

Cardinals

1. INTRODUCTION TO CARDINALS

We work in the base theory ZF. The definitions and many (but not all) of the basic theorems do not require AC; we will indicate explicitly when we are using choice.

Definition 1.1. For sets X, Y , we write $X \preceq Y$ to mean there is a one-to-one function from X to Y . We write $X \sim Y$ to mean there is a bijection from X to Y .

Theorem 1.2. (*Cantor-Schröder-Bernstein*) *If $X \preceq Y$ and $Y \preceq X$ then $X \sim Y$.*

Proof. Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be one-to-one. Let $X_0 = X, Y_0 = Y$ and define recursively $X_n = g(Y_{n-1}), Y_n = f(X_{n-1})$ for $n \geq 1$. Thus, $X_0 = X, X_1 = g(Y), X_2 = g \circ f(X), X_3 = g \circ f \circ g(Y)$, etc., and similarly for the Y_n . Note that $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$, and $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$. Let $X_\infty = \bigcap_n X_n, Y_\infty = \bigcap_n Y_n$. Define $h: X \rightarrow Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in X_n - X_{n+1} \text{ and } n \text{ is even} \\ g^{-1}(x) & \text{if } x \in X_n - X_{n+1} \text{ and } n \text{ is odd} \\ f(x) & \text{if } x \in X_\infty \end{cases}$$

Easily h is a bijection between $X_n - X_{n+2} = (X_n - X_{n+1}) \cup (X_{n+1} - X_{n+2})$ and $Y_n - Y_{n+2} = (Y_n - Y_{n+1}) \cup (Y_{n+1} - Y_{n+2})$ for any even n , as f bijects $(X_n - X_{n+1})$ and $(Y_{n+1} - Y_{n+2})$ and g^{-1} bijects $(X_{n+1} - X_{n+2})$ and $(Y_n - Y_{n+1})$. Thus, h is a bijection between $X - X_\infty$ and $Y - Y_\infty$. It is easy to also see that f induces a bijection from X_∞ to Y_∞ , and thus h is a bijection from X to Y . \square

Note that the above proof did not use AC.

Definition 1.3. κ is a cardinal if $\kappa \in \text{ON}$ and $\forall \alpha < \kappa \neg(\alpha \sim \kappa)$.

Note that this definition is easily formalized into set-theory, and thus the collection of cardinals is a class. Also, note that AC is not used in the definition of cardinal. We frequently use κ, λ, μ for cardinals.

If $\alpha \in \text{ON}$, we let $|\alpha|$ denote the least ordinal β such that $\beta \sim \alpha$. Clearly $|\alpha|$ is a cardinal. Also, $|\alpha| \leq \alpha$. Again, AC is not needed to define $|\alpha|$ for $\alpha \in \text{ON}$. We refer to $|\alpha|$ as the *cardinality* of α .

Exercise 1. Show that every $n \in \omega, \neg(n+1 \preceq n)$. Make sure you are giving a “real” proof from within ZF; in particular, you are using only the official definition of integer.

Exercise 2. Show that every $n \in \omega$ is a cardinal, and ω is a cardinal.

We define addition and multiplication on the cardinals. Note that these are different operations than those defined on the ordinals. Thus, when we write $\kappa \cdot \lambda$, it needs to be specified (or be clear from the context) whether we are talking about multiplication as ordinals or as cardinals.

Definition 1.4. for cardinals κ, λ :

- (1) $\kappa + \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$
- (2) $\kappa \cdot \lambda = |\kappa \times \lambda|$

Note that $\kappa + \lambda$ is just the cardinality of the ordinals sum of κ and λ , and likewise for multiplication. Thus, these operations are clearly associative (also trivial to show directly). They are also clearly commutative, unlike the ordinal operations.

Lemma 1.5. *Any infinite cardinal is a limit ordinal.*

Proof. If κ is an infinite cardinal and $\kappa = \alpha + 1$, then α is also infinite. However, if α is an infinite ordinal then $\alpha = 1 + \alpha \sim \alpha + 1$ and so $|\kappa| = |\alpha|$, a contradiction. \square

Exercise 3. Show that for any cardinals κ, λ that $\kappa + \lambda = \max\{\kappa, \lambda\}$. [hint: Say $\lambda \leq \kappa$. Prove by induction on $\beta \in \text{ON}$ that if $\alpha \leq \beta$ and γ is the mingling of α and β according to $x_\eta < y_\eta < x_{\eta+1}$ (here γ is the order-type of $X \cup Y$, and the x_η, y_η enumerate X, Y respectively), then $\gamma \leq \beta + n$ for some $n \in \omega$ if β is a successor, and $\gamma \leq \beta$ if β is a limit].

