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An Inquiry into the Number of Isomorphism Classes of Boolean Algebras and the Borel Cardinality of Certain Borel Equivalence Relations

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

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to the memory of my father, Earl Davis Oliver

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PUBLICATIONS

Andrew E. Caldwell, Yu Cao, Andrew B. Kahng, Farinaz Koushanfar, Hua Lu, Igor L. Markov, Michael Oliver, Dirk Stroobandt, and Dennis Sylvester, *GTX: The MARCO GSRC technology extrapolation system*, ***Proc. DAC, 2000*** (2000), pp. 693–698

Y. Cao, C. Hu, X. Huang, A. B. Kahng, I. Markov, M. Oliver, D. Stroobandt, and D. Sylvester, *Improved a priori interconnect predictions and technology extrapolation in the GTX system*, ***IEEE Transactions on Very Large Scale Integration Systems*** (2002)

Mike Oliver, *On the restriction of the domain of universals in Loglan*, ***The Loglanist***, vol. 2 (1978), pp. 50–51

———, *Three proposed additions to the Loglan prepositional system*, ***The Loglanist***, vol. 3 (1979), no. 2, pp. 96–98

Abstract of the Dissertation

An Inquiry into the Number of Isomorphism Classes of Boolean Algebras and the Borel Cardinality of Certain Borel Equivalence Relations

by

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The dissertation consists of two chapters, each addressing a separate problem. Both problems relate to the notions of definable cardinality and ideals on the natural numbers.

In the first chapter we examine the question of how many Boolean algebras, distinct up to isomorphism, that are quotients of the powerset of the naturals by Borel ideals, can be proved to exist in ZFC alone. The maximum possible value is easily seen to be the cardinality of the continuum $2^{\aleph_0}$; earlier work by Ilijas Farah had shown that this was the value in models of Martin's Maximum or some similar forcing axiom, but it was open whether there could be fewer in models of the Continuum Hypothesis.

We develop and apply a new technique for constructing many ideals whose quotients must be nonisomorphic in any model of ZFC. The technique depends on isolating a kind of ideal, called *shallow*, that can be distinguished from the ideal of all finite sets even after any isomorphic embedding, and then piecing

together various copies of the ideal of all finite sets using distinct shallow ideals. In this way we are able to demonstrate that there are continuum-many distinct quotients by Borel ideals, indeed by analytic P-ideals, and in fact that there is in an appropriate sense a Borel embedding of the Vitali equivalence relation into the equivalence relation of isomorphism of quotients by analytic P-ideals. We also show that there is a definable wellordered collection of Borel ideals with distinct quotients.

The second chapter addresses the Borel cardinality of the ideal $\mathscr{Z}_0$ of asymptotically zero-density sets, shown to be the same as that of the equivalence relation induced by the classical Banach space c_0, and also shows that a large collection of ideals introduced by Louveau and Veličkovič, with pairwise incomparable Borel cardinality, are all Borel reducible to c_0. This refutes a conjecture of Hjorth and has facilitated further work by Farah.

CHAPTER 1

Cardinality of the Isomorphism Types of Quotients by Borel Ideals

1.1 Introduction and nomenclature

1.1.1 Background

In [Far03], Farah asks the question "How Many Boolean Algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ Are There?", with the understanding that there is some definability criterion being imposed on $\mathcal{I}$, since if no such criterion is imposed then every Boolean algebra of cardinality $2^{\aleph_0}$ is isomorphic to one of the form $\mathcal{P}(\omega)/\mathcal{I}$ and there were known to be $2^{2^{\aleph_0}}$ such (pairwise nonisomorphic) Boolean algebras. In this work we shall address only the case where $\mathcal{I}$ is Borel.

On the face of it the answer might appear likely to be independent of ZFC; certainly it is possible for different models of ZFC to answer differently the question of whether two given Borel ideals have isomorphic quotients, even when the models are both wellfounded and have the same reals (so that the quotients being compared are identical across the models). For example, it follows from CH that $\mathcal{P}(\omega)/\mathrm{Fin} \cong \mathcal{P}(\omega)/(\mathrm{Fin} \times \emptyset)$, because both $\mathcal{P}(\omega)/\mathrm{Fin}$ and $\mathcal{P}(\omega)/(\mathrm{Fin} \times \emptyset)$ are $\aleph_1$-saturated in the model-theoretic sense (see [Far03, Proposition 6.1]). However it follows from OCA+MA that $\mathcal{P}(\omega)/\mathrm{Fin} \ncong \mathcal{P}(\omega)/(\mathrm{Fin} \times \emptyset)$ (see [Far00b, Corollary 3.4.5]). Here OCA stands for the Open Coloring Axiom; see [Far00b, Chapter 2] for definitions.

Thus Farah has addressed the question from two sides: In [Far00b] he looks at set-theoretic propositions consistent with ZFC, such as Martin's Maximum, that tend to minimize the opportunity for given definable ideals to have isomorphic quotients. On the other hand, in [Far03] he examines the question of what quotients must be isomorphic if CH holds, which tends to maximize the opportunity to find isomorphisms between definable structures of cardinality $2^{\aleph_0}$, and therefore (potentially) to minimize the number of isomorphism types. In this latter case he found many partial classification results, showing for example (given CH) that there are exactly two quotients, up to isomorphism, by dense density ideals, but leaving open the question of whether there were $2^{\aleph_0}$ (or indeed even infinitely many) distinct quotients by Borel (or even $\mathbf{\Sigma}^1_1$) ideals.

Steprāns, in [Ste03], uses a variation on Sacks Forcing to show that there is a family of $2^{\aleph_0}$ distinct $\mathbf{\Pi}^0_3$ ideals on a certain *Polish lattice* (that is, a lattice ordering on a Polish space that is closed as a subset of the Cartesian product of the space with itself; an ideal on such a lattice is a subset closed downward and under join) that have pairwise nonisomorphic quotients. The method also works to give ideals on the natural numbers, but apparently at the cost of increasing the complexity to $\mathbf{\Pi}^1_1$. At this writing it is not clear whether the method can be refined to give $\mathbf{\Pi}^0_3$ ideals on the natural numbers; if so, it would provide an alternative proof of much of the content of Theorem 1.3.4.2.

It should be noted that Steprāns' method provides information that the present work does not; namely, he shows that two lattices (or two Boolean algebras, as the case may be) are nonisomorphic by showing that neither can be completely embedded into the other.

1.1.2 Basic definitions and nomenclature

By an *ideal* we shall always mean a collection $\mathcal{I}$ of subsets of a countably infinite index set I (usually $I = \omega$ or $I = \omega \times \omega$) such that

i) if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$

ii) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$

iii) if $A \subseteq I$ and A is finite, then $A \in \mathcal{I}$

Condition (iii) is not part of the standard definition of an ideal, but we include it to avoid trivialities.

Elements of $\mathcal{I}$ are said to be $\mathcal{I}$-*null*; subsets of I that are not $\mathcal{I}$-null are called $\mathcal{I}$-*positive*, and we write $\mathcal{I}^+$ for the collection of all $\mathcal{I}$-positive sets.

$\mathcal{I}$ induces an equivalence relation $\approx_{\mathcal{I}}$ on $\mathcal{P}(I)$ by

$$X \approx_{\mathcal{I}} Y \iff X \bigtriangleup Y \in \mathcal{I}$$

If $X \subseteq I$ we write $[X]_{\mathcal{I}}$ for the $\approx_{\mathcal{I}}$-equivalence class of X. We write $\mathcal{P}(I)/\mathcal{I}$ for the Boolean algebra whose underlying set is the collection of all $\approx_{\mathcal{I}}$-equivalence classes, and whose $\wedge$ and $\vee$ are induced by $\cap$ and $\cup$ respectively.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on index sets I and J respectively, we define

$$\mathcal{I} \times \mathcal{J} \triangleq \{A \subseteq I \times J | \{m \in I | \{n | < m, n > \in A\} \in \mathcal{J}^+\} \in \mathcal{I}\}$$

That is, the $\mathcal{I} \times \mathcal{J}$-positive subsets of $I \times J$ are the ones with $\mathcal{I}$-positively many $\mathcal{J}$-positive vertical sections.

Note that while we do not officially consider $\{\emptyset\}$ to be an ideal, we do define $\mathcal{I} \times \emptyset$ and $\emptyset \times \mathcal{I}$ as though it were (leaving off the braces around $\emptyset$). That is, a subset of $I \times \omega$ is $\mathcal{I} \times \emptyset$-positive if and only if it has $\mathcal{I}$-positively many nonempty vertical sections, whereas a subset of $\omega \times I$ is $\emptyset \times \mathcal{I}$-positive just in case it has no $\mathcal{I}$-positive vertical sections.

Ideals $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler isomorphic*, $\mathcal{I} \approx_{RK} \mathcal{J}$, if (modulo null sets) there is a bijection between the underlying sets I and J that respects the ideals—that is, there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection $h : I \setminus A \to J \setminus B$ such that, for any $X \subseteq I$, $X \in \mathcal{I} \iff h"X \in \mathcal{J}$. If $\mathcal{I} \approx_{RK} \mathcal{J}$ then easily $\mathcal{P}(I)/\mathcal{I} \cong \mathcal{P}(J)/\mathcal{J}$ as Boolean algebras.

If A is $\mathcal{I}$-positive, we write $\mathcal{I} \restriction A$ for $\{X \subseteq I | X \cap A \in \mathcal{I}\}$. If $\mathcal{B}$ is a Boolean algebra and $x \in \mathcal{B}$, $x \neq \emptyset_{\mathcal{B}}$, we write $\mathcal{B} \restriction x$ for the Boolean algebra $\{y | y \leq_{\mathcal{B}} x\}$ with the Boolean operations inherited from $\mathcal{B}$. Clearly

$$\mathcal{P}(I)/\left(\mathcal{I} \restriction A\right) \cong \left(\mathcal{P}(I)/\mathcal{I}\right) \restriction [A]_{\mathcal{I}}$$

Ideals in our context can never be closed under countable unions; the entire underlying set is the union of countably many singletons, and singletons are null. In some sense the closest we can get is the notion of a *P-ideal.* $\mathcal{I}$ is a P-ideal if, for any countable collection of $\mathcal{I}$-null sets $\{B_k | k \in \omega\}$ there's a $\mathcal{I}$-null set B that misses only finitely much of each B_k (that is, $(\forall k) B_k \setminus B \in \mathrm{Fin}$).

An ideal on ω is a subset of $\mathcal{P}(\omega)$; on the latter we take the product topology. Any reference to the descriptive-set-theoretic complexity of an ideal (say, "Borel ideal", "analytic ideal", "$\mathbf{\Pi}^0_3$ ideal") should be understood in terms of that topology, as should any reference to Wadge reducibility between ideals.

Some specific ideals to which we shall make frequent reference:

Fin	the ideal of all finite subsets of ω
$\mathcal{Z}_0$	the ideal of all subsets of ω with asymptotic zero density
$\emptyset \times \mathrm{Fin}$	all subsets of $\omega \times \omega$ with no infinite vertical sections
$\mathrm{Fin} \times \emptyset$	all subsets of $\omega \times \omega$ with only finitely many nonempty vertical sections

1.2 Construction of the ideals

1.2.1 Motivation

The construction is an idea of Hjorth, who noted that it was possible to "integrate" ideals with respect to a partition of the natural numbers in such a way that the original ideal could be recovered from the order-theoretic properties of the quotient algebra corresponding to the "integral".

1.2.2 Formal definition

Definition 1.2.2.1. *Given $\mathcal{I}$ an ideal on ω containing Fin and $\vec{A} = (A_n)_{n\in\omega}$ a sequence of disjoint subsets of ω (some possibly empty), we write:*

$$\mathcal{I}(\vec{A}) = \{X \subseteq \omega | \forall n\ X \cap A_n \in \mathit{Fin} \wedge \{n | X \cap A_n \neq \emptyset\} \in \mathcal{I}\}$$

Definition 1.2.2.2. *Given $\vec{A} = (A_n)_{n\in\omega}$ a sequence of disjoint subsets of ω (some possibly empty), we write:*

$$P_{\vec{A}} = \{n \in \omega | A_n \text{ is infinite}\}$$

It may be easier to think of $\mathcal{I}(\vec{A})$ in terms of the positive sets: A set is $\mathcal{I}(\vec{A})$-positive just in case it either has infinite intersection with some A_n, or meets $\mathcal{I}$-positively many A_n. For example, any A_n is itself an $\mathcal{I}(\vec{A})$-positive set, below which $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$. On the other hand, given any $\mathcal{I}$-positive set C, we can choose one element (say, the least) from A_n for each $n \in C$; the set of these is now a $\mathcal{I}(\vec{A})$-positive set below which $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$ is isomorphic to $\mathcal{P}(\omega)/\mathcal{I}$ restricted to C. By choosing ideals $\mathcal{I}$ with a structural property distinguishing them from Fin (see Section 1.3.2.1 below) we can rule out isomorphisms of certain types between quotient algebras.

