Introduction to Forcing

The forcing method ia a technique for taking a model of ZF (usually ZFC) and enlarging it by adding a new object G, called a generic, to produce a bigger universe $V[G]$. The method gives us enough control over the model $V[G]$ that we can arrange that various statements of interest will be true in $V[G]$, if we choose the G in the right way.

Before discussing the method itself, we first discuss a logical problem. Since V is, by definition, the class of all sets, we cannot enlarge it at all. What exactly do we mean then? This problem is actually not very serious, and can be dealt with in several different ways. We mention one way now, and touch on the others later. First, if we are willing to assume a tiny bit more than ZFC, namely ZFC + CON(ZFC), then the problem disappears altogether. For the theory ZFC + CON(ZFC) proves that there is a set model M of ZFC. Since M is a set, there is now no reason why we cannot enlarge it to a bigger model $M[G]$. Since M is a model of ZFC, it is just as good a starting point as V. We could even (assuming AC in V) take M to be countable by theorem **??**.

If we are not willing to strengthen ZFC, we can still proceed as follows. We would like to argue that there is a model of ZFC + ϕ, say. If not, then ZFC $\vdash \neg\phi$, and so for finitely many of the axioms of ZFC, say, $\{\psi_1, \ldots, \psi_n\}$ we would have $\{\psi_1, \ldots, \psi_n\} \vdash \neg\phi$. However by the reflection theorem and theorem **??** there is a countable set M which satisfies $\{\psi_1, \ldots, \psi_n\}$. We may also assume M satisfies as many additional (but finitely many) axioms of ZFC as we wish, so that any of the arguments needed for the forcing construction are also true in M. Then we may do the forcining construction starting from M to produce $M[G]$, and we will have $\psi_1, \ldots, \psi_n$ and ϕ all holding in $M[G]$ (the latter by the forcing construction). This will contradict the fact that $\{\psi_1, \ldots, \psi_n\} \vdash \neg\phi$.

The argument of the preceeding paragragh took place in the meta-theory, but it will be easily formalizable to give the theorem CON(ZFC) $\vdash$ CON(ZFC + ϕ).

In what follows, we usually suppress mentioning these logical points, and just assume M is a countable transitive model of ZF or ZFC. We now turn to the forcing method itself.

The basic setup for the forcing method will be a transitive model M of (enough of) ZF, and a partial order $\mathbb{P} = \langle P, \leqslant \rangle$ in M. Recall a partial order means a reflexive, transitive, binary relation on a set $\mathbb{P}$. It is sometimes convenient, though not necessary, to assume the partial order has a largest element, which we denote by $\mathbb{1}$ (we can always adjoin a new maximal element to the partial order, and none of the arguments below will be effected).

If P is a partial order and $p, q \in P$, we say p and q are *compatible*, written $p \parallel q$, if there is an $r \in P$ such that $r \leqslant p$ and $r \leqslant q$. Otherwise they are incompatible, written $p \perp q$. If $A \subset P$ consists of pairwise incompatible elements, we say A is an *antichain*. We say $D \subseteq P$ is *dense* if $\forall p \in P\ \exists q \leqslant p\ (q \in D)$. We say $D \subseteq P$ is *predense* if $\forall p \in D\ \exists q \in D\ (q \parallel p)$. This is equivalent to saying the downward closure $\{p \in P \colon \exists q \in D\ p \leqslant q\}$ of D is dense.

Definition 0.1. A filter G on a partial order $\mathbb{P}$ is a $G \subseteq P$ satisfying

(1) $\forall p, q \in G\ (p \parallel q)$.
(2) $\forall p \in G\ \forall q \geqslant p\ (q \in G)$.

Example 1. Let $P = 2^{<\omega}$, and define $p \leqslant q$ iff p extends q. Then any filter G on P defines a $f_G\colon 2^\alpha \to \{0, 1\}$ for some $\alpha \leqslant \omega$. Conversely, any "real" $x \in 2^\omega$ defines a filter over P. We could also consider the variation where P consists of finite partial functions from ω to 2. Again, $p \leqslant q$ iff p extends q. A filter gives a partial function from ω to 2, and any $x \in 2^\omega$ again defines a filter on P.

The next definition is a central one in the theory of forcing.

Definition 0.2. Let M be a model of ZF, and $\mathbb{P} \in M$ a partial order. We say a filter G over $\mathbb{P}$ is M-generic is $G \cap D \neq \emptyset$ for all dense $D \subseteq P$ with $D \in M$.

Exercise 1. Show that G is M-generic iff G meets all of the predense sets $D \subseteq P$ in M iff G meets all of the maximal antichains $A \subseteq P$ in M.

Exercise 2. Show that if A is dense below p, then any generic containing p will meet A.

A generic G is required to meet all of the dese sets D which lie in M. G itself will, in all non-triial cases, not be in M itself. By non-trivial we can take the following.

Definition 0.3. $\mathbb{P}$ is *splitting* if $\forall p \in P\ \exists q \leqslant p\ \exists r \leqslant p\ (q \perp r)$.

Lemma 0.4. *If M is a model of ZF, $\mathbb{P} \in M$, $\mathbb{P}$ is separative, and G is M-generic for $\mathbb{P}$, then $G \notin M$.*

Proof. Suppose $G \in M$. Consider $D = \{p \in P\colon \forall q \in G\ (p \perp q)\}$. To see D is dense, fix $p \in P$. Let $q \leqslant p$, $r \leqslant p$ with $q \perp r$. Then at least one of q, r must be in D as otherwise we could extend q, r to q', r' respectively with $q', r' \in G$. Since G is a filter, we could get an $s \leqslant q'$, $s \leqslant r'$, contradicting $q \perp r$. So, D is dense, and by assumption then $G \cap D \neq \emptyset$. This violates G being a filter. □

Can we find generics? The next lemma shows that we can always do this in the case where M is countable.

Lemma 0.5. *Let M be a countable model of ZF, and $\mathbb{P} \in M$. Then there is a M-generic filter G for P.*

Proof. Let $D_0, D_1, \dots$ enumerate the dense subsets of P which lie in M. Using the denseness of the D_n, define a sequence $p_0 \geqslant p_1 \geqslant p_2 \geqslant \dots$ where $p_n \in D_n$. Let $G = \{p \in P\colon \exists n\ (p \geqslant p_n)\}$. G is easily a filter and by constuction meets all of the D_n. □

We will see the significance of generics shortly. First we describe the constuction of the model $M[G]$ for G a filter on the partial order $\mathbb{P} \in M$ (we don't need G to be generic for this construction). We would like $M[G]$ to be the smallest model containing M and G, in some sense. Every element x of $M[G]$ will have a *name* τ in M. The filter G will define an evaluation map $\tau \to \tau_G$ from the names to the elements of $M[G]$, so every $x \in M[G]$ will be of the form $x = \tau_G$. The names will live in M (i.e., be a class in M), but since $G \notin M$, the evaluation map will not be deininable in M. First we define the class $M^{\mathbb{P}}$ of names.

Definition 0.6. Let M be a model of ZF, and $\mathbb{P} \in M$ a partial order. By transfinite recursion on ON^M we define $M^{\mathbb{P}}_\alpha$ as follows: $M^{\mathbb{P}}_0 = \emptyset$. For α limit, $M^{\mathbb{P}}_\alpha = \bigcup_{\beta<\alpha} M^{\mathbb{P}}_\beta$. We define $\tau \in M_{\alpha+1}$ iff $\tau \in M_\alpha$ or τ is a relation with $\mathrm{dom}(\tau) \subseteq M_\alpha$ and $\mathrm{ran}(\tau) \subseteq P$. We let $M^{\mathbb{P}} = \bigcup_{\alpha \in \mathrm{ON}^M} M^{\mathbb{P}}_\alpha$.

