Forcing and the Continuum

1. Cardinal Arithmetic

Our first goal is to show the independence of CH from ZFC. First consider the partial order for adding a single real number $x \in 2^\omega$. One way would be to take the partial order $\mathcal{P}_1 = \langle 2^{<\omega}, \leqslant \rangle$, that is, conditions are finite sequences of 0's and 1's, and $p \leqslant q$ iff p extends q. $\mathcal{P}_1$ can thus be viewed as the full binary tree. Alternatively, we could consider $\mathcal{P}_2$ which consists of all partial functions $p\colon \omega\{0,1\}$ with finite support (that is, a finite domain), and again ordered by extension. There is clearly a dense embedding (namely inclusion) of $\mathcal{P}_1$ into $\mathcal{P}_2$, and thus forcing with $\mathcal{P}_1$ is equivalent to forcing with $\mathcal{P}_2$ by lemma **??**. It is for the moment a little more convenient to work with $\mathcal{P}_2$. We will call this the Cohen forcing, and refer to a generic G for this forcing as a Cohen real. Technically a generic filter $G \subseteq \mathcal{P}_2$ is not a real, but can be identified with one. Namely, let $x(n) = i$ iff $\exists p \in G\ (p(n) = i)$. This defines a function as any two conditions in G are compatible, and this function has domain ω as $D_n = \{p\colon n \in \mathrm{dom}(p)\}$ is dense for each $n \in \omega$. Clearly $M[G] = M[x]$, and we henceforth identify a generic with the corresponding real x. Although this produces a model $M[x]$ which satisfies $V \neq L$, and in fact $V \neq \mathrm{HOD}$ (as we show later), adding a single real is not enough to change the value of the continuum function. To do this we must add more reals.

Let us work in a model M of (enough of) ZF, and let κ be a cardinal (i.e., $(\kappa \in \mathrm{CARD})^M$). The natural generalization of $\mathcal{P}_2$ to add κ many reals would be the partial order $\mathrm{FN}(\kappa \times \omega, 2)$ = all partial functions from $\kappa \times \omega$ to $\{0,1\}$ with finite support. We order again by extension, that is $p \leqslant q$ iff $p \restriction \mathrm{dom}(q) = q$. Of course, a function $F\colon \kappa \times \omega \to 2$ can be identified with a κ sequence of functions $F_\alpha\colon \omega \to 2$, namely $F_\alpha(n) = F(\alpha, n)$. A filter G on $\mathrm{FN}(\kappa \times \omega, 2)$ can again be identified with a partial function from $\kappa \times \omega$ to 2. If G is generic, then this function (which we will aslo call G) has domain $\kappa \times \omega$. This is because for each $\alpha < \kappa$, $n \in \omega$, the set $D_{\alpha,n} = \{p\colon (\alpha, n) \in \mathrm{dom}(p)\}$ is dense. We summarize this discussion in the following lemma.

Lemma 1.1. *Let M be a transitive model of ZF, κ a cardinal of M, and G an M-generic for $FN(\kappa \times \omega, 2)^M$. Then G can be identified with a function $G\colon \kappa \times \omega \to 2$. If $\{G_\alpha\})\alpha < \kappa$ is the correspoding sequnce of reals, then $G_\alpha \neq G_\beta$ whenever $\alpha \neq \beta$.*

Proof. If $\alpha \neq \beta$, then $D_{\alpha,\beta} = \{p\colon \exists n \in \omega\ p(\alpha, n) \neq p(\beta, n)\}$ is dense (we mean here that both $p(\alpha, n)$, $p(\beta, n)$ are defined and not equal). □

Note that $\mathrm{FN}(\kappa \times \omega, 2)$ is isomorphic to $\mathrm{FN}(\kappa, 2)$, by taking a bijection between $\kappa \times \omega$ and κ. Thus, we may work with $\mathrm{FN}(\kappa, 2)$ which is notationally a little easier.

If G is generic for $\mathrm{FN}(\kappa, 2)$, then from lemma 1.1 in $M[G]$ we have $2^\omega \geqslant \kappa$. So, if we take $\kappa = (\omega_2)^M$, then in $M[G]$ we have $2^\omega \geqslant (\omega_2)^M$. To get a model of $\neg$CH, we also need to know that $(\omega_2)^M = (\omega_2)^{M[G]}$. In fact, we will show that this forcing preserves all cardinals and cofinalities. There is a general principle involved which is useful in many arguments.

Definition 1.2. A partial order $\mathcal{P}$ has the *countable chain condition*, or is said to be c.c.c.if every antichain $A \subseteq P$ is countable. More generally, $\mathcal{P}$ is said to be κ-c.c. if every antichain $A \subseteq P$ has size $< \kappa$. The saturation of a partial order $\mathrm{sat}(\mathcal{P})$, also called its cellularity, is the least κ such that $\mathcal{P}$ has the κ-c.c.