We should note as well that we can consider our ideals as living on any countably infinite set, say $\omega \times \omega$, and that if every A_n is infinite (that is, if $P_{\vec{A}} = \omega$) then the ideal $(\emptyset \times \mathrm{Fin}) \cap (\mathcal{I} \times \emptyset)$ on $\omega \times \omega$ has a quotient isomorphic to $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$.

The following simple fact will come in handy:

Lemma 1.2.2.1. *Let $\mathcal{I}$ be an ideal and $\vec{A} = (A_n)_{n\in\omega}$ a sequence of disjoint subsets of ω. For any $X \subseteq \omega$, if we write X_n for $X \cap A_n$ and $\vec{X}$ for the sequence of X_n, then*

$$\left(\mathcal{P}(\omega)/\mathcal{I}(\vec{A})\right) \restriction [X]_{\mathcal{I}(\vec{A})} \cong \mathcal{P}(\omega)/\mathcal{I}(\vec{X})$$

via a canonical isomorphism.

Proof. Given Y a representative for an element of $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$ below $[X]_{\mathcal{I}(\vec{A})}$, the isomorphism sends $[Y]_{\mathcal{I}(\vec{A})}$ to $[Y]_{\mathcal{I}(\vec{X})}$. It is routine to verify that this is well-defined, a bijection, and respects the Boolean operations.

□

1.3 Non-isomorphism results on the quotients by the ideals $\mathcal{I}(\vec{A})$

1.3.1 Connection between "input" ideals and structure of Boolean algebra

Lemma 1.3.1.1. *Let $\mathcal{I}$ be an ideal on ω containing Fin, let $\vec{A} = (A_n)_{n\in\omega}$ be a sequence of disjoint subsets of ω, and let $\mathcal{I}(\vec{A})$ be as defined above. Then for every $C \subseteq P_{\vec{A}}$, $C \in \mathcal{I}$ if and only if $\{A_n | n \in C\}$ has a least upper bound with respect to $\mathcal{I}(\vec{A})$ (that is, $\{[A_n]_{\mathcal{I}(\vec{A})} | n \in C\}$ has a least upper bound in $\mathcal{P}(\omega)/\mathcal{I}(\vec{A})$).*

Proof.

$\Rightarrow$: The least upper bound will be represented by $X \triangleq \bigcup_{n\in C} A_n$. Clearly this is an upper bound. Suppose that Y is also a representative for an upper bound. Then for each $n \in C$, $(X \setminus Y) \cap A_n = A_n \setminus Y$ must be finite, as otherwise Y would not be above A_n with respect to $\mathcal{I}(\vec{A})$. But these are the only n for which $(X \setminus Y) \cap A_n$ is nonempty, and C is $\mathcal{I}$-null. Therefore $X \setminus Y$ is $\mathcal{I}(\vec{A})$-null.

$\Leftarrow$: Suppose X (no longer defined as above) is a representative for the least upper bound. For each $n \in C$, $A_n \setminus X$ must be finite, so $X \cap A_n$ must be infinite, and in particular nonempty (this uses $n \in P_{\vec{A}}$). Form Y by removing from X one element of $X \cap A_n$ for each $n \in C$. Then Y must still be an upper bound, but if C were $\mathcal{I}$-positive we would have $Y <_{\mathcal{I}(\vec{A})} X$, contradicting the assumption that X is a least upper bound. Therefore C is $\mathcal{I}$-null. $\square$

1.3.2 Wadge reduction

The goal of this section, culminating in Theorem 1.3.2.1, is to show that, given ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ with a certain property (called *shallowness*), and given $\vec{A}$ a partition of ω into countably many infinite sets, if $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{A})$, then $\mathcal{J}_1 \leq_W \mathcal{J}_2$.

1.3.2.1 ω-partitions

In [Far03], Proposition 6.1, Farah gives a list of equivalent conditions for an atomless ideal on ω to have a quotient that is not $\aleph_1$-saturated in the model–theoretic sense. This derives from a result of Just and Mijajlović; see [JM87]. We shall not make use of the model theory in this paper, and in the interest of independent readability we instead isolate the following equivalent:

Definition 1.3.2.1. *Given a Boolean algebra $\mathcal{B}$ and $x \in \mathcal{B}$, $x \neq 0_{\mathcal{B}}$, we say $(x_n)_{n\in\omega}$ is an* ω-partition of x *if:*

i) $\forall n$ ($x_n \leq x$ *and* $x_n \neq 0_{\mathcal{B}}$),

ii) $\forall n \neq m$ $x_n \wedge x_m = 0_{\mathcal{B}}$, *and*

iii) $\forall y \leq x[(\forall n\ y \wedge x_n = 0_{\mathcal{B}}) \Rightarrow y = 0_{\mathcal{B}}]$

Definition 1.3.2.2. *Given an ideal $\mathcal{I}$ on ω and a subset X of ω, X $\mathcal{I}$-positive, we say $(X_n)_{n\in\omega}$ is an* ω-partition of X with respect to $\mathcal{I}$ *if:*

i) $\forall n$ ($X_n \leq_{\mathcal{I}} X$ *and* X_n *is $\mathcal{I}$-positive*),

ii) $\forall n \neq m$ ($X_n \cap X_m$ *is $\mathcal{I}$-null*), *and*

iii) $\forall Y \leq_{\mathcal{I}} X[(\forall n\ Y \cap X_n$ *is $\mathcal{I}$-null*) $\Rightarrow Y$ *is $\mathcal{I}$-null*]

Though not necessary for our current purposes, it is useful to note that the *nonexistence* of an ω-partition is equivalent, at least in our context, to what Farah calls countable saturation and Chang and Keisler (see [CK90], p. 256) call $\aleph_1$-saturation, as this provides an important constraint on nonisomorphism results from ZFC alone. If CH holds, then any two countably saturated Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ are isomorphic.

Also note that (iii) of Definition 1.3.2.2 implies that X is a least upper bound for the X_n in the order given by $\mathcal{I}$, and mutatis mutandis for Definition 1.3.2.1.

Definition 1.3.2.3. *An ideal* $\mathcal{I}$ *is* shallow *if* $\mathcal{I} \neq \mathcal{P}(\omega)$ *and every* $\mathcal{I}$*-positive set has an* ω*-partition with respect to* $\mathcal{I}$*.*

We call these ideals "shallow" because they are the ones with respect to which there are no "deep" sets in the sense of [Far03]. It should be noted that shallowness is *not* a "smallness" or "simplicity" condition—there are shallow ideals of arbitrarily high complexity, and $\mathcal{P}(\omega)/\mathrm{Fin}$ has smaller Borel cardinality than $\mathcal{P}(\omega)/\mathcal{Z}_0$, though Fin is not shallow (in fact *every* set is deep with respect to Fin) and $\mathcal{Z}_0$ is shallow.

Claim 1.3.2.1. $\mathcal{P}(\omega)/Fin$ *has no* ω*-partition below any point.*

Proof. Suppose to the contrary that $(X_n)_{n\in\omega}$ is an ω-partition of some infinite set X with respect to Fin. Let Y consist of the least element of each X_n that is not in any X_m for $m < n$ (such an element must exist because X_n is infinite and has finite intersection with each X_m, $m < n$). Then $\forall n \; Y \cap X_n \in \mathrm{Fin}$ but $Y \notin \mathrm{Fin}$. □

Claim 1.3.2.2. *We can get large collections of shallow ideals.*

The precise meaning of this claim will become clear. We note it here to call attention to the way we intend to establish nonisomorphism results, by showing that pieces of one quotient cannot match up with pieces of another, because the former are quotients by shallow ideals and the latter are isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$. See Section 1.4 below.

1.3.2.2 How an isomorphism must behave on the A_n

Given sequences $\vec{A} = (A_n)_{n\in\omega}$ and $\vec{B} = (B_i)_{i\in\omega}$ of subsets of ω, each sequence pairwise disjoint, and shallow ideals $\mathcal{J}_1$ and $\mathcal{J}_2$, write $\mathcal{B}_1$ for $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ and $\mathcal{B}_2$ for $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, and suppose

$$\phi : \mathcal{B}_1 \to \mathcal{B}_2$$

is an isomorphism.

Let us overload the symbol ϕ by choosing once and for all an arbitrary lift of ϕ to a function from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$, which we shall also call ϕ.

Now let $S \subseteq \omega \times \omega$ be the relation defined by

$$\begin{aligned} nSi &\iff \phi(A_n) \cap B_i \text{ is infinite} \\ &\iff [\phi(A_n) \cap B_i]_{\mathcal{J}_2(\vec{A})} > 0 \\ &\iff [\phi(A_n)]_{\mathcal{J}_2(\vec{A})} \wedge [B_i]_{\mathcal{J}_2(\vec{A})} > 0 \\ &\iff \phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right) \wedge [B_i]_{\mathcal{J}_2(\vec{A})} > 0 \end{aligned}$$

Note by the last equivalent that S does not depend on how we lifted ϕ.

Lemma 1.3.2.1. *$\phi(A_n)$ is essentially the union of its infinite B_i pieces. That is, for each n, $\phi(A_n) \approx_{\mathcal{J}_2(\vec{B})} \bigcup_{i|nSi}[\phi(A_n) \cap B_i]$.*

Proof. Clearly

$$\begin{aligned} \phi(A_n) &= \bigcup_{i|nSi} [\phi(A_n) \cap B_i] \\ &\cup \bigcup_{i|\neg nSi} [\phi(A_n) \cap B_i] \\ &\cup \left[\phi(A_n) \setminus \bigcup_{i\in\omega} B_i\right] \end{aligned}$$

The final summand does not meet any B_i and so is $\mathcal{J}_2(\vec{B})$-null.

Thus if the claim fails, $\bigcup_{i|\neg nSi}[\phi(A_n) \cap B_i]$ is $\mathcal{J}_2(\vec{B})$-positive (call this set T). All the $T \cap B_i$ are finite, by the definition of S, so there must be a $\mathcal{J}_2$-positive set U of indices i on which $T \cap B_i$ is nonempty. For each $i \in U$, choose one element (say, the least) of $T \cap B_i$ and let D be the set of all these.

Now it is easy to see that the subsets of D modulo $\mathcal{J}_2(\vec{B})$ are an isomorphic copy of the restriction of $\mathcal{P}(\omega)/\mathcal{J}_2$ to some nonzero point. Therefore there is an ω-partition of D with respect to $\mathcal{J}_2(\vec{B})$. However, below the $\mathcal{J}_1(\vec{A})$ equivalence class of A_n, $\mathcal{B}_1$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$, and therefore there is no nonzero point below $[A_n]$ in $\mathcal{B}_1$ having an ω-partition. As ϕ is an isomorphism, this is a contradiction. $\square$

Lemma 1.3.2.2. *There are only finitely many such pieces. That is, for each n, $\{i|nSi\}$ is finite.*

Proof. First observe that for any n, the set of i such that $\phi(A_n)$ meets B_i is $\mathcal{J}_2$-null. Otherwise, let D be the set of all least elements of nonempty sets of the form $\phi(A_n) \cap B_i$. D is $\mathcal{J}_2(\vec{B})$-positive, so there is some $\mathcal{J}_1(\vec{A})$-positive $D' \leq_{\mathcal{J}_1(\vec{A})} A_n$ such that $\phi(D') = D$. But $\mathcal{B}_1$ restricted to B' is isomorphic to $\mathcal{P}(\omega)/\mathrm{Fin}$ and therefore there is no ω-partition below D' in $\mathcal{B}_1$, whereas $\mathcal{B}_2$ restricted to D is isomorphic to $\mathcal{P}(\omega)/\mathcal{J}_2$ restricted to the set of all i such that $\phi(A_n)$ meets B_i, so there *is* an ω-partition below D with respect to $\mathcal{J}_2(\vec{B})$. This is a contradiction.