Using AC, we can the notion of cardinality of an arbitrary set.

Definition 1.6. (ZFC) Let $|X|$ denote the least ordinal α such that $\alpha \sim X$.

Using AC, this definition is well-defined, as every set X can be well-ordered, and thus put into bijection with an ordinal. Clearly the least such ordinal, namely $|X|$, is a cardinal.

Exercise 4. Using AC, give another proof of the Cantor-Schröder-Bernstein theorem. (hint: First identify X, Y with cardinals, say κ, λ . Without loss of generality, $\kappa < \lambda$. Show, without using Cantor-Schröder-Bernstein, that if there were a one-to-one map from λ to κ , then $\kappa \sim \lambda$, a contradiction. Use the previous exercise.)

The next result shows that for infinite cardinals κ, λ , $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$, and thus addition and multiplication on the infinite cardinals coincide.

Theorem 1.7. *If κ is an infinite cardinal, then $\kappa \cdot \kappa = \kappa$.*

Proof. We prove this by induction on κ . Consider the following ordering (Gödel ordering) on $\kappa \times \kappa$:

$$(\alpha, \beta) \triangleleft (\alpha', \beta') \leftrightarrow \max\{\alpha, \beta\} < \max\{\alpha', \beta'\} \vee \\ [(\max\{\alpha, \beta\} = \max\{\alpha', \beta'\}) \wedge (\alpha, \beta) <_{\text{lex}} (\alpha', \beta')].$$

This is easily a well-order of $\kappa \times \kappa$. For each $(\alpha, \beta) \in \kappa \times \kappa$, the initial segment of this order determined by (α, β) is in bijection with a subset of $\max\{\alpha, \beta\} \times \max\{\alpha, \beta\}$ which by induction has cardinality at most $\max\{\alpha, \beta\} < \kappa$. Thus, the ordering \triangleleft has length at most κ . \square

Note again that the above theorem, which completely characterizes addition and multiplication on the infinite cardinals, is proved without AC. The next lemma generalizes theorem 1.7, but now requires the axiom of choice.

Theorem 1.8. (ZFC) *Let $\kappa \in \text{CARD}$. Let $X = \bigcup_{\alpha < \kappa} X_\alpha$, where $|X_\alpha| \leq \kappa$ for all $\alpha < \kappa$. Then $|X| \leq \kappa$.*

Proof. Using AC, let f be a function with domain κ such that for all $\alpha < \kappa$, $f(\alpha): \kappa \rightarrow X_\alpha$ is a bijection. Define $F: \kappa \times \kappa \rightarrow X$ by $F(\alpha, \beta) = f(\alpha)(\beta)$. Clearly F is onto, and the result follows by lemma 1.7. \square

Exercise 5. Show that $|\mathbb{R}| = |\omega^\omega| = 2^\omega$.

Definition 1.9. We say a set X is *finite* if $|X| < \omega$, we say X is *countable* if $|X| \leq \omega$.

Exercise 6. Show that \mathbb{Z} , \mathbb{Q} , and the set of algebraic numbers are countable. Show that the set of formulas in the language of set theory is countable.

To show that there is any uncountable cardinal requires the power set axiom (this can be stated as a precise theorem; will mention later). Armed with the power set axiom, however, the following is a basic result.

Theorem 1.10. (*Cantor*) For any set X , there does not exist a map from X onto $\mathcal{P}(X)$. In particular, $\neg(X \sim \mathcal{P}(X))$.