Thus, c.c.c.is the same as ω_1-c.c. This definition is also made for topological spaces: a topological space is κ-c.c. if every collection of pairwise disjoint open sets has size $< \kappa$. If we recall that every partial order $\mathcal{P}$ can also be viwed as a topological space (with basic open sets of the form $N_p = \{q \colon q \leqslant p\}$), then the two definitions coincide.

Exercise 1. Show that for topological spaces 2^{nd} countable $\Rightarrow$ separable $\Rightarrow$ c.c.c.. Give examples to show that the implications are not reversible (hint: see the following exercise).

Exercise 2. Show that $\prod_{\alpha \in I} \mathbb{R}_{\text{std}}$ is c.c.c.(hint:see lemma **??** below).

The significance of the notion of κ-c.c. for forcing is explained in the following lemma.

Lemma 1.3. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$ a partial order which is wellorderable in M and with $(\mathcal{P}$ is $\kappa - c.c.)^M$. Let G be M-generic for $\mathcal{P}$. Then if $A, B \in M$ and $f \colon A \to B$ is a function, $f \in M[G]$, then there is a function $F \colon A \to \mathcal{P}(B)$, $F \in M$, with $f(a) \in F(a)$ for all $a \in A$ and $(\forall a \in A\ |F(a)| < \kappa)^M$.*

Proof. Let $tau \in M^{\mathcal{P}}$ and $p \Vdash (\tau$ is a function from $\check{A}$ to $\check{B})$. Since $\mathcal{P}$ is wellorderable, working in M there is a function $H \colon A \to \mathcal{P}(P)$ satisfying:

(1) $\forall a \in A\ H(a)$ is an antichain in P.
(2) $\forall a \in A\ \forall p \in H(a)\ \exists b \in B\ (p \Vdash \tau(\check{a}) = \check{b}$.
(3) $H(a)$ is maximal subject to (1) and (2).

Working still in M, let $F(a) = \{b \in B \colon \exists p \in H(a)\ (p \Vdash f(\check{a}) = \check{b})\}$. Since $|H(a)| < \kappa$, $|F(a)| < \kappa$ as well. If $f(a) = b \in B$, then for some $p \in P$, $p \Vdash (\tau(\check{a}) = \check{b})$. By (3), there is a $q \in H(a)$ such that $p \parallel q$. Let $r \leqslant p, q$. So, $r \Vdash (\tau(\check{a}) = \check{b})$. Since $q \in H(a)$, $q \Vdash (\tau(\check{a}) = \check{c})$ for some $c \in b$, and we must have $b = c$. So, $b \in H(a)$. $\square$

Definition 1.4. Let M be a transitive model of ZF and $\mathcal{P} \in M$. We say $\mathcal{P}$ preserves cardinals if $(\kappa \in \text{CARD})^M \leftrightarrow (\kappa \in \text{CARD})^{M[G]}$. We say $\mathcal{P}$ preserves cardinals $\leqslant \lambda$ (or $\geqslant \lambda$, $< \lambda$, etc.) if for all $\kappa \leqslant \lambda$, $(\kappa \in \text{CARD})^M \leftrightarrow (\kappa \in \text{CARD})^{M[G]}$. We say $\mathcal{P}$ preserves cofinalities if for all $\alpha \in \text{ON} \cap M$, $(\text{cof}(\alpha)^M = (\text{cof}(\alpha))^{M[G]}$. We say $\mathcal{P}$ preserves cofinalities $\leqslant \lambda$ (or $\geqslant \lambda$, etc.) if whenever $(\text{cof}(\alpha) \leqslant \lambda)^M$ then $\text{cof}(\alpha)^M = \text{cof}(\alpha)^{M[G]}$.

We note two simple facts.

Fact 1.5. If M is a transitive model of ZF and $\mathcal{P} \in M$, then $\mathcal{P}$ preseves cofinalities iff for all $\kappa \in M$, $(\kappa \text{ is regular})^M \leftrightarrow (\kappa \text{ is regular})^{M[G]}$. Similarly, if $(\kappa \text{ is regular})^M \leftrightarrow (\kappa \text{ is regular})^{M[G]}$ for all $\kappa \leqslant \lambda$ (or $\kappa \geqslant \lambda$), then $\mathcal{P}$ preserves cofinalities $\leqslant \lambda$ (or $\geqslant \lambda$).

