

## $V_\kappa$ , $H_\kappa$ , and HOD

Recall that working in ZF (in fact, ZF–Power) we have previously defined  $V_\alpha$  for all  $\alpha \in \text{ON}$ . Recall that  $V_\alpha$  consists of all sets of rank  $< \alpha$  (remember  $V_\alpha$  is a set). The next lemma shows what axioms of set theory hold in these sets.

**Lemma 0.1.** *Assume ZF. Then for any limit ordinal  $\alpha > \omega$ ,  $V_\alpha \models \text{ZF–Replacement}$ . Assuming ZFC, for any limit ordinal  $\alpha > \omega$ ,  $V_\alpha \models \text{ZFC–Replacement}$ .*

*Proof.* The empty set axiom holds in  $V_\alpha$  as  $\emptyset \in V_\alpha$  and the statement  $\phi(x) = \neg \exists y \in x$  is  $\Delta_0$  so absolute. If  $x, y \in V_\alpha$  then  $\{x, y\} \in V_\alpha$  as  $\alpha$  is a limit. Since the statement  $\phi(x, y, z) = [x \in z \wedge y \in z \wedge \forall w \in z (w \approx x \vee w \approx y)]$  is  $\Delta_0$ , hence absolute, it follows that  $V_\alpha \models \text{pairing}$ . The union axiom is similar. By downwards absoluteness, foundation holds in any class (since we assuming foundation holds in  $V$ ), so it holds in  $V_\alpha$ . Since  $\alpha > \omega$ ,  $\omega \in V_\alpha$ . As the statement  $\phi(x) = “x = \omega”$  is absolute,  $V_\alpha \models \text{infinity}$ . Extensionality holds in  $V_\alpha$  as  $V_\alpha$  is transitive. If  $x \in V_\alpha$ , then  $\mathcal{P}(x) \in V_\alpha$  as  $\alpha$  is a limit. The statement  $\phi(x, y) = [y = \mathcal{P}(x)]$ , written out, is  $\Pi_1$  and hence downwards absolute. Since  $v \models \phi(x, \mathcal{P}(x))$ , it follows that  $V_\alpha \models \phi(x, \mathcal{P}(x))$ . Hence  $V_\alpha \models \text{Power set}$ . To see comprehension, let  $\phi(x_1, \dots, x_n, y, z)$  be a formula and let  $a_1, \dots, a_n, b \in V_\alpha$ . By comprehension in  $V$  applied to the formula  $\phi^{V_\alpha}(\vec{x}, y, z)$  (which has the extra parameter  $V_\alpha$  in it) there is set  $w$  such that  $\forall z (z \in w \leftrightarrow z \in b \wedge \phi^{V_\alpha}(\vec{a}, b, z))$ . Since  $b \in V_\alpha$ , we also have  $w \in V_\alpha$ . Hence  $V_\alpha \models \exists w \forall z (z \in w \leftrightarrow z \in b \wedge \phi(\vec{a}, b, z))$ , and so  $V_\alpha \models \text{Comprehension}$ . Finally, suppose AC holds in  $V$ . Let  $a \in V_\alpha$ . By AC in  $V$ , let  $<$  be a wellordering of  $a$ . Since  $< \in \mathcal{P}\mathcal{P}\mathcal{P}(a)$  and  $\alpha$  is limit,  $< \in V_\alpha$ . The statement  $\phi(x, y) = [y \text{ is a wellordering of } x]$  is  $\Pi_1$ , and since it holds in  $V$  it holds in  $V_\alpha$ . Thus,  $V_\alpha \models \text{AC}$ .  $\square$

Thus, in ZF we can prove that there are set models for all of ZF except replacement. More precisely, for any formulas  $\phi$  of ZF–Replacement,  $\text{ZF} \vdash \forall \alpha (\alpha > \omega \wedge \alpha \text{ limit} \rightarrow \phi^{V_\alpha})$ .