Proof. Suppose $f: X \rightarrow \mathcal{P}(X)$ were onto. Define $A \subseteq X$ by $A = \{x \in X: x \notin f(x)\}$. Since $A \in \mathcal{P}(X)$, let $a \in X$ be such that $f(a) = A$. Then $a \in f(a) \leftrightarrow a \in A \leftrightarrow a \notin f(a)$, a contradiction. \square

With the axiom of choice it follows that for all cardinals κ that $\mathcal{P}(\kappa)$ has cardinality greater than κ , and thus a cardinal greater than κ exists. We would like to get this without the axiom of choice. To do this, fix a cardinal κ , and let $W_\kappa = \{S \in \mathcal{P}(\kappa \times \kappa): S \text{ is a well-ordering}\}$. Note the use of the power set axiom here. Using the replacement axiom, let $A = \{\alpha \in \text{ON}: \exists S \in W_\kappa (\text{o.t.}(S) = \alpha)\}$. Then $\cup A \in \text{ON}$, in fact, A is easily a limit ordinal so $\cup A = A$. First note that if $\alpha < A$ then $|\alpha| = \kappa$ since by definition $\alpha = \text{o.t.}(S)$ where S is a well-ordering of a subset of κ . Since any well-ordering of κ clearly has an order-type which is in bijection with κ , we get that $\forall \alpha < A (|\alpha| \leq \kappa)$. Next we claim that $|A| > \kappa$. For suppose $f: \kappa \rightarrow A$ were a bijection. Let \prec be the corresponding well-ordering of κ defined by $\gamma \prec \delta \leftrightarrow f(\gamma) < f(\delta)$. f itself is an order-isomorphism between (κ, \prec) and (A, \in) , and so $\text{o.t.}(\prec) = A$. This is a contradiction, since A is a limit (i.e., we can easily now define a well-order \prec' of length $A + 1$).

We have thus shown in ZF the following:

Lemma 1.11. For each cardinal κ there is a cardinal greater than κ .

The following definition is therefore well-defined in ZF.

Definition 1.12. For κ a cardinal, we let κ^+ denote the least cardinal greater than κ .

Lemma 1.13. If $A \subseteq \text{CARD}$, then $\sup(A) \in \text{CARD}$.

Proof. Let $\mu = \sup(A)$. Let $\alpha < \mu$. Let $\kappa \in A$ be such that $\alpha < \kappa$. Since κ is a cardinal, there does not exist a one-to-one map from κ into α . Since $\kappa \leq \mu$, there does not therefore exist a one-to-one map from μ into α . So, $\forall \alpha < \mu \neg(|\alpha| = \mu)$ and μ is thus a cardinal. \square

We can now describe the general picture of the cardinals. We define by transfinite recursion on the ordinals for each $\alpha \in \text{ON}$ the cardinal \aleph_α which is also denoted ω_α .

Definition 1.14. ω_α is defined by transfinite recursion by:

- (1) $\omega_0 = \omega$
- (2) $\omega_{\alpha+1} = (\omega_\alpha)^+$
- (3) For α limit, $\omega_\alpha = \sup_{\beta < \alpha} \omega_\beta$.

Theorem 1.15. ω_α is defined for all $\alpha \in \text{ON}$ and is a cardinal. Furthermore, every cardinal is of the form ω_α for some $\alpha \in \text{ON}$.

Proof. That ω_α is well-defined and a cardinal follows from the transfinite recursion theorem and lemmas 1.11, 1.13. Suppose $\kappa \in \text{CARD}$. Let $\alpha \in \text{ON}$ be least such that $\omega_\alpha \geq \kappa$. This is well-defined since a trivial induction gives $\omega_\beta \geq \beta$ for all $\beta \in \text{ON}$. If $\alpha = \beta + 1$ is a successor, then $\omega_\beta < \kappa$ and so $\omega_\alpha = (\omega_\beta)^+ \leq \kappa$, and thus $\omega_\alpha = \kappa$. If α is a limit, then $\omega_\alpha = \sup_{\beta < \alpha} \omega_\beta \leq \kappa$ as each ω_β for $\beta < \alpha$ is $< \kappa$. So again $\omega_\alpha = \kappa$. \square

Finally in this section we introduce cardinal exponentiation.

Definition 1.16. For $\alpha, \beta \in \text{ON}$, let ${}^\alpha\beta = \{f : f \text{ is a function from } \alpha \text{ to } \beta\}$. For κ, λ cardinals, let $\kappa^\lambda = |{}^\lambda\kappa|$. Let $\kappa^{<\lambda} = \sup_{\alpha < \lambda} \kappa^\alpha$.

Note that we need AC to discuss κ^λ , and we henceforth assume AC in discussions where cardinal exponentiation is involved.

The following easy lemma says that the “familiar” exponent rules apply to cardinal exponentiation.