Proof. We prove the first statement, the other being essentially the same. Suppose $(\text{cof}(\alpha) = \lambda)^M$, so $(\lambda \text{ is regular })^M$. Let $f \colon \lambda \to \alpha$, $f \in M$, be increasing and cofinal. By assumption, λ is regular in $M[G]$. From lemma **??** we have in $M[G]$ that $\text{cof}(\alpha) = \text{cof}(\lambda = \lambda$. $\square$

Fact 1.6. If M is a transitive model of ZFC and $\mathcal{P} \in M$, then if $\mathcal{P}$ preserves cofinalities then it preserves cardinals. Similarly, if $\mathcal{P}$ preserves cofinalities $\leqslant \lambda$, then it preserves cardinals $\leqslant \lambda$. If $\mathcal{P}$ preserves cardinals $\geqslant \lambda$ and $(\lambda \text{ is regular})^M$, then $\mathcal{P}$ preserves cardinals $\geqslant \lambda$.

Proof. We prove by induction on the cardinals κ of M that they are cardinals of $M[G]$. For limit cardinals κ of M the inductive step is trivial (since a limit or cardinals is a cardinal). If κ is a successor cardinal of M, then since M satisfies ZFC, κ is regular in M. By assumption, κ is regular in $M[G]$, and hence is a cardinal of $M[G]$. □

Theorem 1.7. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$ with $(\mathcal{P}$ is κc.c.$)^M$, where κ is a regular cardinal of M. Then $\mathcal{P}$ preserves cardinals $\geqslant \kappa$ and preserves cofinalities $\geqslant \kappa$.*

Remark 1.8. Tarski's theorem 3.1 says that sat$(\mathcal{P})$ is always a regular cardinal (assuming ZFC or P is wellorderable). Thus, there is no loss of generality in assuming the κ of the lemma is a regular cardinal in M.

Proof. We first show $\mathcal{P}$ preserves cardinals $\geqslant \kappa$. Let $\lambda \geqslant \kappa$ with $(\lambda \in \text{CARD})^M$. Suppose $(\lambda \notin \text{CARD})^{M[G]}$. Let $f : \alpha \to \lambda$ be onto, where $f \in M[G]$ and $\alpha < \lambda$. By lemma 1.3, there is a $F \in M$ with $F : \alpha \to \mathcal{P}(\lambda)$, $f(\beta) \in F(\beta)$ for all $\beta < \alpha$, and $(|F(\beta)| < \kappa)^M$ for all $\beta < \alpha$. If $\lambda = \kappa$, then F expresses, in M, κ as a union of α $(< \kappa)$ union of sets each of which has size $< \kappa$, a contradiction to the regularity of κ in M. If $\lambda > \kappa$, then in M F gives a map from $\alpha \cdot \kappa$ onto λ, a contradiction as λ is a cardinal in M.

To see $\mathcal{P}$ preserves cofinalities $\geqslant \kappa$, let $\lambda \geqslant \kappa$ with $(\lambda$ is a regular cardinal$)^M$. If λ is not regular in $M[G]$, let $f \in M[G]$ with $f : \alpha \to \lambda$ cofinal, for some $\alpha < \lambda$. Let $F \in M$ be as in lemma 1.3. Since ρ is regular in M, each $F(\beta)$ is a bounded subset of λ. By regularity of ρ again, F has bounded range in λ, a contradiction as $f(\beta) \in F(\beta)$ and f is cofinal. □

Corollary 1.9. *If M is a transitive model of ZF, $\mathcal{P} \in M$ and $(\mathcal{P}$ isc.c.$)^M$, then $\mathcal{P}$ preserves all cardinals and cofinalities.*

We now show that the Cohen poset for adding an arbitrary number of real, that is FN$(\kappa, 2)$, is c.c.c. The following combinatorial lemma embodies the heart of the proof. It is known as the Δ-system lemma.

Lemma 1.10. *Let $\alpha \in ON$ and A an uncountable collection of finite subsets of α. Then there is a "root" $r \in A^{<\omega}$ and an uncountable $B \subseteq A$ such that $r \subseteq s$ for all $s \in B$, and $\{s - r : s \in B\}$ are pairwise disjoint.*

Proof. Without loss of generality we may assume $|A| = \omega_1$ (note that A is wellorderable, so there is no problem discussing its cardinality). Replacing α with $\cup A$, we may assume $|\alpha| = \omega_1$, and by taking a bijection with ω_1 we may assume $\alpha = \omega_1$. Write each element of A in the form $s = (\alpha_0(s), \dots, \alpha_k(s))$ where $\alpha_0 < \cdots < \alpha_k < \omega_1$. Thinning out A, we may assume that each $s \in A$ has size $k+1$ for some fixed $k \in \omega$. For $l \leqslant k$, let

$$\beta_l = \sup\{\alpha_l : s = (\alpha_0, \dots, \alpha_l, \dots \alpha_k) \in A\}.$$