**Corollary 0.2.** *ZF–Replacement  $\not\models$  ZF.*

*Proof.* Replacement clearly does not hold in  $V_{\omega \cdot 2}$ . Let  $\phi$  be an instance of replacement so that  $\text{ZF} \vdash (\neg \phi)^{V_{\omega \cdot 2}}$ . If ZF–Replacement  $\vdash$  ZF, then there would be finitely many axioms  $\{\psi_1, \dots, \psi_n\}$  of ZF–Replacement such that  $\{\psi_1, \dots, \psi_n\} \vdash \phi$ . But,  $\text{ZF} \vdash \psi_1^{V_{\omega \cdot 2}} \wedge \dots \wedge \psi_n^{V_{\omega \cdot 2}}$ , and so  $\text{ZF} \vdash \phi^{V_{\omega \cdot 2}}$ , a contradiction.  $\square$

In the proof of corollary 0.2 we have implicitly assumed the consistency of ZF, though we could get by just assuming the consistency of ZF–Replacement.

**Exercise 1.** Just assuming the consistency of ZF–Replacement, show that ZF–Replacement  $\not\models$  ZF. (hint: let  $\phi$  be the instance of replacement which gives the existence of  $\omega \cdot 2$ . If ZF–Replacement  $\not\models \phi$ , we’re done, so assume ZF–Replacement  $\vdash \phi$ . Now follow the above prove for this  $\phi$ .)

**Exercise 2.** Let  $\psi_1, \dots, \psi_n$  be axioms of ZF. Show that ZF–Replacement  $\cup \{\psi_1, \dots, \psi_n\} \not\models$  ZF. (hint: Working in ZF, show that there is a least limit ordinal  $\alpha > \omega$  such that  $V_\alpha \models \psi_1 \wedge \dots \wedge \psi_n$ . Argue then that there must be a smaller limit ordinal  $\beta < \alpha$  such that  $V_\beta \models \psi_1 \wedge \dots \wedge \psi_n$ .)

We now define another family of sets called the  $H_\kappa$ . If  $P$  is any property, we say set  $x$  hereditarily has property  $P$  if every element of the transitive closure of  $x$  has property  $P$ .

**Definition 0.3.** (ZFC) Let  $\kappa$  be an infinite regular cardinal.  $H_\kappa$  is the collection of sets which hereditarily have size less than  $\kappa$ , that is, every element of the transitive closure of  $x$  has cardinality  $< \kappa$ .

We first give a simple reformulation of the definition.

**Lemma 0.4.** For all sets  $x$ ,  $x \in H_\kappa$  iff  $|\text{tr cl}(x)| < \kappa$ .

*Proof.* Clearly if  $|\text{tr cl}(x)| < \kappa$  then  $x \in H_\kappa$ . Suppose now  $x \in H_\kappa$ . Recall  $\text{cl}_n(x)$  is defined by:  $\text{cl}_0(x) = x$ ,  $\text{cl}_{n+1}(x) = \text{cl}_n(x) \cup (\cup \text{cl}_n(x))$ . If we assume inductively that  $|\text{cl}_n(x)| < \kappa$ , then  $\text{cl}_{n+1}(x)$  is a  $< \kappa$  union of sets each of which has size  $< \kappa$  by definition of  $H_\kappa$ . Since  $\kappa$  is regular, it follows that  $|\text{cl}_{n+1}(x)| < \kappa$ . Finally,  $\text{tr cl}(x) = \bigcup_n \text{cl}_n(x)$  and using the regularity of  $\kappa$  again we have  $|\text{tr cl}(x)| < \kappa$ .  $\square$

The definition of  $H_\kappa$  shows it is a class, but we now show that is in fact a set.

**Lemma 0.5.** For all infinite regular cardinals  $\kappa$ ,  $H_\kappa \subseteq V_\kappa$ .

*Proof.* Suppose  $x \notin V_\kappa$ , so  $|x| \geq \kappa$ . We know from lemma ?? that for every  $\alpha < |x|$ , there is a  $y \in \text{tr cl}(x)$  such that  $|y| = \alpha$ . This clearly forces  $|\text{tr cl}(x)| \geq \kappa$ , so  $x \notin H_\kappa$ .  $\square$