Lemma 1.17. For κ, λ, σ cardinals we have $\kappa^{\lambda+\sigma} = \kappa^\lambda \cdot \kappa^\sigma$, $(\kappa^\lambda)^\sigma = \kappa^{\lambda \cdot \sigma}$.

Proof. This follows from the following facts about sets A, B, C :

- 1) If $B \cap C = \emptyset$, then there is an obvious bijection between ${}^{(B \cup C)}A$ and ${}^B A \times {}^C A$.
- 2.) There is an obvious bijection between ${}^C({}^B A)$ and ${}^{(C \times B)}A$. \square

Here is one easy computation:

Lemma 1.18. If $2 \leq \kappa \leq \lambda$ are cardinals, then $\kappa^\lambda = 2^\lambda$.

Proof. Clearly $\kappa^\lambda \geq 2^\lambda$. Also, $2^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$. \square

We also have the following easy basic fact.

Lemma 1.19. For all cardinals κ , $2^\kappa = |\mathcal{P}(\kappa)|$.

Proof. This follows immediately from the fact that we may identify any $A \in \mathcal{P}(\kappa)$

with its characteristic function $f_A(\alpha) = \begin{cases} 1 & \text{if } \alpha \in A \\ 0 & \text{if } \alpha \notin A \end{cases}$. \square

So by Cantor’s theorem we have that $2^\kappa > \kappa$ for all cardinals κ . Exactly how big is 2^κ ? More generally, how do we compute κ^λ ? We will turn to these questions shortly, but first we introduce some terminology.

Definition 1.20. The *continuum hypothesis*, CH, is the assertion that $2^\omega = \omega_1$. The *generalized continuum hypothesis*, GCH, is the assertion that $2^{\omega_\alpha} = \omega_{\alpha+1}$ for all $\alpha \in \text{ON}$.

Exercise 7. Show that there are 2^{2^ω} functions from \mathbb{R} to \mathbb{R} . Show that there are 2^ω many continuous functions from \mathbb{R} to \mathbb{R} .

Exercise 8. Let X be a second countable space. Show that there are 2^ω open (likewise for closed) subsets of X . Show that there are 2^ω many Borel subsets of X .

Exercise 9. Let $f : \omega_1 \rightarrow \mathbb{R}$ be an ω_1 sequence of reals. Show that for some $\alpha_0 < \alpha_1 < \alpha_2 < \dots$ we have $f(\alpha_0) > f(\alpha_1) > f(\alpha_2) > \dots$ (here we mean in the usual ordering on the reals).

2. COFINALITY

The following is a basic, important definition.

Definition 2.1. Let α be a limit ordinal. Then $\text{cof}(\alpha)$ is the least ordinal β such that there is a function $f: \beta \rightarrow \alpha$ which is unbounded (also said to be cofinal) in α . That is, $\forall \alpha' < \alpha \exists \beta' < \beta (f(\beta) \geq \alpha')$.

Clearly $\text{cof}(\alpha) \leq \alpha$ and is an infinite limit ordinal.

Definition 2.2. An ordinal α is said to be *regular* if $\text{cof}(\alpha) = \alpha$. Otherwise α is said to be *singular*.

Sometimes the convention is made that for α a successor that $\text{cof}(\alpha) = 1$, but we generally only refer to $\text{cof}(\alpha)$ when α is a limit.

Lemma 2.3. *Let α be a limit ordinal. Then there is an increasing map $f: \text{cof}(\alpha) \rightarrow \alpha$ which is cofinal.*

Proof. Let $g: \text{cof}(\alpha) \rightarrow \alpha$ be cofinal. Define f by

$$f(\beta) = \max\{g(\beta), \sup_{\gamma < \beta} f(\gamma) + 1\}.$$

□

Thus, cofinalities are always witnessed by strictly increasing maps. The next fact is a sort of converse to this.