Thus a name τ is a set of ordered pairs $\langle\sigma, p\rangle$ where σ is a name of smaller rank and $p \in P$. Note that the construction of the names is exactly like the construction of the universe of sets itself (i.e., the rank hierarchy) except that the elements of a set are "tagged" with elements of the partial order P. Intuitively, the tag p tells us to throw the corresponding element into the set if $p \in G$. We define the evaluation map as follows.

Definition 0.7. Let M be a model of ZF, and $\mathbb{P} \in M$ a partial order. Let $G \subseteq P$ be a filter. for $\tau \in M^{\mathbb{P}}$, τ_G is defined by transfinite recursion on ϵ by: $\tau_G = \{\sigma_G \colon \exists p \in G \colon \langle\sigma, p\rangle \in \tau\}$.

We define some canonical names which always evaluate to certain objects.

Definition 0.8. Let M be a model of ZF, and $\mathbb{P} \in M$ a partial order. By transfinite recursion in M, define for each $x \in M$ the name $\check{x}$ by: $\check{x} = \{\langle\check{y}, p\rangle \colon y \in x \wedge p \in P\}$. Define the name $\dot{G}$ by $\dot{G} = \{\langle\check{p}, p\rangle \colon p \in P\}$.

For any wellfounded model M of ZF, it is a straightforward induction to show that for all $x \in M$ that $\check{x}_G = x$. It follows that $(\dot{G})_G = G$. Thus, we have canonical names for all of the sets in M as well as the filter G. Note that the function $x \mapsto \check{x}$ is a class function in M. We thus have:

Lemma 0.9. *For any wellfounded model M of ZF, partial order $\mathbb{P} \in M$ and filter G on $\mathbb{P}$, $M \subseteq M[G]$ and $G \in M[G]$.*

We would like to say that $M[G]$ is the smallest model of ZF containing M and G, but we don't know that $M[G]$ is a model of ZF. This will not, in fact, be true for all G, but will be true for the generic G. The forcing theorem will give us the tools to show this, as well as relating truth in $M[G]$ to truth in M in general. First we prove a few more simple facts about $M[G]$.

Lemma 0.10. *$M[G]$ is transitive and $ON^{M[G]} = ON^M$.*

Proof. If $x \in M[G]$, then $x = \tau_G$ for some $\tau \in M^{\mathbb{P}}$. If $y \in x$, then by definition of τ_G we have $y = \sigma_G$ for some $\sigma \in M^{\mathbb{P}}$ (in fact, $\sigma \in \operatorname{tr}\operatorname{cl}(x)$). So, $y \in M[G]$. Since M, $M[G]$ are transitive, $\mathrm{ON}^M = \mathrm{ON} \cap M$ and $\mathrm{ON}^{M[G]} = \mathrm{ON} \cap M[G]$. So, to show $\mathrm{ON}^M = \mathrm{ON}^{M[G]}$ it suffices to show that every $x \in M[G]$ has rank less than ON^M. By a straightforward induction on ON^M we have that if $\tau \in M^{\mathbb{P}}_\alpha$, then $|\tau_G| \leqslant \alpha$. □

Since $M[G]$ is transitive, it satisfies foundation and extensionality. One can also show directly that it satisfies pairing and union. For example, if τ_G, $\sigma_G \in M[G]$, then $\rho = \{\langle\tau, \mathbb{1}\rangle, \langle\sigma, \mathbb{1}\rangle\}$ is such that $\rho_G = \{\tau_G, \sigma_G\}$. Note that these fact do not require G to be generic, they are true for any filter G on $\mathbb{P}$. To show the other axioms, and to further explore the model $M[G]$ however requires the forcing theorem which relates truth in $M[G]$ to truth in M for generic G.

Definition 0.11. By the *forcing language* we mean all statements of the form $\phi(\sigma_1, \ldots, \sigma_n)$ where $\phi(x_1, \ldots, x_n)$ is a formula in the language of set theory and $\sigma_1, \ldots, \sigma_n \in M^{\mathbb{P}}$.

We now define the forcing relation, which is the central concept in the theory of forcing. For $p \in \mathbb{P}$ and $\phi(\sigma_1, \ldots, \sigma_n)$ in the forcing language, we define the notion $p \Vdash \phi(\sigma_1, \ldots, \sigma_n)$ (read p forces $\phi(\sigma_1, \ldots, \sigma_n)$). Formally, this is defined by induction on ϕ, where for atomic ϕ it is defined by transfinite recurion in M

on the maximum of the ranks of the σ_i. This definition, it is important to note, takes place entirely inside of M (more precisely, for each formula ϕ there is a class function in M from $(M^{\mathbb{P}})^n$ to $\mathbb{P}$ assigning to each $(\sigma_1, \dots, \sigma_n) \in (M^{\mathbb{P}})^n$ the set $\{p \in P : p \vdash \phi(\sigma_1, \dots, \sigma_n)\}$).

Definition 0.12. Let M be a transitive model of ZF and $\mathbb{P}$ a partial order in M. For $p \in P$ and $\phi(\sigma_1, \dots, \sigma_n)$ in the forcing language, the relation $p \vdash \phi(\sigma_1, \dots, \sigma_n)$ is defined by transfinite recursion through the following cases.

Case 1) $\phi = (\tau_1 \in \tau_2)$.

We define $p \vdash \tau_1 \in \tau_2$ iff $\{q : \exists \langle \sigma, r\rangle \in \tau_2 \ (q \leqslant r \wedge q \vdash (\tau_1 \approx \sigma))\}$ is dense below p.

Case 2) $\phi = (\tau_1 = \tau_2)$.

We define $p \vdash \tau_1 \approx \tau_2$ iff for all $\langle \sigma_1, q\rangle \in \tau_1$, the set

$$\{r : (r \leqslant q \to \exists \langle \sigma_2, s\rangle \in \tau_2 \ r \leqslant s \wedge r \vdash (\sigma_1 \approx \sigma_2))\}$$

is dense below p, and likewise for all $\langle \sigma_2, q\rangle \in \tau_2$ the set

$$\{r : (r \leqslant q \to \exists \langle \sigma_1, s\rangle \in \tau_1 \ r \leqslant s \wedge r \vdash (\sigma_1 \approx \sigma_2))\}$$

is dense below p.

Case 3) $\phi = \alpha \wedge \beta$.

We define $p \vdash \alpha \wedge \beta$ iff $p \vdash \alpha$ and $p \vdash \beta$.

Case 4) $\phi = \neg\psi$.

We define $p \vdash \neg\psi$ iff $\neg\exists q \leqslant p \ q \vdash \psi$.

Case 5) $\phi(\sigma_1, \dots, \sigma_n) = \exists x \psi(\sigma_1, \dots, \sigma_n, x)$.

We define $p \vdash \exists x \psi(\sigma_1, \dots, \sigma_n, x)$ iff $\{q : \exists \tau \in M^{\mathbb{P}} \ q \vdash \psi(\sigma_1, \dots, \sigma_n, \tau)\}$ is dense below p.

The motivation for the definition is our desire to control truth in $M[G]$. We would like to have that $p \vdash \phi(\sigma_1, \dots, \sigma_n)$ iff for every generic G containing p we have $M[G] \models \phi((\sigma_1)_G, \dots, (\sigma_n)_G)$. In other words, for every statement in the forcing language, the truth of the statement, view as a function on the set of generics, is continuous. The reader will note the analogy with a basic theorem in analysis. Namely, every reasonable (say, Borel) function $f : \mathbb{R} \to \{0, 1\}$ (or even $f : \mathbb{R} \to \mathbb{R}$) is continuous when resticted to a certain comeager set. The generics serve the role as the comeager set, for all truth functions for all statements in the forcing language simultaneously.