One could give another proof of lemma 0.5 as follows. Let  $x \in H_\kappa$ . Using AC, let  $\lambda < \kappa$  be the cardinality of  $\text{tr cl}(x)$ , and let  $f: \lambda \rightarrow \{x\} \cup \text{tr cl}(x)$  be a bijection. Define the relation  $E$  on  $\lambda$  by  $\alpha E \beta$  iff  $f(\alpha) \in f(\beta)$ . Thus,  $(\lambda, E)$  “codes” the set  $\{x\} \cup \text{tr cl}(x)$ . Clearly  $\lambda, E \in V_\kappa$ . Let  $\pi$  be the collapse map for  $(\lambda, E)$ . We show by induction on  $E$  that for all  $\alpha \in \lambda$  that  $\pi(\alpha) \in V_\kappa$ . We have  $\pi(\alpha) = \{\pi(\beta) : \beta \in \lambda \wedge \beta E \alpha\}$ . By induction, each  $\pi(\beta)$  in the above expression is in  $V_\kappa$ , so  $\pi(\alpha) \subseteq V_\kappa$ . Since  $\kappa$  is regular and  $\lambda < \kappa$ ,  $\pi(\alpha) \subseteq V_\gamma$  for some  $\gamma < \kappa$ , and hence  $\pi(\alpha) \in V_{\gamma+1} \subseteq V_\kappa$ . In particular,  $x \in V_\kappa$ .

Note that if  $x \subseteq H_\kappa$ , then to show  $x \in H_\kappa$  it is enough to show that  $|x| < \kappa$ . Also, since each ordinal is transitive, clearly  $H_\kappa \cap \text{ON} = \kappa$ .

The next lemma shows that ZFC–Power holds in the  $H_\kappa$ .

**Lemma 0.6.** (ZFC) For every axiom  $\phi$  of ZFC–Power,  $\text{ZFC} \vdash \forall \kappa [\kappa \text{ regular} \rightarrow \phi^{H_\kappa}]$ .

*Proof.* We work in ZFC. Let  $\kappa$  be an infinite regular cardinal. We show that all the axioms of ZFC except power hold in  $H_\kappa$ . Note for the following arguments that by definition  $H_\kappa$  is transitive. Foundation holds in  $H_\kappa$  as it holds in any set or class. Clearly,  $\emptyset \in H_\kappa$ , and by absoluteness then  $H_\kappa$  satisfies the emptyset axiom. If  $x, y \in H_\kappa$ , then  $|\{x, y\}| < \omega < \kappa$ , so  $\{x, y\} \in H_\kappa$ . By absoluteness,  $H_\kappa$  satisfies the pairing axiom. If  $x \in H_\kappa$  then  $\cup x \subseteq \text{tr cl}(x)$ , so  $\text{tr cl}(\cup x) \subseteq \text{tr cl}(x)$ , and so  $|\text{tr cl}(\cup(x))| \leq |\text{tr cl}(x)| < \kappa$ . Thus,  $\cup(x) \in H_\kappa$ . By absoluteness,  $H_\kappa$  satisfies the union axiom. Since  $\kappa > \omega$ ,  $\omega \in H_\kappa$  as we noted above. By absoluteness,  $H_\kappa$  satisfies the infinity axiom. Since  $H_\kappa$  is transitive, it satisfies extensionality (if  $x \neq y$  are in  $H_\kappa$ , then there is a  $z \in x$ , say, with  $z \notin y$ . By transitivity,  $z \in H_\kappa$ ).

To show comprehension, let  $\phi(x_1, \dots, x_n, y, z)$  be a formula, and let  $a_1, \dots, a_n, w \in H_\kappa$ . We must show that  $\exists v \in H_\kappa \forall y \in H_\kappa (y \in v \leftrightarrow (y \in w \wedge \phi^{H_\kappa}(\vec{a}, y, w)))$ . By comprehension in  $V$  applied to the formula  $\phi^{H_\kappa}(\vec{a}, y, z)$ , there is a  $v \in V$  such that  $\forall y (y \in v \leftrightarrow (y \in w \wedge \phi^{H_\kappa}(\vec{a}, y, w)))$ . In particular,  $\forall y \in H_\kappa (y \in v \leftrightarrow (y \in w \wedge \phi^{H_\kappa}(\vec{a}, y, w)))$ . Thus, it suffices to observe that  $v \in H_\kappa$ , which follows as  $v \subseteq w \in H_\kappa$  (any subset of an element of  $H_\kappa$  is in  $H_\kappa$ ).