Now suppose for some n, $\{i|nSi\}$ is infinite (but by the above argument, necessarily $\mathcal{J}_2$-null). Then $\phi(A_n)$ is above (by Lemma 1.3.2.1, actually equivalent to) $\bigcup_{i|nSi}[\phi(A_n) \cap B_i]$ in the order given by $\mathcal{J}_2(\vec{B})$. But a subset of $\bigcup_{i|nSi}[\phi(A_n) \cap B_i]$ is $\mathcal{J}_2(\vec{B})$-positive just in case it is infinite on at least one of the B_i (because the set of indices i being considered is $\mathcal{J}_2$-null). That means that $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$ restricted to $\bigcup_{i|nSi}[\phi(A_n) \cap B_i]$ is isomorphic to $\mathcal{P}(\omega \times \omega)/(\emptyset \times \mathrm{Fin})$. But it is easily seen that $\omega \times \omega$ has an ω-partition with respect to $\emptyset \times \mathrm{Fin}$, whereas there is no ω-partition below A_n with respect to $\mathcal{J}_1(\vec{A})$. This again is a contradiction. $\square$

Lemma 1.3.2.3. *$\phi(A_n)$ is the least upper bound (mod $\mathcal{J}_2(\vec{B})$) of its infinite B_i pieces:*

$$[\phi(A_n)]_{\mathcal{J}_2(\vec{A})} = \bigvee_{i|nSi} [\phi(A_n) \cap B_i]_{\mathcal{J}_2(\vec{A})}$$

Proof. Immediate from Lemmata 1.3.2.1 and 1.3.2.2. $\square$

Lemma 1.3.2.4. *For each i, $B_i \approx_{\mathcal{J}_2(\vec{B})} \bigcup_{n|nSi}[\phi(A_n) \cap B_i]$, and moreover there are only finitely many n such that nSi. Therefore*

$$[B_i]_{\mathcal{J}_2(\vec{A})} = \bigvee_{n|nSi} \phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right) \wedge [B_i]_{\mathcal{J}_2(\vec{A})}$$

(This is the flip side of Lemmata 1.3.2.1, 1.3.2.2 and 1.3.2.3).

Proof. Choose an arbitrary lift of the inverse isomorphism $\phi^{-1} : \mathcal{B}_2 \to \mathcal{B}_1$ to a function from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$, and again overload the symbol ϕ^{-1} by referring to the lift as ϕ^{-1} as well. ϕ and ϕ^{-1} (as functions on $\mathcal{P}(\omega)$) may not be inverses, but we will have, for $X \subseteq \omega$, $\phi(\phi^{-1}(X)) \approx_{\mathcal{J}_2(\vec{B})} X$ and $\phi^{-1}(\phi(X)) \approx_{\mathcal{J}_1(\vec{A})} X$.

Now Lemmata 1.3.2.1, 1.3.2.2 and 1.3.2.3 immediately apply to ϕ^{-1}; it remains only to point out that if we write S' for the relation

$$iS'n \iff \phi^{-1}\left([B_i]_{\mathcal{J}_2(\vec{A})}\right) \wedge [A_n]_{\mathcal{J}_1(\vec{A})} > 0$$

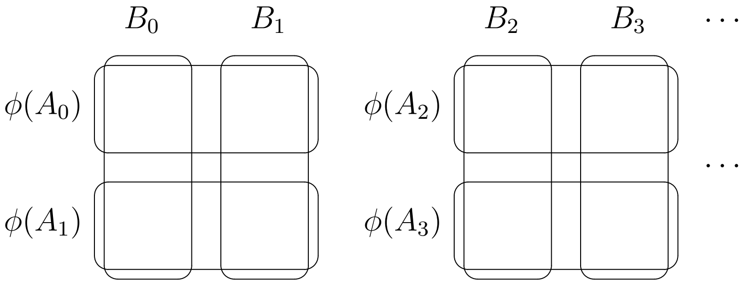
then

$$\begin{aligned} iS'n &\iff [B_i]_{\mathcal{J}_2(\vec{A})} \wedge \phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right) > 0 \\ &\iff nSi \end{aligned}$$

$\square$

1.3.2.3 Example

Figure 1.1: Example isomorphism



Suppose for example that for each n we have

$$\begin{aligned} 2n \quad S \quad & 2n \\ 2n \quad S \quad & 2n+1 \end{aligned}$$

and that no other pairs of natural numbers bear the S relation. That is, $\phi(A_{2n})$ is essentially the union of infinite pieces of B_{2n} and B_{2n+1}, and $\phi(A_{2n+1})$ occupies the "other half" of B_{2n} and B_{2n+1}. This situation is illustrated in Figure 1.1.

Then for a given set of natural numbers C, we know by Lemma 1.3.1.1 that $C \in \mathscr{J}_1$ if and only if $\{A_n | n \in C\}$ has a least upper bound with respect to $\mathscr{J}_1(\vec{A})$, which happens just in case $\{\phi(A_n) | n \in C\}$ has a least upper bound with respect to $\mathscr{J}_2(\vec{B})$; that is, if $D_0 \cup D_1$ has such a least upper bound, where

$$\begin{aligned} D_0 &\triangleq \{\phi(A_{2n}) | 2n \in C\} \\ D_1 &\triangleq \{\phi(A_{2n+1}) | 2n+1 \in C\} \end{aligned}$$

which we can rewrite

$$\begin{aligned} D_0 &= \{(\phi(A_{2n}) \cap A_{2n}) \cup (\phi(A_{2n}) \cap B_{2n+1}) | 2n \in C\} \\ D_1 &= \{(\phi(A_{2n}) \cap A_{2n+1}) \cup (\phi(A_{2n+1}) \cap B_{2n+1}) | 2n+1 \in C\} \end{aligned}$$

Now given any $X \subseteq \omega$, it is easy to see that X is an upper bound for $D_0 \cup D_1$ if and only if X is an upper bound for $D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$, where

$$\begin{aligned} D_{00} &\triangleq \{\phi(A_{2n}) \cap B_{2n} | 2n \in C\} \\ D_{01} &\triangleq \{\phi(A_{2n}) \cap B_{2n+1} | 2n \in C\} \\ D_{10} &\triangleq \{\phi(A_{2n+1}) \cap B_{2n} | 2n+1 \in C\} \\ D_{11} &\triangleq \{\phi(A_{2n+1}) \cap B_{2n+1} | 2n+1 \in C\} \end{aligned}$$

Therefore $C \in \mathcal{J}_1$ if and only if $D_{00} \cup D_{01} \cup D_{10} \cup D_{11}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$. The method used in Lemma 1.3.1.1 shows that such a least upper bound exists just in case the set of all indices represented in the D_{ij}, namely $\{2n, 2n+1 | 2n \in C \vee 2n+1 \in C\}$, is $\mathcal{J}_2$-null.

1.3.2.4 Formal reduction

The example in Section 1.3.2.3 suggests the following claim: for each $C \subseteq P_{\vec{A}}$,

$$C \in \mathcal{J}_1 \iff \{i | \exists n (nSi \wedge n \in C)\} \in \mathcal{J}_2$$

In the case where $P_{\vec{A}} = \omega$, this equivalence gives us a Wadge reduction demonstrating $\mathcal{J}_1 \leq_W \mathcal{J}_2$. To see this, we must check that the function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ given by $f(C) = \{i | \exists n (nSi \wedge n \in C)\}$ is continuous. Choose a basic open set U in the topology on $\mathcal{P}(\omega)$; say, let $\{a_0, \ldots, a_k\}$ and $\{b_0, \ldots, b_\ell\}$ be given disjoint finite sets of natural numbers and let

$$U = \{X \subseteq \omega | (\forall i \leq k)(a_i \in X) \wedge (\forall j \leq \ell)(b_j \notin X)\}$$

What we need is for $f^{-1}[U]$ to be open. Choose an element C of $f^{-1}[U]$. As $f(C) \in U$, there must be $n_0 \ldots n_k$ taken from C such that $(\forall i \leq k)(n_k S a_k)$. Moreover no b_j is an element of $f(C)$, so for each $j \leq \ell$ and any n such that nSb_j, we have $n \notin C$; by Lemma 1.3.2.4 there are only finitely many such n.

Now given any C' such that $(\forall i \leq k)(n_i \in C')$ and such that C' does not contain any of the finitely many n bearing the S relation to any b_j, $j \leq \ell$, we have that $f(C') \in U$, so $C' \in f^{-1}[U]$. The collection of all such C' is an open neighborhood of C included in $f^{-1}[U]$, so $f^{-1}[U]$ is open.

1.3.2.5 Proof of reduction

Theorem 1.3.2.1. *for each* $C \subseteq P_{\vec{A}}$,

$$C \in \mathcal{J}_1 \iff \{i | \exists n(nSi \wedge n \in C)\} \in \mathcal{J}_2$$

Proof. The proof is a generalization of the argument in the example in 1.3.2.3.

We will argue that the following are equivalent:

i) $C \in \mathcal{J}_1$

ii) $\{A_n | n \in C\}$ has a least upper bound with respect to $\mathcal{J}_1(\vec{A})$

iii) $\{\phi(A_n) | n \in C\}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$

iv) $\{\phi(A_n) \cap B_i | nSi \wedge n \in C\}$ has a least upper bound with respect to $\mathcal{J}_2(\vec{B})$

v) $\{i | \exists n(nSi \wedge n \in C)\} \in \mathcal{J}_2$

The equivalence of (i) and (ii) is immediate from Lemma 1.3.1.1. (ii) is equivalent to (iii) because ϕ is an isomorphism.

To see that (iii) is equivalent to (iv), note that $D \triangleq \{\phi(A_n) | n \in C\}$ and $E \triangleq \{\phi(A_n) \cap B_i | nSi \wedge n \in C\}$ have the same collection of upper bounds with respect to $\mathcal{J}_2(\vec{B})$: Every element of E is $\leq_{\mathcal{J}_2(\vec{B})}$ some element of D, so any upper bound for D is an upper bound for E. Conversely, if $X \subseteq \omega$ is an upper bound for E, then fixing $n \in C$ we want to check that $X \geq_{\mathcal{J}_2(\vec{B})} \phi(A_n)$. But we know, for each i such that nSi, that $X \geq_{\mathcal{J}_2(\vec{B})} \phi(A_n) \cap B_i$, so by Lemma 1.3.2.3 we have $X \geq_{\mathcal{J}_2(\vec{B})} \phi(A_n)$.

For (iv)⇒(v), suppose that $F \triangleq \{i | \exists n(nSi \wedge n \in C)\}$ is $\mathcal{J}_2$-positive. Then given any upper bound X for E (as defined above), we can form X' by removing, for each $i \in F$, the least element of $X \cap \phi(A_n) \cap B_i$, where n is least such that nSi. Then $X' <_{\mathcal{J}_2(\vec{B})} X$, but X' is still an upper bound for E, contradicting (iv).

For (v)⇒(iv), we will argue that given (v), the union of all sets in E is the least upper bound for E (call this union X). Suppose X' is any upper bound for E. Then for any n and i such that nSi, X' can miss only finitely many points of $\phi(A_n) \cap B_i$. Fixing i, it follows by Lemma 1.3.2.4 that X' misses only finitely many points of X in B_i. But the only i for which X' can miss *any* points of X in B_i are those $i \in F$ (where F is as in the previous paragraph) and F by hypothesis is $\mathcal{J}_2$-null. Therefore $X \leq_{\mathcal{J}_2(\vec{B})} X'$. □

1.3.3 $\aleph_1$ distinct quotients

Theorem 1.3.3.1. *There is an uncountable collection of Borel ideals on ω such that if $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals from the collection, then $\mathcal{P}(\omega)/\mathcal{I}_1$ is not isomorphic to $\mathcal{P}(\omega)/\mathcal{I}_2$. Moreover, there is a* definable *embedding from ω_1 into the isomorphism types of quotients by Borel ideals, in the sense that there is a Borel map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for $X, Y \subseteq \omega$ coding countable ordinals, $f(X)$ and $f(Y)$ are Borel codes for ideals, and their quotients are isomorphic if and only if X and Y code the same ordinal.*

Proof. In Section 1.4.1 below, we show that there is a (Borel in the codes) map $\alpha \mapsto \mathcal{J}_\alpha$ such that, for α a countable ordinal, $\mathcal{J}_\alpha$ is a shallow ideal, and such that if $\alpha < \beta$, then $\mathcal{J}_\alpha <_W \mathcal{J}_\beta$.

Now let $\vec{A} = (A_n)_{n \in \omega}$ be a partition of ω into infinite sets. Then if $X \subseteq \omega$ codes a countable ordinal α, we will let $f(X)$ be a code for $\mathcal{J}_\alpha(\vec{A})$. (If X does not code a countable ordinal, we do not care what $f(X)$ is.) The fact that there is a Borel such f follows from the Suslin–Kleene theorem.