The motivation for the definitions in the various specific cases should now be fairly clear if we keep in mind the definition of a generic. For example, in Case 1 above, we are really asserting that any generic G which contains p will contain a $q \leqslant p$ such that for some $\langle \sigma, r\rangle \in \tau_2$ with $q \leqslant r$ (hence $\sigma_G \in (\tau_2)_G$) we have $q \vdash \tau_1 = \sigma$ (which by induction should mean that $(\tau_1)_G = \sigma_G$). We have used here exercise 2.

Lemma 0.13. *If $p \vdash \phi$ and $q \leqslant p$, then $q \vdash \phi$. Conversely, if $\{q : q \vdash \phi\}$ is dense below p, then $p \vdash \phi$.*

Proof. Suppose $p \vdash \phi$ and $q \leqslant p$. If ϕ is atomic or existential, the definition of $p \vdash \phi$ is that a certain set A is dense below p, and the definition of A does not involve p. Trivially, A is dense below q as well, so $q \vdash \phi$. The conjuction case follows immediately by induction. The negation case is trivial as if $\neg\exists s \leqslant p(s \vdash \psi)$ then $\neg\exists s \leqslant q(s \vdash \psi)$.

Suppose next that $\{q\colon q \vdash \phi\}$ is dense below p. The atomic and existential cases follow from the fact that if $\{q\colon A$ is dense below $q\}$ is dense below p, then A is dense below p. The conjuction case again follows immediately by induction. For the negation case $\phi = \neg\psi$, our assumption is that $\{q\colon \neg\exists r \leqslant q\ (r \vdash \psi)\}$ is dense below p. Now, if $q \leqslant p$, then $q \not\vdash \psi$, as otherwise from the first paragraph we would have that any extension of q forces ψ, contradicting our assumption. Hence, by definition $p \vdash \neg\psi$. □

Corollary 0.14. *If $p \not\vdash \phi$, then for some $q \leqslant p$, $q \vdash \neg\phi$. In particular, for any ϕ, $\{p\colon (p \vdash \phi) \vee (p \vdash \neg\phi)\}$ is dense.*

Proof. If the conclusion fails, then $\{r\colon r \vdash \phi\}$ is dense below p. From lemma 0.13 it follows that $p \vdash \phi$. □

Exercise 3. Show that $p \vdash \neg\neg\phi$ iff $p \vdash \phi$. (hint: unravel the definition of $p \vdash \neg\neg\phi$ and use the second part of lemma 0.13).

Lemma 0.15. *If G is a generic filter and $p, q \in G$, then $\exists r \in G\ (r \leqslant p \wedge r \leqslant q)$.*

Proof. Let $D = \{r\colon (r \perp p) \vee (r \perp q) \vee (r \leqslant p \wedge r \leqslant q)\}$. Easily D is dense (if $t \in P$ then either $t \perp p$ or there is a $u \leqslant t$ with $u \leqslant p$. Either $u \perp q$ or there is a $v \leqslant u$ with $v \leqslant q$, and also $v \leqslant u \leqslant p$). Let $r \in G \cap D$. Since G is a filter, we cannot have $r \perp p$ or $r \perp q$, so $r \leqslant p$ and $r \leqslant q$. □

We now state precisely and prove the forcing theorem. For the theorem to make sense, we need to know that generic filters exist, and so we will now require the model M to be countable.

Theorem 0.16 (Forcing Theorem)**.** *Let M be a countable, transitive model of ZF and $\mathbb{P} \in M$ a partial order.*

1. *For any $p \in P$ and $\sigma_1, \dots, \sigma_n \in M^{\mathbb{P}}$, $p \vdash \phi(\sigma_1, \dots, \sigma_n)$ iff for all generic G containing p, $\phi((\sigma_1)_G, \dots, (\sigma_n)_G)^{M[G]}$.*
2. *For any generic G, and $\sigma_1, \dots, \sigma_n \in M^{\mathbb{P}}$, $\phi((\sigma_1)_G, \dots, (\sigma_n)_G)^{M[G]}$ iff $\exists p \in G\ p \vdash \phi(\sigma_1, \dots, \sigma_n)$.*

Proof. We first show that (2) implies (1). Assume first that $p \vdash \phi$, and let G be a generic containing p. If $(\neg\phi)^{M[G]}$, then by (2) there would be a $q \in P$ such that $q \vdash \neg\phi$. Since G is a filter, $p \parallel q$, and from lemma 0.13 we may assume $q \leqslant p$. Also by lemma 0.13, $q \vdash \phi$. This contradicts the definition of $q \vdash \neg\phi$. Assume next that for all generic G containing p that $\phi^{M[G]}$. If $p \not\vdash \phi$, then by corollary 0.14 there would be a $q \leqslant p$ such that $q \vdash \neg\phi$. Let G be a generic filter containing q. From (2), $(\neg\phi)^{M[G]}$, which contradicts our assumption.

We turn our attention to (2). We prove this by induction on ϕ, and when $\phi = (\tau_1 \approx \tau_2)$ by induction on the maximum rank of τ_1, τ_2. We consider cases as in the definition of the forcing relation.

Case 1) $\phi = (\tau_1 \in \tau_2)$.

First assume $p \vdash \phi$. By definition, $D = \{q\colon \exists\langle\sigma, r\rangle \in \tau_2\ (q \leqslant r \wedge q \vdash (\tau_1 \approx \sigma))\}$ is dense below p. Since G is generic, $\exists q \in G \cap D\ (q \leqslant p)$. Let $\langle\sigma, r\rangle \in \tau_2$ be such that $q \leqslant r$ and $q \vdash (\tau_1 \approx \sigma)$. By the equality case (proven next) we may assume that $(\tau_1)_G = \sigma_G$. Since $r \in G$, $\sigma_G \in (\tau_2)_G$ by definition of $(\tau_2)_G$. Thus, $(\tau_1)_G \in (\tau_2)_G$.

Next assume that G is generic and $\phi^{M[G]}$, that is, $(\tau_1)_G \in (\tau_2)_G$. Let $\langle \sigma, q\rangle \in \tau_2$ be such that $q \in G$ and $(\tau_1)_G = \sigma_G$. By the equality case, let $r \in G$ be such that $r \vdash (\tau_1 \approx \sigma)$. From lemma 0.15, let $p \in G$ with $p \leqslant q, r$. Then $p \vdash \tau_1 \in \tau_2$ since for any $s \leqslant p$ we have $\langle \sigma, q\rangle \in \tau_2$, $s \leqslant p \leqslant q$, and $s \vdash (\tau_1 \approx \sigma)$ since $s \leqslant r$. Thus, the set required to be dense below p in the definition of $p \vdash \tau_1 \in \tau_2$ is actually all $s \leqslant p$.

Case 2) $\phi = (\tau_1 \approx \tau_2)$.

First assume $p \vdash \tau_1 \approx \tau_2$. It suffices by symmetry to show that $(\tau_1)_G \subseteq (\tau_2)_G$. Let $x \in (\tau_1)_G$, say $x = (\sigma_1)_G$ where $\langle \sigma_1, q\rangle \in \tau_1$ and $q \in G$. By definition, the set $D = \{r \colon (r \leqslant q \to \exists \langle \sigma_2, s\rangle \in \tau_2\ r \leqslant s \wedge r \vdash (\sigma_1 \approx \sigma_2))\}$ is dense below p. $D' = \{r \leqslant p \colon (r \perp q) \vee (r \leqslant q \wedge r \in D)\}$. Then D' is easily dense below p (if $t \leqslant p$ then either $t \perp q$ or we can extend t to a $u \leqslant t$ with $u \leqslant q$. since $u \leqslant p$, we can get $v \leqslant u$ with $v \in D$. Then $v \leqslant t$ and $v \in D'$.) Let $r \in G \cap D'$. We cannot have $r \perp q$ as $q \in G$. So, $r \leqslant q$ and $r \in D$. By definition of D we now have a $\langle \sigma_2, s\rangle \in \tau_2$ with $r \leqslant s$ and $r \vdash (\sigma_1 \approx \sigma_2)$. By induction, $(\sigma_1)_G = (\sigma_2)_G$. Since $r \in G$ and $r \leqslant s$, $(\sigma_2)_G \in (\tau_2)_G$. Hence, $x = (\sigma_1)_G = (\sigma_2)_G \in (\tau_2)_G$.