To show replacement, let  $\phi(x_1, \dots, x_n, y, z, w)$  be a formula,  $a_1, \dots, a_n, A \in H_\kappa$ , and assume  $(\forall y \in A \exists z \phi(\vec{a}, y, z, A))^{H_\kappa}$ . Thus,  $\forall y \in A \exists z \in H_\kappa \phi^{H_\kappa}(\vec{a}, y, z, A)$ . Applying replacement in  $V$  to the formula  $[\phi^{H_\kappa} \wedge z \in H_\kappa]$ , there is a set  $B$  such that  $\forall y \in A \exists z \in B \cap H_\kappa \phi^{H_\kappa}(\vec{a}, y, z, A)$ . By AC, there is a  $B' \subseteq B$  with  $|B'| \leq |A|$  such that  $\forall y \in A \exists z \in B' \cap H_\kappa \phi^{H_\kappa}(\vec{a}, y, z, A)$ . Since  $B' \subseteq H_\kappa$  and  $|B'| < \kappa$ , we have  $B' \in H_\kappa$ . Thus,  $(\exists B \forall y \in A \exists z \in B \phi(\vec{a}, y, z, A))^{H_\kappa}$ .

Finally, to see AC holds in  $H_\kappa$ , let  $a \in H_\kappa$ . By AC in  $V$ , let  $<$  be a wellordering of  $a$ . Since  $< \subseteq a \times a$ ,  $|<| < \kappa$ . Easily  $< \subseteq H_\kappa$ , as every ordered pair  $\langle x, y \rangle$  is in  $H_\kappa$  if  $x, y \in H_\kappa$ . So,  $< \subseteq H_\kappa$ , and hence  $< \in H_\kappa$ . By downward absoluteness,  $(< \text{ is a wellordering})^{H_\kappa}$ .  $\square$

**Corollary 0.7.**  $ZFC\text{-}Power \not\vdash Power$ .

*Proof.* Similar to the proof of corollary 0.2, using now the fact that  $ZFC \vdash (\neg Power)^{H_{\omega_1}}$ .  $\square$

Lemmas 0.1, 0.6 are stated and proved in the metatheory, however we may formalize their statements and proofs within ZFC. Lemma 0.6, for example, then becomes the statement  $ZF \vdash \forall \alpha \forall n ((\alpha > \omega \text{ limit} \wedge \theta(n) \rightarrow \rho(n, V_\alpha))$  where  $\theta$  is the formula which represents the relation  $R(n) \leftrightarrow n$  codes an axiom of ZF–Replacement. [The only thing slightly problematic in the formalizations is where we consider the relativizations  $\phi^{V_\kappa}$ ,  $\phi^{H_\kappa}$ . But lemma ?? gives us a formula  $\rho$  which expresses the formalizations of these notions, e.g., the formalization of  $(\theta_n)^{V_\kappa}$  is the statement  $\rho(n, V_\kappa)$ .]

Here  $\rho$  is the set satisfaction formula of lemma ?. Likewise, lemma 0.6 when formalized becomes the statement  $ZFC \vdash \forall \kappa \forall n ((\kappa \text{ an uncountable regular cardinal}) \wedge \theta'(n) \rightarrow \rho(n, H_\kappa))$ , where now  $\theta'$  represents  $R'(n) \leftrightarrow n$  codes an axiom of ZF–Power.

The completeness theorem of first order logic may also be formalized within ZFC (or ZF for countable theories, which we consider here). It then becomes the statement that  $ZF \vdash \forall T \subseteq \omega (CON(T) \leftrightarrow \exists M \forall n \in T \rho(n, M))$ . Using this, we may reformulate the formalized lemmas 0.1, 0.6 as consistency results as follows.