Lemma 2.4. *If α, β are limit ordinals and $f: \alpha \rightarrow \beta$ is cofinal and monotonically increasing (i.e., if $\alpha_1 < \alpha_2$ then $f(\alpha_1) \leq f(\alpha_2)$), then $\text{cof}(\alpha) = \text{cof}(\beta)$.*

Proof. If $g: \text{cof}(\alpha) \rightarrow \alpha$ is cofinal, then $f \circ g: \text{cof}(\alpha) \rightarrow \beta$ is cofinal, so $\text{cof}(\beta) \leq \text{cof}(\alpha)$. Suppose $h: \text{cof}(\beta) \rightarrow \beta$ is increasing, cofinal. Define $k: \text{cof}(\beta) \rightarrow \alpha$ by $k(\gamma) = \text{least } \eta \text{ such that } f(\eta) > g(\gamma)$. This is well-defined, and easily cofinal into α . Thus, $\text{cof}(\alpha) \leq \text{cof}(\beta)$. □

Lemma 2.5. *If α is a limit ordinal, then $\text{cof}(\alpha)$ is a regular cardinal.*

Proof. We have already observed that $\text{cof}(\alpha)$ is an infinite limit ordinal. We have $\text{cof}(\alpha)$ regular, that is, $\text{cof}(\text{cof}(\alpha)) = \text{cof}(\alpha)$ as otherwise by composing functions (as in lemma 2.4) we would have a cofinal map from $\text{cof}(\text{cof}(\alpha))$ to α . It remains to observe that a regular ordinal is necessarily a cardinal. Suppose β is a regular limit ordinal, but $\kappa = |\beta| < \beta$. Let $f: \kappa \rightarrow \beta$ be a bijection. In particular, f is cofinal, so $\text{cof}(\beta) \leq \kappa < \beta$, a contradiction. □

Thus, $\text{cof}(\text{cof}(\alpha)) = \text{cof}(\alpha)$ for all (limit) α .

The above lemmas did not use AC. The next lemma does use AC, and can fail without it.

Lemma 2.6. (ZFC) *For every cardinal κ , κ^+ is regular.*

Proof. If not, then there would be a cofinal map $f: \lambda \rightarrow \kappa$, where $\lambda \leq \kappa$. This would write κ^+ as a κ union of sets, each of which size $\leq \kappa$. This contradicts theorem 1.8. □

Thus, in ZFC all successor cardinals are regular. What about limit cardinals? (note: by “limit cardinal” we mean a cardinal not of the form κ^+ . Of course, all infinite cardinals are limit ordinals.) The next lemma computes the cofinality of a limit cardinal.

Lemma 2.7. *Let ω_α be a limit cardinal. Then α is a limit ordinal and $\text{cof}(\omega_\alpha) = \text{cof}(\alpha)$.*

Proof. Clearly α must be a limit ordinal as otherwise ω_α would be a successor cardinal. The map $\beta \mapsto \omega_\beta$ is increasing and cofinal from α to ω_α , so by lemma 2.4 it follows that $\text{cof}(\omega_\alpha) = \text{cof}(\alpha)$. \square

Examples. ω is regular. Assuming ZFC, so are $\omega_1, \omega_2, \dots, \omega_n, \dots$. ω_ω is singular of cofinality ω . All limit cardinals below ω_{ω_1} are singular of cofinality ω . ω_{ω_1} is singular of cofinality ω_1 . $\omega_{(\omega_{\omega_2} + \omega_1)}$ is singular of cofinality ω_1 , while $\text{cof}(\omega_{\omega_{\omega_2}}) = \omega_2$.

All the limit cardinals mentioned above are singular. Are there any regular limit cardinals? We will see that this question is independent of ZFC (in fact, the existence of such cardinals has a stronger consistency strength than ZFC). For now, we make this into a definition.

Definition 2.8. κ is *weakly inaccessible* if κ is a regular limit cardinal. κ is *strongly inaccessible* if κ is regular and $\forall \lambda < \kappa (2^\lambda < \kappa)$. κ is a *strong limit cardinal* if $\forall \lambda < \kappa (2^\lambda < \kappa)$.

Exercise 10. Assuming GCH, identify the strong limit cardinals, and show that κ is weakly inaccessible iff κ is strongly inaccessible.

We end this section with a basic result on cardinal arithmetic.

Theorem 2.9. (König) *For every infinite cardinal κ , $\kappa^{\text{cof}(\kappa)} > \kappa$.*

Proof. Let $f: \text{cof}(\kappa) \rightarrow \kappa$ be cofinal. Suppose $\{g_\alpha\}_{\alpha < \kappa}$ enumerated $(\text{cof}(\kappa))^\kappa$. Define $g \in (\text{cof}(\kappa))^\kappa$ by $g(\beta) =$ the least element of $\kappa - \{g_\alpha(\beta) : \beta < f(\alpha)\}$. Clearly $g \neq g_\alpha$ for any α , a contradiction. \square

Corollary 2.10. *For every infinite cardinal κ , $\text{cof}(2^\kappa) > \kappa$. More generally, $\text{cof}(\kappa^\lambda) > \lambda$.*

Proof. Since $(\kappa^\lambda)^\lambda = \kappa^\lambda$, theorem 2.9 gives that $\text{cof}(\kappa^\lambda)$ must be greater than λ . \square

We can also state König's theorem in a slightly more general form. First, we define the infinitary sum and products operations.