Next assume that $\phi^{M[G]}$, that is, $(\tau_1)_G = (\tau_2)_G$. Let

$$\begin{aligned} D =&\{p \colon (p \vdash \tau_1 \approx \tau_2) \vee \\ &\exists \langle \sigma_1, q_1\rangle \in \tau_1\ [p \leqslant q_1 \wedge \forall r \leqslant p\ \forall \langle \sigma_2, q_2\rangle \in \tau_2\ \neg(r \leqslant q_2 \wedge r \vdash \sigma_1 \approx \sigma_2)]\}. \end{aligned}$$

We first claim that D is dense. To see this, let $p \in P$. If $p \vdash \tau_1 \approx \tau_2$, then $p \in D$. Otherwise, for some $\langle \sigma_1, q_1\rangle \in \tau_1$ the set $A = \{s \colon (s \leqslant q_1 \to \exists \langle \sigma_2, q_2\rangle \in \tau_2\ (s \leqslant q_2 \wedge s \vdash \sigma_1 \approx \sigma_2))\}$ is not dense below p. Let $p' \leqslant p$ be such that $\forall s \leqslant p'\ s \notin A$. Thus, for all $s \leqslant p'$ we have $s \leqslant q_1$ (so in particular $p' \leqslant q_1$), and $\neg \exists \langle \sigma_2, q_2\rangle \in \tau_2\ (s \leqslant q_2 \wedge s \vdash \sigma_1 \approx \sigma_2))$. Thus, $p' \in D$. This shows D is dense. Let $p \in G \cap D$. If $p \vdash \tau_1 \approx \tau_2$, we are done, so suppose the second disjunct in the definition of D holds. Let $\langle \sigma_1, q_1\rangle$ with $p \leqslant q_1$ be as in the second disjunct. Since $p \in G$, $(\sigma_1)_G \in (\tau_1)_G = (\tau_2)_G$. So, let $\langle \sigma_2, q_2\rangle \in \tau_2$ be such that $q_2 \in G$ and $(\sigma_1)_G = (\sigma_2)_G$. By induction, let $q \in G$ be such that $q \vdash \sigma_1 \approx \sigma_2$. Let $r \leqslant p, q, q_2$. Then r violates the definition of $p \in D$.

Case 3) $\phi = \alpha \wedge \beta$.

If $p \vdash \phi$, then by definition $p \vdash \alpha$ and $p \vdash \beta$. By induction, $\alpha^{M[G]}$ and $\beta^{M[G]}$, and so $\phi^{M[G]}$. Conversely, if $\phi^{M[G]}$, then $\alpha^{M[G]}$ and $\beta^{M[G]}$. By induction, $\exists p \in G\ p \vdash \alpha$ and $\exists q \in G\ q \vdash \beta$. Let $r \in G$ with $r \leqslant p, q$. Then $r \vdash \alpha$, $r \vdash \beta$, and so $r \vdash \phi$.

Case 4) $\phi = \neg \psi$.

Assume first $p \vdash \phi$. If $\psi^{M[G]}$ then by induction $\exists q \in G\ (q \vdash \psi)$. Let $r \leqslant p, q$. Then $r \vdash \neg\psi$ and $r \vdash \psi$, a contradiction to the definition of $r \vdash \neg\psi$. Thus, $\phi^{M[G]}$. Next assume $phi^{M[G]}$. Let $D = \{p \colon (p \vdash \psi) \vee (p \vdash \neg\psi)\}$. Then D is dense, so let $p \in G \cap D$. If $p \vdash \neg\psi$ we are done, so assume $p \vdash \psi$. By induction, $\psi^{M[G]}$ which contradicts $\phi^{M[G]}$.

Case 5) $\phi(\vec{\sigma}) = \exists x\ \psi(\vec{\sigma}, x)$.

First assume $p \vdash \phi$ and G is generic containing p. By definition $D = \{q \colon \exists \tau \in M^{\mathbb{P}}\ (q \vdash \psi(\vec{\sigma}, \tau)\}$ is dense below p. Let $q \in G \cap D$ with $q \leqslant p$. Let $\tau \in M^{\mathbb{P}}$ witness $q \in D$, so $q \vdash \psi(\vec{\sigma}, \tau)$. By induction, $\psi((\sigma_1)_G, \ldots, (\sigma_n)_G, \tau_G)^{M[G]}$. Thus, $\phi((\sigma_1)_G, \ldots, (\sigma_n)_G)^{M[G]}$.

Next assume that g is generic and $\phi((\sigma_1)_G, \ldots, (\sigma_n)_G)^{M[G]}$. Let $\tau \in M^{\mathbb{P}}$ be such that $\psi((\sigma_1)_G, \ldots, (\sigma_n)_G), \tau_G)^{M[G]}$. By induction, $\exists p \in G\ p \Vdash \psi(\sigma_1, \ldots, \sigma_n, \tau)$. Trivially then $D = \{q \colon \exists \tau \in M^{\mathbb{P}}\ (q \Vdash \psi(\vec{\sigma}, \tau)\}$ is dense below p (it contains all $q \leqslant p$), and so $p \Vdash \phi(\sigma_1, \ldots, \sigma_n)$. □

This completes the proof of the forcing theorem.

We next show that if M satisfies ZF or ZFC, then so does $M[G]$. Although we are mainly interested in what additional properties we can arrange to hold in $M[G]$, the proof will give us some practice in using the forcing theorem.

Theorem 0.17. *Let M be a transitive model of ZF (or ZFC), and $\mathbb{P} \in M$ a partial order. Suppose G is M-generic for $\mathbb{P}$. Then $M[G]$ satisfies ZF (or ZFC).*

Proof. We have already checked that $M[G]$ is transitive and satisfies foundation, extensionality and pairing. Union is also easy to check directly as if $x = \tau_G \in M[G]$, let $\sigma = \{\langle \rho, \mathbb{1} \rangle \colon \exists \pi \in M^{\mathbb{P}}\ \exists p, q \in P\ (\langle \pi, p \rangle \in \tau \wedge \langle \rho, q \rangle \in \pi)\}$. Clearly $\cup x \subseteq \sigma_G$. As $M \subseteq M[G]$, and $\omega \in M$, $\omega \in M[G]$ and by absoluteness $M[G]$ satifies the infinity axiom.

To verify Power Set, let $x = \tau_G \in M[G]$. Let $\rho = \{\langle \sigma, \mathbb{1} \rangle \colon \mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)\}$. Clearly ρ is a set in M and $\rho \in M^{\mathbb{P}}$. We claim that $\mathcal{P}(x)^{M[G]} \subseteq \rho_G$, which suffices. To see this, let $y \subseteq x$, $y \in M[G]$. Say, $y = \mu_G$. Let $\sigma = \{\langle \pi, p \rangle \colon \pi \in \mathrm{dom}(\tau) \wedge p \Vdash (\pi \in \mu)\}$. Clearly $\langle \sigma, \mathbb{1} \rangle \in \rho$ and so $\sigma_G \in \rho_G$. We claim that $\sigma_G = \mu_G$. If $z \in \sigma_G$, then $z = \pi_G$ where $\langle \pi, p \rangle \in \sigma$ and $p \in G$. Since $p \Vdash (\pi \in \mu)$ (from the definition of σ) we have $z = \pi_G \in \mu_G$ by the forcing theorem. If $z \in \mu_G$, then since $\mu_G \subseteq x$ we have $z = \pi_G$ for some $\langle \pi, p \rangle \in \tau$. By the forcing theorem, there is a $q \in G$ such that $q \Vdash \pi \in \mu$, and hence $\langle \pi, q \rangle \in \sigma$ and so $z \in \sigma_G$.