**Corollary 0.8.**  $ZFC \vdash CON(ZFC - Replacement)$ .

**Corollary 0.9.**  $ZFC \vdash CON(ZFC - Power)$ .

From Gödel's theorem, we know that  $ZFC \not\vdash CON(ZFC)$ , but these corollaries show that if we weaken ZFC a bit, we can prove its formal consistency within ZFC.

We showed that  $H_\kappa \subseteq V_\kappa$  for all infinite regular  $\kappa$ . When does equality hold? Now  $V_\kappa \subseteq H_\kappa$  iff  $\forall \alpha < \kappa (V_\alpha \in H_\kappa)$  (since if  $V_\alpha \in H_\kappa$  then any subset of  $V_\alpha$  is in  $H_\kappa$ ). Since each  $V_\alpha$  is transitive, this is then equivalent to  $\forall \alpha < \kappa (|V_\alpha| < \kappa)$ . By definition of the  $\beth$  function,  $|V_{\omega+\alpha}| = \beth(\alpha)$ . So,  $V_\kappa = H_\kappa$  iff  $\forall \alpha < \kappa \beth(\alpha) < \kappa$ . Now  $\beth(\alpha) \geq \alpha$ , so  $\beth(\alpha+1) \geq 2^{|\alpha|}$  for all  $\alpha$ . Thus, for  $V_\kappa$  to equal  $H_\kappa$  we must have  $\forall \lambda < \kappa 2^\lambda < \kappa$ . Since  $\kappa$  is assumed regular, we must then have that  $\kappa$  is strongly inaccessible. Conversely, if  $\kappa$  is strongly inaccessible, the following exercise shows this condition is satisfied.

**Exercise 3.** Show that if  $\kappa$  is strongly inaccessible then  $\forall \alpha < \kappa (\beth(\alpha) < \kappa)$ .

We thus have:

*Fact 0.10.* For all infinite regular cardinals  $\kappa$ ,  $H_\kappa = V_\kappa$  iff  $\kappa$  is strongly inaccessible.

**Corollary 0.11.**  $ZFC \not\vdash \exists \kappa (\kappa \text{ is strongly inaccessible})$ .

*Proof.* If  $ZFC \vdash \exists \kappa$  ( $\kappa$  is strongly inaccessible), then from the formalized lemmas 0.6, 0.1 we would have  $ZFC \vdash \exists \kappa \forall n (\theta(n) \rightarrow \rho(n, H_\kappa))$  where  $\theta$  represents  $R(n) \leftrightarrow \theta_n \in ZFC$ , and  $\rho$  again is the set satisfaction formula. By the formalized completeness theorem, this would give  $ZFC \vdash CON(ZFC)$ , a contradiction to Gödel's theorem.  $\square$

Note that the proof of corollary 0.11 actually showed the following.

**Corollary 0.12.**  $ZFC + \exists \kappa$  ( $\kappa$  is strongly inaccessible)  $\vdash CON(ZFC)$ .

We sometimes state corollary 0.12 by saying that  $ZFC + \exists \kappa$  ( $\kappa$  is strongly inaccessible) has a greater consistency strength than  $ZFC$ .

**Exercise 4.** Show corollary 0.11 without using Gödel's theorem. [Hint: Suppose  $ZFC \vdash \exists \kappa$  ( $\kappa$  is strongly inaccessible), and let  $\psi_1, \dots, \psi_n$  be a finite fragment of  $ZFC$  such that  $\{\psi_1, \dots, \psi_n\} \vdash \exists \kappa$  ( $\kappa$  is strongly inaccessible). Working in  $ZFC$ , let  $\kappa$  be the least strongly inaccessible cardinal. Use lemmas 0.6, 0.1 and an absoluteness argument to show that there is a strongly inaccessible  $\lambda < \kappa$ .]

$V_\kappa$  and  $H_\kappa$  are sets which we showed satisfy certain fragments of  $ZFC$ . We now consider a certain class, HOD, which we show (assuming  $ZFC$ ) satisfies all of  $ZFC$ . In fact, HOD will be a transitive class containing all of the ordinals, a so-called *inner model* of  $ZFC$ .