To see that it works, note that if $\mathcal{P}(\omega)/\mathcal{J}_\alpha(\vec{A})$ were isomorphic to $\mathcal{P}(\omega)/\mathcal{J}_\beta(\vec{A})$, $\beta < \alpha$, then by Theorem 1.3.2.1 we would have $\mathcal{J}_\alpha$ Wadge-reducible to $\mathcal{J}_\beta$, contrary to the construction. $\square$

1.3.4 Continuum many distinct quotients by analytic P-ideals

The goal of this section is to show that there is a collection of $2^{\aleph_0}$ analytic P-ideals (therefore necessarily $\mathbf{\Pi}^0_3$) with pairwise nonisomorphic quotients. This is not simply a strengthening of Theorem 1.3.3.1 because that theorem establishes a "definable" injection from ω_1 into the isomorphism types of quotients by Borel ideals, which the result of this section will not.

1.3.4.1 Preservation of analytic P-property

Thanks to Farah for pointing out that this next result follows directly from the definition, making it unnecessary to appeal to Solecki's result that analytic P-ideals are precisely the exhaustions of lower semicontinuous submeasures (see [Sol99]). (There is also a rather easy proof using that result.)

Lemma 1.3.4.1. *If $\mathcal{I}$ is an analytic P-ideal, then so is $\mathcal{I}(\vec{A})$*

Proof. Let B_0, B_1, … be a sequence of $\mathcal{I}(\vec{A})$-null sets of naturals. For each k let $C_k \triangleq \{n | A_n \cap B_k \neq \emptyset\}$; then C_k is $\mathcal{I}$-null.

As $\mathcal{I}$ is a P-ideal, there is some $\mathcal{I}$-null set C such that $C_k \setminus C$ is finite for every

k. Now define

$$B \triangleq \{m | \exists k [m \in B_k \wedge \forall \ell (m \in A_\ell \Rightarrow (\ell \geq k \wedge \ell \in C))]\}$$

That is, we take a union of the B_k with some stuff left out: We leave out anything that is in A_ℓ for $\ell \notin C$, and we leave out that part of B_1 that is also in A_0, that part of B_2 that is in A_0 or A_1, and so on.

Now $\{\ell | B \cap A_\ell \neq \emptyset\} \subseteq C \in \mathcal{I}$, and the intersection of B with a given A_ℓ is contained in the union of the intersections of finitely many B_k with A_ℓ (namely those with $k \leq \ell$); each of those intersections is finite (since $B_k \in \mathcal{I}(\vec{A})$), so $B \cap A_\ell$ is finite. Thus $B \in \mathcal{I}(\vec{A})$.

For each k, the elements of $B_k \setminus B$ are either in A_ℓ for $\ell < k$ (there can be only finitely many of these), or in A_ℓ for some $\ell \notin C$. However in the latter case we have $\ell \in C_k \setminus C$, and $C_k \setminus C$ is finite. Since each $B_k \cap A_\ell$ is finite, we get that $B_k \setminus B$ is finite. Thus B is the required witness demonstrating that $\mathcal{I}(\vec{A})$ is a P-ideal. (That $\mathcal{I}(\vec{A})$ is analytic is trivial quantifier-counting.) $\square$

(Actually the method gives that if $\mathcal{I}$ is a P-ideal, analytic or not, then so is $\mathcal{I}(\vec{A})$, but we will not make use of this.)

1.3.4.2 Attempt using Borel reducibility

It was originally hoped that the method of Section 1.3.2 would show, given that $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{A})$, not merely that $\mathcal{J}_1 \leq_W \mathcal{J}_2$, but that $\mathcal{J}_1 \leq_B \mathcal{J}_2$ as equivalence relations. As Louveau points out, Borel reducibility of equivalence relations is in some sense the dimension-2 analogue of Wadge reducibility.

Had this held, then we could have used the ideals given by Louveau and Veličković in [LV94] to establish the result immediately (they give a collection of

$2^{\aleph_0}$ analytic P-ideals such that none is Borel reducible to another as equivalence relations).

However there does not seem to be any direct way to establish this implication. The natural thing to try would be to establish, for $X, Y \subseteq \omega$, that

$$X \bigtriangleup Y \in \mathcal{J}_1 \iff f(X) \bigtriangleup f(Y) \in \mathcal{J}_2$$

where

$$f(X) \triangleq \{m \in \omega | (\exists n \in X) mSn\}$$

But that's false. Look again at the example in Section 1.3.2.3, illustrated in Figure 1.1, and take X to be the set of even natural numbers and Y to be the set of odd natural numbers. Then $X \bigtriangleup Y$ is ω, but $f(X) \bigtriangleup f(Y)$ is $\emptyset$.

Note that what goes wrong has to do with the fact that the S relation in this example is neither a function nor one-one. If we knew that S were a bijection, then the proposed reduction would not be merely a Borel reduction of equivalence relations but a Rudin–Keisler isomorphism between the ideals. I am indebted to Farah for the idea that we can make S do what we want by paring down the underlying set.

1.3.4.3 More Technical Lemmata

In this section we prove some easy, yet notationally messy, facts about the possible structure of ideals $\mathcal{I}(\vec{A})$. The reader may wish to skip ahead to Section 1.3.4.4 and refer back to this section as necessary.

Lemma 1.3.4.2. *If $\mathcal{J}_1$ and $\mathcal{J}_2$ are shallow ideals and $\vec{A}$ and $\vec{B}$ are sequences of disjoint subsets of ω such that $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, and if $P_{\vec{A}}$ is $\mathcal{J}_1$-positive, then $P_{\vec{B}}$ is $\mathcal{J}_2$-positive.*

Proof. See below. ☐

Definition 1.3.4.1. *A set $X \subseteq \omega$ is* shallowizing *with respect to an ideal $\mathcal{I}$ if X is $\mathcal{I}$-positive and $\mathcal{I} \restriction X$ is shallow.*

(We say "shallowizing" rather than "shallow" because the latter could be interpreted as "not deep", which is a weaker notion—a shallowizing set is not only not deep; it has no deep sets below it.)

Lemma 1.3.4.3. *If $\mathcal{I}$ is shallow and $P_{\vec{A}}$ is $\mathcal{I}$-null, then one of the following six cases holds:*

i) $\mathcal{I}(\vec{A}) = \mathcal{P}(\omega)$

ii) $\mathcal{I}(\vec{A}) \approx_{RK} Fin$

iii) $\mathcal{I}(\vec{A}) \approx_{RK} \emptyset \times Fin$

iv) $\mathcal{I}(\vec{A})$ *is shallow*

v) $[\omega]_{\mathcal{I}(\vec{A})} = [X]_{\mathcal{I}(\vec{A})} \oplus [Y]_{\mathcal{I}(\vec{A})}$, *where* $\mathcal{I}(\vec{A}) \restriction X \approx_{RK} Fin$ *and* Y *is* $\mathcal{I}(\vec{A})$*-shallowizing.*

vi) $[\omega]_{\mathcal{I}(\vec{A})} = [X]_{\mathcal{I}(\vec{A})} \oplus [Y]_{\mathcal{I}(\vec{A})}$, *where* $\mathcal{I}(\vec{A}) \restriction X \approx_{RK} \emptyset \times Fin$ *and* Y *is* $\mathcal{I}(\vec{A})$*-shallowizing.*

Proof. Let $Q_{\vec{A}} \triangleq \{n | A_n \in \text{Fin} \wedge A_n \neq \emptyset\}$. Then $P_{\vec{A}}$ is either empty, finite nonempty, or infinite, and $Q_{\vec{A}}$ is either $\mathcal{I}$-null or $\mathcal{I}$-positive. The cases break down as follows:

	$P_{\vec{A}} = \emptyset$	$P_{\vec{A}}$ finite nonempty	$P_{\vec{A}}$ infinite
$Q_{\vec{A}} \in \mathcal{I}$	(i)	(ii)	(iii)
$Q_{\vec{A}} \in \mathcal{I}^+$	(iv)	(v)	(vi)

☐

Lemma 1.3.4.4. *Suppose $\mathcal{I}$ is shallow and $P_{\vec{A}}$ is $\mathcal{I}$-positive. Then there is an $\mathcal{I}(\vec{A})$-shallowizing set, and moreover, given any $X \subseteq \omega$ such that X is $\mathcal{I}(\vec{A})$-shallowizing, there is $Y \subseteq \omega$ disjoint from X such that Y is also $\mathcal{I}(\vec{A})$-shallowizing.*

Proof. Let Y equal $\{k | (\exists n \in P_{\vec{A}})(k \text{ is the least element of } A_n)\}$. Then Y is $\mathcal{I}(\vec{A})$-positive, and $\mathcal{I}(\vec{A}) \restriction Y$ is Rudin–Keisler isomorphic to $\mathcal{I}$, which is shallow by hypothesis. Thus Y is $\mathcal{I}(\vec{A})$-shallowizing.

Given an $\mathcal{I}(\vec{A})$-shallowizing set X, for every n, $X \cap A_n$ must be finite, because otherwise $X \cap A_n$ would be an $(\mathcal{I}(\vec{A}) \restriction X)$-positive set without an ω-partition. So for each $n \in P_{\vec{A}}$ let $A'_n \triangleq A_n \setminus X$; each such A'_n is infinite and in particular nonempty. Now, much as before, let

$$Y \triangleq \{k | (\exists n \in P_{\vec{A}})(k \text{ is the least element of } A'_n)\}$$

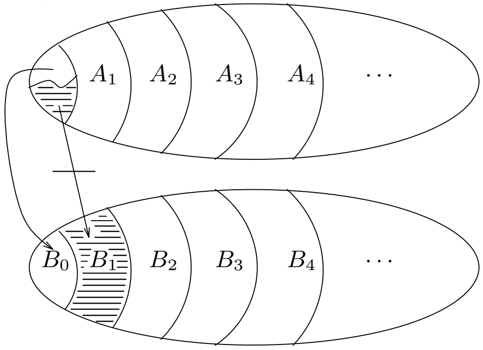
then Y is disjoint from X and $\mathcal{I}(\vec{A})$-shallowizing. □

Proof of Lemma 1.3.4.2. Suppose to the contrary that $P_{\vec{B}}$ is $\mathcal{J}_2$-null. Then $\mathcal{J}_2(\vec{B})$ falls into one of the six cases of Lemma 1.3.4.3. But cases (i), (ii) and (iii) are ruled out because they imply there is no $\mathcal{J}_2(\vec{B})$-shallowizing set, when there must be a $\mathcal{J}_1(\vec{A})$-shallowizing set because $P_{\vec{A}}$ is $\mathcal{J}_1$-positive. Cases (iv), (v) and (vi), pulled back via the isomorphism to $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$, would contradict the "moreover" clause of Lemma 1.3.4.4. □

1.3.4.4 Paring technique

First we explain, given an isomorphism from $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ onto $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, how to cut down the underlying sets to get an isomorphism between pieces of these Boolean algebras, whose corresponding S relation is now a function. The technique will then be run in the other direction to make S into a bijection.

Figure 1.2: Paring an isomorphism



Each A_n is sent by ϕ to some subset of the union of all B_i with nSi (modulo a $\mathcal{J}_2(\vec{B})$-null difference). We take i_n to be the first such i, and we pare away, from the underlying set of $\mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, the image of A_n restricted to each of the *other* i with nSi. Then we must also pare away from the underlying set of $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ the part of A_n that is sent to those other i. See Figure 1.2.

This now leaves an isomorphism between two Boolean algebras that are restrictions of the previous ones, such that in the new S relation i_n is the unique natural number satisfying nSi_n. Moreover the Boolean algebra formed by restricting $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A})$ is in fact Rudin–Keisler isomorphic to the original, because we had to leave an infinite piece of each A_n.

