $\mathcal{M}$. Let $B_\alpha = A_\alpha - \bigcup_{\beta<\alpha} A_\beta$ be the disjointification of the A_α. By induction, each B_α is in $\mathcal{M}$. From σ-finiteness, only countably many of the B_α can have non-zero μ measure. The union of the rest has μ measure 0 by theorem 3.2. Thus, A is the union of countably many sets in $\mathcal{M}$ together with a measure 0 set (which is by definition measurable). Thus $A \in \mathcal{M}$. This show $\mathcal{M}$ is $< 2^\omega$-additive. If the A_α are pairwise disjoint, then again only countably many of them can have non-zero μ measure, and the union of the rest has measure 0. The countable additivity of μ then gives that $\mu(A) = \sum \mu(A_\alpha)$. □

4. Random and Generic Reals

We introduce two new forcings for adding a real, one corresponding to measure and the other to category, $\mathbb{P}_m$ and $\mathbb{P}_c$. The category version turns out to be isomorphic (actually, have an isomorphic completion) to ordinary Cohen forcing, so this is not really a new forcing but just a different point of view. In the measure case, $\mathbb{P}_m$ will be a new forcing, which we call the random real forcing.

Let X be a Polish space (e.g., $X = \mathbb{R}$), and $\mathcal{I}$ an ideal on X. For example, $\mathcal{I}$ could be the ideal $\mathcal{I}_m$ of Lebesgue measure 0 sets, or the ideal $\mathcal{I}_c$ of meager sets. We define an equivalence relation $\sim_I$ of the Borel subsets of X by $A \sim_I B$ iff $A \triangle B \in \mathcal{I}$. For $B \subseteq X$ a Borel set, we let $[B]_\mathcal{I}$, or just $[B]$ if $\mathcal{I}$ is understood, denote the equivalence class of B. We let $\mathbb{P}_\mathcal{I}$ be the partial order whose elements are the equivalence classes $[B]$ (B a Borel subset of X) with $B \notin \mathcal{I}$, ordered by $[A] \leqslant [B]$ iff $A - B \in \mathcal{I}$. This is easily well-defined and a partial order. In fact, $\mathbb{P}_\mathcal{I}$ is actually a Boolean algebra under the obvious operations (for example $[A] + [B] = [A \cup B]$, $-[A] = [X - A]$) provided we add $0 = [\varnothing]$ back in.

We say $\mathcal{I}$ is c.c.c. if the Boolean algebra $\mathbb{P}_\mathcal{I}$ is c.c.c. In other words, there is no ω_1 sequence A_α of Borel subsets of X such that for $\alpha \neq \beta$ we have $A_\alpha \triangle A_\beta \in \mathcal{I}$. For a general ideal, this will not be a complete Boolean algebra, but we have the following.

Lemma 4.1. *If $\mathcal{I}$ is c.c.c. and countably-additive then $\mathbb{P}_\mathcal{I}$ is a complete Boolean algebra.*

Proof. Consider a collection $\{[A_\alpha]\}_{\alpha<\kappa}\}$ of elements of the Boolean algegra. We show that the least upper bound $\sum [A_\alpha]$ exists. Let $\mathcal{B}$ be a maximal collection subject to being an antichain in $\mathcal{P}_\mathcal{I}$ and for each $b \in \mathcal{B}$ there is an α such that $b \leqslant [A_\alpha]$. Since $\mathbb{P}_\mathcal{I}$ is c.c.c. we may write $\mathcal{B} = \{[B_n]\}_{n\in\omega}$. Let $B = \bigcup_n B_n$ (we have implicitly chosen representatives for the equivalence classes). We first claim that $[B] = \sum [B_n]$. Clearly $[B_n] \leqslant [B]$ for each n. On the other hand, if $[B']$ is also such that $[B_n] \leqslant [B']$ for each n, then by countable additivity of $\mathcal{I}$ we get that $B = \bigcup_n B_n \subseteq B' \mod \mathcal{I}$. Hence $[B]$ is the least upper bound of the $[B_n]$.

Next we show that $[B]$ is the least upper bound of the $[A_\alpha]$. Sice for each n, $[B_n] \leqslant [A_\alpha]$ for some α, clearly any upper bound for the $[A_\alpha]$ is an upper bound for the $[B_n]$. So, it suffices to show that B is an upper bound for the $[A_\alpha]$. Fix α, and we show that $[A_\alpha] \leqslant [B]$. If not, then $A_\alpha - B \notin \mathcal{I}$. Then $[A_\alpha - B]$ cound be added to the $[B_n]$, contradicting the maximality of $\mathcal{B}$. □

Exercise 1. Show that for any Polish space X, the category forcing $\mathbb{P}_c$ is c.c.c. show that for any σ-finite Borel measure μ on a separable metric space X, the corresponding measure forcing $\mathbb{P}_m$ is c.c.c.

Corollary 4.2. *$\mathbb{P}_c$ and $\mathbb{P}_m$ are complete Boolean albegras.*