We would like to say a set  $x$  is *ordinal-definable* if there is a formula  $\phi(x_1, \dots, x_n, y)$  and ordinals  $\alpha_1, \dots, \alpha_n$  such that  $x$  is the unique set  $x$  such that  $\phi(\alpha_1, \dots, \alpha_n, y)$ . We would then define HOD to be the sets which are hereditarily ordinal definable. The problem with this is that this definition, at least the way it is stated, is not a legitimate formula of set theory, i.e., doesn't really seem to define a class. The problem, as we mentioned before, is that we cannot define formal satisfaction over classes, only over sets. However, the reflection theorem provides a way to circumvent this logical problem. For any formula  $\phi$  and ordinals  $\alpha_1, \dots, \alpha_n$ , the reflection theorem says that there is a  $\beta > \max\{\alpha_1, \dots, \alpha_n\}$  such that  $\forall y \in V_\beta (\phi(\vec{\alpha}, y)^{V_\beta} \leftrightarrow \phi(\vec{\alpha}, y))$ . Thus, there should be no loss of generality in interpreting our formulas in the  $V_\beta$ . This suggests the following definition.

**Definition 0.13.** OD is the class defined by:

$$x \in OD \leftrightarrow \exists \beta \in ON \exists \alpha_1, \dots, \alpha_n < \beta \exists n \in \omega \forall y \in V_\beta [(y \approx x) \leftrightarrow \rho(n, \langle \vec{\alpha}, y \rangle, V_\beta)]$$

where  $\rho$  is the formula for set satisfaction from lemma ???. Also,

$$x \in HOD \leftrightarrow \forall y \in \text{tr cl}(x) (y \in OD).$$

Note that the quantification over  $\alpha_1, \dots, \alpha_n$  should really be expressed as a quantification over sets  $z = \langle \alpha_1, \dots, \alpha_n \rangle$  coding the sequence.

Our remarks above concerning the reflection theorem now easily give the following.

**Lemma 0.14.** Let  $\phi(x_1, \dots, x_n, y)$  be a formula of set theory. Then

$$ZF \vdash \forall \alpha_1, \dots, \alpha_n \in ON \forall x [\forall y (y \approx x \leftrightarrow \phi(\vec{\alpha}, y)) \rightarrow y \in OD].$$

By definition HOD is transitive, and it trivially contains all of the ordinals. The next theorem shows that working in  $ZF$  we can show that HOD satisfies  $ZFC$ . We will need the following ordering on the finite sequences of ordinals.

**Definition 0.15.** The Gödel ordering  $<_G$  on  $\text{ON}^{<\omega}$  is defined by:

$$\begin{aligned} \langle \alpha_1, \dots, \alpha_n \rangle <_G \langle \beta_1, \dots, \beta_m \rangle &\leftrightarrow \max\{\alpha_1, \dots, \alpha_n\} < \max\{\beta_1, \dots, \beta_m\} \\ &\vee (\max\{\alpha_1, \dots, \alpha_n\} = \max\{\beta_1, \dots, \beta_m\} \wedge n < m) \\ &\vee (\max\{\alpha_1, \dots, \alpha_n\} = \max\{\beta_1, \dots, \beta_m\} \wedge n = m \wedge \vec{\alpha} <_{\text{lex}} \vec{\beta}), \end{aligned}$$

where  $<_{\text{lex}}$  denotes lexicographic order.

The following lemma is clear from the definition.

**Lemma 0.16.** *The Gödel ordering is a class wellordering of  $\text{ON}^{<\omega}$ , all of whose initial segments are sets.*

**Theorem 0.17.** *For all axioms  $\phi$  of ZFC,  $\text{ZF} \vdash \phi^{\text{HOD}}$ .*

*Proof.* Foundation holds in HOD, as it holds in any class with the  $\epsilon$  relation. Extensionality holds as HOD is transitive. Consider pairing. Let  $x, y \in \text{HOD}$ . Fix  $n \in \omega$ ,  $\alpha_1, \dots, \alpha_n \in \text{ON}$ , and  $\beta_1 \in \text{ON}$  such that  $\forall z \in V_{\beta_1} [(z \approx x) \leftrightarrow \rho(n, \langle \vec{\alpha}, y \rangle, V_{\beta_1})]$ , and similarly let  $m, \gamma_1, \dots, \gamma_k, \beta_2$  witness  $y \in \text{HOD}$ . Then,

$$z \in \{x, y\} \leftrightarrow \exists w [w = V_{\beta_1} \wedge \rho(n, \langle \vec{\alpha}, z \rangle, w)] \vee \exists w [w = V_{\beta_2} \wedge \rho(m, \langle \vec{\gamma}, z \rangle, w)]$$