This description is not quite precise because we have not shown that ϕ sends the remaining part of the first algebra to the remaining part of the second, and in fact it may not, exactly. But if we define

$$\begin{aligned}
i_n &\triangleq \text{least } i \text{ such that } nSi \text{ (for each } n\text{)} \\
D_n &\triangleq \phi(A_n) \cap B_{i_n} \\
A'_n &\triangleq A_n \cap \phi^{-1}(D_n) \\
X &\triangleq \bigcup_{n\in\omega} A'_n
\end{aligned}$$

$$B'_i \triangleq \phi(X) \cap B_i \text{ (for each } i)$$

then certainly ϕ restricts to an isomorphism from

$$\mathcal{B}_1 \upharpoonright [X]_{\mathcal{J}_1(\vec{A})}$$

onto

$$\mathcal{B}_2 \upharpoonright [\phi(X)]_{\mathcal{J}_2(\vec{B})}$$

Roughly, X is the part left after paring away the top half of Figure 1.2, and B'_i is what should be left of B_i. The technicality here is that we do not know (at least by the methods so far developed) that B'_i is empty in the case that B_i has been entirely pared away. However we do have:

Claim 1.3.4.1. *If $i = i_n$ for some n then B'_i is infinite, otherwise not.*

Proof. Suppose $i = i_n$. Then $[D_n]_{\mathcal{J}_2(\vec{B})}$ is positive and

$$\begin{aligned} [D_n]_{\mathcal{J}_2(\vec{B})} &\leq [B_i]_{\mathcal{J}_2(\vec{B})} \\ [D_n]_{\mathcal{J}_2(\vec{B})} &\leq [(\phi(A_n)]_{\mathcal{J}_2(\vec{B})} \end{aligned}$$

Thus

$$[\phi(A'_n)]_{\mathcal{J}_2(\vec{B})} = [\phi(A_n)]_{\mathcal{J}_2(\vec{B})} \wedge [D_n]_{\mathcal{J}_2(\vec{B})} = [D_n]_{\mathcal{J}_2(\vec{B})} \leq [B_i]_{\mathcal{J}_2(\vec{B})}$$

so

$$\begin{aligned} [B'_i]_{\mathcal{J}_2(\vec{B})} &= [\phi(X)]_{\mathcal{J}_2(\vec{B})} \wedge [B_i]_{\mathcal{J}_2(\vec{B})} \\ &\geq [\phi(A'_n)]_{\mathcal{J}_2(\vec{B})} \wedge [B_i]_{\mathcal{J}_2(\vec{B})} \\ &= [D_n]_{\mathcal{J}_2(\vec{B})} \\ &> 0 \end{aligned}$$

Otherwise, for each of the finitely many n such that nSi, we have $[A'_n]_{\mathcal{J}_1(\vec{A})} \wedge [\phi^{-1}(B_i)]_{\mathcal{J}_1(\vec{A})} = 0_{\mathcal{B}_1}$, so $[B'_i]_{\mathcal{J}_2(\vec{B})} = 0_{\mathcal{B}_2}$. $\square$

So we know that the S relation of the pared-down isomorphism is a function (it sends n to i_n) and that it is a subset of the original S.

We can now establish

Theorem 1.3.4.1. *Given shallow ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ and sequences $\vec{A}$ and $\vec{B}$ such that $P_{\vec{A}}$ is $\mathcal{J}_1$-positive, and given that $\mathcal{P}(\omega)/\mathcal{J}_1(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B})$, there is an injective partial function $f : \omega \to \omega$ such that the domain of f is $\mathcal{J}_1$-positive, and such that for $X \subseteq \mathrm{dom}(f)$, $X \in \mathcal{J}_1 \iff f\text{"}X \in \mathcal{J}_2$.*

Proof. Given an isomorphism ϕ and working as above, we obtain a restricted isomorphism

$$\phi : \mathcal{P}(\omega)/\mathcal{J}_1(\vec{A'}) \to \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B'})$$

such that the new S relation, call it $S^{(1)}$, given by

$$nS^{(1)}i \iff \phi\left([A'_n]_{\mathcal{J}_1(\vec{A'})}\right) \wedge [B'_i]_{\mathcal{J}_2(\vec{B'})} > 0$$

is a function.

Moreover by Lemma 1.3.4.2 we know that $P_{\vec{B}}$ is $\mathcal{J}_2$-positive; therefore the inverse isomorphism

$$\phi^{-1} : \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B'}) \to \mathcal{P}(\omega)/\mathcal{J}_1(\vec{A'})$$

can similarly be pared down to

$$\phi^{-1} : \mathcal{P}(\omega)/\mathcal{J}_2(\vec{B''}) \to \mathcal{P}(\omega)/\mathcal{J}_1(\vec{A''})$$

and a new inverse S relation, $S^{(2)}$, given by

$$iS^{(2)}n \iff \phi^{-1}\left([B''_i]_{\mathcal{J}_2(\vec{B''})}\right) \wedge [A''_n]_{\mathcal{J}_1(\vec{A''})} > 0$$

Now the desired f is simply the inverse of $S^{(2)}$; we have $\mathrm{dom}(f) = P_{\vec{A''}}$, which again by Lemma 1.3.4.2 must be $\mathcal{J}_1$-positive.

□

The following is the main result of this Section 1.3.4:

Theorem 1.3.4.2. *There are at least E_0-many distinct quotients by analytic P-ideals; that is, there is a Borel function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for any $X, Y \subseteq \omega$, $f(X)$ and $f(Y)$ are Borel codes for analytic P-ideals, and the quotients of $\mathcal{P}(\omega)$ by said ideals are isomorphic as Boolean algebras if and only if $X \bigtriangleup Y$ is finite.*

Proof. For $X \subseteq \omega$ let $f(X)$ be a Borel code for the ideal $\mathcal{I}_X(\vec{A})$, where $\mathcal{I}_X$ is defined in Section 1.4.2 below. (That we can find a Borel—indeed, recursive—function f that accomplishes this is a consequence of the Suslin–Kleene theorem; see the discussion in Section 1.4.1.3.)

If $X \bigtriangleup Y$ is finite, then $\mathcal{I}_X$ and $\mathcal{I}_Y$ are literally the same ideal, so trivially $\mathcal{P}(\omega)/\mathcal{I}_X(\vec{A})$ and $\mathcal{P}(\omega)/\mathcal{I}_Y(\vec{A})$ are isomorphic.

If on the other hand $X \bigtriangleup Y$ is infinite, then without loss of generality suppose $X \setminus Y$ is infinite, and assume there is an (onto) isomorphism $\phi : \mathcal{P}(\omega)/\mathcal{I}_X(\vec{A}) \cong \mathcal{P}(\omega)/\mathcal{I}_Y(\vec{A})$. Then let

$$\begin{aligned} Z &\triangleq \bigcup_{i \in X \setminus Y} I_i \\ \check{Z} &\triangleq \bigcup_{n \in Z} A_n \\ A'_n &\triangleq \begin{cases} A_n & \text{if } n \in Z \\ \emptyset & \text{otherwise} \end{cases} \end{aligned}$$

(where the I_i are as in Section 1.4.2).

Now it is easy to check that $[\mathcal{P}(\omega)/\mathcal{I}_X(\vec{A})] \upharpoonright \check{Z}$ is the same as $\mathcal{P}(\omega)/\mathcal{I}_{X\setminus Y}(\vec{A}')$ and that $[\mathcal{P}(\omega)/\mathcal{I}_Y(\vec{A})] \upharpoonright \phi(\check{Z})$ is the same as $\mathcal{P}(\omega)/\mathcal{I}_Y(\vec{B})$ for some $\vec{B}$, so ϕ restricts to an isomorphism from $\mathcal{P}(\omega)/\mathcal{I}_{X\setminus Y}(\vec{A}')$ onto $\mathcal{P}(\omega)/\mathcal{I}_Y(\vec{B})$. But now

Theorem 1.3.4.1 gives us precisely the injective partial function that, by Claim 1.4.2.1 on page 33, cannot exist.

$\square$

1.3.5 Remarks on the use of the Axiom of Choice

In sections 1.3.2 and 1.3.4 we have used the Axiom of Choice to conclude that any isomorphism $\phi : \mathcal{P}(\omega)/\mathcal{I} \cong \mathcal{P}(\omega)/\mathcal{J}$, for ideals $\mathcal{I}$ and $\mathcal{J}$, must lift to a map $\phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$.

This application of AC is mostly for notational and expository convenience; we do not really need to lift the entire isomorphism to a map from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$ all at once. For example the S relation defined in Section 1.3.2.2 does not require the lifting at all; it can be defined entirely in terms of the behavior of equivalence classes under ϕ. For Lemmata 1.3.2.1 and 1.3.2.2 we do not need any Choice at all; for a given n we need only choose a representative for $\phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right)$, and we never need all these choices for all n at once. Similar considerations work for for Lemmata 1.3.2.3 and 1.3.2.4.

In the proof of Theorem 1.3.2.1, we are finally making use of a fragment of AC in a way that does not seem to be eliminable; to get the equivalence between (iv) and (v) we appear to need representatives for all the $\phi\left([A_n]_{\mathcal{J}_1(\vec{A})}\right)$ at once. However we do not need full AC; countable AC for reals (that is, the proposition that any countable collection of nonempty sets of reals has a choice function) is sufficient. This same fragment also suffices to make the paring technique work as described in Section 1.3.4.4, and therefore all the results of Section 1.3.

1.4 Existence of collections of "input" ideals with desired properties

1.4.1 $\aleph_1$ "input" ideals

In order to get the result of Section 1.3.3, we need to find a "definable" collection of $\aleph_1$ shallow ideals, all of distinct Wadge rank.

1.4.1.1 If $\mathcal{I}$ is shallow, so is $\mathcal{I} \times \mathcal{J}$

Suppose $\mathcal{I}$ is a shallow ideal on ω, and let $\mathcal{J}$ be an arbitrary ideal on ω.

Now choose an $\mathcal{I}\times\mathcal{J}$-positive subset X' of $\omega\times\omega$; we wish to find an ω-partition of X' with respect to $\mathcal{I} \times \mathcal{J}$.

The intuition is simple: $\mathcal{I} \times \mathcal{J}$ is the ideal whose positive sets are the ones that have $\mathcal{I}$-positively many $\mathcal{J}$-positive vertical sections. So we will project everything down to the horizontal axis, the one corresponding to $\mathcal{I}$ and work with the properties of $\mathcal{I}$. Nothing interesting happens in the vertical sections; $\mathcal{J}$ is treated as a black box. A surprising amount of notation goes into formalizing this simple notion.

Let $X \subseteq \omega$ be given by

$$X \triangleq \{n \in \omega | \{m | \langle n, m\rangle \in X'\} \text{ is } \mathcal{J}\text{-positive}\}$$

Now X is an $\mathcal{I}$-positive subset of ω (by the definition of $\mathcal{I} \times \mathcal{J}$), so there is an ω-partition of X with respect to $\mathcal{I}$, call it $(X_n)_{n\in\omega}$.

We will partition X' into $(X'_n)_{n\in\omega}$ by

$$X'_n = \{\langle k, l\rangle \in X' | k \in X_n\}$$

(note that $\bigcup_{n\in\omega} X'_n$ may not be all of X', but the part we have omitted is $\mathcal{I}\times\mathcal{J}$-null; in fact, it has all $\mathcal{J}$-null vertical sections).

Given $Y' \subseteq X'$ such that $\forall n\ (Y'\cap X'_n) \in \mathcal{I}\times\mathcal{J}$, we must now show $Y' \in \mathcal{I}\times\mathcal{J}$. Let

$$\begin{aligned} Y &\triangleq \{k\in\omega|\{m|\langle k,m\rangle \in Y'\} \text{ is } \mathcal{J}\text{-positive}\} \\ Y'' &\triangleq \{\langle k,m\rangle \in Y'|k\in Y\} \end{aligned}$$

That is, Y'' is the union of all $\mathcal{J}$-positive vertical sections of Y', and Y is the projection of Y'' to the horizontal axis. The plan is to argue that, for each n, $Y\cap X_n \in \mathcal{I}$, and therefore $Y\in\mathcal{I}$, so $Y'\in\mathcal{I}\times\mathcal{J}$ by definition of $\mathcal{I}\times\mathcal{J}$. Again by definition of $\mathcal{I}\times\mathcal{J}$, this will follow if we can establish that $\{\langle k,m\rangle \in Y''|k\in Y\cap X_n\}$ is $\mathcal{I}\times\mathcal{J}$-null (because, by definition, all nonempty vertical sections of Y'' are $\mathcal{J}$-positive).

But easily $\{\langle k,m\rangle \in Y''|k\in Y\cap X_n\} \subseteq Y'\cap X'_n$, and $Y'\cap X'_n$ is $\mathcal{I}\times\mathcal{J}$-null by hypothesis. This finishes the proof.

1.4.1.2 The ideal of density $\mathcal{Z}_0$ is shallow

First we show that ω itself has an ω-partition with respect to $\mathcal{Z}_0$. It suffices to find a collection of sets $(X_n)_{n\in\omega}$ such that

i) each X_n is $\mathcal{Z}_0$-positive

ii) the sets are pairwise disjoint, and

iii) $\forall Y\subseteq\omega\ [(\forall n\ Y\cap X_n\in\mathcal{Z}_0) \Rightarrow Y\in\mathcal{Z}_0]$.

We will take

$$X_n \triangleq \{i\in\omega|2^n \text{ divides } i \text{ but } 2^{n+1} \text{ does not}\}$$

(For completeness, take 0 to be an element of X_0.)