Note that $\beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_k$. If $\beta_k < \omega_1$, then every $s \in A$ is a subset of the countable ordinal β_k. This is impossible as there are only countably many finite subsets of β_k. So, let $l \leqslant k$ be least such that $\beta_l = \omega_1$. Inductively choose $A_1 = \{t_\gamma\}_{\gamma<\omega_1} \subseteq A$ such that if $\gamma_1 < \gamma_2$ then $\alpha_k(t_{\gamma_1}) < \alpha_l(t_{\gamma_2})$. Since $\alpha_0(s), \dots, \alpha_{l-1}(s) < \beta_{l-1} < \omega_1$ for all $s \in A_1$, there is an uncountable $B \subseteq A_1$ and $\alpha_0 < \cdots < \alpha_{l-1}$ such that $(\alpha_0(s), \dots, \alpha_{l-1}(s)) = (\alpha_0 < \cdots < \alpha_{l-1})$ for all $s \in B$. Then B is as desired, with the root of the "Δ-system" being $r = \{\alpha_0, \dots, \alpha_{l-1}\}$. □

Lemma 1.11. *For any κ, FN$(\kappa, 2)$ is c.c.c.*

Proof. Suppose $\{p_\alpha\}$ were an uncountable antichain. Let $s_\alpha = \mathrm{dom}(p_\alpha)$. From lemma 1.10 we may assume the s_α form a Δ system with root r. However, there are only finitely many possibilities for $p\restriction r$ and so we may get $\alpha, \beta < \omega_1$ with $p_\alpha \restriction r = p_\beta \restriction r$. Then p_α, p_β are compatible, a contradiction. □

Summarizing, we have now show the following.

Theorem 1.12. *Let M be a transitive model of ZF anf $\mathcal{P} = FN(\omega_1^M, 2)$. Let G be M-generic for $\mathcal{P}$. Then $(2^\omega \geqslant \omega_2)^{M[G]}$.*

Proof. Let $\kappa = \omega_2^M$. Then in $M[G]$ there is a κ sequence of distinct reals. From lemma 1.10 and corollary 1.9 we have that $\omega_2^M = (\omega_2)^{M[G]}$. Thus (There is an ω_2 sequence of distinct reals)$^{M[G]}$. So, $(\neg\mathrm{CH})^{M[G]}$. □

From theorem 1.12 and our previous metamathematical remarks on forcing, we now have the following.

Corollary 1.13. *ZFC does not prove CH.*

Thus, the continuum hypothesis is not provable from within ZFC set theory.

Although Gödel's model L provides a model of CH (in fact GCH), we can also use forcing to get a model of CH. This also introduces another very useful forcing poset.

Definition 1.14. $\mathrm{coll}(\omega, \kappa)$ is the poset of all partial functions $p\colon \omega \to \kappa$ with finite domain. The partial functions are ordered by $p \leqslant q$ iff p extends q. More generally, if $\lambda < \kappa$ are cardinals with λ regular, then $\mathrm{coll}(\lambda, \kappa)$ is the poset of partial functions $p\colon \lambda \to \kappa$ with $|\mathrm{dom}(p)| < \lambda$.

Easily, in any generic extension by $\mathrm{coll}(\lambda, \kappa)$ we will have $|\lambda| = |\kappa|$. Thus, this forcing collapses κ to have cardinality λ. We would like to show, though, that the forcing does not add any bounded subsets of λ. This brings us to another important property of partial orders.

Definition 1.15. We say a poset $\mathcal{P}$ is $\leqslant \lambda$ closed (or $< \lambda$ closed) is whenever $\alpha \leqslant \lambda$ (respectively, $\alpha < \lambda$) and $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_\beta \geqslant \dots$, $\beta < \alpha$, is an α-decreasing sequence in P, then there is a $p \in P$ with $p \leqslant p_\beta$ for all $\beta < \alpha$. We say P is $\leqslant \lambda$ (or $< \lambda$) distributive if whenever $\alpha \leqslant \lambda$ ($\alpha < \lambda$) and $\{D_\beta\}_{\beta<\alpha}$ are dense in P, then $\{p\colon \forall \beta < \alpha\ \exists q \in D_\beta\ (p \leqslant q)\}$ is dense.

The name distributive comes from the fact if P is a complete Boolean algebra, then the definition of $\leqslant \lambda$ distributive is equivalent to saying P satisfies the λ-length distributive law: $\prod_{\alpha<\lambda} \sum_{\beta \in I_\alpha} a_{\alpha,\beta} = \sum_{f \in \prod I_\alpha} \prod_{\alpha<\lambda} a_{\alpha,f(\alpha)}$. [brief sketch: let b the left-hand side, and c the right-hand side. Clearly $c \leqslant b$. Suppose $b - c \neq 0$. For each $\alpha < \lambda$ let $D_\alpha = \{p\colon (\forall \beta \in I_\alpha\ (p \perp a_{\alpha,\beta})) \vee (\exists \beta \in I_\alpha\ (p \leqslant a_{\alpha,\beta}))\}$. Each D_α is clearly dense. By assumption, $D = \{p\colon \forall \alpha < \lambda\ \exists q \in D_\alpha\ (p \leqslant q)\}$ is dense. Let $u \in D$ with $u \leqslant b - c$. Then either for all $\alpha < \lambda$ we have $u \leqslant a_{\alpha,\beta}$ for some $\beta \in I_\alpha$, or else for some α, p is incompatible with all elements $a_{\alpha,\beta}$. In the first case, this defines a function $f \in \prod I_\alpha$ such that $u \leqslant a_{\alpha,f(\alpha)}$ for all $\alpha < \lambda$, and hence $u \leqslant \prod_{\alpha<\lambda} a_{\alpha,f(\alpha)}$. This shows $u \leqslant c$, a contradiction. In the second case, $u \perp \sum_{\beta \in I_\alpha} a_{\alpha,\beta}$ for some α, and thus $u \perp b$, also a contradiction.]