Definition 2.11. If $\{\kappa_\alpha\}_{\alpha < \delta}$ is a sequence of cardinals, we define $\sum_{\alpha < \delta} \kappa_\alpha$ to be the cardinality of the disjoint union of sets X_α , $\alpha < \delta$, where $|X_\alpha| = \kappa_\alpha$. We define $\prod_{\alpha < \delta} \kappa_\alpha$ to be the cardinality of the functions f with domain δ such that $f(\alpha) \in \kappa_\alpha$ for all $\alpha < \delta$.

Exercise 11. Show that $\sum_{\alpha < \delta} \kappa_\alpha = \delta \cdot \sup_{\alpha < \delta} (\kappa_\alpha)$.

The same diagonal argument used in theorem 2.9 also shows the following.

Theorem 2.12. (König) *Let $\kappa_\alpha < \lambda_\alpha$ for all $\alpha < \delta$. Then $\sum_{\alpha < \delta} \kappa_\alpha < \prod_{\alpha < \delta} \lambda_\alpha$.*

Proof. Let $X_\alpha \subseteq \prod_{\alpha < \delta} \lambda_\alpha$, for $\alpha < \delta$, and $|X_\alpha| = \kappa_\alpha$. We must show that $\bigcup_{\alpha < \delta} X_\alpha \neq \prod_{\alpha < \delta} \lambda_\alpha$. For each $\alpha < \delta$, let $f(\alpha)$ be an element of λ_α which is not in $\{g(\alpha) : g \in X_\alpha\}$. This exists since $|X_\alpha| = \kappa_\alpha < \lambda_\alpha$. Clearly, $g \notin \bigcup_{\alpha < \delta} X_\alpha$. \square

Note that theorem 2.12 implies theorem 2.9 by taking $\kappa_\alpha = \kappa$ for all $\alpha < \delta = \text{cof}(\kappa)$.

3. CARDINAL ARITHMETIC UNDER THE GCH

Assuming ZFC + GCH we can readily compute κ^λ for all κ and λ .

Theorem 3.1. *Assume ZFC + GCH. Let κ be an infinite cardinal. Then:*

- (1) *If $\lambda < \text{cof}(\kappa)$, then $\kappa^\lambda = \kappa$.*
- (2) *If $\text{cof}(\kappa) \leq \lambda \leq \kappa$, then $\kappa^\lambda = \kappa^+$.*
- (3) *If $\lambda > \kappa$ then $\kappa^\lambda = \lambda^+$.*

Proof. If $\lambda < \text{cof}(\kappa)$, then any function $f : \lambda \rightarrow \kappa$ has bounded range. Thus, $\kappa^\lambda = \bigcup_{\alpha < \kappa} \alpha^\lambda$. Also, $\alpha^\lambda \leq 2^{\max(\alpha, \lambda)} \leq \kappa$ since $\alpha, \lambda < \kappa$, using the GCH.

If $\text{cof}(\kappa) \leq \lambda \leq \kappa$, then $\kappa^\lambda \geq \kappa$ by theorem 2.9. Also, $\kappa^\lambda \leq \kappa^\kappa = \kappa^+$ by GCH.

If $\lambda > \kappa$, then $\kappa^\lambda \leq 2^\lambda = \lambda^+$. Also, clearly $\kappa^\lambda \geq 2^\lambda = \lambda^+$. \square

4. CARDINAL ARITHMETIC IN GENERAL

We discuss now some properties of cardinal arithmetic assuming only ZFC. We only scratch the surface here, aiming toward one specific result. We first introduce two cardinal functions.

Definition 4.1. The *beth* function \beth_α is defined for $\alpha \in \text{ON}$ by recursion on α as follows: $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and for α limit, $\beth_\alpha = \sup_{\beta < \alpha} \beth_\beta$.

Definition 4.2. The *gimel* function \beth_α is defined by $\beth(\kappa) = \kappa^{\text{cof}(\kappa)}$, for all cardinals κ .