To verify Comprehension, let $\phi(x_1, \ldots, x_n, y, z)$ be a formula and $a_1 = (\sigma_1)_G$, $\ldots, a_n = (\sigma_n)_G$, $b = \tau_G \in M[G]$. We must show that

$$\{z \in b \colon \phi^{M[G]}(a_1, \ldots, a_n, b, z)\} \in M[G].$$

Let $\rho = \{\langle \pi, p \rangle \colon (\pi \in \mathrm{dom}\,\tau) \wedge (p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi) \wedge \pi \in \tau)\}$. We show that ρ_G works. If $z \in \rho_G$, then $z = \pi_G$ where $p \in G$ and $p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi)$ and $p \Vdash \pi \in \tau$. Thus, $\phi^{M[G]}(a_1, \ldots, a_n, b, z)$ and $z = \pi_G \in \tau_G = b$. Conversely, suppose $z \in b$ and $\phi^{M[G]}(a_1, \ldots, a_n, b, z)$. Then $z = \pi_G$ for some $\pi \in \mathrm{dom}(\tau)$. By the forcing theorem, there is a $p \in G$ such that $p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi) \wedge \pi \in \tau$. Then $\langle \pi, p \rangle \in \rho$, and so $z = \pi_G \in \rho_G$.

To verify Replacement, let $\phi(x_1 \ldots, x_n, y, z, w)$ be a formula and $a_1 = (\sigma_1)_G$, $\ldots, a_n = (\sigma_n)_G$, $A = \tau_G \in M[G]$. Assume that

$$\forall z \in A\ \exists w \in M[G]\ \phi^{M[G]}(a_1, \ldots, a_n, A, z, w).$$

By replacement in M there is a set of names $S \in M$ such that for all $\pi \in \mathrm{dom}(\tau)$ and all $p \in P$, if $\exists \mu \in M^{\mathbb{P}}\ (p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi, \mu))$, then $\exists \mu \in S\ (p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi, \mu))$. Let $\rho = \{\langle \mu, \mathbb{1} \rangle \colon \mu \in S\}$. We show that ρ_G verifies this instance of replacement in $M[G]$. For suppose $z \in A$, say $z = \pi_G$ where $\pi \in \mathrm{dom}(\tau)$, and suppose $\exists w \in M[G]\ \phi^{M[G]}(a_1, \ldots, a_n, A, z, w)$. Fix such a $w = \mu_G$. By the forcing theorem, there is a $p \in G$ such that $p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi, \mu)$. From the definition of S it follows that for some $\mu \in S$ that $p \Vdash \phi(\sigma_1, \ldots, \sigma_n, \tau, \pi, \mu)$. Fix such a μ. Then $\mu_G \in \rho_G$ and by the forcing theorem $\phi^{M[G]}(a_1, \ldots, a_n, A, z, \mu_G)$.

We have now verified ZF in $M[G]$. Suppose finally that AC holds in M, and we check it holds in $M[G]$ as well. Let $x = \tau_G \in M[G]$. Let $f \colon \alpha \to \mathrm{dom}(\tau)$, $f \in M$ be a bijection, where $\alpha \in \mathrm{ON}^M$. In $M[G]$ define F with domain α by $F(\beta) = f(\beta)_G$. Then $x \subseteq \mathrm{ran}(F)$, which suffices to show that x can be wellordered in $M[G]$ (recall $M[G]$ already satisfies ZF). □

Exercise 4. Show that if M is a transitive model of ZF, $\mathbb{P} \in M$ is a partial order, and $G \subseteq P$ is a filter (not necessarily generic), then $M[G] \models \forall x\ \exists f\ \exists \alpha\ (\alpha \in \mathrm{ON} \wedge f : \alpha \xrightarrow{\text{onto}} x)$.

1. Forcing and Complete Boolean Algebras

The forcing theorem holds for arbitrary partial orders, but it is an interesting fact that there is no loss of generality (assuming our models M are models of ZF) in considering only partial orders which are complete Boolean algebras (where $p \leqslant q$ recall means $p \cdot q^c = 0$). Although most of the time we deal directly with a partial order, there are times when this point of view is useful. Given a partial order $\mathbb{P} = \langle P, \leqslant \rangle$, we will associate to it a complete Boolean algebra $\mathcal{B}_P$ which we will call the *completion* of $\mathbb{P}$. This construction is just a slight generalization of the construction of the completion of a Boolean algebra, but we give the construction from scratch.

For $p \in P$, let $N_p = \{q \in P : q \leqslant p\}$. The idea of the construction is to identify every set $D \subseteq N_p$ which is dense below p with all of N_p. One way to make this precise is as follows. Define a topology τ, on P by taking as a base sets of the form N_p. Recall $B \subseteq \mathcal{P}(X)$ is a base for a topology on a set X iff whenever $U, V \in B$ and $x \in U \cap V$, then there is a $W \in B$ such that $x \in W \subseteq U \cap V$. This condition is satisfied here since in fact if $r \in N_p \cap N_q$ then $N_r \subseteq N_p \cap N_q$ (by transitivity). So, $A \subseteq P$ is τ-open iff A is downwards closed, that is, $\forall p \in A\ \forall q \leqslant p\ (q \in A)$. Note that N_p is the smallest open set containing p. Recall a set U in a topological space is *regular open* if $U = \mathrm{int}\,\mathrm{cl}(U)$. Thus, an open set is regular open iff it contains any neighborhood in which it is dense. Recall also that for any set $A \subseteq X$ that $\mathrm{int}\,\mathrm{cl}(A)$ is regular open. [One direction, namely $U \subseteq \mathrm{int}\,\mathrm{cl}(U)$ holds for any open set. For the other direction note that $\mathrm{int}\,\mathrm{cl}(\mathrm{int}\ \mathrm{cl}(A)) \subseteq \mathrm{int}\,\mathrm{cl}\,\mathrm{cl}(A) = \mathrm{int}\,\mathrm{cl}(A)$.] It is easy to see that the intersection of two regular open sets is regular open.

Exercise 5. Show directly that an open $U \subseteq P$ is regular open iff whenever U is dense below a $p \in P$, then $p \in U$.

Definition 1.1. Let $\mathbb{P} = \langle P, \leqslant \rangle$ be a partial order. Then $\mathcal{B}_P$ is the collection of all regular open subsets of P with the following operations: $U + V = \mathrm{int}\,\mathrm{cl}(U \cup V)$, $U \cdot V = U \cap V$, $-U = int(P - U)$. Also, define $0_{\mathcal{B}} = \emptyset$ and $1_{\mathcal{B}} = P$ (note: we use here $-U$ for the Boolean complement operation to avoid confusion.)

Lemma 1.2. *For any partial order $\mathbb{P}$, $\mathcal{B}_P$ is a complete Boolean algebra. Moreover, the map $\pi : P \to \mathcal{B}$ defined by $\pi(p) = int\,cl(N_p)$ satisfies the following:*

(1) *If $p \leqslant q$ then $\pi(p) \leqslant \pi(q)$.*
(2) *$p \perp q$ iff $\pi(p) \perp \pi(q)$.*
(3) *$\pi'' \mathbb{P}$ is dense in $\mathcal{B}$.*

Proof. In fact, for any topological space X the regular open sets form a complete Boolean algegra (with the same operations defined above). Checking this is straightforward but tedious. The commutative, identity, and $0-1$ laws are trivial to check. For any open set U, $U \cup \mathrm{int}\,(X - U)$ is dense, and so $U + (-U) = 1$.