The righthand is a formula in set theory with ordinal parameters  $n, m, \vec{\alpha}, \vec{\gamma}, \beta_1, \beta_2$ . Thus,  $\{x, y\} \in \text{OD}$ . Since  $\{x, y\} \subseteq \text{HOD}$ , we have  $\{x, y\} \in \text{HOD}$ .

Infinity holds in HOD by absoluteness, since  $\omega \in \text{HOD}$  (HOD contains all the ordinals).

Consider the union axiom. Let  $x \in \text{HOD}$ , and fix witnesses  $n, \vec{\alpha}, \beta$ . Then

$$z \in \cup x \leftrightarrow \exists w \exists y [w = V_{\beta} \wedge y \in w \wedge \rho(n, \langle \vec{\alpha}, y \rangle, w) \wedge \exists u \in y (z \in u)].$$

Again, the righthand side is a formula with ordinal parameters  $n, \vec{\alpha}, \beta$ , and so  $\cup x \in \text{OD}$ . Since  $\cup x \subseteq \text{tr cl}(x) \subseteq \text{HOD}$ , this shows  $\cup x \in \text{HOD}$ . By absoluteness, HOD satisfies the union axiom.

To show comprehension, let  $\phi(x_1, \dots, x_n, y, z)$  be a formula, and  $a_1, \dots, a_n, A \in \text{HOD}$ . We must show that  $B \in \text{HOD}$ , where

$$B = \{z \in A : \phi^{\text{HOD}}(a_1, \dots, a_n, A, z)\}.$$

Note that  $B$  exists in  $V$  by comprehension in  $V$ . Then,

$$\begin{aligned} x = B &\leftrightarrow \forall z [z \in B \leftrightarrow \exists u_1, \dots, u_n \exists v (u_1 = a_1 \wedge \dots \wedge u_n = a_n \wedge v = A \\ &\wedge \phi^{\text{HOD}}(u_1, \dots, u_n, v, z))], \end{aligned}$$

where in place of “ $u_1 = a_1$ ” we use a formula with only ordinal parameters (the witnesses for  $a_1 \in \text{HOD}$ ) and likewise for the other  $a_i$  and  $A$ . This shows  $B \in \text{OD}$ , and since  $B \subseteq \text{HOD}$  we have  $B \in \text{HOD}$ . Clearly,  $(\forall z (z \in B \leftrightarrow z \in A \wedge \phi^{\text{HOD}}(a_1, \dots, a_n, A, z)))^{\text{HOD}}$ , and so HOD satisfies comprehension.

To show power set, let  $x \in \text{HOD}$ . Let  $\alpha \in \text{ON}$  be large enough so that  $\mathcal{P}(x) \in V_{\alpha}$ . Note that  $V_{\alpha} \in \text{OD}$  (it is definable from  $\alpha$ ). Since HOD is a class,  $y = V_{\alpha} \cap \text{HOD} \in \text{OD}$  as well, and so  $y \in \text{HOD}$ . Clearly  $(\forall z (z \subseteq x \rightarrow z \in y))^{\text{HOD}}$ , so HOD satisfies power set.

To show replacement, let  $\phi(x_1, \dots, x_n, y, z, w)$  be a formula,  $a_1, \dots, a_n, A \in \text{HOD}$ , and assume  $(\forall y \in A \exists z \phi^{\text{HOD}}(\vec{a}, y, z, A))^{\text{HOD}}$ . Let  $\alpha \in \text{ON}$  be such that  $\forall y \in A \exists z \in V_{\alpha} \phi^{\text{HOD}}(\vec{a}, y, z, A)^{\text{HOD}}$ . As  $V_{\alpha} \in \text{OD}$ ,  $B = V_{\alpha} \cap \text{HOD} \in \text{HOD}$ . Then  $(\forall y \in A \exists z \in B \phi^{\text{HOD}}(\vec{a}, y, z, A))^{\text{HOD}}$ .