Exercise 12. Show that $\beth(\kappa) > \kappa$ and $\text{cof}(\beth(\kappa)) > \text{cof}(\kappa)$.

We now show that cardinal exponentiation is determined entirely from the gimel function and the cofinality function. From these two functions we compute κ^λ by induction on κ as follows.

Theorem 4.3. *For all infinite cardinals κ, λ we have:*

- (1) *If $\lambda \geq \kappa$ then $\kappa^\lambda = 2^\lambda$.*
- (2) *If for some $\mu < \kappa$ we have $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
- (3) *If for all $\mu < \kappa$ we have $\mu^\lambda < \kappa$, then:*
 - *If $\lambda < \text{cof}(\kappa)$ then $\kappa^\lambda = \kappa$.*
 - *If $\text{cof}(\kappa) \leq \lambda < \kappa$, then $\kappa^\lambda = \beth(\kappa)$.*

Proof. (1) is clear, and (2) follows from $\kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$. The first case of (3) follows from the fact that $\kappa^\lambda = \sup_{\mu < \kappa} \mu^\lambda = \kappa$ from the assumption of the case. For the second case of (3), clearly we have $\kappa^\lambda \geq \kappa^{\text{cof}(\kappa)} = \beth(\kappa)$. The upper bound follows from the following lemma. \square

Lemma 4.4. *If $\lambda \geq \text{cof}(\kappa)$, then $\kappa^\lambda = (\sup_{\alpha < \kappa} \alpha^\lambda)^{\text{cof}(\kappa)}$.*

Proof. Fix a cofinal map $h : \text{cof}(\kappa) \rightarrow \kappa$. To each $f : \lambda \rightarrow \kappa$, and each $\beta < \text{cof}(\kappa)$ we associate a partial function $f_\beta : \lambda \rightarrow h(\beta)$. Namely, let $f_\beta = f \cap (\lambda \times h(\beta))$. The map $f \rightarrow (f_\beta)_{\beta < \text{cof}(\kappa)}$ is clearly a one-to-one map from κ^λ to $(\sup_{\alpha < \kappa} \alpha^\lambda)^{\text{cof}(\kappa)}$. \square

The next lemma gives some information about the continuum function at singular cardinals.

Lemma 4.5. *Let κ be a singular cardinal and suppose the continuum function below κ is eventually constant, that is, there is a $\gamma < \kappa$ such that $2^{<\kappa} = 2^\gamma$. Then $2^\kappa = 2^\gamma$.*

Proof. An argument similar to lemma 4.4 shows that for any κ we have $2^\kappa \leq (2^{<\kappa})^{\text{cof}(\kappa)}$, and hence easily $2^\kappa = (2^{<\kappa})^{\text{cof}(\kappa)}$. Thus, $2^\kappa = (2^{<\kappa})^{\text{cof}(\kappa)} = (2^\gamma)^{\text{cof}(\kappa)} = (2^\delta)^{\text{cof}(\kappa)} = 2^\delta = 2^\gamma$, where $\gamma < \delta < \kappa$ and $\delta \geq \text{cof}(\kappa)$. \square

We thus have the following theorem, which shows that the continuum function can be computed entirely from the gimel function.

Theorem 4.6. *For any cardinal κ :*

- (1) *If κ is a successor cardinal, then $2^\kappa = \mathfrak{J}(\kappa)$.*
- (2) *If κ is a limit cardinal and the continuum below κ is eventually constant, then $2^\kappa = 2^{<\kappa} \cdot \mathfrak{J}(\kappa)$.*
- (3) *If κ is a limit cardinal and the continuum below κ is not eventually constant, then $2^\kappa = \mathfrak{J}(2^{<\kappa})$.*