It is easy to see that $\leqslant \lambda$ closed (or $< \lambda$ closed) implies $\leqslant \lambda$ (or $< \lambda$) distributive. To see this, let D_β, $\beta < \alpha$ be dense. Let $p \in P$. Define inductively for $\beta \leqslant \alpha$ an element p_β such that $p \geqslant p_0 \geqslant p_1 \geqslant \cdots \geqslant p_\beta$ where $p_\beta \in D_\beta$. We can do this as P is $\leqslant \lambda$ closed, say. Then $p_\alpha \leqslant p$ and p_α extends elements in each of the D_β.

Exercise 3. Show that id $\mathcal{P}$ is $\leqslant \lambda$ distributive and D_α, $\alpha < \lambda$ are all dense below $p \in P$, then any generic filter containing p contains a $q \in G$ such that $\forall \alpha < \lambda\ \exists r \in D_\alpha\ (q \leqslant r)\}$.

In practice, most of the posets we see which are λ-distributive are actually λ-closed, but the proof of the main fact goes through for the notion of λ-distributive. This is expressed in the following lemma.

Lemma 1.16. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$ be $\leqslant \lambda$ distributive in M. Let G be M generic for $\mathcal{P}$. If $f\colon \lambda \to M$ is a function in $M[G]$, then $f \in M$.*

Proof. Let $f = \tau_G$. Let $q \Vdash$ (τ is a function with domain $\check{\lambda}$), with $q \in G$. For each $\alpha < \lambda$, let $D_\alpha = \{p\colon \exists x \in M\ (p \Vdash \tau(\check{\alpha}) = \check{x})\}$. Clearly each D_α is dense below q. Thus there is a $r \in G$ which extends elements of each D_α. So, $\forall \alpha < \lambda\ \exists! x \in M\ (r \Vdash \tau(\check{\alpha}) = \check{x})$. This shows f is definable in M and so is in M (by replacement in $M[G]$ we may assume the given f has range in a set in M). □

We have the following fact whose proof is immediate.

Lemma 1.17. *If λ is regular and $\lambda < \kappa$, then coll(λ, κ) is $< \lambda$-closed.*

Theorem 1.18. *Let M be a transitive model of ZFC, and $\mathcal{P} = (\text{coll}(\omega_1, 2^\omega))^M$. Then for any G which is M-generic over $\mathcal{P}$, $(2^\omega = \omega_1)^{M[G]}$.*

Proof. Since G gives a function from ω_1^M onto $(2^\omega)^M$, we have $(|(2^\omega)^M| \leqslant |\omega_1|^M \leqslant \omega_1)^{M[G]}$. Since $\mathcal{P}$ is $< \omega_1$ closed in M, we have that $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$. So, $(2^\omega \leqslant \omega_1)^{M[G]}$. □

Corollary 1.19. *ZFC, if it is consistent, cannot prove $\neg$CH.*

We now know that CH is independent of ZFC; that is, ZFC set theory can neither prove CH nor prove $\neg$CH.

2. More on the Continuum Function

We will show a little later that starting with a model M of ZFC we may force over M to get a model of ZFC + GCH. Furthermore, starting with a model of ZFC+GCH we may then force to get a model of ZFC where the continuum function takes any values on the regular cardinals consistent with monotonicity (if $\kappa < \lambda$ then $2^\kappa \leqslant 2^\lambda$), Cantor's theorem ($2^\kappa > \kappa$) and König's theorem ($\text{cof}(2^\kappa) > \kappa$). In this section, we prove a warm-up version.

First we show that we may force to get the GCH up to any ω_n.