Since $-U \subseteq (X - U)$, we have $U \cdot (-U) = 0$. This checks the negation laws. The associative law for $\cdot$ is trivial. To check the associative law for addition, let U, V, W be regular open. Since addition is commutative, it is enough to check that $U + (V + W) \subseteq (U + V) + W$. Since $V \cup W \subseteq (U + V) + W$, $V + W = \operatorname{int}\operatorname{cl}(V \cup W) \subseteq \operatorname{int}\operatorname{cl}((U + V) + W) = (U + V) + W$ since $(U + V) + W$ is regular open. Thus, $U \cup (V + W) \subseteq (U + V) + W$, and again $U + (V + W) \subseteq \operatorname{int}\operatorname{cl}((U + V) + W) = (U + V) + W$, and we are done.

To verify the distributive law $U \cdot (V + W) = U \cdot V + U \cdot W$, first note that $U \cdot V \cup U \cdot W \subseteq U \cdot (V + W)$, and since the latter is regular open we have $U \cdot V + U \cdot W = \operatorname{int}\operatorname{cl}(U \cdot V \cup U \cdot W) \subseteq U \cdot (V + W)$. For the other direction, let $x \in U \cdot (V + W)$. Let O be a neighborhood in which $V \cup W$ is dense. Since U is open, we may assume $O \subseteq U$. But then $U \cap V \cup U \cap W$ is dense in O, and so $x \in \operatorname{int}\operatorname{cl}(U \cdot V \cup U \cdot W) = U \cdot V + U \cdot W$.

To verify the distributive law $U + (V \cdot W) = (U + V) \cdot (U + W)$, first note that $U \cup (V \cdot W) \subseteq (U + V) \cdot (U + W)$, and since the latter is regular open we have $U + (V \cdot W) \subseteq (U + V) \cdot (U + W)$. For the other direction, let $x \in (U + V) \cdot (U + W)$. Let O_1 be a neighborhood of x in which $U \cup V$ is dense, and O_2 a neighborhood of x in which $U \cup W$ is dense. Let $O = O_1 \cap O_2$. We show that $U \cup (V \cap W)$ is dense in O, which suffices. Let $O_3 \subseteq O$ be open. If $O_3 \cap U \neq \emptyset$, we are done. Otherwise, V and W are each dense in O_3. Since V, W are regular open, $O_3 \subseteq V \cap W$ and we are done.

For U a regular open set we claim that $-(-U) = U$. Clearly $U \subseteq -(-U)$. For the other direction, let $x \in -(-U)$. Then there is a neighborhood O of x such that $O \cap (-U) = \emptyset$. This means every neighborhood of every point in O intersects U, that is, U is dense in O. Since U is regular open, $O \subseteq U$, so $x \in U$. From this it follows that we need only check one of de Morgan's laws. We check $-(U \cdot V) = (-U) + (-V)$. Since $U^c \subseteq (U \cdot V)^c$, $\operatorname{int}(U^c) \subseteq \operatorname{int}((U \cdot V)^c)$. So, $-U \subseteq -(U \cdot V)$. Likewise, $-V \subseteq -(U \cdot V)$, and so $(-U) \cup (-V) \subseteq -(U \cdot V)$. Since the latter is regular open, $(-U) + (-V) \subseteq -(U \cdot V)$. Let $x \in -(U \cdot V)$. Let O be a neighborhood of x missing $U \cap V$. We claim that $(-U) \cup (-V)$ is dense in O, which suffices. Let $O_1 \subseteq O$ be open. If $-U$ misses O_1, then U is dense in O_1, and hence $O_1 \subseteq U$ as U is regular open. Likewise, if $-V$ also misses O_1 then $O_1 \subseteq V$ and so $O_1 \subseteq U \cap V$, a contradiction.

We have now shown that that the regular open sets in any topological space form a Boolean algebra. To see they form a complete Boolean algebra, it suffices to show that if $\{U_\alpha\}$ are regular open, then there is a least upper bound for the U_α. Let $U = \operatorname{int}\operatorname{cl}(\bigcup_\alpha U_\alpha)$, so U is regular open. Clearly $U_\alpha \leqslant U$ for each α. For U, V regular open it is easy to check that $U \leqslant V$ (i.e., $U \cdot (-V) = 0$) iff $U \subseteq V$. So if W is regular open and $U_\alpha \leqslant W$ for all α, then $U_\alpha \subseteq W$ for each α, so $\bigcup_\alpha U_\alpha \subseteq W$. Then $U = \operatorname{int}\operatorname{cl}(\bigcup_\alpha U_\alpha) \subseteq \operatorname{int}\operatorname{cl}(W) = W$. Thus $U \leqslant W$. This shows U is the least upper bound of the U_α.

Finally, we verify π satisfies (1)-(3). If $p \leqslant q$ then $N_p \subseteq N_q$ so $\operatorname{int}\operatorname{cl}(N_p) \subseteq \operatorname{int}\operatorname{cl}(N_q)$. Hence $\pi(p) \leqslant \pi(q)$. If $p \parallel q$, then (1) implies that $\pi(p) \parallel \pi(q)$. Suppose $p \perp q$ but $\pi(p) \parallel \pi(q)$, that is, $\pi(p) \cdot \pi(q) \neq 0$. Let $r \in \operatorname{int}\operatorname{cl}(N_p) \cap \operatorname{int}\operatorname{cl}(N_q)$. Thus, $N_r \subseteq \operatorname{cl}(N_p)$ and $N_r \subseteq \operatorname{cl}(N_q)$. The first says that r can be extended to a $s \leqslant r$, $s \in N_p$, and the second says that we may extend s to a $t \leqslant s$, $t \in N_q$. Then $t \leqslant p, q$, a contradiction. □

The map π of lemma 1.2 is not necessarily one-to-one, but it is under a very mild condition on P which is always satisfied in practice. Namely, suppose whenever

$p \nleqslant q$ then $\exists r \leqslant p\ (r \perp q)$. Then if $p \neq q$, say $p \nleqslant q$, and we let $r \leqslant p$, $r \perp q$, then $r \in N_p$ but $r \notin \operatorname{int}\operatorname{cl}(N_q)$. Sometimes the word *separative* is used to denote this condition.

We next show that forcing with $\mathbb{P}$ is equivalent to forcing with its completion $\mathcal{B}_P$. The proof of this does not use the fact that $\mathcal{B}_P$ is a complete Boolean algebra, just properties (1)-(3) above. We abstract these properties into a definition.

Definition 1.3. Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders. We say $\pi\colon P \to Q$ is a *dense embedding* if:

(1) If $p \leqslant q$ then $\pi(p) \leqslant \pi(q)$.
(2) $p \perp q$ iff $\pi(p) \perp \pi(q)$.
(3) $\pi'' \mathbb{P}$ is dense in $\mathbb{Q}$.

Lemma 1.4. *Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders in a transitive model M of ZF. Let $\pi\colon P \to Q$ be a dense embedding, $\pi \in M$. Then:*

(1) *If $H \subseteq Q$ is M-generic for $\mathbb{Q}$, then $G \doteq \pi^{-1}(H)$ is M-generic for $\mathbb{P}$.*
(2) *If $G \subseteq P$ is M-generic for $\mathbb{P}$, then $H \doteq \{q \in Q\colon \exists p \in G\ (\pi(p) \leqslant q)\}$ is M-generic for Q.*

In either case, $M[G] = M[H]$.