Suppose $Y \cap X_n \in \mathcal{Z}_0$ for all n, and let $\epsilon > 0$. We must find a natural number M such that, for all $k > M$, $|Y \cap k|/k < \epsilon$.

Observe that, for each n, X_n has a limiting density, which equals $1/2^{n+1}$, and that the sum of these densities is a convergent series summing to 1. So we can choose N_0 such that $\sum_{n<N_0} \rho(X_n) > 1 - \epsilon/4$. Now we can choose M_0 such that, for all $k > M_0$ and all $n < N_0$ we have $\sum_{n<N_0} |X_n \cap k|/k > 1 - \epsilon/2$ (because the sequence whose k^{th} element is $\sum_{n<N_0} |X_n \cap k|/k$ converges to a value greater than $1 - \epsilon/4$). This implies that for all $k > M_0$, $|\bigcup_{n \geq N_0} X_n \cap k|/k < \epsilon/2$.

Now choose $M > M_0$ such that, for all $k > M$ and all $n < N_0$, we have $|Y \cap X_n \cap k|/k < \epsilon/(2N_0)$, which we can do because $Y \cap X_n \in \mathcal{Z}_0$. Then for $k > M$ we have

$$\begin{aligned} |Y \cap k| &= \left|(\bigcup_{n<N_0} Y \cap X_n \cap k) \cup (\bigcup_{n \geq N_0} Y \cap X_n \cap k)\right| \\ &= \sum_{n<N_0} |Y \cap X_n \cap k| + \sum_{n \geq N_0} |Y \cap X_n \cap k| \\ &< kN_0(\epsilon/(2N_0)) + k\epsilon/2 = k\epsilon \end{aligned}$$

Therefore $|Y \cap k|/k < \epsilon$ as desired. Since ϵ was arbitrary, $Y \in \mathcal{Z}_0$, and we have satisfied condition (iii) above. Conditions (i) and (ii) are trivial, so we have an ω-partition of ω with respect to $\mathcal{Z}_0$.

For the general case, suppose X is a $\mathcal{Z}_0$-positive set, therefore in particular infinite. Let $f : X \twoheadrightarrow \omega$ be the collapse function (i.e.

$$f(i) = n \iff i \text{ is the } n^{\text{th}} \text{ element of } X$$

where of course we start counting with 0.)

Now we take the partition to be

$$X_n \triangleq \{i \in X | 2^n \text{ divides } f(i) \text{ but } 2^{n+1} \text{ does not}\}$$

and again throw the smallest (zeroth) element of X into X_0 for completeness.

Condition (i) is no longer *quite* trivial; we observe that, as X is $\mathcal{Z}_0$-positive, there are $\epsilon > 0$ and infinitely many k such that $|X \cap k| > k\epsilon$. We further note that for each n, $|X_n \cap k| \geq \lfloor\ |X \cap k|/2^{n+1}\ \rfloor$ (where $\lfloor \cdot \rfloor$ is the greatest integer function). From this it easily follows that there are $\epsilon' > 0$ and infinitely many k such that $|X_n \cap k|/k > \epsilon'$, so X_n is $\mathcal{Z}_0$-positive. Condition (ii) is obvious; the proof of condition (iii) (which now reads $\forall Y \subseteq X\ [(\forall n\ Y \cap X_n \in \mathcal{Z}_0) \Rightarrow Y \in \mathcal{Z}_0]$) follows closely the argument above for the special case $X = \omega$.

1.4.1.3 There are $\mathcal{J}$ of arbitrarily high Borel rank

We apply an elementary argument found in [Kec95], Exercise 23.4 on page 180, with the hint on page 362, and amplified by a personal communication from Kechris. Here it is shown that for any set of reals A there is an ideal $\mathcal{I}_A$ such that A is Wadge–reducible to $\mathcal{I}_A$. We reproduce the argument: For $x \in 2^\omega$, let $C_x \subseteq 2^{<\omega}$ be $\{\ x \upharpoonright n \mid n \in \omega\}$, and given $A \subseteq 2^\omega$, let $\mathcal{I}_A$ be the ideal on $2^{<\omega}$ generated by all sets C_x for $x \in A$, and all finite subsets of $2^{<\omega}$. That is, for $B \subseteq 2^{<\omega}$, $B \in \mathcal{I}_A \iff (\exists \{x_0, x_1, \ldots, x_{n-1}\} \subseteq A)(B \setminus \cup_{i<n} C_{x_i}$ is finite).

If $x \neq y$, then $C_x \cap C_y$ is finite. The map $x \mapsto C_x$ is a continuous function from 2^ω to $\mathcal{P}(2^{<\omega})$ reducing A to $\mathcal{I}_A$.

A finer analysis shows that if $\xi \geq 3$ is a countable ordinal and A is $\mathbf{\Sigma}^0_\xi$ then $\mathcal{I}_A$ is also $\mathbf{\Sigma}^0_\xi$; therefore, if A is $\mathbf{\Sigma}^0_\xi$-complete, then so is $\mathcal{I}_A$. This analysis proceeds by defining a $\mathbf{\Pi}^0_2$ subset P of $2^{<\omega} \times \mathcal{P}(2^{<\omega})$ such that $(s, \Theta) \in P$ if and only if s determines a unique branch through Θ, which will be designated $f(s, \Theta)$. I.e.

$$\begin{aligned} P &\triangleq \{(s, \Theta) | \exists^\infty t \in \Theta (t \supseteq s) \\ &\quad \wedge \forall t_1, t_2 \in \Theta[(t_1 \supseteq s \wedge t_2 \supseteq s) \Rightarrow (t_1 \subseteq t_2 \vee t_2 \subseteq t_1)]\} \end{aligned}$$

$$\begin{aligned} f(s,\Theta) &\triangleq \text{the unique } x \in 2^\omega \text{ such that} \\ &\qquad \exists^\infty t \in \Theta(t \supseteq s \wedge t \text{ is an initial segment of } x) \end{aligned}$$

where the latter definition is applied only when $(s,\Theta) \in P$.

Now f is continuous on P (that is, for any open $\mathcal{U} \subseteq 2^\omega$, $\{(s,\Theta) \in P | f(s,\Theta) \in \mathcal{U}\}$ is the intersection of P with some open subset of $2^{<\omega} \times \mathcal{P}(2^{<\omega})$). By induction, if $X \subseteq 2^\omega$ is $\mathbf{\Sigma}^0_\xi$ for $\xi \geq 3$, then the pullback of X by f, namely $\{(s,\Theta) \in P | f(s,\Theta) \in X\}$, is also $\mathbf{\Sigma}^0_\xi$. Now a set $B \subseteq 2^{<\omega}$ is in $\mathcal{I}_A$ just in case $B \setminus \cup_{i<n} C_{x_i}$ is finite for some $x_0, \ldots, x_{n-1} \in A$. For such x_i's, let k be greater than the lengths of each of the finitely many elements of $B \setminus \cup_{i<n} C_{x_i}$. Then, for every $s \in 2^{<\omega}$ with length$(s) \geq k$, either $s \in C_{x_i}$ for some $i < n$, in which case $(s,B) \in P$ and $f(s,B) = x_i$, or $s \notin C_x$, in which case $t \notin B$ for every $t \supseteq s$. Thus

$$\begin{aligned} B \in \mathcal{I}_A \quad \Rightarrow \quad & (\exists k)(\forall s)\Big[\text{length}(s) \geq k \Rightarrow \\ & ((\forall t \supseteq s)(t \notin B) \vee [(s,B) \in P \wedge f(s,B) \in A]\Big] \end{aligned}$$

However the converse holds as well, because given k as in the RHS above, there are only finitely many s of length k. The $x_0, \ldots, x_{n-1}$ are the values of $f(s,B)$ for those s of length k satisfying the second disjunct above. Thus the RHS *characterizes* membership in $\mathcal{I}_A$, so $\mathcal{I}_A$ is $\mathbf{\Sigma}^0_\xi$ given that A is.

This finishes the part of the argument taken from Kechris' book.

Now for a given countable ordinal β, let $\mathrm{WO}_{<\beta}$ be the set of all elements of 2^ω that code an ordinal less than β. By [Ste78], $\mathrm{WO}_{<\omega^\alpha}$ is $\mathbf{\Sigma}^0_{2\cdot\alpha}$-complete. For each α let $A_\alpha \triangleq \mathrm{WO}_{<\omega^{\alpha\cdot\omega}}$ and let $\mathcal{I}_\alpha \triangleq \mathcal{I}_{A_\alpha}$ in the sense of the Kechris argument above. Then $\mathcal{I}_\alpha$ is $\mathbf{\Sigma}^0_{\alpha\cdot\omega}$-complete.

(See also Zafrany ([Zaf89]), who proves the complexity claims about $\mathrm{WO}_{<\omega^\alpha}$ by first defining ideals of unbounded Borel complexity; such ideals are exactly what we need here. However Zafrany does not define these ideals in a uniform way on ω, but rather on different countable sets as α increases; it is not clear to me whether any slight modification of his work would provide the ideals we want directly, without going through the Kechris argument.)

Now let $\mathcal{J}_\alpha \triangleq \mathcal{Z}_0 \times \mathcal{I}_\alpha$. Then $\mathcal{J}_\alpha$ is a shallow ideal and is $\mathbf{\Sigma}^0_{\alpha\cdot\omega+n}$ for some n, but is also $\mathbf{\Sigma}^0_{\alpha\cdot\omega}$-hard. Therefore, for any countable α, β with $\alpha < \beta$ we have that $\mathcal{J}_\alpha <_W \mathcal{J}_\beta$.

It remains to check that there is a Borel function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, if X is a code for a countable ordinal α, then $f(X)$ is a Borel code for $\mathcal{J}_\alpha$. For this we appeal to the Suslin–Kleene theorem; see [Mos80, 7B], and observe that we can find recursive functions that, given a code for a countable α, return $\mathbf{\Sigma}^1_1$-codes for $\mathcal{J}_\alpha$ and its complement.

1.4.2 Mutually RK-irreducible analytic P-ideals

In this section we define continuum-many analytic P-ideals with the property that every positive set has an ω-partition, such that if $\mathcal{J}_1$, $\mathcal{J}_2$ are distinct ideals from the collection and $X \subseteq \omega$ is $\mathcal{J}_1$-positive, there is no injection $h : X \to \omega$ such that

$$(\forall Y \subseteq X)\; Y \in \mathcal{J}_1 \iff h\text{"}Y \in \mathcal{J}_2$$

Let (a_i) be a sequence increasing fast enough that the ratio $a_{i+1}/(\sum_{k=0}^{i} a_k)$ goes to infinity as i goes to infinity. Let $n_i = \sum_{k=0}^{i-1} a_i$, and let $I_i = [n_i, n_{i+1})$, so that $|I_i| = a_i$.

For any $A \subseteq \omega$, let $\mathcal{I}_A$ be the ideal of sets whose density on I_i goes to zero as i goes to infinity for $i \in A$.

Claim 1.4.2.1. *If A and B are almost disjoint, then there is no $\mathcal{I}_A$-positive set*

X such that $\mathcal{P}(X)/\mathcal{I}_A$ is Rudin–Keisler isomorphic to any piece of $\mathcal{P}(\omega)/\mathcal{I}_B$.

Proof. Suppose to the contrary that X is $\mathcal{I}_A$-positive and $h : X \to \omega$ is an injection such that for all $Y \subseteq X$,

$$Y \in \mathcal{I}_A \iff h\text{"}Y \in \mathcal{I}_B$$

Now since X is positive, there is $\epsilon > 0$ such that for infinitely many $i \in A$, $\mu_i(X) > \epsilon$, where $\mu_i(X)$ is the density of X on I_i.

Choose M_0 such that for $i > M_0$, $a_i/(\sum_{k=0}^{i-1} a_k) > 3/\epsilon$. Define:

$$\begin{aligned} I_i^- &= \{n \in I_i | h(n) \in I_j, j < i\} \\ I_i^+ &= \{n \in I_i | h(n) \in I_j, j > i\} \\ I_i^= &= \{n \in I_i | h(n) \in I_i\} \end{aligned}$$

$$\begin{aligned} X^- &= \bigcup_i (X \cap I_i^-) \\ X^+ &= \bigcup_i (X \cap I_i^+) \\ X^= &= \bigcup_i (X \cap I_i^=) \end{aligned}$$

Now for $i > M_0$ we have

$$|I_i^-| \leq \sum_{k=0}^{i-1} a_k \leq (\epsilon/3)a_i$$

so $\mu_i(X^-) \leq \mu_i(I_i^-) \leq \epsilon/3$.