Lemma 2.1. *Let M be a countable transitive model of ZFC. Then for every $n \in \omega$ there is a generic extension $M[G]$ such that $(\forall k \leqslant n\ 2^{\omega_k} = \omega_{k+1})^{M[G]}$.*

Proof. Let M be a countable transitive model of ZFC. Let $\mathcal{P}_0 = (\text{coll}(2^\omega, \omega_1)^M$. So, $\mathcal{P}_0 \in M$ and is $< \omega_1$ closed in M. Let G_0 be M generic for $\mathcal{P}_0$. Let $M_1 = M[G_0]$. Then $(2^\omega = \omega_1)^{M_1}$. Let $\mathcal{P}_1 = (\text{coll}(2^{\omega_1}, \omega_2))^{M_1}$. So, $\mathcal{P}_1$ is $< (\omega_2)^{M_1}$ closed in M_1. Let G_1 be M_1 generic for $\mathcal{P}_1$. Let $M_2 = M_1[G_1]$. Since forcing with $\mathcal{P}_1$ does not add any new subsets of ω, we have $(2^\omega)^{M_1} = (2^\omega)^{M_2}$ and $(\omega_1)^{M_1} = (\omega_1)^{M_2}$. Thus, we still have $(2^\omega = \omega_1)^{M_2}$. We clearly have $(|2^{\omega_1}|^{M_1} \leqslant |(\omega_2)^{M_1}| \leqslant \omega_2)^{M_2}$, and thus $((2^{\omega_1})^{M_1} \leqslant \omega_2)^{M_2}$. Since forcing with $\mathcal{P}_1$ does not add any new subsets of ω_1, we also have $(2^{\omega_1})^{M_1} = (2^{\omega_1})^{M_2}$. We now have $(2^{\omega_1} \leqslant \omega_2)^{M_2}$, and so $(2^\omega = \omega_1 \wedge 2^{\omega_1} = \omega_2)^{M_2}$. Continuing in this manner we finish. □

The partial orders $\mathrm{FN}(\kappa, 2)$ and $\mathrm{coll}(\kappa, \lambda)$ are both special cases of the following.

Definition 2.2. $\mathrm{FN}(\kappa, \lambda, \rho)$ is the partial order of partial functions $p \colon \kappa \to \lambda$ with $|\operatorname{dom}(p)| < \rho$. We define $p \leqslant q$ iff p extends q.

Thus, $\mathrm{FN}(\kappa, 2) = \mathrm{FN}(\kappa, 2, \omega)$, and $\mathrm{coll}(\kappa, \lambda) = \mathrm{coll}(\kappa, \lambda, \kappa)$.

The following generalizes lemma 1.11.

Lemma 2.3. *$FN(\kappa, \lambda, \rho)$ is $(\lambda^{<\rho})^+$ c.c.*

Proof. Suppose $A \subseteq \mathrm{FN}(\kappa, \lambda, \rho)$ is an antichain of size $(\lambda^{<\rho})^+$. Let $\delta = (\lambda^{<\rho})^+$, so δ is regular. Note that $\delta > \rho$. Let $A = \{p_\alpha\}_{\alpha<\delta}$, and let $d_\alpha = \operatorname{dom}(p_\alpha)$. Without loss of generality we may assume that $\kappa = \delta$ (there are at most $\rho \cdot \delta = \delta$ elements of $\bigcup_{\alpha<\delta} d_\alpha$). Since δ is regular and $\delta > \rho$, there is a $\tau < \rho$ such that for δ many α we have $\mathrm{o.t.}(d_\alpha) = \tau$. So, we may assume $\mathrm{o.t.}(d_\alpha) = \tau$ for all $\alpha < \delta$. For $\beta < \tau$ let $\eta(\beta) = \sup\{d_\alpha(\beta) \colon \alpha < \delta\}$, where $d_\alpha(\beta)$ denotes the β^{th} element of d_α. For some $\beta < \tau$ we must have $\eta(\beta) = \delta$, for otherwise, since $\delta > \tau$ is regular, we would have that $\sup_{\beta<\tau} \eta(\beta) < \delta$. This would say that all of the d_α are subsets of some $\delta' < \delta$. Hence, $|\{d_\alpha\}| \leqslant (\delta')^\tau$. But, $|\delta'| \leqslant \lambda^{<\rho}$, so $(\delta')^\tau \leqslant \lambda^{<\rho\cdot\tau} = \lambda^{<\rho} < \delta$. Hence $|A| \leqslant \lambda^{<\rho} \cdot \lambda^\tau < \delta$, a contradiction.

Let $\beta_0 < \tau$ be least such that $\eta(\beta_0) = \delta$. Since δ is regular, it is straightforward using the definition of β_0 to thin $\{d_\alpha\}$ to a subsequence of size δ such that if $\alpha_1 < \alpha_2$ then $\sup(d_{\alpha_1}) < d_{\alpha_2}(\beta_0)$. We assume that the d_α now have this property. Let $\eta = \sup\{\eta(\beta) \colon \beta < \beta_0\}$. By regularity of δ, $\eta < \delta$. We have $d_\alpha \restriction \beta_0 \subseteq \eta$. However there are then at most $\eta^\tau < \delta$ (as argued above) many possibilities for d_α. So, we may assume the d_α are are equal to some fixed d. But then there are only $\lambda^\tau < \delta$ many possibilities for $p_\alpha \restriction d$. Thus we may assume all the p_α agree on d. But any two such p_α are compatible, a contradiction. □

To compute the cardinality of the subsets of κ in $M[G]$, we need to have a reasonable representation for the subsets of κ in $M[G]$. The following lemma, which does that, is a refinement of the argument used to show the power set axiom in $M[G]$.