Proof. The first part is clear as successor cardinals are regular. Suppose that κ is a limit cardinal, and the continuum function is eventually constant below κ , say $2^{<\kappa} = 2^\gamma$. If κ is singular, then $2^\kappa = 2^\gamma$ by the previous lemma. If κ is regular, then $2^\kappa = \kappa^\kappa = \kappa^{\text{cof}(\kappa)} = \mathfrak{J}(\kappa)$. In the singular case, note that $2^\gamma = 2^{<\kappa} \geq \kappa$. Thus, $\mathfrak{J}(\kappa) = \kappa^{\text{cof}(\kappa)} \leq (2^\gamma)^{\text{cof}(\kappa)} = (2^\delta)^{\text{cof}(\kappa)} = 2^\delta = 2^\gamma$ (where $\text{cof}(\kappa) < \delta < \kappa$). Thus, $2^{<\kappa} \cdot \mathfrak{J}(\kappa) = 2^{<\kappa}$. If κ is regular, then $2^{<\kappa} \leq 2^\kappa = \kappa^\kappa = \mathfrak{J}(\kappa)$. So, $2^{<\kappa} \cdot \mathfrak{J}(\kappa) = \mathfrak{J}(\kappa)$. In either case, $2^\kappa = 2^{<\kappa} \cdot \mathfrak{J}(\kappa)$. Finally, suppose κ is a limit cardinal and the continuum function below κ is not eventually constant. Then $\text{cof}(2^{<\kappa}) = \text{cof}(\kappa)$. Then $2^\kappa = (2^{<\kappa})^{\text{cof}(\kappa)} = \mathfrak{J}(2^{<\kappa})$. \square

Note that if κ is a strong limit cardinal (i.e., $2^{<\kappa} = \kappa$) then $2^\kappa = \mathfrak{J}(\kappa)$, so the continuum function and gimel function coincide in the main case of interest.

What can be said about the value of 2^κ or $\mathfrak{J}(\kappa)$ in general? We shall see that if κ is regular, then nothing more can be said that $2^\kappa > \kappa$ and $\text{cof}(2^\kappa) > \kappa$ (and of course if $\kappa_1 < \kappa_2$ then $2^{\kappa_1} \leq 2^{\kappa_2}$). So, for κ regular, $2^\kappa = \mathfrak{J}(\kappa)$ is basically arbitrary subject to Cantor's and König's theorems.

If κ is a singular limit cardinal, the situation is much more difficult. Here there are restrictions on 2^κ . For now, let's just observe one more fact about the gimel function. Note that if there is a $\lambda < \kappa$ such that $\lambda^{\text{cof}(\kappa)} \geq \kappa$, then $\mathfrak{J}(\kappa) = \lambda^{\text{cof}(\kappa)}$, and so is computed by exponentiation on smaller cardinals.

Exercise 13. Suppose κ is a singular limit cardinal. Suppose there is a $\lambda < \kappa$ such that $\lambda^{\text{cof}(\kappa)} \geq \kappa$, and let λ be the least such. Show that $\text{cof}(\lambda) \leq \text{cof}(\kappa)$.

More generally, if there is a $\lambda < \kappa$ with $\text{cof}(\lambda) \geq \text{cof}(\kappa)$ and $\mathfrak{J}(\lambda) \geq \kappa$, then $\mathfrak{J}(\kappa) = \kappa^{\text{cof}(\kappa)} \leq \lambda^{\text{cof}(\lambda) \cdot \text{cof}(\kappa)} = \lambda^{\text{cof}(\lambda)} = \mathfrak{J}(\lambda)$.

Finally, suppose κ is a singular limit cardinal and there is no $\lambda < \kappa$ such that $\lambda^{\text{cof}(\kappa)} \geq \kappa$. We claim that $\text{cof}(\mathfrak{J}(\kappa)) \geq$ the the least α such that $\lambda^\alpha \geq \kappa$ for some $\lambda < \kappa$. For suppose $\lambda^\alpha < \kappa$ for all $\lambda < \kappa$, we show that $\text{cof}(\mathfrak{J}(\kappa)) > \alpha$. Then $\mathfrak{J}(\kappa)^\alpha = \kappa^{\text{cof}(\kappa) \cdot \alpha}$. We may assume $\alpha \geq \text{cof}(\kappa)$ as we know that $\text{cof}(\mathfrak{J}(\kappa)) \geq \text{cof}(\kappa)$. Thus, $\mathfrak{J}(\kappa)^\alpha = \kappa^\alpha = (\sup_{\lambda < \kappa} \lambda^\alpha)^{\text{cof}(\kappa)} = \kappa^{\text{cof}(\kappa)} = \mathfrak{J}(\kappa)$. Thus, $\text{cof}(\mathfrak{J}(\kappa)) > \alpha$ by König.

We have thus shown:

Lemma 4.7. *Suppose κ is a singular strong limit cardinal. Then $\text{cof}(\mathfrak{J}(\kappa)) > \kappa$.*