Proof. Suppose $H \subseteq Q$ is generic for $\mathbb{Q}$. Then G is easily closed upwards. Also, if $p, q \in G$, then $\pi(p) \parallel \pi(q)$ as they both lie in H. From (2) it follows that $p \parallel q$, hence G is a filter. To see it is generic, let $D \subseteq P$ be dense. Let $E = \pi'' D$. Then E is dense in Q from (3) [If $p \in Q$, let $q \leqslant p$ be in the range of π, say $q = \pi(t)$. As D is dense, let $u \leqslant t$, $u \in D$. Then $\pi(u) \in E$, $\pi(u) \leqslant \pi(t) \leqslant p$ using (1).] Let $p \in H \cap E$. Say $p = \pi(t)$, where $t \in D$. Then $t \in G \cap D$.

Suppose next that $G \subseteq P$ is M-generic for P. Define $H \subseteq Q$ as above. H is upwards closed by definition, and it is easy to check it is a filter using (1). To see it is generic, let $E \subseteq Q$ be dense. Let $D = \{p \in P\colon \exists q \in E\ (\pi(p) \leqslant q)\}$. To see D is dense, let $p \in P$. Since E is dense, let $q \in E$, $q \leqslant \pi(p)$. By (3), let $r \leqslant q$ with $r \in \operatorname{ran}(\pi)$, say $r = \pi(t)$. Since $\pi(t) = r \leqslant \pi(p)$, by (2) we have $p \parallel t$. Let $u \leqslant p$, $u \leqslant t$. By (1), $\pi(u) \leqslant \pi(p)$, $\pi(u) \leqslant \pi(t) \leqslant q$. As $q \in E$, this shows $u \in D$, so D is dense. Let now $p \in G \cap D$. Since $p \in D$, let $q \geqslant \pi(p)$ with $q \in E$. Then $q \in H \cap E$ and we are done.

To see that $M[G] = M[H]$, simply note that in $M[G]$ we may define H since $\pi \in M$, so $M[H] \subseteq M[G]$. Likewise $M[G] \subseteq M[H]$. □

Thus, there is no difference between forcing with a partial order $\mathbb{P}$ and forcing with its completion $\mathcal{B}_P$, which is a complete Boolean algebra.

If $\pi\colon P \to Q$ is a dense embedding, then lemma 1.4 says that generics for P and Q correspond; they are essentially the same thing. If we relax condition (3) of definition 1.3, then Q may be much larger than P, and we do not expect generics for P to give generics for Q. However, we can still get that generics for Q give P generics if we replace (3) by a suitable condition. This is the concept of a *complete embedding*, defined next.

Definition 1.5. Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders. We say $\pi\colon P \to Q$ is a complete embedding if it satisfies the following.

(1) If $p \leqslant q$ then $\pi(p) \leqslant \pi(q)$.
(2) $p \perp q$ iff $\pi(p) \perp \pi(q)$.

(3) For every $q \in Q$ there is a $p \in P$ such that $\forall r \leqslant p\ (\pi(r) \parallel q)$.

To motivate (3), note that if $\pi \colon P \to Q$, and whenever $H \subseteq Q$ is M-generic for Q, then $P \doteq \pi^{-1}(H)$ is also generic for P, then we must have (3). Otherwise, for some $q \in Q$, $\{p \in P \colon \pi(p) \perp q\}$ is dense in P. Let H be generic for Q with $q \in H$. If $G = \pi^{-1}(H)$ is also generic, then $\exists p \in G \cap D$. But then $q, \pi(p) \in H$ and are incompatible, a contradiction.

The definition of complete embedding in more natural when P, Q are complete Boolean algebras. In this case, $\pi \colon P \to Q$ is a complete embedding iff π is a ono-to-one homomorphisms of Boolean algebras (i.e., π preserves the Boolean algebra operations) which is *complete*, that is, it preserves arbitrary sups and infs: for $\{p_\alpha\} \subseteq P$, $\pi(\sup(\{p_\alpha\})) = \sup(\{\pi(p_\alpha)\})$. To see this, suppose P, Q are complete Boolean algebras and (1)-(3) hold. Let $\{p_\alpha\} \subseteq P$ and $p = \sup(\{p_\alpha\})$. By (1), $\pi(p) \geqslant \pi(p_\alpha)$ for all α, so $\pi(p) \geqslant \sup(\{\pi(p_\alpha)\})$. Suppose $q = \pi(p) - \sup(\{\pi(p_\alpha)\}) = neq 0_Q$. By (3), let $r \in P$ be such that $\forall s \leqslant r\ (\pi(s) \parallel q)$. In particular $\pi(r) \parallel q$, so $\pi(r) \parallel \pi(p)$. By (2), $r \parallel p$. Let $s \leqslant p, r$. We claim that $s \perp p_\alpha$ for each α, a contradiction (since then $p - s$ is an upper bound for the p_α). To see this, suppose $s \parallel p_\alpha$, say $t \leqslant s$, $t \leqslant p_\alpha$. Then $\pi(t) \leqslant \pi(p_\alpha)$ and since $t \leqslant r$, $\pi(t) \parallel q$, a contradiction to the definition of q. This show (1)-(3) imply that π preserves sups. A similar argument shows that π preserves arbitrary infs (exercise **??**). Since for any Boolean algebra $p+q = \sup\{p, q\}$ and $p \cdot q = \inf\{p, q\}$ it follows that π preserves $+$ and $\cdot$. To see π preserves Boolean complements, note that $\pi(p) \cdot \pi(-p) = 0$ by (2). If $\pi(-p) \neq -\pi(p)$, then $q \doteq -(\pi(p) + \pi(-p)) \neq 0_Q$. Let $r \in P$ be as in (3) for q. Without loss of generality suppose $r \parallel p$, and let $s \leqslant r, p$. Then by (1) $\pi(s) \leqslant \pi(p)$, and by definition of r, $\pi(s) \parallel q$. This contradicts the definition of q. It is immediate from (1) that $\pi(0_P) = 0_Q$ and $\pi(1_P) = 1_Q$. Finally, π is one-to-one, for suppose $p \mp q$ but $\pi(p) = \pi(q)$. Without loss of generality assume $r \doteq p - q \neq 0_P$. Then $r \perp q$ but $\pi(r) \parallel \pi(q)$, contradicting (2).

Conversely, suppose P, Q are complete Boolean algebras and $\pi \colon P \to Q$ is a one-to-one homomorphism which is complete (i.e., preserves arbitrary sups and infs). Properties (1) and (2) are immediate as π is a homomorphism. To see (3), fix $q \in Q$. Let $p = \inf\{r \in P \colon \pi(r) \geqslant q\}$. Suppose $s \leqslant p$. Since $p - s \lneq p$, we must have $\pi(p - s) \ngeqslant q$. As $\pi(p) = \pi(s) + \pi(p - s)$, we have $\pi(s) \parallel q$.

Exercise 6. Show that if $\pi \colon P \to Q$ satisies (1)-(3) where P, Q are complete Boolean albebras, then π preserves arbitrary infimums.

Theorem 1.6. *Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders in a transitive model M of ZF. Let $\pi \colon P \to Q$, $\pi \in M$, be a complete embedding. If $H \subseteq Q$ is M-generic for Q, then $G = \pi^{-1}(H)$ is M-generic for P. Furthermore, $M[G] \subseteq M[H]$.*

Proof. Let $H \subseteq Q$ be M-generic for Q, and let $G = \pi^{-1}(H)$. Let $D \subseteq P$ be dense. Now $\pi'' D$ is not necessarily dense in Q, but $E = \{q \colon \forall p \in D\ (q \perp \pi(p)) \vee \exists p \in D\ (q \leqslant \pi(p))\}$ is dense in Q. Let $q \in H \cap E$. We cannot have $\forall p \in D\ (q \perp \pi(p))$, for if so, let $p \in P$ be as in (3) for q. Let $r \in D$, $r \leqslant p$. Then $\pi(r) \parallel q$, a contradiction. So, $\exists p \in D\ (q \leqslant \pi(p))$. Thus, $\pi(p) \in H$, so $p \in G$.