Lemma 2.4. *Let M be a transitive model of ZF, and $\mathcal{P} \in M$. Let $\tau \in M^{\mathcal{P}}$. Let $\mu \in M^{\mathcal{P}}$. Then there is a* nice name *(for a subset of τ) $\sigma \in M^{\mathcal{P}}$ such that $\mathbb{1} \Vdash (\mu \subseteq \tau \to \mu = \sigma)$. By a nice name we mean a name of the form $\bigcup_{\pi\in\operatorname{dom}(\tau)} \pi \times A_\pi$, where $A_\pi \subseteq P$ is an antichain.*

Proof. Let $\sigma = \{\langle \pi, p\rangle \colon \pi \in \operatorname{dom}(\tau) \wedge (p \Vdash \pi \in \mu)\}$. Let G be M generic for $\mathcal{P}$. We must show that $(\mu_G \subseteq \tau_G \to \mu_G = \sigma_G)$. So, assume $\mu_G \subseteq \tau_G$. If $z \in \sigma_G$, then $z = \pi_G$ where $p \Vdash \pi \in \mu$ and $p \in G$. So, $z = \pi_G \in \mu_G$. For the other direction, assume $z \in \mu_G$. Since $\mu_G \subseteq \tau_G$, $z = \pi_G$ for some $\langle \pi, p\rangle \in \tau$ with $p \in G$. Let $q \leqslant p$ with $q \Vdash \pi \in \mu$. Then $\langle \pi, q\rangle \in \sigma$ and $q \in G$, so $z = \pi_G \in \sigma_G$. □

In particular, every subset of an ordinal α in $M[G]$ has a name of the form $\bigcup_{\beta<\alpha}(\check{\beta} \times A_\beta)$, where the A_β are antichains.

Lemma 2.5. *Let M be a transitive model of ZFC and $\mathcal{P} \in M$ be λ-c.c. Let G be M generic for P, and κ a cardinal of $M[G]$. Then in $M[G]$ we have $2^\kappa \leqslant [(|P|^{<\lambda})^\kappa]^M$.*

Proof. Since P is λ-c.c., there are at most $(|P|^{<\lambda})^M$ many antichains of P which lie in M. Thus, there are at most $[(|P|^{<\lambda})^\kappa]^M$ many nice names for a subset of κ which lie in M. In $M[G]$, there is a map from this set of nice names onto $\mathcal{P}(\kappa)$ by lemma 2.4. □

We now show that starting with a model of GCH, we may force to get the continuum function to take arbitrary values on finitely many regular cardinals, subject to monotonicity and Cantor's and König's theorems.

Lemma 2.6. *Let M be a countable transitive model of ZFC + GCH. Let $\kappa_1 < \cdots < \kappa_n$ be regular in M. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be cardinals of M with $\operatorname{cof}(\lambda_i) > \kappa_i$. Then there is a forcing extension $M[G]$ of M which preserves all the cardinals and cofinalities of M and such that $(2^{\kappa_1} = \lambda_1 \wedge \cdots \wedge 2^{\kappa_n} = \lambda_n)^{M[G]}$.*

Proof. Let $M_0 = M$. We construct a sequence of forcing extensions $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n$, each of which will preserve cardinals and cofinalities. Assume inductively that M_i has been defined and

(1) M and M_i have the same functions $f\colon \kappa \to \mathrm{ON}$, for all $\kappa < \kappa_{n-i+1}$.
(2) $(\forall \kappa < \kappa_{n-i+1}\ (2^\kappa = \kappa^+))^{M_i}$.
(3) $(2^{\kappa_{n-i+1}} = \lambda_{n-i+1} \wedge \cdots \wedge 2^{\kappa_n} = \lambda_n)^{M_i}$.