Now there must be $M_1 > M_0$ such that for all $i > M_0$, $i \in A$, we have $\mu_i(X^=) \leq \epsilon/3$, because if for infinitely many $i \in A$ this inequality failed, we could take Y to be $X^=$ on those infinitely many intervals, and then $\mu_i(Y)$ would not approach 0 as $i \to \infty$ in A, but $\mu_i(h"Y)$ *would* equal zero for all but finitely many $i \in B$. (A and B are almost disjoint, and h sends elements of each $Y \cap I_i$ into the same I_i.)

So now for infinitely many $i > M_1$ we have that $\mu_i(X) > \epsilon$, but $\mu_i(X^-) \leq \epsilon/3$ and $\mu_i(X^=) \leq \epsilon/3$. So for such i we have $\mu_i(X^+) > \epsilon/3$.

Now let Y be the union of $X^+ \cap I_i$ for the infinitely many i mentioned in the previous paragraph. Then Y is *not* in $\mathcal{I}_A$, because $\mu_i(Y)$ is infinitely often greater than $\epsilon/3$. However $h"Y$ *is* in $\mathcal{I}_B$, because everything in Y gets sent by h to a higher interval, and the condition $a_{i+1}/(\sum_{k=0}^{i} a_k) \to \infty$ now guarantees that $\mu_i(h"Y) \to 0$ as $i \to \infty$ (in fact we need not even restrict to $i \in B$). $\square$

Each $\mathcal{I}_A$ (for A infinite) is shallow because it is the restriction of $\mathcal{Z}_0$ to a positive set, and is an analytic P-ideal because it is the exhaustion of a lower semicontinuous submeasure ϕ given by

$$\phi(X) \triangleq \sup_{i \in A} \frac{|X \cap I_i|}{a_i}$$

(See [Sol99] for the result that the analytic P-ideals are precisely the exhaustions of lower semicontinuous submeasures.)

CHAPTER 2

Borel Cardinalities Below c_0

In this chapter we present two results that were useful to Farah in his study of "c_0-equalities"—see [Far01].

2.1 The Ideal of Density is equireducible with c_0

2.1.1 Origin of the Question

When investigating whether Borel reductions (see Definition 2.2.1.1 below on page 41) exist between given equivalence relations, it is sometimes convenient to replace one equivalence relation with a combinatorially simpler one which is known to be reducible in both directions with the given equivalence relation. For example, consider the equivalence relation on $\mathbb{R}^\omega$ induced by the action of ℓ_1 by pointwise addition (let us refer to this equivalence relation simply as ℓ_1). Hjorth [Hjo00] has shown that if $E \leq_B \ell_1$, then either $\ell_1 \leq_B E$, or E is reducible to an equivalence relation all of whose equivalence classes are countable. In his exposition, he replaces ℓ_1 with the equivalence relation given by the *summable ideal* $\mathcal{I}_{1/n}$ on the power set of ω: If $A \subseteq \omega$, then $A \in \mathcal{I}_{1/n}$ just in case $\sum_{n \in A} 1/(n+1) < \infty$. Since $\ell_1 \leq_B \mathcal{I}_{1/n}$ and $\mathcal{I}_{1/n} \leq_B \ell_1$, this substitution is legitimate. (Here we refer to as $\mathcal{I}_{1/n}$ the equivalence relation $\sim$ on $P(\omega)$ given by $A \sim B \leftrightarrow A \triangle B \in \mathcal{I}_{1/n}$.)

Consider instead the equivalence relation on $\mathbb{R}^\omega$ generated by the action by pointwise addition of c_0 (the set of all sequences of reals that approach zero). Once again, we will refer to this equivalence relation simply as c_0. Kechris had suggested that c_0 might similarly be equivalent to the *ideal of density* $\mathcal{Z}_0$:

Definition 2.1.1.1. *For* $A \subseteq \omega$,

$$A \in \mathcal{Z}_0 \iff \frac{|A \cap n|}{n} \to 0 \text{ as } n \to \omega$$

(Once again, we use $\mathcal{Z}_0$ to refer both to this ideal and to the equivalence relation generated thereby.)

In this section we shall demonstrate that, as Kechris had conjectured, c_0 and $\mathcal{Z}_0$ are mutually Borel reducible.

2.1.2 Easy direction

We want to see that c_0 reduces $\mathcal{Z}_0$ (i.e. $\mathcal{Z}_0 \leq_B c_0$); that is, there is a Borel function θ: $\mathcal{P}(\omega) \to \mathbb{R}^\omega$ such that for any $X, Y \in \mathcal{P}(\omega)$,

$$X \triangle Y \in \mathcal{Z}_0$$

if and only if

$$\theta(X)_n - \theta(Y)_n \to 0 \qquad \text{as} \qquad n \to \omega$$

2.1.2.1 Obvious approach

We might try to let the n^{th} real of the sequence $\theta(X)$ be the density of X up to n, i.e.

$$\theta(X)_n = |X \cap n|/n$$

which has the virtue of trivially making $\theta(X)$ approach 0 if and only if $X \in \mathcal{Z}_0$.

The problem with this option is that the empty set is not the only possibility for Y. E.g. if X is the set of odd natural numbers and Y the set of even naturals, then both $\theta(X)$ and $\theta(Y)$ will approach 1/2, but $X \triangle Y$ is certainly not in $\mathscr{Z}_0$; in fact, $X \triangle Y = \omega$.

2.1.2.2 How do we fix it?

We have to make $\theta(X)$ encode not only the densities of X itself, but of X compared with all possible comparison sets Y. Since there are $2^{\aleph_0}$ of these Y, this would seem a daunting task, except that we only have to compare finite pieces of them at any one time.

That is to say: Enumerate all pairs $\langle S, k\rangle$ where k is a natural number and $S \subseteq k$, and write $\langle S_n, k_n\rangle$ for the n^{th} such pair. Note that $k_n \to \omega$ as $n \to \omega$.

Now given $X \in \mathcal{P}(\omega)$, we let

$$\theta(X)_n = \frac{|(X \cap k_n) \triangle S_n|}{k_n}$$

Now if $X \triangle Y \in \mathscr{Z}_0$, then for each n we have

$$\begin{aligned} k_n \cdot |\theta(X)_n - \theta(Y)_n| &= \big|\, |(X \cap k_n) \triangle S_n| - |(Y \cap k_n) \triangle S_n| \,\big| \\ &\leq |((X \cap k_n) \triangle S_n) \triangle ((Y \cap k_n) \triangle S_n)| \\ &= |(X \triangle Y) \cap k_n| \end{aligned}$$

So $\lim_{n\to\omega} \theta(X)_n - \theta(Y)_n = 0$, so $\theta(X) - \theta(Y) \in c_0$.

On the other hand, suppose $X \triangle Y \notin \mathscr{Z}_0$, and let us consider the sequence n_0, n_1, $\ldots$ determined by Y in the sense that, for each ℓ, $S_{n_\ell} = Y \cap k_{n_\ell}$. Then for every ℓ we have $\theta(Y)_{n_\ell} = 0$, and $\theta(X)_{n_\ell} = \frac{|(X \triangle Y) \cap k_{n_\ell}|}{k_{n_\ell}}$. But as ℓ ranges over ω, k_ℓ takes on every value in ω; since $X \triangle Y \notin \mathscr{Z}_0$, we know that $\frac{|(X \triangle Y) \cap k_{n_\ell}|}{k_{n_\ell}}$ does not go to zero. Therefore $\theta(X) - \theta(Y) \notin c_0$.

We have established

Claim 2.1.2.1. $\mathcal{Z}_0 \leq_B c_0$.

Proof. See above discussion. $\square$

2.1.3 Harder direction

Claim 2.1.3.1. $c_0 \leq_B \mathcal{Z}_0$.

i.e. there is a Borel function $\theta\colon \mathbb{R}^\omega \to \mathcal{P}(\omega)$ *such that for any* $\vec{x}, \vec{y} \in \mathbb{R}^\omega$,

$$x_n - y_n \to 0 \qquad as \qquad n \to \omega$$

if and only if

$$\theta(\vec{x}) \triangle \theta(\vec{y}) \in \mathcal{Z}_0$$

Proof. It is enough to get such a function from $[0,1]^\omega \to \mathcal{P}(\omega)$. For, suppose we have a Borel function $\theta : [0,1]^\omega \to \mathcal{P}(\omega)$ satisfying the above criterion; we can recover a reduction $\theta' : \mathbb{R}^\omega \to \mathcal{P}(\omega)$ as follows: Given $f \in \mathbb{R}^\omega$ and a bijective pairing function $\langle , \rangle : \omega \times \omega \to \omega$, define $\hat{f} : [0,1]^\omega \to \mathcal{P}(\omega)$ by

$$\hat{f}(\langle n, k\rangle) \triangleq \begin{cases} 0 & \text{if } f(k) < n \\ 1 & \text{if } f(k) > n+1 \\ f(k) - n & \text{otherwise} \end{cases}$$

Now take $\theta'(f) \triangleq \theta(\hat{f})$.

Definition 2.1.3.1. *a rational is* i-dyadic *for a natural number* i *if it equals* $j/2^i$*, for some integer* j*.*

Definition 2.1.3.2. *For a real number* x*, we write*

$$\|x\|_i \triangleq \lfloor x \cdot 2^i \rfloor / 2^i$$

where $\lfloor \cdot \rfloor$ *is the greatest-integer function. That is,* $\|x\|_i$ *is the greatest* i*-dyadic* $\leq x$*.*

For $W \in \mathcal{P}(\omega)$, we let $\mu_n(W) \triangleq |W \cap n|/n$, the density of W up to n. We want to fix in advance a sequence $k_0, k_1, ...$ such that for all i,

i) k_i is a multiple of 2^i

ii) we always have $k_i \ll k_{i+1}$ in the sense that if $W \in \mathcal{P}(\omega)$ and

$$|W \cap [k_i, k_{i+1})| = j \cdot (k_{i+1} - k_i)/2^i$$

then $\|\mu_{k_{i+1}}(W)\|_i = j/2^i$. That is, if you fill in the space from k_i to k_{i+1} with density $j/2^i$, then the part up to k_i is negligible up to finding the nearest i-dyadic. More simply, what we need is $k_{i+1} > 2 \cdot 2^i \cdot k_i$.

iii) We also have $2^i \ll k_i$. Here what we want is that, beyond k_i, anything you do to 2^i or fewer coordinates cannot change the density up to that point by more than 2^{-i}. For this it is enough that $k_i > 2^{2i}$.

Clearly $k_i = 2^{2^{i+47}}$ works.

Now we are ready to define θ, as follows: Given $\vec{x} \in [0,1]^\omega$ and a particular i, let $j = \|x_i\|_i \cdot 2^i$, and then write

$$B_{\vec{x},i} \triangleq \left\{ k \in [k_i, k_{i+1}) \;\middle|\; k \bmod 2^i \geq 2^i - j \right\}$$

(note that the j on the right-hand side depends on $\vec{x}$ and i). Then we take

$$\theta(\vec{x}) \triangleq \bigcup_{i \in \omega} B_{\vec{x},i}$$

That is, we break the interval $[k_i, k_{i+1})$ into blocks of length 2^i, from each of which we accept the final j numbers. Thus the density in the interval $[k_i, k_{i+1})$ is the nearest i-dyadic to x_i, and in fact by (ii) the density up to k_{i+1} is near x_i.

If $x_n - y_n$ does *not* approach zero, then there will be some ℓ for which, for infinitely many $n > \ell$, x_n and y_n differ by more than $2/2^\ell$. For such n, supposing $x_n > y_n$, note:

$$\begin{aligned}
\left|\theta(\vec{x}) \bigtriangleup \theta(\vec{y}) \cap [k_n, k_{n+1})\right| &= |B_{\vec{x},n} \bigtriangleup B_{\vec{y},n}| \\
&= |B_{\vec{x},n}| - |B_{\vec{y},n}| \\
&= (\|x_n\|_n - \|y_n\|_n)(k_{n+1} - k_n) \\
&\geq (x_n - y_n - 1/2^n)(k_{n+1} - k_n) \\
&\geq (1/2^\ell) \cdot k_{n+1} \cdot (1 - 2^{-(n+1)}) \\
&\geq (1/2^{\ell+1}) \cdot k_{n+1}
\end{aligned}$$

Thus for infinitely many n, $\mu_{k_{n+1}}(\theta(\vec{x}) \triangle \theta(\vec{y})) \geq 1/2^{\ell+1}$ where ℓ is fixed. Therefore $\theta(\vec{x}) \bigtriangleup \theta(\vec{y})$ is *not* in $\mathscr{Z}_0$.