Since $\pi \in M$, working in $M[H]$ we may clearly define G, so $G \in M[H]$. Thus, $M[G] \subseteq M[H]$. □

If we deal with partial orders which are complete Boolean algebras (which we can do without loss of generality by lemma **??**), then the definition of forcing simplifies considerably. This is because if $A \subseteq N_p = \{q : q \leqslant p\}$ is dense below p, then $\sup(A) = p$. Also, if $q \Vdash \phi$ for all $q \in A$, then $\sup(A) \Vdash \phi$. Thus, the various dense sets that appear in the general definition will "collapse" when dealing with complete Boolean algebras. We make this more precise in the following definition.

Definition 1.7. Let M be a transitive model of ZF and $\mathcal{B} \in M$ with ($\mathcal{B}$ is a complete Boolean algebra)^M. For $\phi(\tau_1, \ldots, \tau_n)$ a statement in the forcing language, we define its *Boolean value* $[\![\phi]\!]$ by induction on ϕ (and for the equality case by induction on the ranks of the names) as follows. (note: we use $\sum$, $\prod$ for supremums and infimums).

(1) $[\![\tau_1 \in \tau_2]\!] = \sum_{\langle \sigma, p \rangle \in \tau_2} (p \cdot [\![\tau_1 \approx \sigma]\!])$.
(2) $[\![\tau_1 \approx \tau_2]\!] = \prod_{\sigma \in \mathrm{dom}(\tau_1)} (-[\![\sigma \in \tau_1]\!] + [\![\sigma \in \tau_2) + \prod_{\sigma \in \mathrm{dom}(\tau_2)} (-[\![\sigma \in \tau_2]\!] + [\![\sigma \in \tau_1)]\!]$.
(3) $[\![\phi \wedge \psi]\!] = [\![\phi]\!] + [\![\psi]\!]$.
(4) $[\![\neg \phi]\!] = -[\![\phi]\!]$.
(5) $[\![\exists x\ \phi(\vec{\tau}, x)]\!] = \sum_{\sigma \in M^{\mathcal{B}}} [\![\phi(\vec{\tau}, \sigma)]\!]$.

When $\mathcal{B}$ is a complete Boolean algebra, the notion of generic filter can be reformulated and simplified a bit as well. First note that a generic filter on $\mathcal{B}$ is necessarily an ultrafilter, that is, for any $p \in \mathcal{B}$, either $p \in G$ or $-p \in \mathcal{B}$. This follows from the fact that $\{q \in \mathcal{B} : q \leqslant p \vee q \leqslant (-p)\}$ is dense in $\mathcal{B}$. Also, if $p, q \in \mathcal{B}$ then $p \cdot q \in \mathcal{B}$. This follows since $\{r : (r \leqslant p \cdot q) \vee (r \cdot p = 0) \vee (r \cdot q = 0)\}$ is dense. Thus, when dealing with complete Boolean algebras we speak of generic ultrafilters on $\mathcal{B}$. We can reformulate the density condition in the definition of generic as follows.

Lemma 1.8. *Let M be a transitive model of ZF and $\mathcal{B} \in M$ with ($\mathcal{B}$ is a complete Boolean algebra)^M. Then G is an M-generic filter on $\mathcal{B}$ iff G is an ultrafilter on $\mathcal{B}$ which respects supremums and infimums from M. That is, if $S \subseteq \mathcal{B}$, then $\sum S \in G$ iff $\exists p \in S\ (p \in G)$ and $\prod S \in G$ iff $\forall p \in G\ (p \in G)$.*

Proof. We have already seen that if $G \subseteq \mathcal{B}$ is a generic filter on $\mathcal{B}$ then it is an ultrafilter. If $S \subseteq \mathcal{B}$, then $D = \{p \in \mathcal{B}(: p \cdot \sum S = 0) \vee (\exists s \in S\ p \leqslant s)\}$ is dense. If $\sum S \in G$, and if we let $p \in G \cap D$, then we have $p \leqslant s$ for some $s \in S$. Thus $s \in G$ as well. The proof that G respects infimums is similar.

Finally, if G respects supremums and infimums, then in particular whenever $D \subseteq \mathcal{B}$ is dense, that is $\sum D = 1_{\mathcal{B}}$, we must have $p \in G$ for some $p \in D$. So, G is generic. □

The connection between the forcing relation and Boolean values is given in the next lemma.

Lemma 1.9. *Let M be a transitive model of ZF, $\mathcal{B} \in M$, and ($\mathcal{B}$ is a complete Boolean algebra)^M. Then for any statement ϕ in the forcing language, $p \Vdash \phi$ iff $p \leqslant [\![\phi]\!]$.*

Proof. The proof by induction on ϕ is straightforward, we consider just one case here, say $\phi = (\tau_1 \in \tau_2)$. Now

$$\begin{aligned} p \leqslant [\![\phi]\!] = \sum_{\langle\sigma,q\rangle\in\tau_2\rangle} (q \cdot [\![\tau_1 \approx \sigma]\!]) \\ \leftrightarrow \{r \leqslant p \colon \exists\langle\sigma, q\rangle \in \tau_2 \; r \leqslant q \cdot [\![\tau_1 \approx \sigma]\!]\} \text{ is dense below } p \\ \leftrightarrow \{r \leqslant p \colon \exists\langle\sigma, q\rangle \in \tau_2 \; r \leqslant q \wedge r \leqslant [\![\tau_1 \approx \sigma]\!]\} \text{ is dense below } p \\ \leftrightarrow \{r \leqslant p \colon \exists\langle\sigma, q\rangle \in \tau_2 \; r \leqslant q \wedge r \vdash \tau_1 \approx \sigma\} \text{ is dense below } p. \\ \leftrightarrow p \vdash (\tau_1 \in \tau_2) \end{aligned}$$

The first equivalence is the fact that $p \leqslant \sum S$ iff $\{q \leqslant p \colon \exists s \in S \; (q \leqslant s)\}$ is dense below p. The next to last eqivalence is by induction, and the last equivalence is the definition of $p \vdash (\tau_1 \in \tau_2)$. □

An alternative possibility when considering forcing with complete Boolean algebras is that we introduce definition 1.7, and then *define* the forcing relation by $p \vdash \phi$ iff $p \leqslant [\![\phi]\!]$. This has the advantage that the definition of forcing is somewhat simpler and more natural. Of course, in this approach one must now prove the forcing theorem directly: for all generic ultrafilters G on $\mathcal{B}$, $\phi^{M[G]}$ iff $[\![\phi]\!] \in G$. The proof is essentially the same as before. For example, consider the case $\phi = (\tau_1 \in \tau_2)$. Suppose first $[\![\phi]\!] = \sum_{\langle\sigma,q\rangle\in\tau_2\rangle}(q \cdot [\![\tau_1 \approx \sigma]\!]) \in G$. From lemma 1.8 it follows that for some $\langle\sigma, q\rangle \in \tau_2\rangle \in \tau_2$ that $q \cdot [\![\tau_1 \approx \sigma]\!] \in G$. Thus, $q \in G$ and $[\![\tau_1 \approx \sigma]\!] \in G$. By induction, $(\tau_1)_G \approx \sigma_G$. Since $q \in G$, $\sigma_G \in (\tau_2)$, and we are done. For the other direction, suppose $(\tau_1)_G = (\tau_2)_G$. Fix $\langle\sigma, q\rangle \in \tau_2$ such that $q \in G$ and $(\tau_1)_G = \sigma_G$. By induction, $[\![\tau_1 \in \sigma]\!] \in G$. Since G is a filter, $q \cdot [\![\tau_1 \in \sigma]\!] \in G$. Thus, $\sum_{\langle\sigma,q\rangle\in\tau_2\rangle}(q \cdot [\![\tau_1 \approx \sigma]\!]) \in G$.