Note that (1) implies (2) given that cardinals are preserved from M to M_i. Let $\mathcal{P} = \mathrm{FN}(\kappa_{n-i} \times \lambda_{n-i}, 2, \kappa_{n-i})^{M_i} \cong \mathrm{FN}(\lambda_{n-i}, 2, \kappa_{n-i})^{M_i}$. Thus, in M_i, $\mathcal{P}$ is $(2^{<\kappa_{n-i}})^+ = \kappa_{n-i}^+$-c.c. Since $\mathcal{P}$ is $< \kappa_{n-i}$ closed, it preseves all cardinals and cofinalities $\leqslant \kappa_{n-i}$. Since $\mathcal{P}$ is κ_{n-i}^+-c.c. it preserves all cardinals and cofinalities $> \kappa_{n-i}$. Thus, $\mathcal{P}$ preserves all cardinals and cofinalities. Since $\mathcal{P}$ is again $< \kappa_{n-i}$ closed, it adds no new functions $f\colon \kappa \to \mathrm{ON}$ for $\kappa < \kappa_{n-i}$. Thus, (1) and (2) hold for M_{i+1}. We clearly have $(2^{\kappa_{n-i}} \geqslant \lambda_{n-i})^{M_{i+1}}$. By (1), $(\lambda_{n-i}^{\kappa_{n-i}})^{M_i} = (\lambda_{n-i}^{\kappa_{n-i}})^M$. Since $(\operatorname{cof}(\lambda_{n-i}) > \kappa_{n-i})^M$ it follows from the GCH in M that $(\lambda_{n-i}^{\kappa_{n-i}} = \lambda_{n-i})^M$. Thus, $(\lambda_{n-i}^{\kappa_{n-i}} = \lambda_{n-i})^{M_i}$. In M_i, $\mathcal{P}$ has cardinality at most $\sum_{\kappa<\kappa_{n-i}} \lambda_{n-i}^\kappa = \lambda_{n-i}$. From lemma 2.5 $(2^{\kappa_{n-i}} \leqslant (\lambda_{n-i}^{\kappa_{n-i}})^{\kappa_{n-i}} = \lambda_{n-i})^{M_{i+1}}$. □

As a corollary of lemmas 2.6 and 2.1, it follows that starting with a countable transitive model M of ZFC, we may get a forcing extension $M[G]$ in which the continuum function moves the first n many infinite cardinals to any other ω_m's, subject to monotonicity and Cantor's theorem (König's theorem is irrelevant as all of the ω_m are regular). For example:

Corollary 2.7. *If ZFC is consistent, then so is ZFC + $2^\omega = \omega_5 \wedge 2^{\omega_1} = \omega_8 \wedge 2^{\omega_2} = \omega_8 \wedge 2^{\omega_7} = \omega_9$.*

Exercise 4. Compute the continuum function in the model $M[G]$ of lemma 2.6.

3. Tarski's Theorem

We prove the following theorem, valid for any partial order.

Theorem 3.1. *(ZFC) Let $\mathcal{P} = \langle P, \leqslant \rangle$ be a partial order. Then sat(P) is a regular cardinal.*

Proof. Let $\kappa = \text{sat}(P)$, so every antichain of P has size $< \kappa$, but for every $\lambda < \kappa$ there is an antichain of P of size λ. Suppose $\text{cof}(\kappa) = \rho < \kappa$. Let $B = \{p \colon \forall q \leqslant p\ (\text{sat}(q) = \kappa)\}$, where $\text{sat}(q) = \sup\{|A| \colon A \subseteq N_q \text{ is an antichain }\}$ (recall $N_q = \{r \colon r \leqslant q\}$). Suppose first that $B \neq \varnothing$, and let $p \in B$. Let $A \subseteq N_p$ be an antichain of size $|A| = \rho$. For each $q \in A$, let $A_q \subseteq N_q$ be an antichain of size $f(q) < \kappa$, where $f \colon A \to \kappa$ is cofinal (we may do this since $p \in B$). Then $\bigcup_{q \in A} A_q$ is an antichain of P of size κ, a contradicion.

Suppose next that $B = \varnothing$. Let $D = \{p \colon \text{sat}(p) < \kappa\}$. Thus, D is dense. Let $A \subseteq P$ be maximal subject to A being an antichain and $A \subseteq D$. Since D is dense, A is easily a maximal antichain, that is, for every $q \in P$ there is a $p \in A$ such that $q \parallel p$. By assumption, $|A| < \kappa$. Also, we must have $\sup\{\text{sat}(p) \colon p \in A\} = \kappa$. This is because for any regular $\lambda < \kappa$ with $\lambda > |A|$ there is an antichain of P of size λ, and there must be a single $p \in A$ such that λ many elements of this antichain are compatible with p (this then gives an antichain of size λ in N_p). We may now construct $\{p_\alpha\}_{\alpha < \rho} \subseteq A$ such that $\text{sat}(p_\alpha) > \sup\{\text{sat}(p_\beta) \colon \beta < \alpha\}$ and $\sup_{\alpha < \rho} \text{sat}(p_\alpha) = \kappa$. For each $\alpha < \rho$, let $A_\alpha \subseteq N_{p_\alpha}$ be an antichain of size $\geqslant \sup_{\beta < \alpha} \text{sat}(p_\beta)$. Then $\bigcup_{\alpha < \rho} A_\alpha$ is an antichain of P of size κ, a contradiction. □