On the other hand, suppose $x_n - y_n$ does approach zero, and let

$$\rho_n \triangleq \mu_n(\theta(\vec{x}) \bigtriangleup \theta(\vec{y}))$$

We need to see that $\rho_n \to 0$ as $n \to \omega$.

First note that for any i, we will have

$$\|\rho_{k_{i+1}}\|_i = \Big| \ \|x_i\|_i - \|y_i\|_i \ \Big|$$

by construction and by property (ii) of the k_i's. This value clearly goes to zero as i goes to infinity, so all we need to do is make sure that the value of ρ_n does not get too large for values of n between the k_i's.

If n between k_i and k_{i+1} is a multiple of 2^i, writing $\rho' \triangleq | \ \|x_i\|_i - \|y_i\|_i \ |$, then ρ_n is a weighted average of ρ_{k_i} and ρ', because all the blocks of length 2^i starting at k_i have density ρ'. Thus, for such n, $\rho_n \leq \max(\rho_{k_i}, \rho')$. But now by property (iii) of the k_i's, it is true for *any* n between k_i and k_{i+1} that

$$\rho_n \leq \max(\rho_{k_i}, \rho') + 2^{-i}$$

But all terms above go to zero, so ρ_n goes to zero. Thus the reduction is established. $\square$

2.2 Borel Upper Bounds for the Louveau–Veličkovič and Mazur Towers

2.2.1 Introduction and nomenclature

2.2.1.1 Basic definitions

Definition 2.2.1.1. *As usual, for X and Y Polish spaces, E and F Borel equivalence relations on X and Y respectively, we write $E \leq_B F$, and say E* is Borel reducible to *F, just in case there exists a Borel function $\theta : X \to Y$ such that for all $x_0, x_1 \in X$, $x_0 E x_1$ if and only if $\theta(x_0) F \theta(x_1)$. In this case we say that F* reduces *E.*

2.2.1.2 Background

In [LV94], it is shown that the $\leq_B$ ordering is very rich, that in particular the partial order of almost–inclusion on sets of naturals may be embedded into $\leq_B$ restricted to the $\mathbf{\Pi}^0_3$ equivalence relations. In [Maz00] this result is improved to $\mathbf{\Sigma}^0_2$; however, Mazur's equivalence relations, unlike Louveau's and Veličkovič's, may not be regarded as induced by the action of a Polish group on a Polish space.

For each subset X of ω, the authors (in effect) define an equivalence relation E_X such that $E_X \leq_B E_Y$ if and only if X is almost included in Y (i.e. $X \setminus Y$ is finite).

In this Section 2.2 we show that all these relations E_X are Borel reducible to

c_0. This refutes a conjecture of Hjorth that all equivalence relations reducible to c_0 were either Borel equivalent to c_0 or Borel–reducible to the equivalence relation of equality on countable sets of reals.

Similarly, we show that all the equivalence relations defined by Mazur are reducible to ℓ^∞.

2.2.1.3 The equivalence relations in question

Louveau and Veličkovič fix two sequences $\{a_n | n \in \omega\}$ and $\{b_n | n \in \omega\}$, where the exact values are not important except that the a_n grow very fast and the b_n grow much faster than that. They then divide the natural numbers up into intervals $I_n = [m_n, m_{n+1})$ where $m_n = \sum_{k<n} b_k$. Then for $A \subseteq \omega$, $X, Y \in \mathcal{P}(\omega)$, they define $X E_A Y$ if and only if $\frac{\log(|(X \triangle Y) \cap I_n|+1)}{a_n} \to 0$ as $n \to \omega, n \in A$.

It should be clear that given any $A \subseteq \omega$, we can obtain the equivalence relation E_A by altering the sequence of a_n's and b_n's and then looking at E_ω. Therefore we shall suppress the dependence on A and simply define, for $X, Y \in \mathcal{P}(\omega)$,

Definition 2.2.1.2.

$$X \sim_{LV} Y \iff \frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n} \to 0 \text{ as } n \to \omega$$

Mazur, on the other hand, defines equivalence relations of exactly the same form except that instead of asking whether the sequence $\frac{\log(|(X \triangle Y) \cap I_n|+1)}{a_n}$ goes to zero, one asks whether it is bounded. Thus we define:

Definition 2.2.1.3.

$$X \sim_M Y \iff \exists M \forall n \left(\frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n} < M \right)$$

Now we can state that the purpose of this section is to show that

$$\sim_{LV} \leq_B c_0$$

and

$$\sim_M \leq_B \ell^\infty$$

2.2.2 Reduction for the Louveau–Veličkovič Tower

2.2.2.1 The idea

We have that two sets of natural numbers are Louveau–Veličkovič–equivalent just in case a certain sequence of reals approaches zero, and we wish to reduce it to the equivalence relation whereby two sequences of reals are equivalent just in case their difference approaches zero. Thus we might naturally wish to send a set of natural numbers to its corresponding sequence of reals, thus:

$$\theta : \mathcal{P}(\omega) \to \mathbb{R}^\omega$$
$$(\theta(X))_n = \frac{\log(|X \cap I_n| + 1)}{a_n}$$

2.2.2.2 Why doesn't it work?

As in Section 2.1 on the reduction of the ideal of density to c_0, we may consider something like $X = \{n \in \omega | n \text{ is even}\}$, $Y = \{n \in \omega | n \text{ is odd}\}$. Then $(\theta(X))_n - (\theta(Y))_n$ certainly goes to zero as n goes to ∞, but $X \bigtriangleup Y = \omega$, so $X \nsim_{LV} Y$ provided only that the sequences a_n and b_n are so chosen that $\sim_{LV}$ is not trivial.

2.2.2.3 How to fix it

The reduction proposed above compares, so to speak, the set x with the empty set. We need to compare it with all possible sets of natural numbers. However, at any one moment (i.e. as far as concerns any element of the sequence of reals produced), we need consider only finite initial segments of the natural numbers.

Therefore consider a bijective pairing function

$$\langle , \rangle : \{(n,k) : k < 2^{|I_n|}\} \to \omega$$

and let $\{S_{n,k} | n \in \omega, k \in 2^{|I_n|}\}$ enumerate all the subsets of all the I_n.

Now the corrected reduction is:

$$\begin{aligned} \theta &: \mathcal{P}(\omega) \to \mathbb{R}^\omega \\ (\theta(X))_{\langle n,k \rangle} &= \frac{\log(|(X \bigtriangleup S_{n,k}) \cap I_n| + 1)}{a_n} \end{aligned}$$

Actually, since $S_{n,k} \subseteq I_n$, the above reduces to

$$(\theta(X))_{\langle n,k \rangle} = \frac{\log(|((X \cap I_n) \bigtriangleup S_{n,k})| + 1)}{a_n}$$

2.2.3 Proof that the reduction works

We need to see

Claim 2.2.3.1. *For $X, Y \in \mathcal{P}(\omega), X \sim_{LV} Y \iff (\theta(X))_n - (\theta(Y))_n \to 0$ as $n \to \omega$*

Proof. $\Longrightarrow$: First observe that for natural numbers a,b, we have $(a+1)(b+1) = ab + a + b + 1 \geq a + b + 1$. Taking logs of both sides,

$$\log(a + b + 1) \leq \log(a + 1) + \log(b + 1)$$

Also if u,v,w are finite sets of natural numbers, then $|u \bigtriangleup v| \leq |u \bigtriangleup w| + |v \bigtriangleup w|$ (e.g. because $u \bigtriangleup v = (u \bigtriangleup w) \bigtriangleup (v \bigtriangleup w) \subseteq (u \bigtriangleup w) \cup (v \bigtriangleup w)$).

Also write

$$\epsilon_n = \frac{\log(|(X \bigtriangleup Y) \cap I_n| + 1)}{a_n}$$

Then since $X \sim_{LV} Y$, we know $\epsilon_n \to 0$.

Letting $u = X \cap I_n, v = S_{n,k}, w = Y \cap I_n$, we obtain

$$\begin{aligned}
\theta(X)_{\langle n,k\rangle} &= \frac{\log(|u \triangle v| + 1)}{a_n} \\
&\leq \frac{\log(|u \triangle w| + 1)}{a_n} + \frac{\log(|v \triangle w| + 1)}{a_n} \\
&= \epsilon_n + \theta(Y)_{\langle n,k\rangle} \\
\theta(X)_{\langle n,k\rangle} - \theta(Y)_{\langle n,k\rangle} &\leq \epsilon_n
\end{aligned}$$

Symmetrically, we get the same inequality with X and Y reversed, so

$$|\theta(X)_{\langle n,k\rangle} - \theta(Y)_{\langle n,k\rangle}| \leq \epsilon_n$$

Now $\epsilon_n \to 0$ and for each n there are only finitely many k such that $\langle n, k\rangle$ exists (i.e. such that (n, k) is in the domain of the pairing function); therefore

$$\theta(X)_n - \theta(Y)_n \to 0$$

$\Longleftarrow$: choose a sequence k_n such that for each n, $S_{n,k_n} = Y \cap I_n$. Then the subsequence $(\theta(X))_{\langle n,k_n\rangle} - (\theta(Y))_{\langle n,k_n\rangle}$ goes to zero as n goes to ω. However $(\theta(Y))_{\langle n,k_n\rangle}$ is always zero by construction, and

$$(\theta(X))_{\langle n,k_n\rangle} = \frac{\log(|(X \triangle Y) \cap I_n| + 1)}{a_n}$$

Therefore since the left-hand side goes to zero, the right–hand side does as well, so $X \sim_{LV} Y$. □

2.2.4 The Mazur Tower

2.2.4.1 The Reduction

The reduction $\theta : \mathcal{P}(\omega) \to \mathbb{R}^\omega$ to reduce $\sim_M$ to ℓ^∞ is precisely the same as the θ defined above to reduce $\sim_{LV}$ to c_0.

2.2.5 Proof that the reduction works

The proof is essentially identical to the Louveau–Veličkovič case; replacing the phrase "goes to zero" with "is bounded".

REFERENCES

[CK90] C. C. CHANG and H. JEROME KEISLER, *Model theory*, third ed., North–Holland, 1990.

[Far00a] ILIJAS FARAH, *Analytic quotients: Theory of liftings for quotients over analytic ideals on the integers*, Memoirs of the AMS, no. 702, American Mathematical Society, November 2000, Volume 148.

[Far00b] ______, *Rigidity conjectures*, ***Proceedings of Logic Colloquium 2000,*** (2000).

[Far01] ______, *Basis problem for turbulent actions II: c_0-equalities*, ***Proceedings of the London Mathematical Society. Third Series,*** vol. 82 (2001), no. 1, pp. 1–30.

[Far03] ______, *How many Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are there?*, ***Illinois Journal of Mathematics,*** (2003?).

[Hjo00] GREG HJORTH, *Actions of the classical Banach spaces*, ***Journal of Symbolic Logic,*** vol. 65 (2000), no. 1, pp. 392–420.

[JM87] WINFRIED JUST and ŽARKO MIJAJLOVIĆ, *Separation properties of ideals over ω*, ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,*** vol. 33 (1987), no. 3, pp. 267–276.

[Kec95] ALEXANDER S. KECHRIS, *Classical descriptive set theory*, Springer–Verlag, 1995.

[LV94] ALAIN LOUVEAU and BOBAN VELIČKOVIČ, *A note on Borel equivalence relations*, ***Proceedings of the American Mathematical Society,*** vol. 120 (1994), no. 1, pp. 255–259.

[Maz00] KRZYSZTOF MAZUR, *A modification of Louveau and Veličkovič's construction for F_σ-ideals*, ***Proceedings of the American Mathematical Society,*** vol. 128 (2000), no. 5, pp. 1475–1479.

[Mos80] YIANNIS MOSCHOVAKIS, *Descriptive set theory*, North–Holland, 1980.

[Sol99] SŁAWOMIR SOLECKI, *Analytic ideals and their applications*, ***Annals of Pure and Applied Logic,*** vol. 99 (1999), no. 1–3, pp. 51–72.

[Ste78] JACQUES STERN, *Évaluation du rang de Borel de certains ensembles*, ***Comptes Rendus Hebdomaires des Séances de l'Académie des Sciences,*** vol. 286 (1978), no. 20, pp. A855–857, Série A–B.

[Ste03] Juris Steprāns, *Many quotient algebras of the integers modulo co-analytic ideals*, preprint, York University, 2003.

[Zaf89] Samy Zafrany, *Borel ideals vs. Borel sets of countable relations and trees*, ***Annals of Pure and Applied Logic,*** vol. 43 (1989), no. 2, pp. 161–195.