Finally, to show AC holds in HOD, let  $x \in \text{HOD}$ . Every  $y \in x$  is in HOD and has a witness sequence  $n, \vec{\alpha}, \beta$ . To compare two elements  $y, z$  of  $x$ , we take the  $<_G$  least witnesses for  $y$  and  $z$ , and compare them in the  $<_G$  ordering. In more detail, define a wellordering of  $x$  by:

$$\begin{aligned}
y < z &\leftrightarrow y \in x \wedge z \in x \wedge \exists n \in \omega \exists \langle \vec{\alpha} \rangle \in \text{ON}^{<\omega} \exists \beta_1 \in \text{ON} \\
&\quad \exists m \in \omega \exists \langle \vec{\gamma} \rangle \in \text{ON}^{<\omega} \exists \beta_2 \in \text{ON} \\
&\quad [\forall u (u \approx y \leftrightarrow \exists w (w \approx V_{\beta_1} \wedge \rho(n, \langle \vec{\alpha}, u \rangle, w))) \\
&\quad \wedge \forall \langle n', \vec{\alpha}', \beta'_1 \rangle <_G \langle n, \vec{\alpha}, \beta_1 \rangle \neg \forall u (u \approx y \leftrightarrow \exists w (w \approx V_{\beta'_1} \wedge \rho(n', \langle \vec{\alpha}', u \rangle, w)))] \\
&\quad \wedge [\forall u (u \approx z \leftrightarrow \exists w (w \approx V_{\beta_2} \wedge \rho(m, \langle \vec{\gamma}, u \rangle, w))) \\
&\quad \wedge \forall \langle m', \vec{\gamma}', \beta'_2 \rangle <_G \langle m, \vec{\gamma}, \beta_2 \rangle \neg \forall u (u \approx z \leftrightarrow \exists w (w \approx V_{\beta'_2} \wedge \rho(m', \langle \vec{\gamma}', u \rangle, w)))] \\
&\quad \wedge \langle n, \vec{\alpha}, \beta_1 \rangle <_G \langle m, \vec{\gamma}, \beta_2 \rangle
\end{aligned}$$

Here for “ $y \in x$ ” we substitute an appropriate formula with only ordinal parameters, and likewise for  $z \in x$ .

Easily  $<$  is a wellordering, and the above formula shows  $< \in \text{OD}$ . Since  $< \subseteq \text{HOD}$ , we have  $< \in \text{HOD}$ , and by downwards absoluteness ( $<$  is a wellordering of  $x$ )<sup>HOD</sup>.  $\square$

**Corollary 0.18.** *If ZF is consistent, then so is ZFC.*

*Proof.* Suppose ZF is consistent, but ZFC is not. Then  $\text{ZF} \vdash \neg \text{AC}$ . Let  $\psi_1, \dots, \psi_n$  be finitely many axioms of ZF such that  $\{\psi_1, \dots, \psi_n\} \vdash \neg \text{AC}$ . From theorem 0.17,  $\text{ZF} \vdash \psi_1^{\text{HOD}} \wedge \dots \wedge \psi_n^{\text{HOD}}$ . Hence,  $\text{ZF} \vdash (\neg \text{AC})^{\text{HOD}}$ . But,  $\text{ZF} \vdash \text{AC}^{\text{HOD}}$  from theorem 0.17, a contraction to the consistency of ZF.  $\square$

Thus, the axiom of choice cannot introduce a contradiction into mathematics, unless one was already present. This is a significant statement in view of the many “pathological” sets that may be constructed with AC. For example, with AC a standard construction gives a non-measurable set of reals, and it can be shown that AC is necessary for the construction (more on this later).

Finally, we note that although corollary 0.18 was proved in the meta-theory, the argument is readily formalized, which gives the following version of the corollary.

**Corollary 0.19.**  $\text{CON}(\text{ZF}) \vdash \text{CON}(\text{ZFC})$ .

Thus, the theories ZF and ZFC have the same consistency strength.